

Straggler Mitigation in Distributed Matrix Multiplication: Fundamental Limits and Optimal Coding

Qian Yu*, Mohammad Ali Maddah-Ali[†], and A. Salman Avestimehr*

* Department of Electrical Engineering, University of Southern California, Los Angeles, CA, USA

[†] Nokia Bell Labs, Holmdel, NJ, USA

Abstract

We consider the problem of massive matrix multiplication, which underlies many data analytic applications, in a large-scale distributed system comprising a group of worker nodes. We target the stragglers' delay performance bottleneck, which is due to the unpredictable latency in waiting for slowest nodes (or stragglers) to finish their tasks. We propose a novel coding strategy, named *entangled polynomial code*, for designing the intermediate computations at the worker nodes in order to minimize the recovery threshold (i.e., the number of workers that we need to wait for in order to compute the final output). We demonstrate the optimality of entangled polynomial code in several cases, and show that it provides orderwise improvement over the conventional schemes for straggler mitigation. Furthermore, using *bilinear complexity*, we characterize the optimal recovery threshold among all linear coding strategies within a factor of 2. In particular, while evaluating bilinear complexity is a well-known challenging problem, we show that optimal recovery threshold for linear coding strategies can be approximated within a factor of 2 of this fundamental quantity. Finally, we show that the techniques developed in this paper can also be extended to several other problems such as coded convolution and fault tolerance computing, leading to tight characterizations.

I. INTRODUCTION

Matrix multiplication is one of the key operations underlying many data analytics applications in various fields such as machine learning, scientific computing, and graph processing. Many such applications require processing terabytes or even petabytes of data, which needs massive computation and storage resources that cannot be provided by a single machine. Hence, deploying matrix computation tasks on large-scale distributed systems has received wide interests [1]–[4].

There is, however, a major performance bottleneck that arises as we scale out computations across many distributed nodes: stragglers' delay bottleneck, which is due to the unpredictable latency in waiting for slowest nodes (or stragglers) to finish their tasks [5]. The conventional approach for mitigating straggler effects involves injecting some form of "computation redundancy" such as repetition (e.g., [6]). Interestingly, it has been shown recently that *coding theoretic* concepts can also play a transformational role in this problem, by efficiently creating "computational redundancy" to mitigate the stragglers [7]–[10].

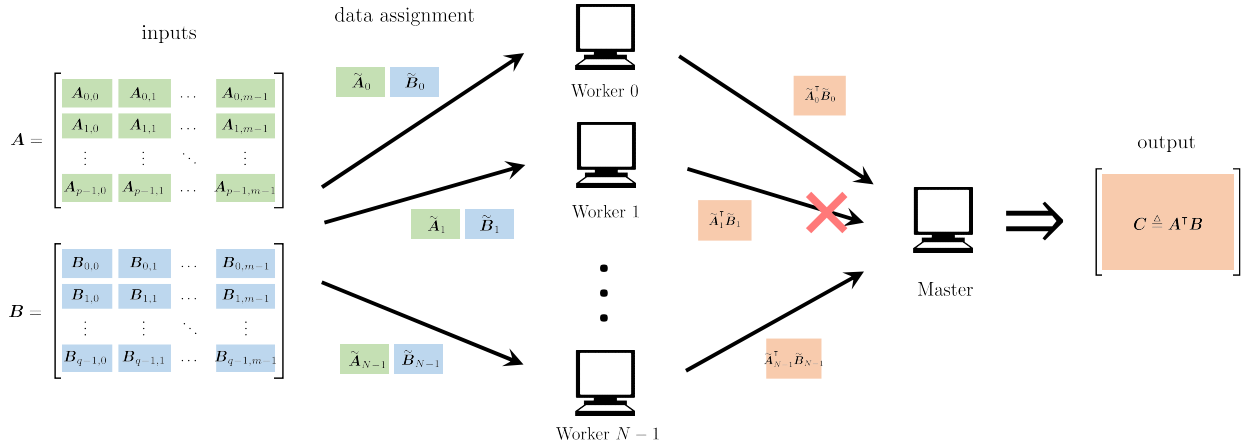


Fig. 1: Overview of the distributed matrix multiplication problem. Each worker computes the product of the two stored encoded submatrices (\tilde{A}_i and \tilde{B}_i) and returns the result to the master. By carefully designing the coding strategy, the master can decode the multiplication result of the input matrices from a subset of workers, without having to wait for stragglers (worker 1 in this example).

In this paper, we consider a general formulation of distributed matrix multiplication, study information-theoretic limits, and develop optimal coding designs for straggler effect mitigation. We consider the canonical master-worker distributed setting, where a group of N workers aim to collaboratively compute the product of two large matrices A and B , and return the result $C = A^T B$ to the master. As shown in Figure 1, the two input matrices are partitioned (arbitrarily) into p -by- m and p -by- n blocks of submatrices respectively, where all submatrices within the same input are of equal size. Each worker has a local memory that can be used to store any coded function of each matrix, denoted by \tilde{A}_i 's and \tilde{B}_i 's, each with a size equal to that of the corresponding submatrices. The workers then multiply their two stored (coded) submatrices and return the results to

the master. By carefully designing the coding functions, the master can decode the final result without having to wait for the slowest workers, which providing robustness against stragglers.

Note that by allowing different values of parameters p , m , and n , we allow flexible partitioning of input matrices, which in return enables different utilization of system resources (i.e., the required amount of storage at each worker and the amount of communication from worker to master). Hence, considering the system constraints on available storage and communication resources, one can choose p , m , and n accordingly. We aim to find optimal coding and computation designs for *any* choice of parameters p , m and n , to provide optimum straggler effect mitigation for various situations.

With a careful design of the coded submatrices \tilde{A}_i and \tilde{B}_i at each worker, the master only needs results from the fastest workers before it can recover the final output, which effectively mitigates straggler issues. To measure the robustness against straggler effects of a given coding strategy, we use the metric *recovery threshold*, defined previously in [10], which is equal to the minimum number of workers that the master needs to wait for in order to compute the output C . Given this terminology, our main problem is as follows: What is the minimum possible recovery threshold and the corresponding coding scheme, for any choice of parameters p , m , n , and N ?

We propose a novel coding technique, referred to as *entangled polynomial code*, which achieves the recovery threshold of $pmn + p - 1$ for all possible parameter values. The construction of the entangled polynomial code is based on the observation that when multiplying an m -by- p matrix and a p -by- n matrix, we essentially evaluate a subspace of bilinear functions, spanned by the pairwise product of the elements from the two matrices. Although potentially there are a total of p^2mn pairs of elements, at most pmn pairs are directly related to the matrix product, which is an order of p less. The particular structure of the proposed code entangles the input matrices to the output such that the system almost avoids unnecessary multiplications and achieves a recovery threshold in the order of pmn , while allowing robust straggler mitigation for arbitrarily large systems. This allows orderwise improvement upon conventional uncoded approaches, random linear codes, and MDS-coding type approaches for straggler mitigation [7], [8].

Entangled polynomial code generalizes our previously proposed polynomial code for distributed matrix multiplication [10], which was designed for the special case of $p = 1$ (i.e., allowing only column-wise partitioning of matrices A and B). However, as we move to arbitrary partitioning of the input matrices (i.e., arbitrary values of m , n , and p), a key challenge is to design the coding strategy at each worker such that its computation best *aligns* with the final computation C . In particular, to recover the product C , the master needs mn components that each involve summing p products of submatrices of A and B . Entangled polynomial code effectively aligns the workers' computations with the master's need, which is its key distinguishing feature from polynomial code.

We show that entangled polynomial code achieves the optimal recovery threshold among all linear coding strategies in the cases of $m = 1$ or $n = 1$. It also achieves the optimal recovery threshold among all possible schemes within a factor of 2 when $m = 1$ or $n = 1$.

Furthermore, for *all* partitionings of input matrices (i.e., all values of p , m , n , and N), we characterize the optimal recovery threshold among all linear coding strategies within a factor of 2 of $R(p, m, n)$, which denotes the *bilinear complexity* of multiplying an m -by- p matrix to a p -by- n matrix (see Definition 3 later in the paper). While evaluating bilinear complexity is a well-known challenging problem in the computer science literature (see [11]), we show that the optimal recovery threshold for linear coding strategies can be approximated within a factor of 2 of this fundamental quantity.

We establish this result by constructing a computation strategy, which achieves a recovery threshold of $2R(p, m, n) - 1$ using a variation of the entangled polynomial code. Specifically, this coding construction exploits the fact that any matrix multiplication problem can be converted into a problem of computing the element-wise product of two arrays of length $R(p, m, n)$. Then we show that this augmented computing task can be optimally handled using a variation of the entangled polynomial code, and the corresponding optimal code achieves the recovery threshold $2R(p, m, n) - 1$.

Finally, we show that the coding construction and converse bounding techniques developed for proving the above results can also be directly extended to several other problems. For example, we show that the converse bounding technique can be extended to the problem of coded convolution, which was originally considered in [12]. We prove that the state-of-the-art scheme we proposed in [10] for this problem is in fact optimal among all linear coding schemes. These techniques can also be applied in the context of fault tolerance computing, where we provide tight characterizations on the maximum number of detectable or correctable errors.

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a problem of matrix multiplication with two input matrices $A \in \mathbb{F}^{s \times r}$ and $B \in \mathbb{F}^{s \times t}$, for some integers r , s , t and a sufficiently large field \mathbb{F} .¹ We are interested in computing the product $C \triangleq A^T B$ in a distributed computing environment with a master node and N worker nodes, where each worker can store $\frac{1}{pm}$ fraction of A and $\frac{1}{pn}$ fraction of B , based on some integer parameters p , m , and n (see Fig. 1).

¹Here we consider the general class of fields, which includes finite fields, the field of real numbers, and the field of complex numbers.

Specifically, each worker i can store two coded matrices $\tilde{A}_i \in \mathbb{F}_p^{\frac{s}{p} \times \frac{r}{m}}$ and $\tilde{B}_i \in \mathbb{F}_p^{\frac{s}{p} \times \frac{t}{n}}$, computed based on A and B respectively. Each worker can compute the product $\tilde{C}_i \triangleq \tilde{A}_i^\top \tilde{B}_i$, and return it to the master. The master waits only for the results from a subset of workers before proceeding to recover the final output C using certain *decoding functions*.

Given the above system model, we formulate the *distributed matrix multiplication problem* based on the following terminology: We define the *computation strategy* as a collection of $2N$ *encoding functions*, denoted by

$$\mathbf{f} = (f_0, f_1, \dots, f_{N-1}), \quad \mathbf{g} = (g_0, g_1, \dots, g_{N-1}), \quad (1)$$

that are used by the workers to compute each \tilde{A}_i and \tilde{B}_i , and a class of *decoding functions*, denoted by

$$\mathbf{d} = \{d_{\mathcal{K}}\}_{\mathcal{K} \subseteq \{0,1,\dots,N-1\}}, \quad (2)$$

that are used by the master to recover C given results from any subset \mathcal{K} of the workers. Each worker i stores matrices

$$\tilde{A}_i = f_i(A), \quad \tilde{B}_i = g_i(B), \quad (3)$$

and the master can compute an estimate \hat{C} of matrix C using results from a subset \mathcal{K} of the workers by computing

$$\hat{C} = d_{\mathcal{K}}(\{\tilde{C}_i\}_{i \in \mathcal{K}}). \quad (4)$$

For any integer k , we say a computation strategy is k -*recoverable* if the master can recover C given the computing results from *any* k workers. Specifically, a computation strategy is k -*recoverable* if for any subset \mathcal{K} of k users, the final output \hat{C} from the master equals C for all possible input values. We define the *recovery threshold* of a computation strategy, denoted by $K(\mathbf{f}, \mathbf{g}, \mathbf{d})$, as the minimum integer k such that computation strategy $(\mathbf{f}, \mathbf{g}, \mathbf{d})$ is k -recoverable.

We aim to find a computation strategy that requires the minimum possible recovery threshold and allows efficient decoding at the master. Among all possible computation strategies, we are particularly interested in a certain class of designs, referred to as the *linear codes* and defined as follows:

Definition 1. For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, we say a computation strategy is a *linear code* given parameters p , m , and n , if there is a partitioning of the input matrices A and B where each matrix is divided into the following submatrices of equal sizes

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p-1,0} & A_{p-1,1} & \cdots & A_{p-1,m-1} \end{bmatrix}, \quad (5)$$

$$B = \begin{bmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,n-1} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{p-1,0} & B_{p-1,1} & \cdots & B_{p-1,n-1} \end{bmatrix}, \quad (6)$$

such that the encoding functions of each worker i can be written as

$$\tilde{A}_i = \sum_{j,k} A_{j,k} a_{ijk}, \quad \tilde{B}_i = \sum_{j,k} B_{j,k} b_{ijk}, \quad (7)$$

for some tensors a and b , and the decoding function given each subset \mathcal{K} can be written as ²

$$\hat{C}_{j,k} = \sum_{i \in \mathcal{K}} \tilde{C}_i c_{ijk}, \quad (8)$$

for some tensor c . For brevity, we denote the set of linear codes as \mathcal{L} .

The major advantage of linear codes is that they guarantee both the encoding and the decoding complexities of the scheme scale linearly with respect to the size of the input matrices. Furthermore, as we have proved in [10], linear codes are optimal for $p = 1$. Given the above terminology, we define the following concept.

Definition 2. For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, we define the *optimum linear recovery threshold* as a function of the problem parameters p , m , n , and N , denoted by K_{linear}^* , as the minimum achievable recovery threshold among all linear codes. Specifically,

$$K_{\text{linear}}^* \triangleq \min_{(\mathbf{f}, \mathbf{g}, \mathbf{d}) \in \mathcal{L}} K(\mathbf{f}, \mathbf{g}, \mathbf{d}). \quad (9)$$

²Here $\hat{C}_{j,k}$ denotes the master's estimate of the subblock of C that corresponds to $\sum_{\ell} A_{\ell,j} B_{\ell,k}$.

Our goal is to characterize the optimum linear recovery threshold K_{linear}^* , and to find computation strategies to achieve such optimum threshold. Note that if the number of workers N is too small, obviously no valid computation strategy exists even without requiring straggler tolerance. Hence, in the rest of the paper, we only consider the meaningful case where N is large enough to support at least one valid computation strategy. Equivalently, N has to be at least the bilinear complexity of multiplying an m -by- p matrix and a p -by- n matrix.

We are also interested in characterizing the minimum recovery threshold achievable using general coding strategies (including non-linear codes). Similar to [10], we define this value as the *optimum recovery threshold* and denote it by K^* .

III. MAIN RESULTS

We state our main results in the following theorems:

Theorem 1. *For a distributed matrix multiplication problem of computing $A^T B$ using N workers, with parameters p , m , and n , the following recovery threshold can be achieved by a linear code, referred to as the entangled polynomial code.³*

$$K_{\text{entangled-poly}} \triangleq pmn + p - 1. \quad (10)$$

Furthermore, the entangled polynomial code can be decoded at the master node with at most the complexity of polynomial interpolation given $pmn + p - 1$ points.

Remark 1. Compared to some other possible approaches, our proposed entangled polynomial code provides orderwise improvement in the recovery threshold (see Fig. 2). One conventional approach (referred to as the *uncoded repetition scheme*) is to let each worker store and multiply uncoded submatrices. With the additional computation redundancy through repetition, the scheme can robustly tolerate some stragglers. However, its recovery threshold grows linearly with respect to the number of workers. Another approach is to let each worker store two random linear combinations of the input submatrices (referred to as the *random linear code*). With high probability, this achieves recovery threshold $K_{\text{RL}} \triangleq p^2 mn$, which does not scale with N . However, to calculate C , we need the result of pmn sub-matrix multiplications. Indeed, the lack of structure in the random coding forces the system to wait for p times more than what is essentially needed. One surprising aspect of the proposed entangled polynomial code is that, due to its particular structure which aligns the workers' computations with the master's need, it avoids unnecessary multiplications of submatrices. As a result, it achieves a recovery threshold that does not scale with N , and is orderwise smaller than that of the random linear code. Furthermore, it allows efficient decoding at the master, which requires at most an almost linear complexity.

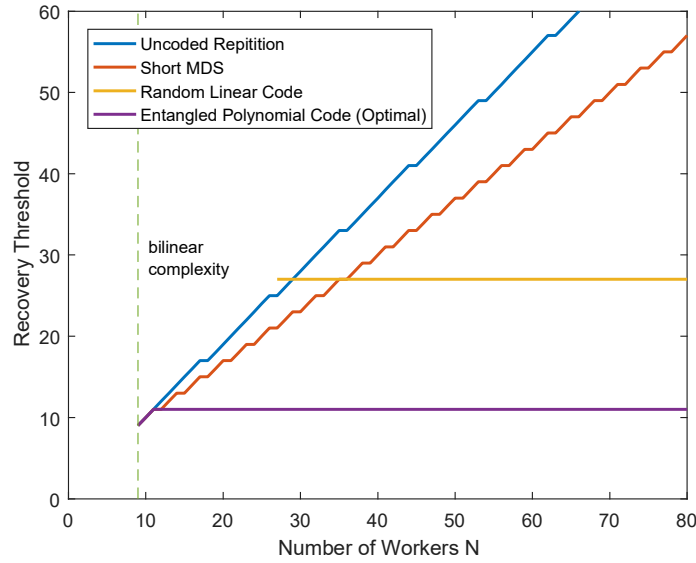


Fig. 2: Comparison of the recovery thresholds achieved by the uncoded repetition scheme, the random linear code, the short-MDS (or short-dot) [7], [8] and our proposed entangled polynomial code, given problem parameters $p = m = 3$, $n = 1$. The entangled polynomial code orderwise improves upon all other approaches. It also achieves the optimum linear recovery threshold in this scenario.

Remark 2. There have been several works in prior literature investigating the $p = 1$ case [7], [10], [13]. For this special case, the entangled polynomial code reduces to our previously proposed polynomial code, which achieves the optimum recovery threshold mn and orderwise improves upon other designs. On the other hand, there has been some investigation on matrix-by-vector type multiplication [7], [8], which can be viewed as the special case of $m = 1$ or $n = 1$ in our proposed problem. The short-MDS code (or short-dot) has been proposed, achieving a recovery threshold of $N - \lfloor \frac{N}{p} \rfloor + m$, which scales linearly with N . Our proposed entangled polynomial code also strictly and orderwise improves upon that (see Fig. 2).

³For $N < pmn + p - 1$, we define $K_{\text{entangled-poly}} \triangleq N$.

Our second result is the optimality of the entangled polynomial code when $m = 1$ or $n = 1$. Specifically, we prove that entangled polynomial code is optimal in this scenario among all linear codes. Furthermore, if the base field \mathbb{F} is finite, it also achieves the optimum recovery threshold K^* within a factor of 2, with non-linear coding strategies into account.

Theorem 2. *For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, with parameters p , m , and n , if $m = 1$ or $n = 1$, we have*

$$K_{\text{linear}}^* = K_{\text{entangled-poly}}. \quad (11)$$

Moreover, if the base field \mathbb{F} is finite,

$$\frac{1}{2} K_{\text{entangled-poly}} < K^* \leq K_{\text{entangled-poly}}. \quad (12)$$

Remark 3. We prove Theorem 2 by first exploiting the algebraic structure of matrix multiplication to develop a linear algebraic converse for equation (11), and then constructing an information theoretic converse to prove inequality (12). The linear algebraic converse only relies on two properties of the matrix multiplication operation: 1) bilinearity, and 2) uniqueness of zero element. This technique can be extended to any other bilinear operations with similar properties, such as convolution, as mentioned later (see Theorem 4). On the other hand, the information theoretic converse is obtained through a cut-set type argument, which allows a lower bound on the recovery thresholds even for non-linear codes.

Our final result on the main problem is characterizing the optimum linear recovery threshold K^* within a factor of 2 for all possible p , m , n , and N , using the fundamental concept of bilinear complexity [11]:

Definition 3. We define the *bilinear complexity* of multiplying an m -by- p matrix and a p -by- n matrix, denoted by $R(p, m, n)$, as the minimum number of element-wise multiplication required to complete such an operation. Rigorously, $R(p, m, n)$ denotes the minimum integer R , such that we can find tensors $a \in \mathbb{F}^{R \times p \times m}$, $b \in \mathbb{F}^{R \times p \times n}$, and $c \in \mathbb{F}^{R \times m \times n}$, satisfying

$$\sum_i c_{ijk} \left(\sum_{j', k'} A_{j'k'} a_{ij'k'} \right) \left(\sum_{j'', k''} B_{j''k''} b_{ij''k''} \right) = \sum_\ell A_{\ell j} B_{\ell k}. \quad (13)$$

for any input matrices $A \in \mathbb{F}^{p \times m}$, $B \in \mathbb{F}^{p \times n}$.

Using this concept, we state our result as follows.

Theorem 3. *For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, with parameters p , m , and n , the optimum linear recovery threshold is characterized by*

$$R(p, m, n) \leq K_{\text{linear}}^* \leq 2R(p, m, n) - 1, \quad (14)$$

where $R(p, m, n)$ denotes the bilinear complexity of multiplying an m -by- p matrix and a p -by- n matrix.

Remark 4. The key idea of proving Theorem 3 is to first convert any matrix multiplication problem into computing the element-wise product of two vectors of length $R(p, m, n)$. Then we show that an optimal computation strategy can be developed for this augmented problem using similar ideas we developed for the entangled polynomial code. Similarly, factor-of-2 characterization can also be obtained for non-linear codes, as discussed in Section VI.

The techniques we developed in this paper can also be extended to several other problems, such as coded convolution [12] and fault tolerance computing [14], [15], leading to tight characterizations. For coded convolution, we present our result in the following theorem.

Theorem 4. *For the distributed convolution problem of computing $\mathbf{a} * \mathbf{b}$ using N workers that can each store $\frac{1}{m}$ fraction of \mathbf{a} and $\frac{1}{n}$ fraction of \mathbf{b} , the optimum recovery threshold that can be achieved using linear codes, denoted by $K_{\text{conv-linear}}^*$, is exactly characterized by the following equation*

$$K_{\text{conv-linear}}^* = K_{\text{conv-poly}} \triangleq m + n - 1. \quad (15)$$

Remark 5. Theorem 4 is proved based on our previously developed coded computing scheme for convolution, which is a variation of the polynomial code [10]. As mentioned before, we extend the proof idea of Theorem 2 to prove the matching converse. This theorem proves the optimality of the computation scheme in [10] among all computation strategies where the encoding functions are linear. For detailed problem formulation and proof, see Appendix A.

Our second extension is in the fault tolerance computing setting, where, unlike the straggler effects we studied in this paper, arbitrary errors can be injected into the computation, and the master has no information about which subset of workers are returning errors. We show that the techniques we developed for straggler mitigation can also be applied in this setting to improve robustness against computing failures, and the optimality of any encoding function in terms of recovery threshold also preserves when applied in the fault tolerance computing setting. As an example, we present the following theorem, demonstrating this connection.

Theorem 5. For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, with parameters p , m , and n , if $m = 1$, or $n = 1$, or $p = 1$, the entangled polynomial code can detect up to

$$E_{\text{detect}}^* = N - K_{\text{entangled-poly}} \quad (16)$$

errors, and correct up to

$$E_{\text{correct}}^* = \left\lfloor \frac{N - K_{\text{entangled-poly}}}{2} \right\rfloor \quad (17)$$

errors. This can not be improved using any other linear coding strategies.

Remark 6. The proof idea for Theorem 5 is to connect the straggler mitigation problem and the fault tolerance problem by extending the concept of Hamming distance to coded computing. Specifically, we map the straggler mitigation problem to the problem of correcting erasure errors, and the fault tolerant problem to the problem of correcting arbitrary errors. The solution to these two communication problems are deeply connected by the Hamming distance, and we show that this result extends to coded computing (see Lemma 3 in Appendix B). Since the concept of Hamming distance is not exclusively defined for linear codes, this connection also holds for arbitrary computation strategies. Furthermore, this approach can be easily extended to the hybrid settings where both stragglers and computing errors exist, and similar results can be proved. The detailed formulation and proof can be found in Appendix B.

In Section IV, we prove Theorem 1 by describing the entangled polynomial code. Then in Section V, we prove Theorem 2 by deriving the converses. Finally, we present the coding construction and converse for proving Theorem 3 in Section VI.

IV. ENTANGLED POLYNOMIAL CODE

In this section, we prove Theorem 1 by formally describing the entangled polynomial code and its decoding procedure. We start with an illustrating example.

A. Illustrating Example

Consider a distributed matrix multiplication task of computing $A^\top B$ using $N = 5$ workers that can each store half of the rows (i.e., $p = 2$ and $m = n = 1$). We evenly divide each input matrix along the row side into 2 submatrices:

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \quad (18)$$

Given this notation, we essentially want to compute

$$C = A^\top B = [A_0^\top B_0 + A_1^\top B_1]. \quad (19)$$

A naive computation strategy is to let the 5 workers compute each $A_i^\top B_i$ uncodedly with repetition. Specifically we can let 3 workers compute $A_0^\top B_0$ and 2 workers compute $A_1^\top B_1$. However, this approach can only robustly tolerate 1 straggler, achieving a recovery threshold of 4. Another naive approach is to use random linear codes, i.e., let each worker store a random linear combination of A_0 , A_1 , and a combination of B_0 , B_1 . However, the resulting computation result of each worker is a random linear combination of 4 variables $A_0^\top B_0$, $A_0^\top B_1$, $A_1^\top B_0$, and $A_1^\top B_1$, which also results in a recovery threshold of 4.

Surprisingly, there is a simple computation strategy for this example that achieves the optimum linear recovery threshold of 3. The main idea is to instead inject structured redundancy tailored to the matrix multiplication operation. We present this proposed strategy as follows:

Suppose elements of A, B are in \mathbb{R} . Let each worker $i \in \{0, 1, \dots, 4\}$ store the following two coded submatrices:

$$\tilde{A}_i = A_0 + iA_1, \quad \tilde{B}_i = iB_0 + B_1. \quad (20)$$

To prove that this design gives a recovery threshold of 3, we need to find a valid decoding function for any subset of 3 workers. We demonstrate this decodability through a representative scenario, where the master receives the computation results from workers 1, 2, and 4, as shown in Figure 3. The decodability for the other 9 possible scenarios can be proved similarly.

According to the designed computation strategy, we have

$$\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_4 \end{bmatrix} = \begin{bmatrix} 1^0 & 1^1 & 1^2 \\ 2^0 & 2^1 & 2^2 \\ 4^0 & 4^1 & 4^2 \end{bmatrix} \begin{bmatrix} A_0^\top B_1 \\ A_0^\top B_0 + A_1^\top B_1 \\ A_1^\top B_0 \end{bmatrix}. \quad (21)$$

The coefficient matrix in the above equation is a Vandermonde matrix, which is invertible because its parameters 1, 2, 4 are distinct in \mathbb{R} . So one decoding approach is to directly invert equation (21), of which the returned result includes the needed matrix $C = A_0^\top B_0 + A_1^\top B_1$. This proves the decodability.

However, as we will explain in the general coding design, directly computing this inverse problem using the classical inversion algorithm might be expensive in some more general cases. Quite interestingly, because of the algebraic structure we designed

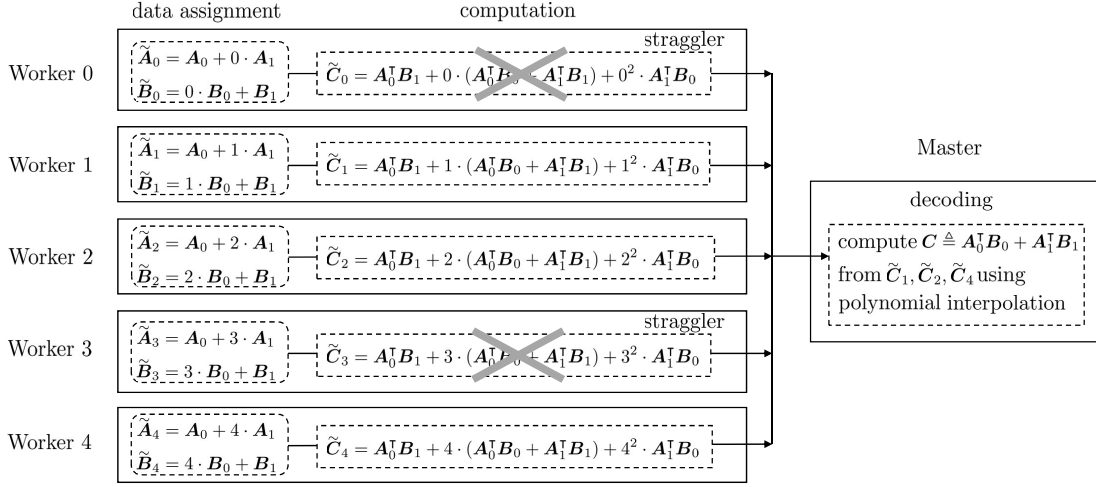


Fig. 3: Example using entangled polynomial code, with 5 workers that can each store half of each input matrix. (a) Computation strategy: each worker i stores $A_0 + iA_1$ and $iB_0 + B_1$, and computes their product. (b) Decoding: master waits for results from any 3 workers, and decodes the output using polynomial interpolation.

for the computation strategy (i.e., equation (20)), the decoding process can be viewed as a polynomial interpolation problem (or equivalently, decoding a Reed-Solomon code).

Specifically, in this example each worker i returns

$$\tilde{C}_i = \tilde{A}_i^T \tilde{B}_i = A_0^T B_1 + i(A_0^T B_0 + A_1^T B_1) + i^2 A_1^T B_0, \quad (22)$$

which is essentially the value of the following polynomial at point $x = i$:

$$h(x) \triangleq \tilde{A}_i^T \tilde{B}_i = A_0^T B_1 + x(A_0^T B_0 + A_1^T B_1) + x^2 A_1^T B_0. \quad (23)$$

Hence, recovering C using computation results from 3 workers is equivalent to recovering the linear term coefficient of a quadratic function given its values at 3 points. Later in this section, we will show that by mapping the decoding process to polynomial interpolation, we can achieve almost-linear decoding complexity even for arbitrary parameter values.

B. General Coding Design

Now we present the entangled polynomial code, which achieves a recovery threshold $pmn + p - 1$ for any p, m, n and N as stated in Theorem 1.⁴ First of all, we evenly divide each input matrix into pm and pn submatrices according to equations (5) and (6). We then assign each worker $i \in \{0, 1, \dots, N-1\}$ an element in \mathbb{F} , denoted by x_i , and make sure that all x_i 's are distinct. Under this setting, we define the following class of computation strategies.

Definition 4. Given parameters $\alpha, \beta, \theta \in \mathbb{N}$, we define the (α, β, θ) -polynomial code as

$$\tilde{A}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} A_{j,k} x_i^{j\alpha+k\beta}, \quad \tilde{B}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{n-1} B_{j,k} x_i^{(p-1-j)\alpha+k\theta}, \quad \forall i \in \{0, 1, \dots, N-1\}. \quad (24)$$

In an (α, β, θ) -polynomial code, each worker essentially evaluates a polynomial whose coefficients are fixed linear combinations of the products $A_{j,k}^T B_{j',k'}$. Specifically, each worker i returns

$$\tilde{C}_i = \tilde{A}_i^T \tilde{B}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} \sum_{j'=0}^{p-1} \sum_{k'=0}^{n-1} A_{j,k}^T B_{j',k'} x_i^{(p-1+j-j')\alpha+k\beta+k'\theta}. \quad (25)$$

Consequently, when the master receives results from enough workers, it can recover all these linear combinations using polynomial interpolation. Recall that we aim to recover

$$C = \begin{bmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,n-1} \\ C_{1,0} & C_{1,1} & \cdots & C_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m-1,0} & C_{m-1,1} & \cdots & C_{m-1,n-1} \end{bmatrix}, \quad (26)$$

⁴For $N < pmn + p - 1$, a recovery threshold of N is achievable by definition. Hence we focus on the case where $N \geq pmn + p - 1$.

where each submatrix $C_{k,k'} \triangleq \sum_{j=0}^{p-1} A_{j,k}^\top B_{j,k'}$ is also a fixed linear combination of these products. We design the values of parameters (α, β, θ) such that all these linear combinations appear in (25) separately as coefficients of terms of different degrees. Furthermore, we want to minimize the degree of the polynomial \tilde{C}_i , in order to reduce the recovery threshold.

One design satisfying these properties is $(\alpha, \beta, \theta) = (1, p, pm)$, i.e.,

$$\tilde{A}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} A_{j,k} x_i^{j+kp}, \quad \tilde{B}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{n-1} B_{j,k} x_i^{p-1-j+kpm}. \quad (27)$$

Hence, each worker returns the value of the following degree $pmn + p - 2$ polynomial at point $x = x_i$:

$$h_i(x) \triangleq \tilde{A}_i^\top \tilde{B}_i = \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} \sum_{j'=0}^{p-1} \sum_{k'=0}^{n-1} A_{j,k}^\top B_{j',k'} x_i^{(p-1+j-j')+kp+k'pm}, \quad (28)$$

where each $C_{k,k'}$ is exactly the coefficient of the $(p-1+kp+k'pm)$ -th degree term. Since all x_i 's are selected to be distinct, recovering C given results from any $pmn + p - 1$ workers is essentially interpolating $h(x)$ using $pmn + p - 1$ distinct points. Because the degree of $h(x)$ is $pmn + p - 2$, the output C can always be uniquely decoded.

In terms of complexity, this decoding process can be viewed as interpolating a degree $pmn + p - 2$ polynomial for $\frac{rt}{mn}$ times. It is well known that polynomial interpolation of degree k has a complexity of $O(k \log^2 k \log \log k)$ [16]. Therefore, decoding entangled polynomial code only requires at most a complexity of $O(prt \log^2(pmn) \log \log(pmn))$, which is almost linear to the input size of the decoder ($\Theta(prt)$ elements). This complexity can be reduced by simply swapping in any faster polynomial interpolation algorithm or Reed-Solomon decoding algorithm. In addition, this decoding complexity can also be further improved by exploiting the fact that only a subset of the coefficients are needed for recovering the output matrix.

V. CONVERSES

In this section, we provide the proof of Theorem 2. We first prove equation (11) by developing a linear algebraic converse. Then we prove inequality (12) through an information theoretic lower bound.

A. Matching Converses for Linear Codes

To prove equation (11), we start by developing a converse bound on recovery threshold for general parameter values, then we specialize it to the settings where $m = 1$ or $n = 1$. We state this converse bound in the following lemma:

Lemma 1. *For a distributed matrix multiplication problem with parameters p, m, n , and N , we have*

$$K_{\text{linear}}^* \geq \min\{N, pm + pn - 1\}. \quad (29)$$

When $m = 1$ or $n = 1$, the RHS of inequality (29) is exactly $K_{\text{entangled-poly}}$. Hence equation (11) directly follows from Lemma 1. So it only suffices to prove Lemma 1, and we prove it as follows:

Proof. To prove Lemma 1, we only need to consider the following two scenarios:

(1) If $K_{\text{linear}}^* = N$, then (29) is trivial.

(2) If $K_{\text{linear}}^* < N$, then we essentially need to show that for any parameter values p, m, n , and N , we have $K_{\text{linear}}^* \geq pm + pn - 1$. By definition, if such a linear recovery threshold is achievable, we can find a computation strategy, i.e., tensors a, b , and a class of decoding functions $\mathcal{d} \triangleq \{d_{\mathcal{K}}\}$, such that

$$d_{\mathcal{K}} \left(\left\{ \left(\sum_{j',k'} A_{j',k'}^\top a_{ij'k'} \right) \left(\sum_{j'',k''} B_{j'',k''} b_{ij''k''} \right) \right\}_{i \in \mathcal{K}} \right) = A^\top B \quad (30)$$

for any input matrices A and B , and for any subset \mathcal{K} of K_{linear}^* workers.

We choose the values of A and B , such that each $A_{j,k}$ and $B_{j,k}$ satisfies

$$A_{j,k} = \alpha_{jk} A_c, \quad (31)$$

$$B_{j,k} = \beta_{jk} B_c, \quad (32)$$

for some matrices $\alpha \in \mathbb{F}^{p \times m}$, $\beta \in \mathbb{F}^{p \times n}$, and constants $A_c \in \mathbb{F}^{\frac{s}{p} \times \frac{r}{m}}$, $B_c \in \mathbb{F}^{\frac{s}{p} \times \frac{r}{n}}$ satisfying $A_c^\top B_c \neq 0$. Consequently, we have

$$d_{\mathcal{K}} \left(\left\{ \left(\sum_{j',k'} \alpha_{j'k'} a_{ij'k'} \right) \left(\sum_{j'',k''} \beta_{j''k''} b_{ij''k''} \right) A_c^\top B_c \right\}_{i \in \mathcal{K}} \right) = A^\top B \quad (33)$$

for all possible values of α, β , and \mathcal{K} .

Fixing the value i , we can view each subtensor a_{ijk} as a vector of length pm , and each subtensor b_{ijk} as a vector of length pn . For brevity, we denote each such vector by \mathbf{a}_i and \mathbf{b}_i respectively. Similarly, we can also view matrices α and β as vectors

of length pm and pn , and we denote these vectors by α and β . Furthermore, we can define dot products within these vector spaces following the conventions. Using these notations, (33) can be written as

$$d_{\mathcal{K}}(\{(\alpha \cdot \mathbf{a}_i)(\beta \cdot \mathbf{b}_i)A_c^\top B_c\}_{i \in \mathcal{K}}) = A^\top B. \quad (34)$$

Given the above definitions, we now prove that within each subset \mathcal{K} of size K_{linear}^* , the vectors $\{\mathbf{a}_i\}_{i \in \mathcal{K}}$ span the space \mathbb{F}^{pm} . Essentially, we need to prove that for a given subset \mathcal{K} , there does not exist a non-zero α such that $\alpha \cdot \mathbf{a}_i = 0$ for all $i \in \mathcal{K}$. Assume the opposite that such an α exists so that $\alpha \cdot \mathbf{a}_i$ is always 0, then the LHS of (34) becomes a fixed value. On the other hand, since α is non-zero, we can always find different values of β such that $\alpha^\top \beta$ is variable. Recalling (31) and (32), the RHS of (34) cannot be fixed if $\alpha^\top \beta$ is variable, which results in a contradiction.

Now we use this conclusion to prove (29). For any fixed \mathcal{K} , let \mathcal{B} be a subset of indices such that $\{\mathbf{a}_i\}_{i \in \mathcal{B}}$ form a basis. Recall that we are considering the case where $K_{\text{linear}}^* < N$, meaning that we can find a worker $\tilde{k} \notin \mathcal{K}$. For convenience, we define $\mathcal{K}^+ = \mathcal{K} \cup \{\tilde{k}\}$, and $\mathcal{K}^- \triangleq \mathcal{K}^+ \setminus \mathcal{B}$. Obviously, $|\mathcal{B}| = pm$, and $|\mathcal{K}^-| = |\mathcal{K}^+| - |\mathcal{B}| = K_{\text{linear}}^* + 1 - pm$. Hence, it suffices to prove that $|\mathcal{K}^-| \geq pn$, which only requires that $\{\mathbf{b}_i\}_{i \in \mathcal{K}^-}$ forms a basis of \mathbb{F}^{pn} . Equivalently, we only need to prove that any β satisfying $\beta \cdot \mathbf{b}_i = 0$ for any $i \in \mathcal{K}^-$ must satisfy $\beta = 0$.

Recall that $\{\mathbf{a}_i\}_{i \in \mathcal{B}}$ forms a basis, so we can find a matrix $\alpha_i \in \mathbb{F}^{p \times m}$ for each $i \in \mathcal{B}$ such that $\alpha_i \cdot \mathbf{a}_{i'} = \delta_{i,i'}$.⁵ From elementary linear algebra, the vectors $\{\alpha_i\}_{i \in \mathcal{B}}$ also form a basis of $\mathbb{F}^{p \times m}$. Correspondingly, their matrix version $\{\alpha_i\}_{i \in \mathcal{B}}$ form a basis of $\mathbb{F}^{p \times m}$.

For any $k \in \mathcal{B}$, we define $\mathcal{K}_k = \mathcal{K}^+ \setminus \{k\}$. Note that $|\mathcal{K}_k| = K_{\text{linear}}^*$, so equation (34) also holds for any subset \mathcal{K}_k . Note that if $\alpha = \alpha_k$, and β satisfies $\beta \cdot \mathbf{b}_i = 0$ for any $i \in \mathcal{K}^-$, then the corresponding LHS of (34) remains fixed. As a result, $A^\top B$ must also be fixed. Similar to the above discussion, this requires that the value of $\alpha_k^\top \beta$ be fixed. This value has to be 0 because $\beta = 0$ satisfies our stated condition.

Now we have proved that any β satisfying $\beta \cdot \mathbf{b}_i = 0$ must also satisfy $\alpha_k^\top \beta = 0$ for $k \in \mathcal{B}$. Because $\{\alpha_k\}_{k \in \mathcal{B}}$ form a basis of $\mathbb{F}^{p \times m}$, such β acting on $\mathbb{F}^{p \times m}$ through matrix product has to be the zero operator, so $\beta = 0$. As mentioned above, this results in $K_{\text{linear}}^* \geq pm + pn - 1$, which completes the proof of Lemma 1 and equation (11). \square

Remark 7. Note that in the above proof, we never used the fact that the decoding functions are linear. Hence, the converse does not require the linearity of the decoder. This fact will be used later in our discussion regarding the fault tolerance computing in Appendix B.

B. Information Theoretic Converse for Nonlinear Codes

Now we prove inequality (12) through an information theoretic converse bound. Similar to the proof of equation (11), we start by proving a general converse.

Lemma 2. *For a distributed matrix multiplication problem with parameters p , m , n , and N , if the base field \mathbb{F} is finite, we have*

$$K^* \geq \max\{pm, pn\}. \quad (35)$$

When $m = 1$ or $n = 1$, the RHS of inequality (35) is greater than $\frac{1}{2}K_{\text{entangled-poly}}$. Hence inequality (12) directly results from Lemma 2, which we prove as follows.

Proof. Without loss of generality, we assume $m \geq n$, and aim to prove $K^* \geq pm$. Specifically, we need to show that, for any computation strategy and any subset \mathcal{K} of workers, if the master can recover C given the results from workers in \mathcal{K} , then we must have $|\mathcal{K}| \geq pm$.

Suppose the condition in the above statement holds. Given each input A , the workers can compute $\{\tilde{A}_i\}_{i \in \mathcal{K}}$ using the encoding functions. On the other hand, for any possible value of B , the workers can compute $\{\tilde{C}_i\}_{i \in \mathcal{K}}$ based on B and $\{\tilde{A}_i\}_{i \in \mathcal{K}}$. Hence, if we view A as a random variable and $\{\tilde{C}_i\}_{i \in \mathcal{K}}$ as random functions of B , we have the following Markov chain:

$$A \rightarrow \{\tilde{A}_i\}_{i \in \mathcal{K}} \rightarrow \{\tilde{C}_i\}_{i \in \mathcal{K}}. \quad (36)$$

Because the master can decode C as a function of $\{\tilde{C}_i\}_{i \in \mathcal{K}}$, it can also obtain C as a random function if the values of $\{\tilde{C}_i\}_{i \in \mathcal{K}}$ for all $B \in \mathbb{F}^{s \times t}$ are known. This random function is exactly a linear map from $\mathbb{F}^{s \times t}$ to $\mathbb{F}^{r \times t}$, which returns the matrix product of A^\top and the input. For brevity, we denote this function by A_{op} . Consequently, we have the following extended Markov chain

$$A \rightarrow \{\tilde{A}_i\}_{i \in \mathcal{K}} \rightarrow \{\tilde{C}_i\}_{i \in \mathcal{K}} \rightarrow A_{\text{op}}. \quad (37)$$

Given the basic property of matrix multiplication, there is a natural one-to-one correspondence between A and A_{op} . Hence, the input matrix A can be exactly determined from A_{op} , i.e., $H(A|A_{\text{op}}) = 0$. Using the data processing inequality, we also

⁵Here $\delta_{i,j}$ denotes the discrete delta function, i.e., $\delta_{i,i} = 1$, and $\delta_{i,j} = 0$ for $i \neq j$.

have $H(A|\{\tilde{A}_i\}_{i \in \mathcal{K}}) = 0$. Now let A be uniformly randomly sampled from $\mathbb{F}^{s \times r}$, and we have $H(A) = sr \log_2 |\mathbb{F}|$ bits. On the other hand, each \tilde{A}_i consists of $\frac{sr}{pm}$ elements, which has an entropy of at most $\frac{sr}{pm} \log_2 |\mathbb{F}|$ bits. Consequently, we have

$$|\mathcal{K}| \geq \frac{H(A)}{\max_{i \in \mathcal{K}} H(\tilde{A}_i)} \geq pm. \quad (38)$$

This concludes the proof of Lemma 2 and inequality (12). \square

VI. FACTOR OF 2 CHARACTERIZATION OF OPTIMUM LINEAR RECOVERY THRESHOLD

In this section, we provide the proof of Theorem 3. Note that the converse bound in this theorem is a natural consequence of Definition 3.⁶ Hence, we focus on proving the upper bound by developing an achievability scheme. Specifically, we need to provide a computation strategy that achieves a recovery threshold of at most $2R(p, m, n) - 1$ for all possible values of p, m, n , and N . As we have explained in Section IV-B, a recovery threshold of N is always achievable, so we only need to focus on the scenario where $N \geq 2R(p, m, n) - 1$.

Our proof consists of 2 steps. In Step 1, we show that any matrix multiplication is essentially computing the element-wise product of two vectors of length $R(p, m, n)$. Then in Step 2, we show that we can find an optimal computation strategy for this augmented computing task using similar ideas we developed for entangled polynomial code, which only requires a recovery threshold of $2R(p, m, n) - 1$.

For Step 1, recall the definition of bilinear complexity in Section III. We can always find tensors $a \in \mathbb{F}^{R(p, m, n) \times p \times m}$, $b \in \mathbb{F}^{R(p, m, n) \times p \times n}$, and $c \in \mathbb{F}^{R(p, m, n) \times m \times n}$ such that any block of the final output C can be computed as

$$C_{j,k} = \sum_i c_{ijk} \tilde{A}_{i,\text{vec}}^\top \tilde{B}_{i,\text{vec}}, \quad (39)$$

where $\tilde{A}_{i,\text{vec}}$ and $\tilde{B}_{i,\text{vec}}$ are linear combinations of the blocks of A and B , defined as

$$\tilde{A}_{i,\text{vec}} \triangleq \sum_{j,k} A_{j,k} a_{ijk}, \quad \tilde{B}_{i,\text{vec}} \triangleq \sum_{j,k} B_{j,k} b_{ijk}. \quad (40)$$

This essentially converts matrix multiplication into a problem of computing the element-wise product of two “vectors” $\tilde{A}_{i,\text{vec}}$ and $\tilde{B}_{i,\text{vec}}$, each of length $R(p, m, n)$. Specifically, the master only needs $\tilde{A}_{i,\text{vec}}^\top \tilde{B}_{i,\text{vec}}$ for decoding the final output.

Now in Step 2, we want to find the optimum computation strategy for this augmented computation task. Specifically, given two arbitrary vectors $\tilde{A}_{i,\text{vec}}$ and $\tilde{B}_{i,\text{vec}}$ of length $R(p, m, n)$, we want to achieve a recovery threshold of $2R(p, m, n) - 1$ for computing their element-wise product using N workers, each of which can multiply two coded vectors of length 1.

This problem can be solved using similar techniques we developed for entangled polynomial code. The main coding idea is to first view the elements in each vector as values of a degree $R(p, m, n) - 1$ polynomial at $R(p, m, n)$ different points. Specifically, given $R(p, m, n)$ distinct elements in the field \mathbb{F} , denoted by $x_0, x_1, \dots, x_{R(p, m, n)-1}$, we find polynomials \tilde{f} and \tilde{g} of degree $R(p, m, n) - 1$, whose coefficients are matrices, such that

$$\tilde{f}(x_i) = \tilde{A}_{i,\text{vec}} \quad (41)$$

$$\tilde{g}(x_i) = \tilde{B}_{i,\text{vec}}. \quad (42)$$

Recall that we want to recover $\tilde{A}_{i,\text{vec}}^\top \tilde{B}_{i,\text{vec}}$, which is essentially recovering the values of the degree $2R(p, m, n) - 2$ polynomial $\tilde{h} \triangleq \tilde{f}^\top \tilde{g}$ at these $R(p, m, n)$ points. Earlier in this paper, we already developed a coding structure that allows us to recover polynomials of this form. We now reuse the idea in this construction.

Let y_0, y_1, \dots, y_{N-1} be distinct elements of \mathbb{F} . We let each worker i store

$$\tilde{A}_i = \tilde{f}(y_i), \quad (43)$$

$$\tilde{B}_i = \tilde{g}(y_i), \quad (44)$$

which are linear combinations of the input submatrices. After computing the product, each worker essentially evaluates $\tilde{h}(y_i)$. Hence, from the results of any $2R(p, m, n) - 1$ workers, the master can recover \tilde{h} , which has degree $2R(p, m, n) - 2$, and proceed with decoding the output matrix C . This construction achieves a recovery threshold of $2R(p, m, n) - 1$, which proves the upper bound in Theorem 3.

Remark 8. The computation strategy we developed in Step 2 provides a tight upper bound on the characterization of the optimum linear recovery threshold for computing element-wise product of two arbitrary vectors using N machines. Its optimality naturally follows from Theorem 2, given that the element-wise product of two vectors contains all the information needed to compute the dot-product, which is a special case of matrix multiplication. We formally state this result in the following corollary.

⁶Rigorously, it also requires the linear independence of the $A_i^\top B_j$'s, which can be easily proved.

Corollary 1. Consider the problem of computing the element-wise product of two vectors of length R using N workers, each of which can store a linearly coded element of each vector and return their product to the master. The optimum linear recovery threshold, denoted as $K_{\text{e-prod-linear}}^*$ is given by the following equation:⁷

$$K_{\text{e-prod-linear}}^* = \min\{N, 2R - 1\}. \quad (45)$$

Remark 9. Note that Step 2 of this proof does not require the computation strategy to be linear. Hence, using exactly the same coding approach, we can easily extend this result to non-linear codes, and prove a similar factor-of-2 characterization for the optimum recovery threshold K^* , formally stated in the following corollary.

Corollary 2. For a distributed matrix multiplication problem with parameters p , m , and n , let $N^*(p, m, n)$ denotes the minimum number of workers such that a valid (possibly non-linear) computation strategy exists. Then for all possible values of N , we have

$$N^*(p, m, n) \leq K^* \leq 2N^*(p, m, n) - 1. \quad (46)$$

VII. CONCLUDING REMARKS

In this paper, we studied the coded distributed matrix multiplication problem and proposed the entangled polynomial code, which allows optimal straggler mitigation and orderwise improves upon the prior arts. Based on our proposed coding idea, we proved a fundamental connection between the optimum linear recovery threshold and the bilinear complexity, which characterizes the optimum linear recovery threshold within a factor of 2 for all possible parameter values. The techniques developed in this paper can be directly applied to many other problems, including coded convolution and fault tolerance computing, providing matching characterizations.

One interesting follow-up direction is to find better characterization of the optimum linear recovery threshold. Although this problem is completely solved for cases including $m = 1$, $n = 1$, or $p = 1$, there is room for improvement in general cases. Another interesting question is whether there exist non-linear coding strategies that strictly out-perform linear codes, especially for the important case where the input matrices are large ($s, r, t \gg p, m, n$), while allowing for efficient decoding algorithms with almost linear complexity.

APPENDIX A

THE OPTIMUM LINEAR RECOVERY THRESHOLD FOR CODED CONVOLUTION

In this appendix, we first provide the problem formulation for coded convolution, then we prove Theorem 4, which shows the optimality of Polynomial Code for Coded Convolution.

A. System Model and Problem Formulation

Consider a convolution task with two input vectors

$$\mathbf{a} = [\mathbf{a}_0 \ \mathbf{a}_1 \ \dots \ \mathbf{a}_{m-1}], \quad \mathbf{b} = [\mathbf{b}_0 \ \mathbf{b}_1 \ \dots \ \mathbf{b}_{n-1}], \quad (47)$$

where all \mathbf{a}_i 's and \mathbf{b}_i 's are vectors of length s over a sufficiently large field \mathbb{F} . We want to compute $\mathbf{c} \triangleq \mathbf{a} * \mathbf{b}$ using a master and N workers. Each worker can store two vectors of length s , which are functions of \mathbf{a} and \mathbf{b} respectively. We refer to these functions as the *encoding functions*, denoted by (\mathbf{f}, \mathbf{g}) similar to the matrix multiplication problem.

Each worker computes the convolution of its stored vectors, and returns it to the master. The master only waits for the fastest subset of workers, before proceeding to decode \mathbf{c} . Similar to the matrix multiplication problem, we define the *recovery threshold* given the *encoding functions*, denoted by $K(\mathbf{f}, \mathbf{g})$, as the minimum number of workers that the master needs to wait that guarantees the existence of valid decoding functions. We aim to characterize the optimum recovery threshold achievable by any linear encoding functions, denoted by $K_{\text{conv-linear}}^*$, and identify an optimal computation strategie that achieves this optimum threshold.

B. Proof of Theorem 4

Now we prove Theorem 4, which completely solves the above problem. As we have shown in [10], the recovery threshold stated in Theorem 4 is achievable using a variation of polynomial code. This result proves an upperbound of $K_{\text{conv-linear}}^*$. It also identifies an optimal computation strategy. Hence, in this section we focus on proving the matching converse.

Specifically, we aim to prove that given any problem parameters m , n , and N , for any computation strategy, if the encoding functions (\mathbf{f}, \mathbf{g}) are linear, then its recovery threshold is at least $m + n - 1$. We prove it by contradiction.

Assume the opposite, then the master can recover \mathbf{c} using results from a subset of at most $m + n - 2$ workers. We denote this subset by \mathcal{K} . Obviously, we can find a partition of \mathcal{K} into two subsets, denoted by \mathcal{K}_a and \mathcal{K}_b , such that $|\mathcal{K}_a| \leq m - 1$ and

⁷Obviously, we need $N \geq R$ to guarantee the existence of a valid computation strategy.

$|\mathcal{K}_b| \leq n - 1$. Note that the encoding functions of workers in \mathcal{K}_a collaboratively and linearly maps \mathbb{F}^{ms} to $\mathbb{F}^{(m-1)s}$, which has a non-zero kernel. Hence, we can find a non-zero input vector \mathbf{a} such that all workers in \mathcal{K}_a returns 0. Similarly, we can find a non-zero \mathbf{b} such that all workers in \mathcal{K}_b returns 0. Recall that $\mathcal{K}_a \cup \mathcal{K}_b = \mathcal{K}$. Consequently, when the master receives 0 from all workers in \mathcal{K} , the decoding function returns $\mathbf{a} * \mathbf{b}$.

This convolution product must be the $\mathbf{0}$ vector, given that the workers return the same results under zero inputs. However, note that the convolution operator has no zero-divisor. Either \mathbf{a} or \mathbf{b} has to be zero, which contradicts the non-zero assumptions. Hence, we have $K(\mathbf{f}, \mathbf{g}) \geq m + n - 1$. This concludes the proof of Theorem 4.

APPENDIX B

AN EQUIVALENCE BETWEEN FAULT TOLERANCE AND STRAGGLER MITIGATION

In this appendix, we start by formulating a fault tolerance computing problem for matrix multiplication, then we prove Theorem 5 by building a connection between straggler mitigation and fault tolerance, by extending the concept of Hamming distance to coded computing.

A. Problem Formulation

We consider a matrix multiplication problem with two input matrices $A \in \mathbb{F}^{s \times r}$ and $B \in \mathbb{F}^{s \times t}$, and we are interested in computing $C \triangleq A^\top B$ using a master node and N worker nodes, where each worker can store $\frac{1}{pm}$ fraction of A and $\frac{1}{pn}$ fraction of B . Similar to the straggler mitigation problem, each worker i can store two coded matrices $\tilde{A}_i \in \mathbb{F}^{\frac{s}{p} \times \frac{r}{m}}$ and $\tilde{B}_i \in \mathbb{F}^{\frac{s}{p} \times \frac{t}{n}}$, computed based on A and B respectively. Each worker can compute the product $\tilde{C}_i \triangleq \tilde{A}_i^\top \tilde{B}_i$, and return it to the master. Unlike the straggler setting, the master waits for all workers before proceeding to recover the final output C . However, a subset of workers can return error results, and the master has no information on which subset of results are false. Under this setting, the master wants to: (1) determine if there is an error in the workers' outputs, and (2) try to recover the final output C using the possibly false computing results from the workers.

Given the above system model, we formulate this fault tolerance computing problem based on the following terminology. Similar to our main problem in this paper, we define the *encoding functions* and denote them by (\mathbf{f}, \mathbf{g}) . We also define the *decoding function* for the master, however in this problem it can either return an estimate of C , or report an error. We only consider the *valid* decoding functions, which always correctly decodes C when no worker is making mistakes.

For any integer E , we say the encoding functions can *detect* E errors if we can find a decoding function that either returns the correct value of C or reports an error, when no more than E workers are making mistakes. Moreover, we say the encoding functions can *correct* E errors, if the decoding function always correctly decodes C . We denote the maximum possible integer E given these two criteria by $E_{\text{detect}}(\mathbf{f}, \mathbf{g})$ and $E_{\text{correct}}(\mathbf{f}, \mathbf{g})$ respectively.

We aim to find encoding functions that allows detecting and correcting the maximum possible number of errors. Among all possible computation strategies, we are particularly interested in the *linear codes*, as defined in Section II. Given the above terminology, we define the following concepts.

Definition 5. For a distributed matrix multiplication problem of computing $A^\top B$ using N workers, we define the *maximum detectable errors* and the *maximum correctable errors* for linear codes, denoted by E_{detect}^* and E_{correct}^* respectively, as the maximum possible values of $E_{\text{detect}}(\mathbf{f}, \mathbf{g})$ and $E_{\text{correct}}(\mathbf{f}, \mathbf{g})$ over the set of all encoding functions that are linear codes.

Our goal is to characterize the values of E_{detect}^* and E_{correct}^* , and to find optimal computation strategies to achieve these values. We are also interested in extending these characterizations to non-linear codes.

B. Proof of Theorem 5

We start by defining some concepts, which allows connecting the fault tolerance computing problem to the straggler mitigation problem.

Definition 6. We define the *Hamming distance* of any encoding functions (\mathbf{f}, \mathbf{g}) , denoted by $d(\mathbf{f}, \mathbf{g})$, as the maximum integer d such that for any two pairs of input matrices whose products C are different, at least d workers compute different values of \tilde{C}_i .

Definition 7. We define the *Recovery threshold* of any encoding functions (\mathbf{f}, \mathbf{g}) , denoted by $K(\mathbf{f}, \mathbf{g})$, as the minimum possible recovery threshold given any decoding functions.

We prove that all these three mentioned criterion for designing encoding functions are directly connected by the Hamming distance, which is formally stated as follows.

Lemma 3. For any (possibly non-linear) computation strategy, we have

$$K(\mathbf{f}, \mathbf{g}) = N - d(\mathbf{f}, \mathbf{g}) + 1, \quad (48)$$

$$E_{\text{detect}}(\mathbf{f}, \mathbf{g}) = d(\mathbf{f}, \mathbf{g}) - 1, \quad (49)$$

$$E_{\text{correct}}(\mathbf{f}, \mathbf{g}) = \left\lfloor \frac{d(\mathbf{f}, \mathbf{g}) - 1}{2} \right\rfloor. \quad (50)$$

Remark 10. Lemma 3 essentially indicates that optimizing the straggler mitigation performance over any class of coding designs is equivalently optimizing its performance in the fault tolerance setting. Furthermore, all these previously mentioned metrics can be simultaneously optimized by the codes with the maximum possible Hamming distance. Hence, there is no tension among these metrics. This result bridges the rich literature of coding theory and distributed computing.

Proof of Lemma 3. Lemma 3 is a direct consequence of the classical coding theory, given that mitigating straggler effects is essentially correcting erasure errors, and tolerating false results in computing is essentially correcting arbitrary error. Hence, we only provide the proof of (48), where equations (49) and (50) can be proved using similar approaches.

Specifically, we want to prove that for any integer K , a recovery threshold of K is achievable by some encoding functions if and only if their Hamming distance is greater or equal to $N - K + 1$. If K is achievable, it means that we can find decoding functions that uniquely determines the value of C given results from any K workers. Equivalently, for distinct values of C , at least $N - K + 1$ workers has to return distinct results. Recall that the recovery threshold is the minimum of such integer K , and the Hamming distance is the maximum integer that corresponds to $N - K + 1$. We have $K(\mathbf{f}, \mathbf{g}) = N - d(\mathbf{f}, \mathbf{g}) + 1$. \square

Now we continue to prove Theorem 5 using Lemma 3. As mentioned in Remark 7, the proof of Theorem 2 essentially completely characterizes the optimum recovery threshold over all linear encoding functions for $m = 1$ or $n = 1$, which is given by $K_{\text{entangled-poly}}$. On the other hand, this optimum threshold also coincide with $K_{\text{entangled-poly}}$ for $p = 1$, which can be easily proved based on our prior results in [10]. Hence, using Lemma 3, we directly obtain that if $m = 1$ or $n = 1$ or $p = 1$, we have

$$E_{\text{detect}}^* = N - K_{\text{entangled-poly}}, \quad (51)$$

$$E_{\text{correct}}^* = \left\lfloor \frac{N - K_{\text{entangled-poly}}}{2} \right\rfloor. \quad (52)$$

This concludes the proof of Theorem 5.

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