

Project Report - Analyzing Educational Data

Daniel Posmik, Aristofanis Rontogiannis, Jizhou Tian

2024-10-30

Table of contents

Introduction (Daniel)	1
Part 1: Linear Regression Analysis (Daniel)	3
The Least Squares Estimators	3
Hypothesis Testing	5
Part 2: Principal Component Analysis (Aristofanis)	8
Part 3: Non-linear Regression Analysis (Jizhou)	8
Conclusion	8
Code Appendix	9

Introduction (Daniel)

For this project, we will be analyzing educational data. We are interested in understanding the predictors of student performance as measured by exam scores. We will be using a publicly available dataset from Kaggle that contains information about students and their exam scores.

Table 1: Variable Summary for the Educational Data

Variable Name	Variable Type	Description
Hours_Studied	numeric	Hours Studied
Attendance	numeric	Attendance
Parental_Involvement	factor	Parental Involvement
Access_to_Resources	factor	Access to Resources
Extracurricular_Activities	factor	Extracurricular Activities
Sleep_Hours	numeric	Sleep Hours
Previous_Scores	numeric	Previous Scores
Motivation_Level	factor	Motivation Level

Variable Name	Variable Type	Description
Internet_Access	factor	Internet Access
Tutoring_Sessions	numeric	Tutoring Sessions
Family_Income	factor	Family Income
Teacher_Quality	factor	Teacher Quality
School_Type	factor	School Type
Peer_Influence	factor	Peer Influence
Physical_Activity	numeric	Physical Activity
Learning_Disabilities	factor	Learning Disability
Parental_Education_Level	factor	Parental Education Level
Distance_from_Home	factor	Distance from Home
Gender	factor	Gender
Exam_Score	numeric	Exam Score

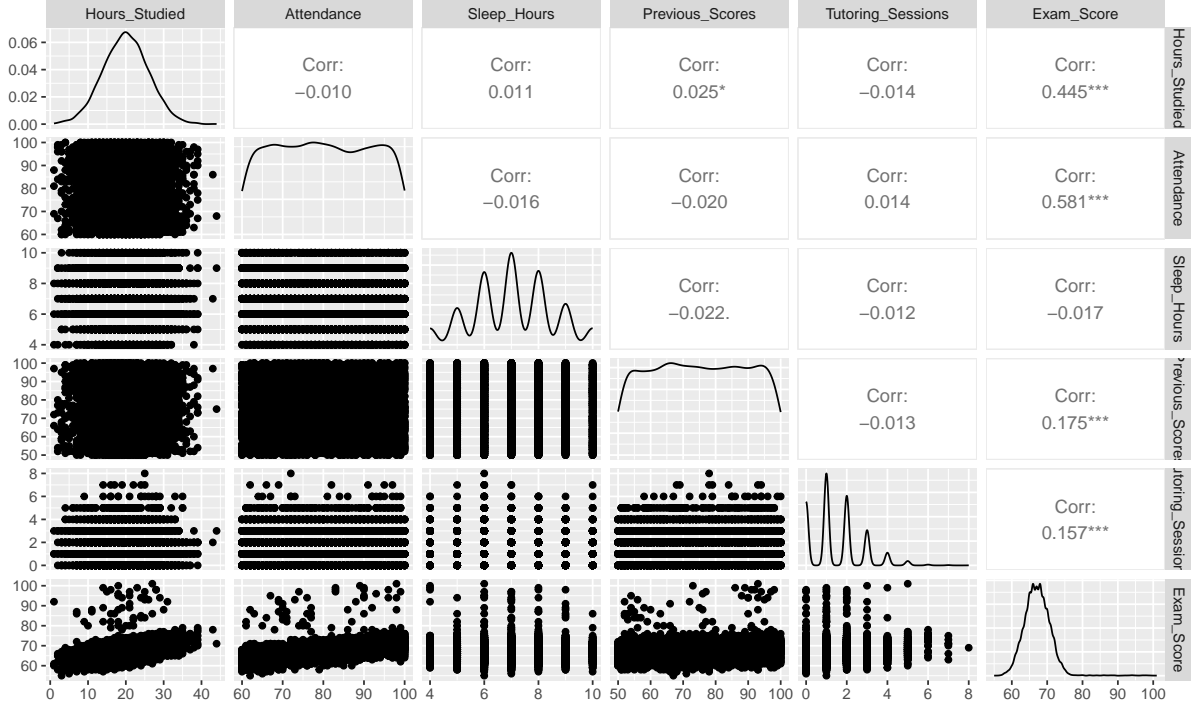
Now, we want to further explore a specific hypothesis about a subset of predictor variables. Suppose we maintain that the following variables are significant predictors:

- Hours Studied
- Attendance
- Sleep Hours
- Previous Scores
- Tutoring Sessions

We can formalize this question as follows:

- $H_0 : [1_{[0, \dots, p+1]}, 0_{[p+2, \dots, P]}] \cdot [\beta_0 \ \dots \ \beta_P]^T = \beta_0 + \dots + \beta_{p+1} = 0$
- $H_A : \{\beta_1 \neq 0\} \cap \dots \cap \{\beta_5 \neq 0\}$

Before we begin our analysis, let us take a look at the dependencies across these data:



Part 1: Linear Regression Analysis (Daniel)

The Least Squares Estimators

Let us begin by discussing the assumptions of linear regression model. In a Gauss-Markov setting, we assume that our linear model is of the form:

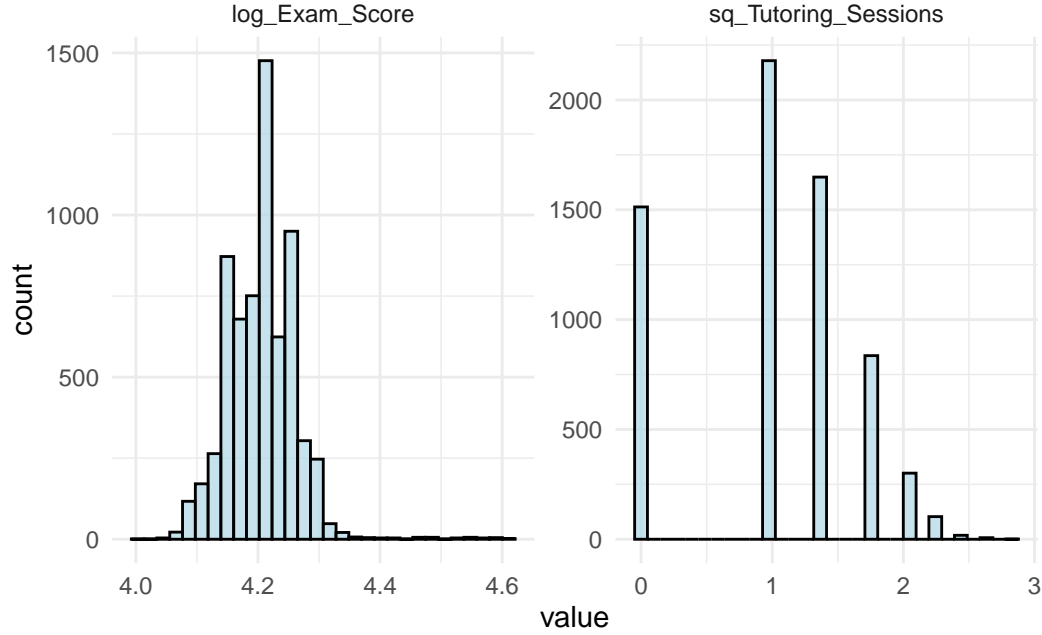
$$\mathbf{Y} = \mathbf{X}\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I)$$

where $\mathbb{E}[\epsilon] = 0$ and $\text{Var}[\epsilon] = \sigma^2 I$ denote the zero-mean and constant variance assumptions. In our case, we begin with $p = 5$, i.e. our design matrix has $p + 1$ columns, accounting for the intercept term. Then, we can write the model as matrices:

That being said, what the Gauss-Markov model boasts in theoretical simplicity, it often lacks in practical validity. If we refer to the exploratory analysis above, we can see that the assumptions may not hold. For one, we have one predictor variable, **Tutoring_Sessions**, and our dependent variable, **Exam_Score**, that are right-skewed. This violates the assumption of normality. Moreover, the Gauss-Markov model assumes constant variance with zeros on the off-diagonal elements of the covariance matrix. In practice, this is an assumption that is frequently violated. Interestingly, in our case the correlation between our predictor variables is

indeed close to 0. If we had more substantial correlations on the off-diagonal elements, we could have solved our estimation problem with the generalized least squares estimator.

In our case, we will remedy the normality assumption by transforming our data. We will use logarithmic transformations on the Exam score variable to achieve a normal distribution (`log_Exam_Score`) and a square root transformation of the Tutoring sessions variable to achieve a distribution that more closely resembles a normal distribution (`log_Tutoring_Sessions`). We chose the square root transformation for the tutoring sessions variable because it contains a lot of 0s, making the logarithmic transformation less suitable.



$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{12} & X_{13} & \cdots & X_{1(p+1)} \\ 1 & X_{22} & X_{23} & \cdots & X_{2(p+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n2} & X_{n3} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

Note that we are using the generalized matrix inverse in case the design matrix is not of full rank. The canonical matrix inverse of the form X^{-1} exists iff X is of full rank. Next, we can solve for $\hat{\beta}$ via the normal equations:

$$\begin{aligned}
\hat{\beta} &= (X^T X)^g X^T Y \\
&= \left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1(p+1)} & X_{2(p+1)} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} 1 & X_{12} & \cdots & X_{1(p+1)} \\ 1 & X_{22} & \cdots & X_{2(p+1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n2} & \cdots & X_{n(p+1)} \end{bmatrix} \right)^g \\
&\quad \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1(p+1)} & X_{2(p+1)} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}
\end{aligned}$$

Using the R `lm()` function, we can estimate the coefficients of the linear model:

We can see that all variables except for **Sleep_Hours** are significant predictors of exam scores, even at a 1% significance level. So what does this tell us about our hypothesis? We will further examine this question in the next subsection.

Hypothesis Testing

Our estimation question is a hypothesis testing problem. In the following, we will rigorously treat it as such, testing whether our subset of predictors (see above) is jointly significant in the prediction of exam scores. Before we proceed, let us introduce additional notation in our hypothesis testing problem:

$$\mathbf{K}^T \beta = [1_{[0, \dots, p+1]}, \quad 0_{[p+2, \dots, P]}] \cdot [\beta_0 \quad \cdots \quad \beta_P]^T = \beta_0 + \cdots + \beta_{p+1} = \mathbf{M}_{1, (p+1)}$$

where $\{\beta_0, \dots, \beta_{p+1}\}$ are the coefficients of the predictors we are interested in and $\{\beta_{p+2}, \dots, \beta_P\}$ are the coefficients of the remaining predictors. Naturally, $p \leq P$.

A necessary condition for the hypothesis to be testable is that $\mathbf{K}^T \beta$ is estimable. We say $\exists A$ s.t. $X^T A = K^T$, i.e. the rows of K are linearly dependent on the rows of X . Indeed, we can verify this without the calculation because we can see that \mathbf{L}^T can be expressed as a linear combination of the design matrix, i.e. the column space of X , $\mathbb{C}(X)$, contains the column space of K , $\mathbb{C}(K)$. A counterexample would be if one of our predictors consisted of 0's only, rendering us unable to estimate \mathbf{K}^T with \mathbf{X} .

We are now ready to state an important intermediate distributional result. Since $\mathbf{K}^T \beta$ is estimable, its best linear unbiased estimator (BLUE) is given by:

$$\begin{aligned}
\mathbf{K}_i^T \hat{\beta} &\sim N(\mathbf{K}_i^T (X^T X)^g X^T X \beta, \sigma^2 \mathbf{K}_i^T (X^T X)^g \mathbf{K}_i) \quad \text{and} \\
\mathbf{K}^T \hat{\beta} &\sim N(\mathbf{K}^T (X^T X)^g X^T X \beta, \sigma^2 \mathbf{K}^T (X^T X)^g \mathbf{K})
\end{aligned}$$

	(1)
(Intercept)	3.818*** (0.004)
Hours_Studied	0.004*** (0.000)
Attendance	0.003*** (0.000)
Sleep_Hours	0.000 (0.000)
Previous_Scores	0.001*** (0.000)
sq_Tutoring_Sessions	0.013*** (0.001)
Num.Obs.	6607
R2	0.639
R2 Adj.	0.638
AIC	−26 058.3
BIC	−26 010.7
Log.Lik.	13 036.154
F	2334.162
RMSE	0.03
+ p < 0.1, * p < 0.05, ** p < 0.01, *** p < 0.001	

Indeed, we can test our hypothesis by constructing a quadratic form. While this is certainly not the only way to test our hypothesis, it is a tractable method to incorporate the precision of each $\hat{\beta}_i$ into our hypothesis testing framework. We will see momentarily that this quadratic form results in favorable distributional properties thanks to the previous normal distributional result. Now, defining $H := K(X^T X)^g K^T$,

$$\begin{aligned} \mathbf{K}^T \hat{\beta} &\sim N(\mathbf{K}^T (X^T X)^g X^T X \beta, \sigma^2 \mathbf{K}^T (X^T X)^g \mathbf{K}) \\ \Leftrightarrow \mathbf{K}^T \hat{\beta} &\sim N(\mathbf{K} \beta, \sigma^2 H) \end{aligned}$$

we can construct the quadratic form

$$(K\beta)^T (\sigma^2 H)^{-1} (K\hat{\beta}) \sim \chi_{\text{df}=\text{rank}(H)}^2(\lambda)$$

where the non-centrality parameter $\lambda = \frac{1}{2}(K\beta)^T (\sigma^2 H)^{-1} (K\beta)$ by the well-known distributional result of a normal quadratic form. We are now ready to construct the F-test statistic as follows:

$$F := \frac{((K\beta)^T (\sigma^2 H)^{-1} (K\hat{\beta})) / \text{rank}(H)}{\text{RSS} / (n - p)} \sim \frac{\chi^2(\lambda)}{\chi^2} \sim F_{\text{rank}(H), n-p}(\lambda)$$

We have successfully constructed a statistical test that allows us to test our hypothesis with a simple F-test. This is very attractive seeing how this test incorporates the precision of our estimates into the hypothesis testing framework, yet is computationally simple. In R, we can use the `anova()` function to perform this test.

Table 2: F-Test Results for the Hypothesis Test

Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
6606	20.696858	NA	NA	NA	NA
6601	7.477088	5	13.21977	2334.162	0

The result shows that under the null hypothesis, the probability of getting a more extreme result than our calculate F-test statistics $\text{Pr}(>F)$ is $2.2e - 16$. This evidence would lead us to reject the null hypothesis and conclude that our subset of predictors is indeed a significant predictor of exam scores. The observed test statistics agree exactly with the ones reported in the regression table. Using the F-statistic is important in settings like ours when we are interested in joint model significance rather than individual predictor significance. It is noteworthy that this analysis could certainly be extended to different model specifications, however, this is beyond the scope of this project.

Part 2: Principal Component Analysis (Aristofanis)

Part 3: Non-linear Regression Analysis (Jizhou)

Conclusion

Code Appendix

```
# Set up knit environment
knitr::opts_chunk$set(echo = F)
knitr::opts_chunk$set(error = F)
knitr::opts_chunk$set(warning = F)
knitr::opts_chunk$set(message = F)

# Load necessary packages
library(tidyverse)
library(kableExtra)
library(knitr)
library(broom)
library(ggplot2)
library(naniar)
library(gtsummary)
library(GGally)
library(MASS)

# Load the data
educ_dta <- read_csv("student_performance.csv") %>%
  mutate(
    Parental_Involvement = as.factor(Parental_Involvement),
    Access_to_Resources = as.factor(Access_to_Resources),
    Extracurricular_Activities = as.factor(Extracurricular_Activities),
    Motivation_Level = as.factor(Motivation_Level),
    Internet_Access = as.factor(Internet_Access),
    Family_Income = as.factor(Family_Income),
    Teacher_Quality = as.factor(Teacher_Quality),
    School_Type = as.factor(School_Type),
    Peer_Influence = as.factor(Peer_Influence),
    Learning_Disabilities = as.factor(Learning_Disabilities),
    Parental_Education_Level = as.factor(Parental_Education_Level),
    Distance_from_Home = as.factor(Distance_from_Home),
    Gender = as.factor(Gender)
  )

# Summary table
table_summary <- tibble(
  "Variable Name" = colnames(educ_dta),
  "Variable Type" = sapply(educ_dta, class),
  "Description" = c(
```

```

    "Hours Studied", #1
    "Attendance", #2
    "Parental Involvement", #3
    "Access to Resources", #4
    "Extracurricular Activities", #5
    "Sleep Hours", #6
    "Previous Scores", #7
    "Motivation Level", #8
    "Internet Access", #9
    "Tutoring Sessions", #10
    "Family Income", #11
    "Teacher Quality", #12
    "School Type", #13
    "Peer Influence", #14
    "Physical Activity", #15
    "Learning Disability", #16
    "Parental Education Level", #17
    "Distance from Home", #18
    "Gender", #19
    "Exam Score" #20
  )
)

# Display the table
knitr::kable(table_summary,
  caption = "Variable Summary for the Educational Data")

pred_var <- c("Hours_Studied",
  "Attendance",
  "Sleep_Hours",
  "Previous_Scores",
  "Tutoring_Sessions")

pred_dta <- educ_dta %>%
  dplyr::select(all_of(pred_var), "Exam_Score")

GGally::ggpairs(pred_dta)
# Log-transform skewed variables
educ_dta <- educ_dta %>%
  mutate(
    log_Exam_Score = ifelse(Exam_Score == 0, 0, log(Exam_Score)),
    sq_Tutoring_Sessions = sqrt(Tutoring_Sessions)
  )

```

```

    )

# Display histograms next to each other
educ_dta %>%
  dplyr::select(log_Exam_Score, sq_Tutoring_Sessions) %>%
  gather() %>%
  ggplot(aes(value)) +
  geom_histogram(bins = 30, color = "black",
    fill = "lightblue", alpha = 0.7) +
  facet_wrap(~key, scales = "free") +
  theme_minimal()

# Fit the linear model
lm_model <-
  lm(log_Exam_Score ~ Hours_Studied + Attendance +
    Sleep_Hours + Previous_Scores + sq_Tutoring_Sessions,
    data = educ_dta)

# Summary
lm.tbl <- summary(lm_model)

# Print with modelsummary
modelsummary::modelsummary(lm_model, stars = TRUE, caption = "Linear Regression Results")

# Null model
null_model <- lm(log_Exam_Score ~ 1, data = educ_dta)

# Perform the F-test
anova.tbl <- anova(null_model, lm_model)

# Print
kableExtra::kable(anova.tbl,
  caption = "F-Test Results for the Hypothesis Test")

```