Predicting Student Performance with Educational Data

PHP 2601 (Linear Models) Final Project

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Introduction

For this project, we will be analyzing educational data. We are interested in understanding the predictors of student performance as measured by exam scores. We will be using a publicly available dataset from Kaggle that contains information about students and their exam scores.

Table 1: Variable Summary for the Educational Data

Variable Name	Variable Type	Description
Hours_Studied	numeric	Hours Studied
Attendance	numeric	Attendance
Parental_Involvement	factor	Parental Involvement
Access_to_Resources	factor	Access to Resources
Extracurricular_Activities	factor	Extracurricular Activities
Sleep_Hours	numeric	Sleep Hours
Previous_Scores	numeric	Previous Scores
Motivation_Level	factor	Motivation Level
Internet_Access	factor	Internet Access
Tutoring_Sessions	numeric	Tutoring Sessions
Family_Income	factor	Family Income
Teacher_Quality	factor	Teacher Quality
School_Type	factor	School Type
Peer_Influence	factor	Peer Influence
Physical_Activity	numeric	Physical Activity
Learning_Disabilities	factor	Learning Disability
Parental_Education_Level	factor	Parental Education Level
Distance_from_Home	factor	Distance from Home
Gender	factor	Gender
Exam_Score	numeric	Exam Score

Now, we want to further explore a specific hypothesis about a subset of predictor variables. Suppose we maintain that the following variables are significant predictors:

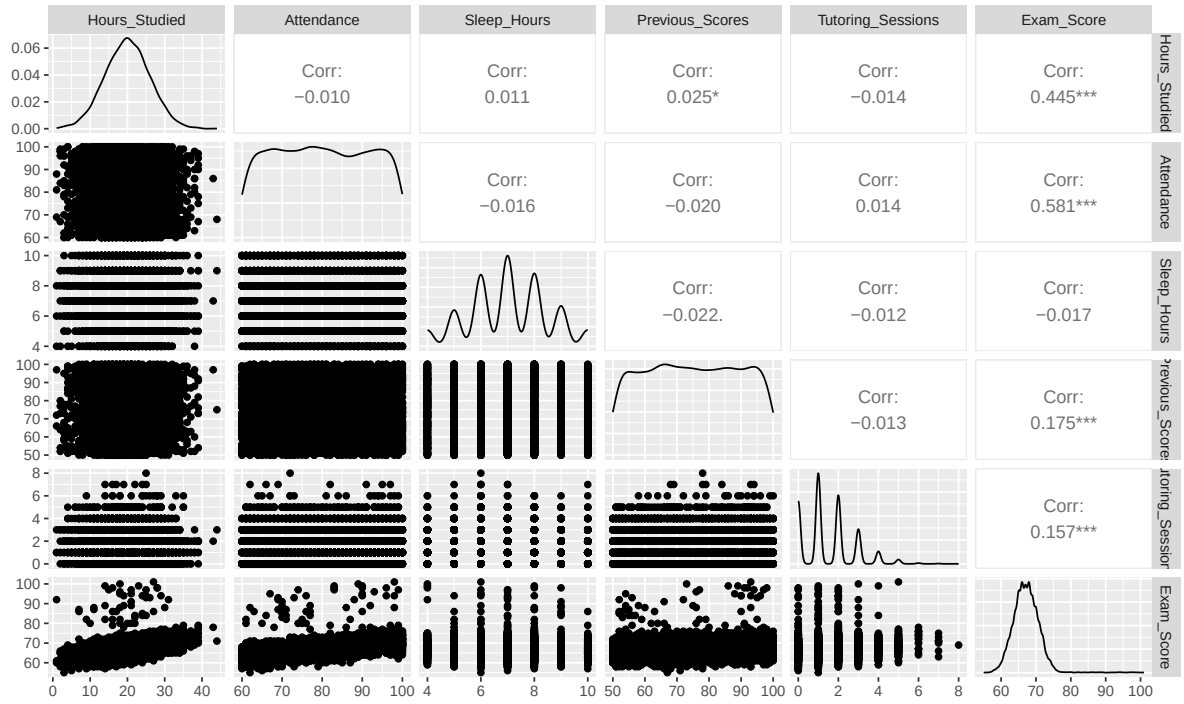
- Hours Studied
- Attendance
- Sleep Hours
- Previous Scores
- Tutoring Sessions

We can formalize this question as follows:

- $H_0 : \begin{bmatrix} 1_{[0,\cdots,p+1]}, & 0_{[p+2,\cdots,P]} \end{bmatrix} \cdot \begin{bmatrix} \beta_0 & \cdots & \beta_P \end{bmatrix}^T = \beta_0 + \cdots + \beta_{p+1} = 0$

- $H_A : \{\beta_1 \neq 0\} \cap \cdots \cap \{\beta_5 \neq 0\}$

Before we begin our analysis, let us take a look at the dependencies across these data:



Part 1: Linear Regression Analysis

The Least Squares Estimators

Let us begin by discussing the assumptions of linear regression model. In a Gauss-Markov setting, we assume that our linear model is of the form:

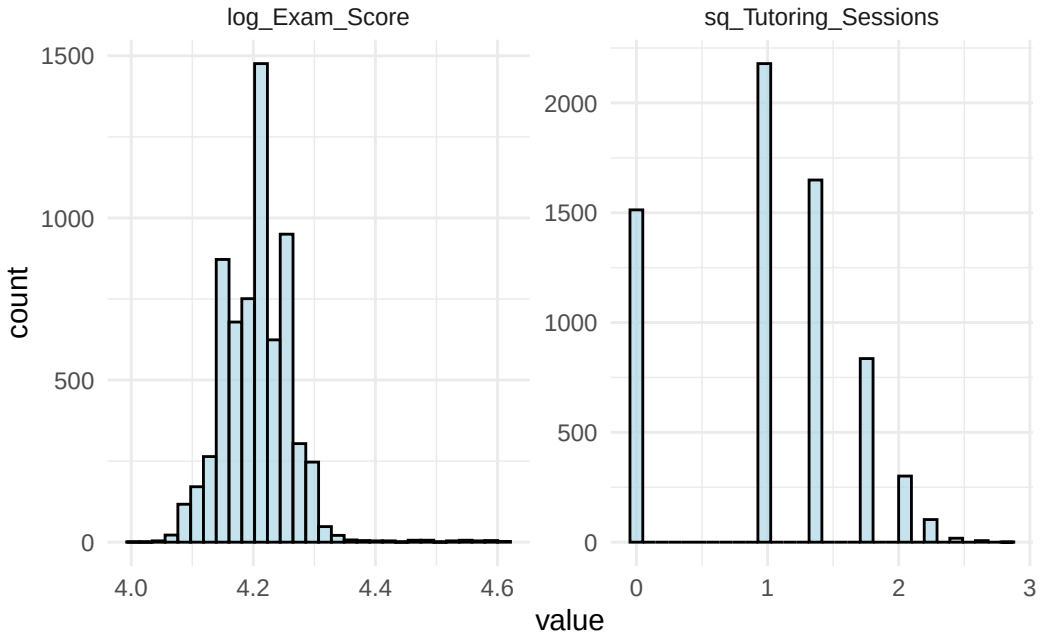
$$\mathbf{Y} = \mathbf{X}\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I)$$

where $\mathbb{E}[\epsilon] = 0$ and $\text{Var}[\epsilon] = \sigma^2 I$ denote the zero-mean and constant variance assumptions. In our case, we begin with $p = 5$, i.e. our design matrix has $p + 1$ columns, accounting for the intercept term. Then, we can write the model as matrices:

That being said, what the Gauss-Markov model boasts in theoretical simplicity, it often lacks in practical validity. If we refer to the exploratory analysis above, we can see that the assumptions may not hold. For one, we have one predictor variable, `Tutoring_Sessions`, and our dependent variable, `Exam_Score`, that are right-skewed. This violates the assumption of normality. Moreover, the Gauss-Markov model assumes constant variance with zeros on the

off-diagonal elements of the covariance matrix. In practice, this is an assumption that is frequently violated. Interestingly, in our case the correlation between our predictor variables is indeed close to 0. If we had more substantial correlations on the off-diagonal elements, we could have solved our estimation problem with the generalized least squares estimator.

In our case, we will rememedy the normality assumption by transforming our data. We will use logaritmic transformations on the Exam score variable to achieve a normal distribution (`log_Exam_Score`) and a square root transformation of the Tutoring sessions variable to achieve a distribution that more closely resembles a normal distribution (`log_Tutoring_Sessions`). We chose the square root transformation for the tutoring sessions variable because it contains a lot of 0s, making the logarithmic transformation less suitable.



$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{12} & X_{13} & \cdots & X_{1(p+1)} \\ 1 & X_{22} & X_{23} & \cdots & X_{2(p+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n2} & X_{n3} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

Note that we are using the generalized matrix inverse in case the design matrix is not of full rank. The canonical matrix inverse of the form X^{-1} exists iff X is of full rank. Next, we can solve for $\hat{\beta}$ via the normal equations:

$$
\begin{aligned}
\hat{\beta} =& (X^T X)^g X^T Y \\
=& \left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1(p+1)} & X_{2(p+1)} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} 1 & X_{12} & \cdots & X_{1(p+1)} \\ 1 & X_{22} & \cdots & X_{2(p+1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n2} & \cdots & X_{n(p+1)} \end{bmatrix} \right)^g \cdot \\
& \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1(p+1)} & X_{2(p+1)} & \cdots & X_{n(p+1)} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}
\end{aligned}
$$

Using the R `lm()` function, we can estimate the coefficients of the linear model:

```
Call:
lm(formula = log_Exam_Score ~ Hours_Studied + Attendance + Sleep_Hours +
    Previous_Scores + sq_Tutoring_Sessions, data = educ_dta)

Residuals:
     Min       1Q   Median       3Q      Max
-0.08391 -0.01675 -0.00173  0.01336  0.37210

Coefficients:
                       Estimate Std. Error t value Pr(>|t|)
(Intercept)           3.818e+00  4.460e-03 856.076   <2e-16 ***
Hours_Studied         4.350e-03  6.916e-05  62.899   <2e-16 ***
Attendance            2.951e-03  3.588e-05  82.256   <2e-16 ***
Sleep_Hours          -2.646e-04  2.822e-04  -0.938    0.348
Previous_Scores       7.146e-04  2.878e-05  24.827   <2e-16 ***
sq_Tutoring_Sessions  1.339e-02  6.417e-04  20.865   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.03366 on 6601 degrees of freedom
Multiple R-squared:  0.6387,    Adjusted R-squared:  0.6385
F-statistic:  2334 on 5 and 6601 DF,  p-value: < 2.2e-16
```

We can see that all variables except for `Sleep_Hours` are significant predictors of exam scores, even at a 1% significance level. So what does this tell us about our hypothesis? We will further examine this question in the next subsection.

	(1)
(Intercept)	3.818***
	(0.004)
Hours_Studied	0.004***
	(0.000)
Attendance	0.003***
	(0.000)
Sleep_Hours	0.000
	(0.000)
Previous_Scores	0.001***
	(0.000)
sq_Tutoring_Sessions	0.013***
	(0.001)
Num.Obs.	6607
R2	0.639
R2 Adj.	0.638
AIC	−26 058.3
BIC	−26 010.7
Log.Lik.	13 036.154
F	2334.162
RMSE	0.03

+ p <0.1, * p <0.05, ** p <0.01, *** p <0.001

Hypothesis Testing

Our estimation question is a hypothesis testing problem. In the following, we will rigorously treat is such, testing whether our subset of predictors (see above) is jointly significant in the prediction of exam scores. Before we proceed, let us introduce additional notation in our hypothesis testing problem:

$$\mathbf{K}^T\beta = \begin{bmatrix}1_{[0,\cdots,p+1]}, & 0_{[p+2,\cdots,P]}\end{bmatrix} \cdot \begin{bmatrix}\beta_0 & \cdots & \beta_P\end{bmatrix}^T = \beta_0 + \cdots + \beta_{p+1} = \mathbf{M}_{1,(p+1)}$$

where $\{\beta_0, \dots, \beta_{p+1}\}$ are the coefficients of the predictors we are interested in and $\{\beta_{p+2}, \dots, \beta_P\}$ are the coefficients of the remaining predictors. Naturally, $p \leq P$.

A necessary condition for the hypothesis to be testable is that $\mathbf{K}^T\beta$ is estimable. We say $\exists\ A$ s.t. $X^TA = K^T$, i.e. the rows of K are linearly dependent on the rows of X. Indeed, we can verify this without the calculation because we can see that $\mathbf{L}^T$ can be expressed as a linear combination of the design matrix, i.e. the columns space of X, $\mathbb{C}(X)$, contains the column space of K, $\mathbb{C}(\mathbf{K})$. A counterexample would be if one of our predictors consisted of 0's only, rendering us unable to estimate $\mathbf{K}^T$ with $\mathbf{X}$.

We are now ready to state an important intermediate distributional result. Since $\mathbf{K}^T\beta$ is estimable, its best linear unbiased estimator (BLUE) is given by:

$$\begin{aligned}\mathbf{K_i}^T\hat{\beta} &\sim N(\mathbf{K_i}^T(X^TX)^gX^TX\beta, \sigma^2\mathbf{K_i}^T(X^TX)^g\mathbf{K_i}) \quad \text{and} \\ \mathbf{K}^T\hat{\beta} &\sim N(\mathbf{K}^T(X^TX)^gX^TX\beta, \sigma^2\mathbf{K}^T(X^TX)^g\mathbf{K})\end{aligned}$$

Indeed, we can can test our hypothesis by constructing a quadratic form. While this is certainly not the only way to test our hypothesis, it is a tractable method to incorporate the precision of each $\hat{\beta}_i$ into our hypothesis testing framework. We will see momentarily that this quadratic form results in favorable distributional properties thanks to the previous normal distributional result. Now, defining $H \coloneqq K(X^TX)^gK^T$,

$$\begin{aligned}&\mathbf{K}^T\hat{\beta} \sim N(\mathbf{K}^T(X^TX)^gX^TX\beta, \sigma^2\mathbf{K}^T(X^TX)^g\mathbf{K}) \\ \iff\ &\mathbf{K}^T\hat{\beta} \sim N(\mathbf{K}\beta, \sigma^2H)\end{aligned}$$

we can construct the quadratic form

$$(K\beta)^T(\sigma^2H)^{-1}(K\hat{\beta}) \sim \chi^2_{\text{df=rank}(H)}(\lambda)$$

where the non-centrality parameter $\lambda = \frac{1}{2}(K\beta)^T(\sigma^2H)^{-1}(K\beta)$ by the well-known distributional result of a normal quadratic form. We are now ready to construct the F-test statistic as follows:

$$F := \frac{\left((K\beta)^T(\sigma^2 H)^{-1}(K\beta)\right)/\text{rank}(H)}{\text{RSS}/(n-p)} \sim \frac{\chi^2(\lambda)}{\chi^2} \sim F_{\text{rank}(H),n-p}(\lambda)$$

We have successfully constructed a statistical test that allows us to test our hypothesis with a simple F-test. This is very attractive seeing how this test incorporates the precision of our estimates into the hypothesis testing framework, yet is computationally simple. In `R`, we can use the `anova()` function to perform this test.

```
Analysis of Variance Table

Model 1: log_Exam_Score ~ 1
Model 2: log_Exam_Score ~ Hours_Studied + Attendance + Sleep_Hours + Previous_Scores +
    sq_Tutoring_Sessions
  Res.Df     RSS Df Sum of Sq      F    Pr(>F)
1   6606 20.6969
2   6601  7.4771  5     13.22 2334.2 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The result shows that under the null hypothesis, the probability of getting a more extreme result than our calculate F-test statistics `Pr(>F)` is $2.2e-16$. This evidence would lead us to reject the null hypothesis and conclude that our subset of predictors is indeed a significant predictor of exam scores. The observed test statistics agree exactly with the ones reported in the regression table. Using the F-statistic is important in settings like ours when we are interested in joint model significance rather than individual predictor significance. It is noteworthy that this analysis could certainly be extended to different model specifications, however, this is beyond the scope of this project.

Part 2: Principal Component Analysis (Aristofanis)

Part 3: Non-linear Regression Analysis (Jizhou)

Random Forest is a powerful and flexible ensemble learning method used for regression, classification, and other tasks. It is based on the idea of combining multiple decision trees to improve predictive performance and robustness, aiming to overcome the over-fitting problem of individual decision tree.

Each decision tree in the Random Forest is trained on a bootstrap sample of the original data, meaning that each tree is trained using approximately two-thirds (63.2%) of the entire training data set. Furthermore, at each split in a tree, a subset of the predictor variables is selected **randomly** to determine the split, further reducing overfitting and encouraging model variance.

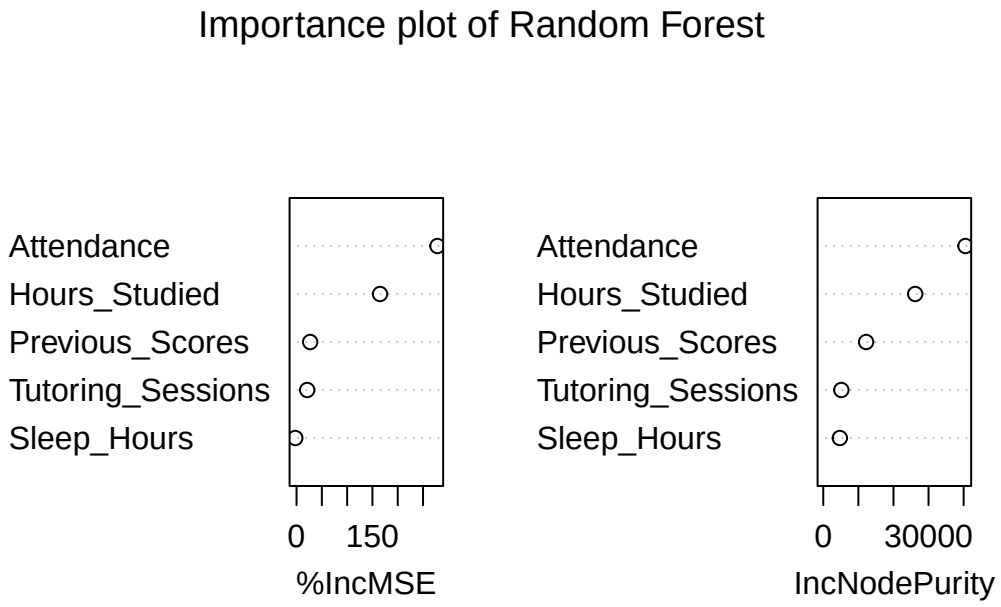
This process ensures that the individual trees in the ensemble are decorrelated, making the model robust to noise and capable of generalizing well.

One of the key advantages of Random Forest is its ability to handle high-dimensional data and complex non-linear relationships between variables. It is also less prone to overfitting compared to individual decision trees, due to the averaging effect across multiple trees. Additionally, Random Forest provides measures of variable importance, enabling insights into the contribution of predictors to the model.

In this section, we will utilize Random Forest to analyze the data set and capture complex non-linear relationships between the variables while assessing its predictive performance.

Random Forest Model

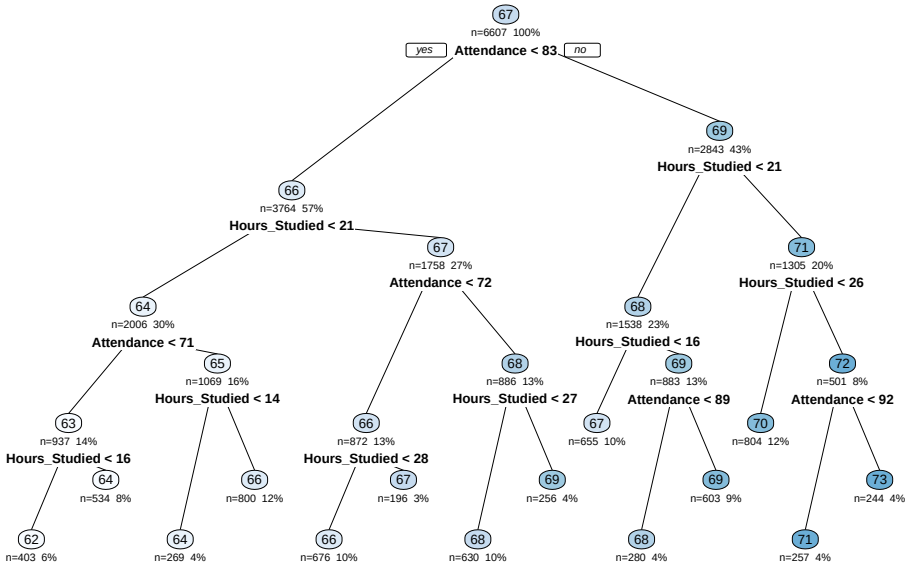
We construct a random forest model with 1,000 trees, 2 variables randomly sampled as a candidate at each split, and a minimum terminal node size of 5. The model's out-of-bag (OOB) MSE is 6.638. From the importance plot, we observe that Attendance is the most important variable, while Sleep_Hours is the least important. This aligns with our findings from the linear regression analysis, where Sleep_Hours is found to be insignificant.



Approximation by a Single Regression Tree

Despite its strengths, Random Forest is computationally intensive, especially when applied to large data sets with a large number of trees. Moreover, it can be less interpretable compared to simpler models, as the ensemble structure makes it challenging to directly understand the decision-making process.

To make the Random Forest more interpretable, we fit a single regression tree to approximate and visualize the results obtained from the Random Forest. The predicted response values from the Random Forest based on OOB samples are used as the response variable, and the same covariates from the Random Forest are employed to fit the regression tree. To assess how well the regression tree approximates the Random Forest, we calculate the Pearson correlation between the predicted response values from the Random Forest and those from the single regression tree.



The correlation between the Random Forest and the single regression tree achieves 0.898, indicating that the single tree provides a good approximation of the Random Forest. This regression tree highlights the factors influencing the target outcome exam scores, with **Attendance** emerging as the most critical predictor. The tree's root splits data based on whether Attendance is below or above 83, reflecting its overall importance. For lower Attendance values, further splits are influenced by **Hours_Studied**. Both lower Attendance and fewer Hours_Studied lead to lower exam scores, aligning with expectations. The tree reveals **interaction effects** between **Attendance** and **Hours_Studied**, as their thresholds creating different groups with varying predicted values.

Conclusion

Code Appendix

```
# Set up knit environment
knitr::opts_chunk$set(echo = F)
knitr::opts_chunk$set(error = F)
knitr::opts_chunk$set(warning = F)
knitr::opts_chunk$set(message = F)

# Load necessary packages
library(tidyverse)
library(kableExtra)
library(knitr)
library(broom)
library(ggplot2)
library(naniar)
library(gtsummary)
library(GGally)
library(MASS)
library(modelsummary)
library(randomForest)
library(rpart.plot)

# Load the data
educ_dta <- read_csv("student_performance.csv") %>%
  mutate(
    Parental_Involvement = as.factor(Parental_Involvement),
    Access_to_Resources = as.factor(Access_to_Resources),
    Extracurricular_Activities = as.factor(Extracurricular_Activities),
    Motivation_Level = as.factor(Motivation_Level),
    Internet_Access = as.factor(Internet_Access),
    Family_Income = as.factor(Family_Income),
    Teacher_Quality = as.factor(Teacher_Quality),
    School_Type = as.factor(School_Type),
    Peer_Influence = as.factor(Peer_Influence),
    Learning_Disabilities = as.factor(Learning_Disabilities),
    Parental_Education_Level = as.factor(Parental_Education_Level),
    Distance_from_Home = as.factor(Distance_from_Home),
    Gender = as.factor(Gender)
  )

# Summary table
table_summary <- tibble(
```

```
  "Variable Name" = colnames(educ_dta),
  "Variable Type" = sapply(educ_dta, class),
  "Description" = c(
    "Hours Studied", #1
    "Attendance", #2
    "Parental Involvement", #3
    "Access to Resources", #4
    "Extracurricular Activities", #5
    "Sleep Hours", #6
    "Previous Scores", #7
    "Motivation Level", #8
    "Internet Access", #9
    "Tutoring Sessions", #10
    "Family Income", #11
    "Teacher Quality", #12
    "School Type", #13
    "Peer Influence", #14
    "Physical Activity", #15
    "Learning Disability", #16
    "Parental Education Level", #17
    "Distance from Home", #18
    "Gender", #19
    "Exam Score" #20
  )
)

# Display the table
knitr::kable(table_summary,
  caption = "Variable Summary for the Educational Data")

# Save the table
write.csv(table_summary, "table_summary.csv", row.names = F)

pred_var <- c("Hours_Studied",
              "Attendance",
              "Sleep_Hours",
              "Previous_Scores",
              "Tutoring_Sessions")

pred_dta <- educ_dta %>%
  dplyr::select(all_of(pred_var), "Exam_Score")
```

```r
p.pairs <- GGally::ggpairs(pred_dta)

p.pairs

ggsave("correlation_matrix.png", p.pairs, width = 10, height = 6)

# Log-transform skewed variables
educ_dta <- educ_dta %>%
  mutate(
    log_Exam_Score = ifelse(Exam_Score == 0, 0, log(Exam_Score)),
    sq_Tutoring_Sessions = sqrt(Tutoring_Sessions)
    )

# Display histograms next to each other
p.hist <- educ_dta %>%
  dplyr::select(log_Exam_Score, sq_Tutoring_Sessions) %>%
  gather() %>%
  ggplot(aes(value)) +
  geom_histogram(bins = 30, color = "black",
    fill = "lightblue", alpha = 0.7) +
  facet_wrap(~key, scales = "free") +
  theme_minimal()

p.hist

ggsave("histograms.png", p.hist, width = 10, height = 6)

# Fit the linear model
lm_model <-
  lm(log_Exam_Score ~ Hours_Studied + Attendance +
                      Sleep_Hours + Previous_Scores + sq_Tutoring_Sessions,
                      data = educ_dta)

# Save results
write.csv(tidy(lm_model), "lm_results.csv", row.names = F)

# Summary
summary(lm_model)

# Print with modelsummary
modelsummary::modelsummary(lm_model,
  stars = TRUE,
```

```
  caption = "Linear Regression Results")

# Null model
null_model <- lm(log_Exam_Score ~ 1, data = educ_dta)

# Perform the F-test
anova.tbl <- anova(null_model, lm_model)

anova.tbl

# Save the table
write.csv(anova.tbl, "anova_results.csv", row.names = F)

# Random Forest

pred_dta_x = pred_dta[,c("Hours_Studied", "Attendance", "Sleep_Hours",
                                "Previous_Scores", "Tutoring_Sessions")] %>%
  as.data.frame
pred_dta_y = pred_dta$Exam_Score

# Hyperparameter tuning for number of variables randomly sampled
# as candidates at each split
set.seed(1)
tune_rf = tuneRF(x = pred_dta_x,
                 y = pred_dta_y,
                 ntreeTry=500)
m = tune_rf[which.min(tune_rf[,2]), 1]
m

# Cobstruct Random Forest
set.seed(123)
rf_m = randomForest(Exam_Score ~., data=pred_dta,
                    mtry=m, ntree = 1000,
                    keep.forest=TRUE, importance=TRUE)
# Mean prediction for each observation
pred_rf = predict(rf_m, type="response")
# OOB MSE from RF
plot(rf_m$mse)
rf_m$mse[1000]
# importance measure from RF
varImpPlot(rf_m, main="Importance plot of Random Forest")
# Single tree for RF
```

```r
dat_rf_singletree = pred_dta %>% dplyr::select(-Exam_Score)
dat_rf_singletree$pred_rf = pred_rf

set.seed(1)
tree0_for_rf = rpart(pred_rf~., data=dat_rf_singletree, method="anova",
                     control=rpart.control(minsplit=5, cp=0.008))
#The two parameters in rpart.control() can be adjusted
#plotcp(tree0_for_rf)
#printcp(tree0_for_rf)
pred_single_tree_rf = predict(tree0_for_rf, newdata=dat_rf_singletree)
cor(pred_rf, pred_single_tree_rf) #0.898
# Visualize the single regression tree
rpart.plot(tree0_for_rf, fallen.leaves = FALSE, tweak=1.1, extra=101, under=TRUE)
```