BOUNDED FUNCTIONS WITH LARGE CIRCULAR VARIATION

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In 1951, S. N. Mergeljan [1] proved that *there exists a bounded holomorphic function f for which*

$$\iint_{|z|<1} |f'(z)|\, dS = \infty. \tag{1}$$

An obvious geometric interpretation of (1) is that the length $l(r)$ of the image of the circle $|z| = r$ grows so rapidly, as $r \to 1$, that $l(r)$ is not an integrable function of r.

An alternate geometric interpretation of (1) is that the length $V(f, \theta)$ of the image of the radius of $e^{i\theta}$ is not an integrable function of θ. W. Rudin [2, Theorem III] has proved a proposition stronger than Mergeljan's, namely, that there exist Blaschke products $B(z)$ such that $V(B, \theta) = \infty$ for almost all θ. It follows that there exists a function f, holomorphic in the unit disk D and continuous in the closure of D, such that $V(f, \theta) = \infty$ for almost all θ [2, Theorem IV].

Both Mergeljan's and Rudin's arguments involve nonconstructive steps, and therefore they do not allow us to visualize the functions f in terms of any of the customary representations. In this note, I give two explicit constructions that prove Mergeljan's result. Unfortunately, my examples are inadequate for Rudin's theorem.

We begin with the function $(a^n - z^n)/(1 - a^n z^n)$, where $2^{-1/n} < a < 1$. We write $a^n = \alpha$ and $z^n = \zeta$, and we observe that for $0 < \rho < \alpha$, the maximum and minimum values of $|(\alpha - \zeta)/(1 - \alpha\zeta)|$ on the circle $|\zeta| = \rho$ are

$$(\alpha + \rho)/(1 + \alpha\rho) \quad \text{and} \quad (\alpha - \rho)/(1 - \alpha\rho),$$

respectively. The difference between the two moduli is $2\rho(1 - \alpha^2)/(1 - \alpha^2\rho^2)$. Therefore the function $(a^n - z^n)/(1 - a^n z^n)$, whose $2n$ points of maximum and minimum modulus on the circle $|z| = r$ separate each other, maps that circle onto a curve of length greater than

$$2n \cdot 2r^n(1 - a^{2n})/(1 - a^{2n} r^{2n}) \qquad (0 < r < a).$$

The integral of this quantity, taken over the interval $3^{-1/n} < r < a$,

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