we make the substitution $s=(1-r^n)/(1+r^n)$, and of the resulting integral

$$4\int_0^1 (e^{-as}-e^{-a/s})(1-s)^{-1+1/n}(1+s)^{-1-1/n}ds$$

we discard everything except the portion over $(0, a^{1/2})$. We may then replace the algebraic factors by a constant, and the quantity to be determined is greater than

$$K_3\int_0^{a^{1/2}} (e^{-a}-e^{-a/s})ds.$$

Consider separately each of the intervals $[(j-1)a, ja]$ $(j=1, 2, \cdots, [a^{-1/2}])$. Since the minimum of the integrand in the jth interval is

$$e^{-a}-e^{-1/j} > -a+j^{-1}-j^{-2},$$

the value of the integral is greater than

$$a\sum_{j=1}^{[a^{-1/2}]} [-a+j^{-1}-j^{-2}] > K_4 a\,|\log a|\,.$$

Now, for $k=2, 3, \cdots$, let $a_k=k^{-1}(\log k)^{-3/2}$, and let

$$f(z) = \exp\left(-\sum a_k\frac{1+z^{n_k}}{1-z^{n_k}}\right).$$

If $n_k\to\infty$ fast enough, then f again has the desired properties.

References

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