Mathematical Image Processing

Exercise sheet 4 - due on Monday, July 8th 2024, 2pm

Exercise 1: Deconvolution **8 P.**

In this exercise two different methods should be tested for deconvolution assuming a given point spread function. The first one performs the van-Cittert algorithm while the second one performs total variation deconvolution using the primal-dual minimization method.

Recall that we modeled our observed image f as result of a continuous convolution with a point spread function $k_A\colon \Omega \to \mathbb{R}$ as:

$$f(x) \;=\; (Au)(x) \;=\; \int_\Omega k_A(x-y)u(y)\,dy \tag{1}$$

1. (2P.) van-Cittert iterations
   A simple method to perform deconvolution are van-Cittert iterations which are based on the idea of a Taylor approximation of the Fourier transform of the assumed point spread function k_A. One can iteratively increase a sharpening effect on the image f by using the Horner scheme:

$$\begin{aligned} u_0 &= f \\ u_1 &= f + q * u_0 \\ &\dots \\ u_{n+1} &= f + q * u_n \;= (\mathcal{I} + q + \cdots + q^{n+1}) * f \end{aligned} \tag{2}$$

   for which q is given by the following relationship: $\hat{q} = 1 + \hat{k_A}$ and $\mathcal{I}$ is a δ-pulse, i.e., $\mathcal{I}(0) = 1$ and $\mathcal{I}(x) = 0$ for all $x \neq 0$.

   Write a function `function u = deconv_van_Cittert(f,k_A,iterations)` that performs a specified amount of van-Cittert iterations following the scheme in (2) based on a given point spread function k_A.

   Note:

   - There is no explicit stopping criterion for the van-Cittert iterations. You need to visually choose the optimal solution (hence the iterative procedure).
   - Use the provided example script `deconvolution.m` and the provided blurring operators given by the function `[Op, Op_adj, fftFilter] = createGaussianBlurringOperator(size_I,hsize,sigma)` for the implementation.

2. (6P.) TV-regularized deconvolution:
   A mathematically more elegant way is to consider the model (1) for a given point spread function k_A and to formulate a variational optimization problem of the form:

$$u^* = \operatorname*{argmin}_{u \in BV} \frac{1}{2}\|Au - f\|^2 + \lambda\,\|\nabla u\|_1 = \operatorname*{argmin}_{u \in BV} \frac{1}{2}\|k_A * u - f\|^2 + \lambda\,\|\nabla u\|_1. \tag{3}$$

This consists in minimizing a data-fidelity term that takes into account the convolution operator k_A and the total variation of the image that ensures regularity of the solution. Compared to the ROF model, the data fidelity term has changed so one has to adapt the primal-dual formulation of the problem in [?] accordingly.

You can reformulate the minimization problem (3) into the following primal-dual form:

$$\begin{aligned} & \min_u \quad \frac{1}{2\lambda}\|Au - f\|_2^2 + \|\nabla u\|_1 \\ \Leftrightarrow \quad & \min_u \quad \max_{p,q} \langle q, Au - f\rangle - \langle u, \mathrm{div} p\rangle + \chi_{\mathcal{K}}(p) - \frac{\lambda}{2}\|q\|_2^2. \end{aligned} \tag{4}$$

In that case, we reformulate problem (3) under the form

$$\begin{aligned} &\min_u \quad F(Ku) + G(u) \\ &\text{with} \quad F(Ku) = \frac{1}{2\lambda}\|Au - f\|_2^2 + \|\nabla u\|_1, \text{ and } G(u) = 0. \end{aligned}$$

This means that $K = (\nabla; A)$, so $Ku = (\nabla u; Au)$ and $F(v) = F(v_1, v_2) = F_1(v_1) + F_2(v_2)$, with $F_1(v_1) = \|v_1\|_1$ and $F_2(v_2) = \frac{1}{2\lambda}\|v_2 - f\|_2^2$.

Hence, $F^*(p, q) = F_1^*(p) + F_2^*(q)$. We already know $(I + \gamma\partial F_1^*)^{-1}$ from the minimization of the ROF model:

$$p = (I + \gamma\partial F_1^*)^{-1}(\widetilde{p}) \ \Leftrightarrow\ p_i = \frac{\widetilde{p}_i}{\max(1, |\widetilde{p}_i|)}. \tag{5}$$

We can explicitely compute the dual proximal operator associated to the l^2-norm:

$$q = (I + \gamma\partial F_2^*)^{-1}(\widetilde{q}) \ \Leftrightarrow\ q = \frac{\widetilde{q} - \gamma f}{1 + \gamma\lambda}. \tag{6}$$

On the other hand, $G(u) = 0$ so $(I + \tau\partial G)^{-1} = Id$.

Write a function `function u = deconv_TV_pd(f,A,lambda)` that performs the TV deconvolution of the given image f assuming a given point spread function A based on the primal-dual minimization method and return the deconvolved solution of problem (3).

Note:

- Use the provided example script `deconvolution.m` and the provided blurring operators given by the function `[Op, Op_adj, fftFilter] = createGaussianBlurringOperator(size_I,hsize,sigma)` for the implementation.
- For the primal-dual algorithm we use $\theta = 1$. The step parameters γ and τ have to fulfill the inequality $\gamma\tau L^2 < 1$, where L is the norm of the operator K. In our case K is the gradient operator, and $L^2 = \|K\| = 8$.