Numerical Solution of Hyperbolic PDEs

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Outline

Outline

Convection Equation

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,t) &= u_0(x), \quad x \in \mathbb{R} \end{aligned}$$

Theoretical solution:

$$u(x,t) = u_0(x - at)$$

where, characteristic line

$$x - at = constant$$

simple schemes

upwind scheme

if $a > 0$,

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_j^n - u_{j-1}^n}{h} = 0$$

if $a < 0$,

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_{j+1}^n - u_j^n}{h} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{a + |a|}{2}\frac{u_j^n - u_{j-1}^n}{h} + \frac{a - |a|}{2}\frac{u_{j+1}^n - u_j^n}{h} = 0$$

conditional stable, $\lambda|a| \leq 1$

simple schemes

Lax-Friedrichs scheme

FTCS scheme:

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0$$

It is unconditional unstable!

$$\frac{u_j^{n+1}-u_j^n}{\tau}\to\frac{u_j^{n+1}-\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)}{\tau}$$

Lax-Friedrichs scheme

$$\frac{u_j^{n+1}-\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0$$

conditional stable, $\lambda|a|\le 1$

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=\frac{h}{2\lambda}\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}$$

simple schemes

Lax-Wendroff scheme

Taylor series expansion:

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \tau \left[\frac{\partial u}{\partial t}\right]_j^n + \frac{\tau^2}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + O(\tau^3)$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$u_j^{n+1} = u_j^n - \frac{\lambda a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\lambda^2 a^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = \frac{a^2\tau}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

conditional stable, $\lambda|a| \leq 1$

simple schemes
Lax-Wendroff scheme

von Neumann analysis:

$$
\begin{aligned}
G &= 1 - \frac{\sigma}{2}(e^{I\phi} - e^{-I\phi}) + \frac{\sigma^2}{2}(e^{I\phi} - 2 + e^{-I\phi}) \\
&= 1 - I\sigma sin\phi - \sigma^2(1 - cos\phi)
\end{aligned}
$$

$$\sigma = a\tau/h$$

simple schemes

characteristic line method

$$u(x_j, t^{n+1}) = u(x_j - a\tau, t^n)$$

provided, $x_{j-1} < x_j - a\tau < x_j$

Lagrangian interpolation:

$$f(x) \approx L_n(x) = \sum_{k=1}^{n} \prod_{\substack{i=0 \\ i \neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i} \right) y_k$$

simple schemes
characteristic line method

- $u_{j-1}^n, u_j^n \to u(x_Q, t_n)$, upwind

$$u_Q^n = \frac{a\tau}{h} u_{j-1}^n + \frac{h - a\tau}{h} u_j^n$$

- $u_{j-1}^n, u_{j+1}^n \to u(x_Q, t_n)$, Lax-Friedrichs

$$u_Q^n = \frac{h + a\tau}{2h} u_{j-1}^n + \frac{h - a\tau}{2h} u_{j+1}^n$$

- $u_{j-1}^n, u_j^n, u_{j+1}^n \to u(x_Q, t_n)$, Lax-Wendroff

$$u_Q^n = \frac{(a\tau)(h + a\tau)}{2h^2} u_{j-1}^n + \frac{(h - a\tau)(h + a\tau)}{h^2} u_j^n - \frac{(h - a\tau)(a\tau)}{2h^2} u_{j+1}^n$$

- $u^n_{j-2}, u^n_{j-1}, u^n_j \to u(x_Q, t_n)$, Beam-Warming

$$
\begin{aligned}
u^n_Q = & -\frac{(a\tau)(h-a\tau)}{2h^2}u^n_{j-2} + \frac{(2h-a\tau)(a\tau)}{h^2}u^n_{j-1} \\
& + \frac{(2h-a\tau)(h+a\tau)}{2h^2}u^n_j
\end{aligned}
$$

$$
u^{n+1}_j = u^n_Q = u^n_j - a\tau\frac{3u^n_j - 4u^n_{j-1} + u^n_{j_2}}{2h} + \frac{a^2\tau^2}{2}\frac{u^n_j - u^n_{j-1} + u^n_{j-2}}{h^2}
$$

conditional stable, $\lambda a \leq 2$

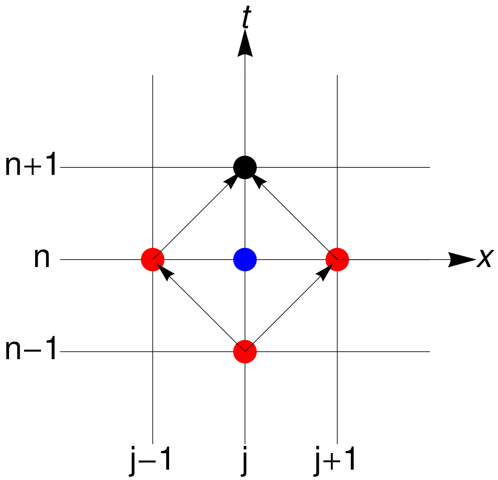
simple schemes

leap frog

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

conditional stable, $\lambda|a| \leq 1$

Computational molecule



Outline

1. First-order Hyperbolic Partial Differential Equation
   - Linear Case
   - quasi-linear case
2. Numerical tests
3. linear hyperbolic equations
4. Consistency condition, Truncation Error and Equivalent Differential Equation
5. CFL condition

$$
\begin{aligned}
\frac{\partial u}{\partial t} + a(x,t,u)\frac{\partial u}{\partial x} &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\
u(x,t) &= u_0(x), \quad x \in \mathbb{R}
\end{aligned}
$$

characteristic lines, $\frac{dx}{dt} = a(x,t,u)$ is curves.
Lax-Friedrichs scheme:

$$
\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\tau} + a_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0
$$

Burgers's equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

Test cases

Test Case 1 Find $u(x, 30)$ where

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,$$
$$u(x, 0) = -sin(\pi x),$$

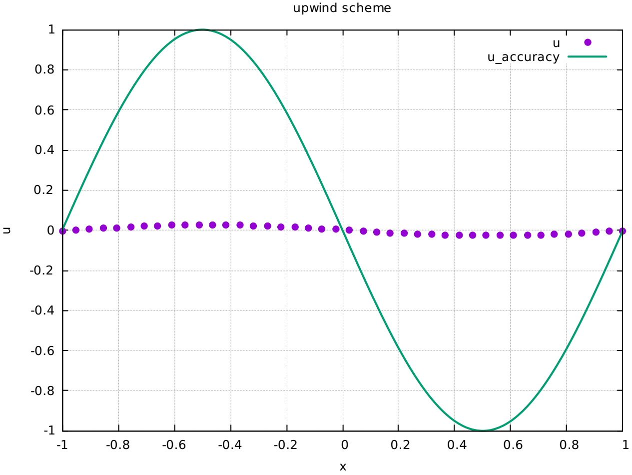
which is a linear advection of one period of a sinusoid.

Test Case 2 Find $u(x, 4)$ where

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= 0, \\ u(x,0) &= \begin{cases} 1 & for \quad |x| < \frac{1}{3} \\ 0 & for \quad \frac{1}{3} < |x| \leq 1 \end{cases}, \end{aligned}$$

which is linear convection with square wave.

Test1


upwind scheme
u
u_accuracy
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
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0
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0.4
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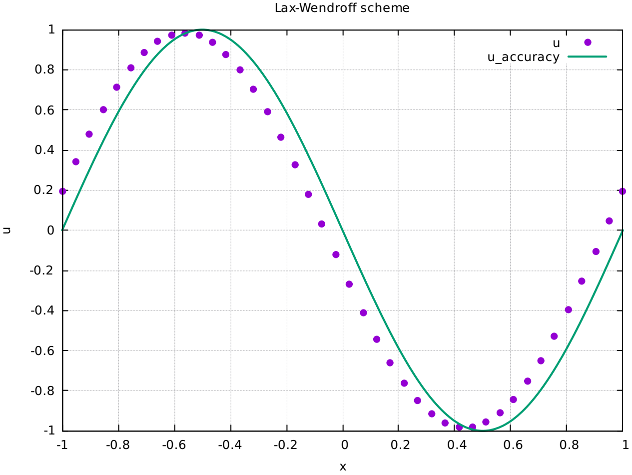
0.8
1
x
u

Test1


Lax-Wendroff scheme
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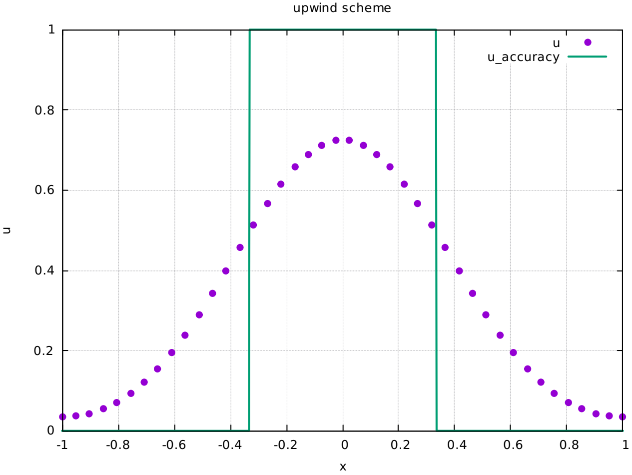
1
x

Test2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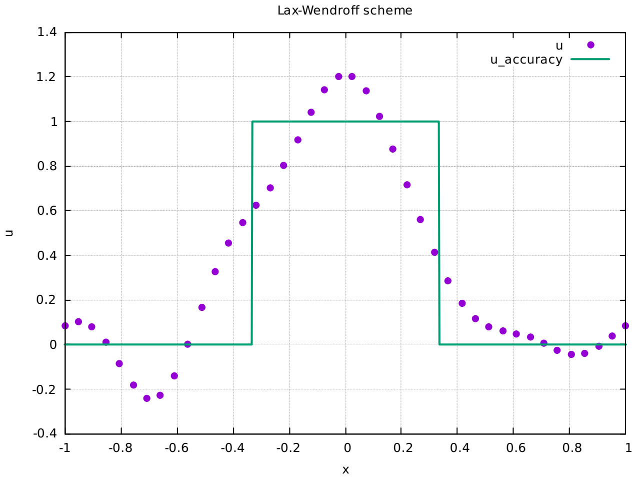
Test2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-0.4
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-0.8
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-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x
u

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0$$

definition: if all the eigenvalues of **A** are real, and

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = diag[\lambda_1, \lambda_2, \cdots, \lambda_p]$$

Lax-Friedriches schemes

$$\frac{\mathbf{u}_j^{n+1} - \frac{1}{2}(\mathbf{u}_{j+1}^n) + \mathbf{u}_{j-1}^n)}{\tau} + \mathbf{A}\frac{\mathbf{u}_{j+1}^n) + \mathbf{u}_{j-1}^n}{2h} = 0$$

upwind scheme

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

let $\mathbf{w} = \mathbf{S}^{-1}\mathbf{u}$

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}$$

characteristic form, decoupling:

$$\frac{\partial w_m}{\partial t} + \lambda_m \frac{\partial w_m}{\partial x} = 0, m = 1, \cdots, p$$

upwinding,

$$\mathbf{w}_j^{n+1} = \mathbf{w}_j^n - \frac{\tau}{2h}\mathbf{\Lambda}(\mathbf{w}_{j+1}^n - \mathbf{w}_{j-1}^n) + \frac{\tau}{2h}|\mathbf{\Lambda}|(\mathbf{w}_{j+1}^n - 2\mathbf{w}_j^n + \mathbf{w}_{j-1}^n)$$

Methology

- The function values u_{i+j}^{n+k} in the numerical scheme are developed in a Taylor series around the value u_i^n and the high order terms are maintained in substituting these developments back in the numerical equation.
- An equation is hereby obtained expressing the numerical scheme as the mathematical model equation plus additional terms, resulting from the Taylor series.These additional terms are called the truncation error, and noted ε_T
- The truncation error will have the form:

$$\varepsilon_T = O(\tau^p, h^q)$$

where p and q are the lowest values occurring in the development of the truncation error. This defines the order of accuracy of the scheme.

Equivalent Equation

$$u_t + au_x = 0$$

FTCS

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{a}{2h}(u_{j+1}^n - u_{j-1}^n) = 0$$

we can select u_j^n as representing either the exact solution of the mathematical model, or as the exact solution of the numerical scheme.

Taylor series expansion

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{a}{2h}(u_{j+1}^n-u_{j-1}^n)-(u_t+au_x)_j^n=\frac{\tau}{2}(u_{tt})_i^n + \frac{h^2}{6}a(u_{xxx})_j^n+O(\tau^2,h^4)$$

Truncation error:

$$\varepsilon_T=\frac{\tau}{2}(u_{tt})_i^n+\frac{h^2}{6}a(u_{xxx})_j^n+O(\tau^2,h^4)$$

The truncation error (TE) is therefore defined as the difference between the numerical scheme and the differential equation.

First interpretation of the consistency condition

The Taylor expansion is performed around the exact solution of the differential equation, i.e. around $\tilde{u}(jh, n\tau) \equiv \tilde{u}_j^n$, where $\tilde{u}(x,t)$ is the analytical solution.

$$\frac{\tilde{u}_j^{n+1} - \tilde{u}_j^n}{\tau} + a\frac{\tilde{u}_{j+1}^n - \tilde{u}_{j-1}^n}{2h} = \tilde{\varepsilon}_T$$

This relation shows that the exact solution $\tilde{u}_j^n$ does not satisfy the difference equation exactly, but is solution of a modified scheme, with the truncation error in the right-hand side.

Second interpretation of the consistency condition

The Taylor expansion is performed around the exact solution of the discretized equation $\bar{u}_j^n$.

$$(\bar{u}_t + a\bar{u}_x)_j^n = -\frac{\tau}{2}(\bar{u}_{tt})_j^n - a\frac{h^2}{6}(\bar{u}_{xxx})_j^n - O(\tau^2, h^4) \equiv -\bar{\varepsilon}_T$$

This relation shows that the exact solution of the discretized equation does not satisfy exactly the differential equation at finite values of t and x

However, the solution of the numerical scheme satisfies an equivalent differential equation (EDE), also sometimes called modified differential equation, which differs from the original (differential) equation by a truncation error represented by the terms on the right-hand side.

The truncation error:

$$\bar{\varepsilon}_T = a^2 \frac{\tau}{2} (\bar{u}_{xx})_j^n + a \frac{h^2}{6} (\bar{u}_{xxx})_j^n + O(\tau^2, h^4)$$

Equivalent differential equation

$$(\bar{u}_t + a\bar{u}_x)_j^n = -a^2 \frac{\tau}{2} (\bar{u}_{xx})_j^n + O(\tau^2, h^2)$$

FOU Scheme

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{a}{h}(u_j^n - u_{j-1}^n) = 0$$

EDE:

$$\frac{\partial \bar{u}}{\partial t} + a\frac{\partial \bar{u}}{\partial x} = \frac{ah}{2}\left(1 - \frac{a\tau}{h}\right)\frac{\partial^2 \bar{u}}{\partial x^2} \equiv \nu_{num}\frac{\partial^2 \bar{u}}{\partial x^2}$$

numerical diffusion or numerical viscosity:

$$\nu_{num} \equiv \frac{ah}{2}\left(1 - \frac{a\tau}{h}\right)$$

negative diffusion is unstable, hence

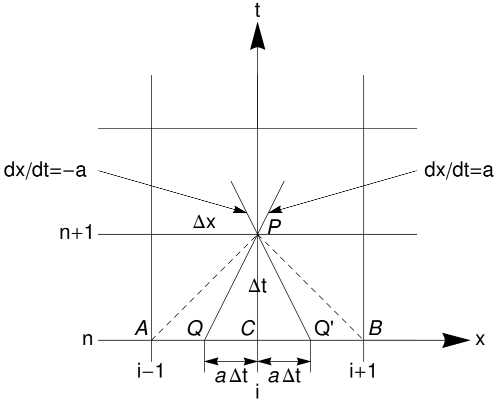
$$0 \leq \frac{a\tau}{h} \leq 1$$

the Courant or CFL (Courant-Friedrichs-Lewy) number $\sigma = \frac{a\tau}{h}$

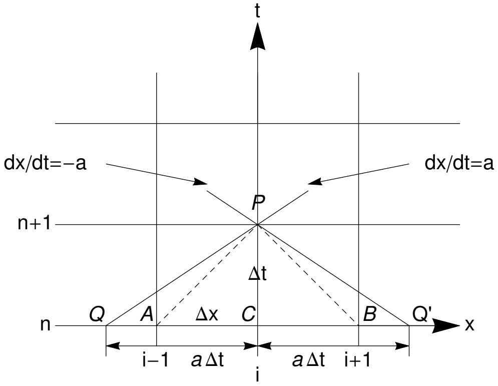
The CFL stability condition $\sigma < 1$ expresses that the mesh ratio τ/h has to be chosen in such a way that **the domain of dependence of differential equation should be entirely contained in the numerical domain of dependence of the discretized equations**.

In other words, the numerical scheme defining the approximation u_j^{n+1} in mesh point j must be able to include all the physical information which influences the behavior of the system in this point.

$\sigma < 1$



$\sigma > 1$



Implicit scheme