Energy-Minimal Scheduling of Divisible Loads

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ABSTRACT

It is known that energy-minimal scheduling is strictly harder than minimal-makespan scheduling, but it is not well understood. This paper is a study of the problems of scheduling to minimize the energy consumed by a system of dissimilar machines (with no restriction on the makespan). The different system types considered are for machines with identical speeds but different power ratings, and for machines with different speeds and power ratings but no idle power consumption. The results give insights for further studies classifying various types of energy-minimal scheduling problems, and for deriving algorithms for exact or approximate solutions for such problems.

Categories and Subject Descriptors

D.4 [**Operating Systems**]: Process Management—*Scheduling*; C.2.4 [**Computer-Communication Networks**]: Distributed Systems; D.4.8 [**Operating Systems**]: Performance—*Modeling and prediction*

General Terms

Theory, Algorithms, Performance

Keywords

scheduling; energy minimization; distributed systems; divisible loads

1. INTRODUCTION

Energy is a precious resource, and it is a desideratum in the current industrial economy that every system be as energy efficient as possible.

In this scenario, though the energy efficiency of individual machines of a system is important, greater energy efficiencies can be achieved by proper scheduling of jobs over the system of machines. Clearly, scheduling jobs in such a way that the total energy consumption of the system is minimal, or at least low, is of the essence. Computing schedules given input conditions including the energy specifications of machines in a given system, such that the energy consumption is reduced, is of importance.

Our work focuses on systems whose machines are similar in their capabilities but may have different working and idle power ratings and working speeds. Based on different types of machines cooperatively running similar loads, we classify the problems of energy-minimal offline scheduling with non-identical interconnected machines and independent jobs, as follows:

1. *Identical speeds, divisible loads*:
   We have discussed this class of problem in Section 3.1.

2. *Identical speeds, non-divisible loads*:
   Our results for the previous case show that all working machines should have equal working time. But in the case of non-divisible jobs, this problem is an instance of the k-partition problem [2]. Hence we can say that energy-minimal scheduling in this case is an NP-hard problem. We give a detailed proof and approximation algorithm for the same in later work.

3. *Different speeds, divisible loads*:
   In Section 3.2 we have discussed a special case of this class of problem, where the idle power consumption of the machines is zero.

4. *Different speeds, non-divisible loads*:
   Scheduling for non-divisible loads, even on a system with identical-speed machines, is NP-hard. So this is also a hard problem, and only approximation algorithms can be given.

Since one has to take other parameters into account, energy-minimal scheduling is much more complex than the classical makespan problem. Agrawal and Rao [1] show that scheduling to minimize energy consumption is strictly harder (more general) than makespan scheduling. They present a model for scheduling considering the working power and idle power of each machine, which is a step in the direction of developing a generic theory (one not specific to a particular domain) for scheduling under energy constraints. They propose three heuristic algorithms for energy-aware scheduling.

Surprisingly, there is relatively little literature by the engineering and scientific community on scheduling for minimizing or reducing energy. The prime focus of general works on scheduling [5, 9] has been to optimize on objectives related to time—such as makespan, earliness, and avoidance of tardiness. Some literature focuses on energy reduction in specific domains, particularly high-performance computing or embedded systems, and communication networks. Such work invariably relies on specific features and technologies of those domains [12], such as DVFS. One branch of scheduling which is relatively closer is multi-objective scheduling, but works dealing with it [4, 7, 10] do not offer a general solution to energy minimal scheduling.

In the present paper we derive results that can guide the design of scheduling algorithms for systems which deal with independent jobs on interconnected machines. The results indicate conditions that, if satisfied, mean that such a schedule will be energy-minimal.

There is of course a lot of work on scheduling divisible or non-divisible loads in general, and even in the context of data centers, such approaches are seen in some recent work; e.g., Wang *et al.* [11] consider both divisible job and non-divisible jobs in the context of profit optimization for a data center, and Lin *et al.* [8] consider divisible load scheduling and given a bandwidth-aware algorithm for cloud computing to obtain better performance.

In our system model we consider divisible and independent jobs to develop a theory and this theory can be applied to any domain. We can use this theory to solve other, more complex classes of problems.

The novelty of this work is in considering an energy-specific objective with completely generic machine specifications. No other theoretical work until now has considered energy consumption of machines while idle, which is known to be significant from real life; e.g., in data centers [3, 6]. Our system model also shows the absolute limits of energy-minimal scheduling for specific types of systems. We have derived results (for divisible jobs) which serve as an upper-bound to maximum achievable efficiency—because a system with divisible jobs can always have higher efficiency as compared to system with non-divisible jobs [9].

In the system as we have considered it, machines are connected to one another and execute similar jobs, so that a job from one machine can be transferred to any other machine. There is no set limit to the number of machines in a system. Machines and jobs to be executed are independent, and hence any job can be executed in any order on any machine.

Our model and approach are extensible and hence can be developed to include systems having timing or precedence constraints or machines with different capabilities. Based on our results and algorithms, the theory of energy-minimal scheduling can be further developed.

The remainder of the paper is organized as follows. The system model, classification of energy-minimal scheduling problems, and notation used are presented in Section 2. The proposed method along with relevant results are explained in Section 3. Finally, Section 4 describes the conclusion and scope for future work.

2. PROBLEM FORMULATION

In this section we formally introduce our system model and define the energy-minimal scheduling problem. We also describe the type of systems we have worked upon in this paper. In the first subsection we describe the system model along with the notations used in this paper, and then the scheduling problem in the second subsection.

2.1 System Model

Consider machines numbered 1 to m. The working power of machine i is denoted as μ_i, and the idle power as γ_i. The sum of the idle powers of all the machines is given by Γ. The speed of machine i is denoted as v_i. The speed (throughput of work per unit time) of a machine is fixed throughout its working tenure.

When $v_i = 1$, with $1 \leq i \leq m$, then we can say that the machines are identical in their working capacities or speed, implying that they can execute and complete any job given to them in equal time. And if in that case all jobs are executed sequentially on one machine, then that time taken is W.

All machines in the system work in parallel, and the maximum working time of the system to execute a given set of jobs is T, which is the makespan of the system for that set of jobs. If machine i works only for time t_i, then the idle time of machine i is given by $T - t_i$. The amount of work done by machine i is represented by w_i.

$$w_i = t_i v_i \qquad (1)$$

The sum of the work done by all machines is equal to the total work to be done, i.e., $\sum_{i=1}^{m} w_i = W$. E represents the energy consumption of the complete system.

Since we consider the energy consumption in working as well as idle states, the energy consumed by machine i is the sum of the energy consumed in the working state and that consumed in the idle state. The energy consumed by machine i in the working state is given by $\mu_i t_i$, and the energy consumed in the idle state by $\gamma_i (T - t_i)$.

We assume the following of our system:

1. All jobs are independent. It means a job need not wait for completion of any other particular job to start its execution. This implies that

$$T = \max t_i, \forall i, 1 \leq i \leq m \qquad (2)$$

2. Jobs are divisible, so that jobs can be arbitrarily divided.

3. All machines stay on for the duration of the makespan of the whole set of jobs. (This is reasonable considering that in many systems the cycle time to stop/restart a machine is large; stopping idle machines can be covered by setting idle power to zero.)

4. In the identical-speed model, all machines are arranged in increasing order of the differences of their working and idle powers. For the different-speed but zero idle power consumption model, all machines are arranged in increasing order of the ratios of their working powers and speeds.

5. In the identical-speed model, one unit of work takes one unit of time for execution, irrespective of the machine on which it is executed.

2.2 Problem Definition

We deal with two classes of energy-minimal scheduling problems:

1. *Identical speeds, divisible loads*:
   The machines have different working and idle power consumption. All machines have equal speeds in execution of jobs. The aim for this type of system is to minimize E given by,

$$E = \sum_{i=1}^{m} [\mu_i t_i + \gamma_i (T - t_i)] \quad (3)$$

2. *Different speeds, divisible loads*:
   Machines can have different working power consumptions as well as different speeds, but power consumptions of all machines is zero. The energy equation for this type of problem can be formed by replacing t_i with $\frac{w_i}{v_i}$ (from (1)) and dropping the term associated with γ in (3). The objective in this type of problem is to minimize E given by (4).

$$E = \sum_{i=1}^{m} \mu_i \frac{w_i}{v_i} \quad (4)$$

After formulating the problems, we now proceed to solve them in the following sections.

3. ENERGY MINIMIZATION PROBLEM

Here we derive the basic results that govern energy-minimal scheduling. We have first derived results for the case where machines have identical speeds. Though any scheduling algorithm which fulfills the criteria arrived at in these results will provide optimal schedules, we give an algorithm which does so. Later on, we deal with the case where machines have different speeds but zero idle power consumptions.

3.1 Identical Speed

In this subsection we consider the energy-minimal scheduling of machines with identical speeds and address two questions in relation to expending the least energy in a system: (i) Given the set of m machines, which subset of machines should be allowed to work and which should remain idle for all times during the makespan? and (ii) What should be the distribution of work among the working machines?

As indicated previously, we assume that all machines are arranged in the increasing order of the differences between their working and idle powers. We claim that when the machines are arranged in such an order, then the loading of machines should be such that the working times of these machines are in non-increasing order.

LEMMA 3.1. *Given $\mu_k - \gamma_k \leq \mu_l - \gamma_l$, then for energy optimality of the system, $t_k \geq t_l \; \forall k, l$, where $1 \leq k \leq l \leq m$.*

PROOF. We prove this by contradiction. Given $\mu_k - \gamma_k \leq \mu_l - \gamma_l$, then let us say that for minimal energy consumption, $t_k < t_l$.

With a little rearrangement of (3) and expanding it, we get,

$$E = \sum_{i=1}^{k-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + [(\mu_k - \gamma_k)t_i + \gamma_k T]$$
$$+ \sum_{i=k+1}^{l-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + [(\mu_l - \gamma_l)t_i + \gamma_l T] \quad (5)$$
$$+ \sum_{i=l+1}^{m} [(\mu_i - \gamma_i)t_i + \gamma_i T]$$

There are two possible cases to be considered: when $t_k \geq t_l$, and when $t_k < t_l$. We derive the energy equations for both cases and then compare them. By comparing the energy equations in both, we arrive at a condition under which the energy consumed in one case is less, and find a contradiction.

Case 1: $t_k \geq t_l$. Take $t_k = t + \epsilon_1$ and $t_l = t - \epsilon_1$, where $\epsilon_1 \geq 0$. Putting these values in (5) we get,

$$E = \sum_{i=1}^{k-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + [(\mu_k - \gamma_k)(t + \epsilon_1) + \gamma_k T]$$
$$+ \sum_{i=k+1}^{l-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + \sum_{i=l+1}^{m} [(\mu_i - \gamma_i)t_i + \gamma_i T] \quad (6)$$
$$+ [(\mu_l - \gamma_l)(t - \epsilon_1) + \gamma_l T]$$

Case 2: $t_k < t_l$. Take $t_k = t - \epsilon_2$ and $t_l = t + \epsilon_2$, where $\epsilon_2 > 0$. Putting these values in (5) we get,

$$E' = \sum_{i=1}^{k-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + [(\mu_k - \gamma_k)(t - \epsilon_2)$$
$$+ \gamma_k T] + \sum_{i=k+1}^{l-1} [(\mu_i - \gamma_i)t_i + \gamma_i T] + [(\mu_l - \gamma_l)(t \quad (7)$$
$$+ \epsilon_2) + \gamma_l T] + \sum_{i=l+1}^{m} [(\mu_i - \gamma_i)t_i + \gamma_i T]$$

If we say that in Case 2 the energy consumed is less, then using (6) and (7), we get,

$$(\mu_k - \gamma_k) > (\mu_l - \gamma_l) \quad (8)$$

Clearly (8) is in contradiction to our assumption. QED. □

We have shown which machines should be given comparatively more work then others. For energy optimality it may well be suitable to give work to only some of the machines while letting others run completely idle. We now state the condition showing which machines should be used when using only a subset of all the machines is beneficial.

LEMMA 3.2. *If we give work to only some r machines, where $1 \leq r \leq m$, then, for reduced energy consumption, these are the r machines that form the set $\{1, 2, \ldots, r\}$.*

PROOF. For the r machines working, $t_i > 0$, and for the other $m - r$ machines $t_i = 0$. If $t_{r+1} > 0$ then there must be any t_i from the set $\{t_1, t_2, \ldots, t_r\}$ which is equal to 0. But by Lemma 3.1, $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_r > 0$ since $t_{r+1} > 0$. Clearly these two statements are contradictory and so it is not possible that $t_{r+1} > 0$. Hence it stands proved that for energy optimality, if we give work to only some r machines where $1 \leq r \leq m$, then these r machines are of the set $\{1, 2, \ldots, r\}$. □

Using Lemma 3.2, given the number of machines to be used, we can find which machines should be assigned jobs. We now state and prove the condition which decides how many machines should be used so that energy consumption of the system is minimal.

Let the energy consumption of a system of m machines, when jobs are distributed to r machines, be given by $E_{m,r}$. In the next theorem, we use a term s_i which represents the amount of work taken away from machine i and given to other machine(s).

THEOREM 3.3. *When $r-1$ machines are working, for machine r to be given work (i.e., $t_r \neq 0$) and result in reduced energy consumption, the following must hold.*

$$\sum_{i=1}^{r-1}[(\mu_i - \gamma_i - \mu_r + \gamma_r)s_i] + s_1\sum_{i=1}^{m}\gamma_i > 0 \qquad (9)$$

PROOF. We prove this by construction. If it is known that only r of the m machines are used and the rest remain idle all the time, the above energy equation can be re-written as follows, where E is replaced by $E_{m,r}$:

$$E_{m,r} = \sum_{i=1}^{r-1}[\mu_i t_i + \gamma_i(T - t_i)] + \mu_r t_r \qquad (10)$$
$$+ \gamma_r(T - t_r) + T\sum_{i=r+1}^{m}\gamma_i$$

Putting $T = t_1$, [by (2), Lemma 3.1 and Lemma 3.2] and $t_r = W - \sum_{i=1}^{r-1} t_i$ in (10), we get,

$$E_{m,r} = \sum_{i=1}^{r-1}[(\mu_i - \gamma_i - \mu_r + \gamma_r)t_i] + (\mu_r - \gamma_r)W + t_1\Gamma \quad (11)$$

This is a general equation for the energy of a system with m machines, with jobs given to r machines, where $1 \leq r \leq m$. If we give jobs to $r-1$ machines in a m machine system, its energy can be derived by putting $r = r - 1$ in (11). Also the t_is are changed to t_i's as in the following:

$$E_{m,r-1} = \sum_{i=1}^{r-2}[\mu_i - \gamma_i - \mu_{r-1} + \gamma_{r-1}]t_i' \qquad (12)$$
$$+ (\mu_{r-1} - \gamma_{r-1})W + t_1'\Gamma$$

If $E_{m,r-1} > E_{m,r}$, then only we should give work to r machines, else we give to $r - 1$ machines only. Using (11) and (12) in this condition and putting $s_i = t_i' - t_i$, we get,

$$\sum_{i=1}^{r-1}[(\mu_i - \gamma_i - \mu_r + \gamma_r)s_i] + s_1\Gamma > 0 \qquad (13)$$

QED. □

Having derived the condition describing which machines to use for doing jobs, all we need to know now for scheduling is the amount of work to be given to each of these machines. We now state a result governing the distribution of amount of work among machines.

THEOREM 3.4. *When we give jobs to r number of machines, then the distribution of jobs on all r machines should be equal and given by $\frac{W}{r}$.*

PROOF. We prove this by contradiction. Consider two cases, one in which the distribution of work among the machines which qualify to work according to Theorem 3.3 is equal, and the other in which the work distribution is unequal. We claim, to show the contradiction, that in the case in which the distribution is unequal, the energy consumption is greater as compared to the other, and that such is hence a non-optimal distribution.

Case 1: Equal distribution, $t_i = \frac{W}{r}, \forall i, 1 \leq i \leq r$.

Case 2: Unequal distribution, $t_1 = \frac{W}{r} + \epsilon$, $t_2 = t_3 = \ldots = t_{r-1} = \frac{W}{r}$ and $t_r = \frac{W}{r} - \epsilon$.

Here we assign more work to machine 1 compared to that required by equal distribution of work among r machines (by Lemma 3.1, a bias has to favor smaller-numbered machines, and therefore machine 1 most of all). We have chosen only to alter the work distributions of machines 1 and r to keep the proof simple. We could have chosen any machine k for doing extra work and any machine l for doing lesser work, where $k > l$. But for allotting extra work ϵ to machine k, we have to increase the amount of work assigned to all machines in the set $\{1, 2, \ldots, k\}$ by at least ϵ, so that Lemma 3.1 is satisfied.

Similarly for reducing the work of machine l by ϵ, we have to reduce the amount of work assigned to all machines in the set $\{l+1, l+2, \ldots, r\}$ by at least ϵ so that Lemma 3.1 is satisfied. Hence if we want to alter the amount of work of just two machines (to keep the proof simple) from an equal distribution, then we have alter it for machine 1 and r. It may be noted that our proof is generalizable for imbalances involving any numbers of machines.

Let the energy in Case 1 be denoted by $E_{m,r}'$ and the energy in Case 2 be $E_{m,r}''$. Then according to our proposition:

$$E_{m,r}'' - E_{m,r}' < 0 \qquad (14)$$

Putting the respective values of t_i's in (11) to derive $E_{m,r}'$,

$$E_{m,r}' = (\mu_1 - \gamma_1 - \mu_r + \gamma_r)(\frac{W}{r}) + \sum_{i=2}^{r-1}[(\mu_i - \gamma_i \qquad (15)$$
$$- \mu_r + \gamma_r)(\frac{W}{r})] + (\mu_r - \gamma_r)W + (\frac{W}{r})\Gamma$$

Similarly, we may derive $E_{m,r}''$ as:

$$E_{m,r}'' = (\mu_1 - \gamma_1 - \mu_r + \gamma_r)(\frac{W}{r} + \epsilon) + \sum_{i=2}^{r-1}[(\mu_i - \gamma_i \qquad (16)$$
$$- \mu_r + \gamma_r)(\frac{W}{r})] + (\mu_r - \gamma_r)W + (\frac{W}{r} + \epsilon)\Gamma$$

Using (14), (15) and (16) we get

$$(\mu_1 - \gamma_1 - \mu_r + \gamma_r) + \Gamma < 0 \qquad (17)$$

Re-writing (9) from Theorem 3.3, we get,

$$(\mu_1 - \gamma_1 - \mu_r + \gamma_r)s_1 + \sum_{i=2}^{r-1}[(\mu_i - \gamma_i - \mu_r + \gamma_r)s_i] + s_1\Gamma > 0 \quad (18)$$

Now $s_i \geq 0, \forall i \in \{1, 2, \ldots, r\}$ and according to our assumption 4 in Section 2.1, $(\mu_i - \gamma_i - \mu_r + \gamma_r) \leq 0, \forall i \in \{1, 2, \ldots, r\}$. Hence the first and the second term of (18) are negative, and the third term is positive. So for the condition in (18) to hold, the following is necessary:

$$(\mu_1 - \gamma_1 - \mu_r + \gamma_r) + \Gamma > 0. \qquad (19)$$

But (17) is in contradiction with (19). Hence, $E''_{m,r} - E'_{m,r} <$ 0 is false, which means, $E'_{m,r} < E''_{m,r}$. Hence it stands proved that we should give equal fractions of jobs to all the machines to get minimum energy. QED. $\square$

Thus, for energy-minimal scheduling in this setting, work has to be divided equally among r machines, where r is chosen to satisfy (9). If the energy consumption overhead due to division/breaking of jobs overhead is null or negligible, then any schedule which complies with Theorems 3.3 and 3.4 is energy-optimal.

Algorithm 1 finds out machines which should be assigned work for the energy-minimal scheduling when the machines of the system are of identical speed.

Algorithm 1: Find the minimum energy consumption of the system when the speeds of machine are identical

input : m, μ_i, γ_i, W
output: r, T, E

for $i = 1$ *to* m **do**
   | calculate $\mu_i - \gamma_i$
end
sort $(\mu_i - \gamma_i)$;
$low \leftarrow 1$, $high \leftarrow m$;
while $low \leq high$ **do**
   $mid \leftarrow low + \frac{(high - low)}{2}$;
   Calculate $E(mid - 1)$, $E(mid)$, $E(mid + 1)$;
   if
   $(E(mid - 1) > E(mid))$&&$(E(mid) \leq E(mid + 1))$
   then
      | $r \leftarrow mid$;
      | return r;
   else
      if $E(mid - 1) > E(mid)$ **then**
         | $low \leftarrow mid$;
      else
         | $high \leftarrow mid$;
      end
   end
end
$T \leftarrow \frac{W}{r}$;
$E \leftarrow T[\sum_{i=1}^{r} \mu_i + \sum_{i=r+1}^{m} \gamma_i]$;

Algorithm 1 takes the number of machines, the working power and idle power of the machines, and the total work to be done as input, and gives the value of r, i.e., the number of working machines, as output. Along with the number of working machines, it also calculates the value of makespan and total energy consumption of the system. The algorithm is based on the previously given results in a way that should be absolutely clear.

We now proceed to we analyse the case when the speed of machines are also different.

3.2 Different Speeds When Idle Power is Zero

We now consider the scheduling problem in a system where machines can have different speeds of working, i.e. different machines can execute the same job over different durations. In this model, the idle power consumptions of all the machines are zero, i.e. $\gamma_i = 0, \forall i$, as would happen if idle machines were switched off. Finding energy-minimal sched-

ule in this case can be solved in polynomial time, as we now show.

We start with a lemma for prioritization of work among machines.

LEMMA 3.5. *Given* $\frac{\mu_k}{v_k} \leq \frac{\mu_l}{v_l}$, *then for minimal energy,* $w_k \geq w_l$ $\forall k, l$, *where* $1 \leq k \leq l \leq m$.

PROOF. We prove this by contradiction. Given $\frac{\mu_k}{v_k} \leq \frac{\mu_l}{v_l}$, then by contradiction, $w_k < w_l$.
Rearranging (4), we get,

$$E = \mu_k \frac{w_k}{v_k} + \mu_l \frac{w_l}{v_l} + \sum_{i \neq k,l} \mu_i \frac{w_i}{v_i} \qquad (20)$$

To analyze just the two machines k and l, assume that the work distribution among the rest of the machines is constant. Since the idle power consumptions of all the machines is zero, the power consumed by a machine completely depends only upon the amount of work it is doing. We thus take the amount of work done by all machines other than k and l be equal to w' and the power dissipated by them be C, i.e.,

$$C = \sum_{i \neq k,l} \mu_i \frac{w_i}{v_i} \qquad (21)$$

Substituting C in (20), we get,

$$E = \frac{\mu_k w_k}{v_k} + \frac{\mu_l w_l}{v_l} + C \qquad (22)$$

Clearly $w_k + w_l + w' = W$. Now there are two cases to be considered: when $w_k \geq w_l$, and when $w_k < w_l$.

Case 1: $w_k \geq w_l$. Take $w_k = w + \epsilon_1$ and $w_l = w - \epsilon_1$, where $\epsilon_1 \geq 0$. Putting these values in (22), we get,

$$E = \frac{\mu_k w}{v_k}(1 + \epsilon_1) + \frac{\mu_l w}{v_l}(1 - \epsilon_1) + C \qquad (23)$$

Case 2: $w_k < w_l$. Take $w_k = w - \epsilon_2$ and $w_l = w + \epsilon_2$, where $\epsilon_2 > 0$. Putting these values in (22) we get,

$$E' = \frac{\mu_k w}{v_k}(1 - \epsilon_2) + \frac{\mu_l w}{v_l}(1 + \epsilon_2) + C \qquad (24)$$

If we say that in Case 2 the energy consumed is less, this means $E - E' > 0$. Using (23) from (24) in this condition, we get,

$$\frac{\mu_k}{v_k} > \frac{\mu_l}{v_l} \qquad (25)$$

(25) is in contradiction to our assumption 4. QED. $\square$

As for the identical-speed machine case, here also it is possible that just giving work to a subset of machines might prove more energy efficient than all the machines. Our next lemma tells which subset of machines should be working.

LEMMA 3.6. *If we give work to only some* r *machines, where* $1 \leq r \leq m$, *then, for minimal energy consumption, these are the* r *machines that form the set* $\{1, 2, \ldots, r\}$.

This lemma is the analog of Lemma 3.2 and the proof is similar. Now we proceed to the main theorem which indicates the set of machines to be used.

THEOREM 3.7. *When the speeds of all machines are different, and their idle power consumption is zero, to minimize the energy, we give all the load to the machine whose* $\frac{\mu_i}{v_i}$ *is minimum.*

PROOF. We prove this by contradiction. Since we have arranged the machines in increasing order of $\frac{\mu_i}{v_i}$, the first machine has minimum $\frac{\mu_i}{v_i}$. We denote the energy spent when first i machines are working as $E_{m,i}$. From Lemma 3.6, we know that if $E_{m,i} > E_{m,i-1}$ then $E_{m,i+1} > E_{m,i-1}$. Hence if we prove that $E_{m,1} < E_{m,2}$, then it is implicit that $E_{m,1} < E_{m,i}, \forall i, 1 < i \leq m$. Consider two cases:

Case 1: When all the work is given to first machine, i.e. $w_1 = W$. Using (4), the energy consumed by system is given by:

$$E_{m,1} = \frac{\mu_1 W}{v_1} \qquad (26)$$

Case 2: When work is given to machines 1 and 2. Using (4), the energy consumed by system is given by:

$$E_{m,2} = \frac{\mu_1 w_1}{v_1} + \frac{\mu_2 w_2}{v_2} \qquad (27)$$

As a contradiction to our theorem, we try to prove that $E_{m,1} > E_{m,2}$. Using (26) and (27), and substituting $w_1 = W - w2$ we get,

$$\frac{\mu_1}{v_1} > \frac{\mu_2}{v_2} \qquad (28)$$

Here (28) is in contradiction to our assumption 4. QED. □

It is shown that only one machine has to be given all work and this machine can easily be identified. Clearly the solution to the problem of energy-minimal scheduling for machines with different speeds and zero idle power can be found in polynomial time. Algorithm 2 gives the makespan and the energy consumption for the same.

Algorithm 2: Find the minimum energy consumption of the system with variable speeds but when idle power consumption of machines is not considered

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input : m, μ_i, v_i, W
output: r, T, E

for $i = 1$ *to* m **do**
$\quad$ calculate $\frac{\mu_i}{v_i}$
end
$\min(\frac{\mu_i}{v_i})$;
$(\frac{\mu_1}{v_1}) \leftarrow min(\frac{\mu_i}{v_i})$;
$r \leftarrow 1$;
$T \leftarrow \frac{W}{v_1}$;
$E \leftarrow \mu_1 T$;

This algorithm is based on the result given above. The output of the theorem is the total energy consumption of the system, and the makespan. The complexity for Algorithm 2 is $\mathcal{O}(m)$, where m is the number of machines in the system.

4. CONCLUSIONS

This paper provides a generic formulation of energy-minimal scheduling of divisible jobs. We have derived results which govern the work distribution among machines of different energy specifications, for keeping overall energy consumption to a minimum. We have also suggested algorithms in line with our theoretical findings. Though energy-minimal scheduling of non-divisible loads is NP-hard, we can strive to achieve a close to ideal performance, using approximation algorithms. Our results also give insights to how many machines are actually useful for doing assigned work in an energy-optimal manner, and can be combined with constraints on the system such as timing constraints to derive energy-minimal schedules for the problem in hand. Our aim is for this work to serve as a building block for a generalized theory of energy-minimal and energy-aware scheduling.

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