Modeling a Zombie Outbreak (following Munz et al., 2009)

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MATH3006

When zombies attack!: Mathematical modelling of an outbreak of zombie infection

Philip Munz[1], Ioan Hudea[2], Joe Imad[3], Robert J. Smith?[4*]

Abstract

Zombies are a popular figure in pop culture/entertainment and they are usually portrayed as being brought about through an outbreak or epidemic. Consequently, we model a zombie attack, using biological assumptions based on popular zombie movies. We introduce a basic model for zombie infection, determine equilibria and their stability, and illustrate the outcome with numerical solutions. We then refine the model to introduce a latent period of zombification, whereby humans are infected, but not infectious, before becoming undead. We then modify the model to include the effects of possible quarantine or a cure. Finally, we examine the impact of regular, impulsive reductions in the number of zombies and derive conditions under which eradication can occur. We show that only quick, aggressive attacks can stave off the doomsday scenario: the collapse of society as zombies overtake us all.

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1) Extending systems of ODE's into multidimensional space

A Refresher: Solving a 2x2 system of ODEs

a) Solving a Linear 2x2 System

Solve for λ in: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda I\right)$

b) Solving a Non-linear 2x2 System

Solve for λ in: $\det\left(\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} - \lambda I\right)$

We evaluate eigenvalues at each steady state; for the linearized system ***all*** *eigenvalues must be* ***hyperbolic*** *(real part is non-zero) to infer stability on original system.*

Eigenvalue	Stability/Behavior		
	Stability	Oscillatory behavior	Notation
All real and +	unstable	none	unstable node
All real, + and coincide	unstable	none	unstable inflected node
All real and -	stable	none	stable node
All real, - and coincide	stable	none	stable inflected node
Mixed + and -,real	unstable	none	unstable saddle node
Complex, $\lambda_r > 0$	unstable	undamped	unstable spiral
Complex, $\lambda_r < 0$	stable	damped	stable spiral
Complex, $\lambda_r = 0$	un-/stable	un-/damped	un-/stable limit cycle

Solving multidimensional Systems of ODEs (1/3)

Munz et al.'s (2009) basic model:

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S \\ Z' &= \beta SZ + \zeta R - \alpha SZ \\ R' &= \delta S + \alpha SZ - \zeta R\,. \end{aligned}$$

$$\begin{aligned} -\beta SZ &= 0 \\ \beta SZ + \zeta R - \alpha SZ &= 0 \\ \alpha SZ - \zeta R &= 0\,. \end{aligned}$$

$$(\bar{S}, \bar{Z}, \bar{R}) = (N, 0, 0)\,.$$

$$(\bar{S}, \bar{Z}, \bar{R}) = (0, \bar{Z}, 0)\,.$$

- Setting up the system stays the same, here we would obtain a 3x3 matrix.
- ***Note:*** The authors let pi = 0 and delta = 0 for simplicity (They call it the 'short-term' scenario).
- Set Equations equal to zero to obtain steady states

Solving multidimensional Systems of ODEs (2/3)

Munz et al.'s (2009) basic model:

$$J(N,0,0) = \begin{bmatrix} 0 & -\beta N & 0 \\ 0 & \beta N - \alpha N & \zeta \\ 0 & \alpha N & -\zeta \end{bmatrix}.$$

$$\det(J - \lambda I) = -\lambda\{\lambda^2 + [\zeta - (\beta - \alpha)N]\lambda - \beta\zeta N\}.$$

$$J(0,\bar{Z},0) = \begin{bmatrix} -\beta\bar{Z} & 0 & 0 \\ \beta\bar{Z} - \alpha\bar{Z} & 0 & \zeta \\ \alpha\bar{Z} & 0 & -\zeta \end{bmatrix}.$$

$$\det(J - \lambda I) = -\lambda(-\beta\bar{Z} - \lambda)(-\zeta - \lambda).$$

- Determine Eigenvalues at each steady state (Determinant of J)
  - See Appendix A for method!
- Evaluate the eigenvalues at each steady state (See Slide 5)
- Conclude stability of steady state
- ***Remember***: Steady states of linearized systems must be hyperbolic to infer stability on original system!

Solving multidimensional Systems of ODEs (3/3)

Munz et al.'s (2009) basic model:

i.e. $$\boxed{\det(J - \lambda I) = -\lambda(-\beta\bar{Z} - \lambda)(-\zeta - \lambda).}$$

A system of n'th dimension produces a polynomial of n'th degree!

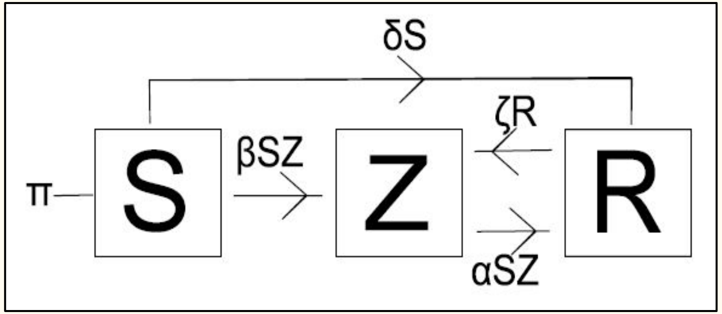
- **Remember:** Solving for the determinant of an NxN matrix is easy
- **The Caveat:** Solving for the polynomial becomes computationally intensive
  - Per the Abel-Ruffini Theorem, there exists no quintic formula (or anything beyond quintic).
  - We must rely numerical and iterative methods to solve these polynomials (i.e. Newton's method).
- Higher-dimensional systems do not change our methodology, **but** they become very complex to solve; hard to visualize, and computationally intensive.

2) Understanding the work of Munz et al. (2009)

The Base (SZR) Model (1/2)

<u>S: Population of Susceptibles; Z: Population of Zombies; R: Population of Removed</u>

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S \\
Z' &= \beta SZ + \zeta R - \alpha SZ \\
R' &= \delta S + \alpha SZ - \zeta R\,.
\end{aligned}
$$



Pi: *Overall population growth constant*

Delta: *Non-zombie death rate*

Alpha: *Zombie destruction rate*

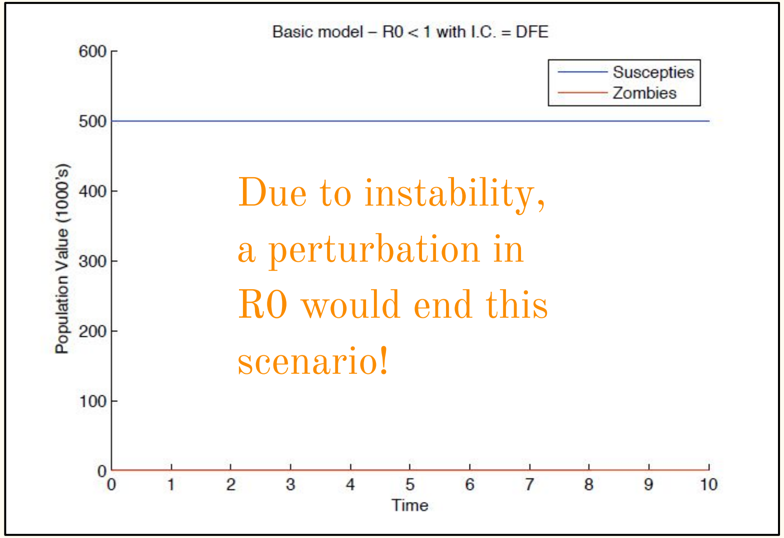
Beta: *transmission parameter*

Zeta: *Resurrection rate from R*

Note: Pi = 0 and Delta = 0 is the ‘short-term’ scenario (Static populations).

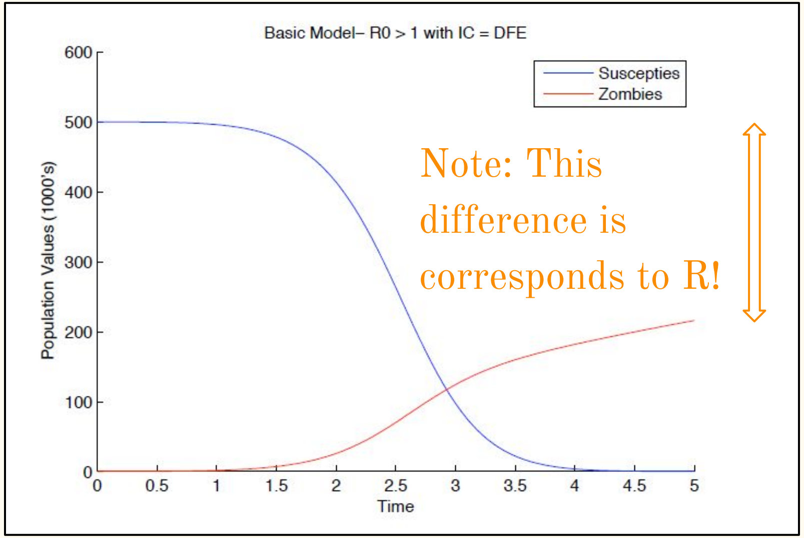
The Base (SZR) Model (2/2)

The Numerical Solution(s) (obtained through Euler's method): Alpha = 0.005; Beta = 0.0095; Zeta = 0.0001; and Delta = 0.0001.



With R0 < 1, no change over time

(However, unstable equilibrium)



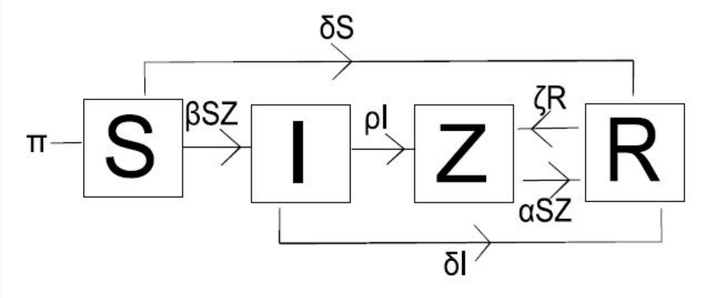
With R0 > 1, Zombies quickly take over

(Stable equilibrium)

The Latent Infection (SIZR) Model (1/2)

I: <u>Population of Infected (A new class that is between S and Z states)</u>

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S \\
I' &= \beta SZ - \rho I - \delta I \\
Z' &= \rho I + \zeta R - \alpha SZ \\
R' &= \delta S + \delta I + \alpha SZ - \zeta R
\end{aligned}
$$



***Pi:** Overall population growth constant*

***Delta:** Non-zombie death rate*

***Alpha:** Zombie destruction rate*

***Beta:** transmission parameter*

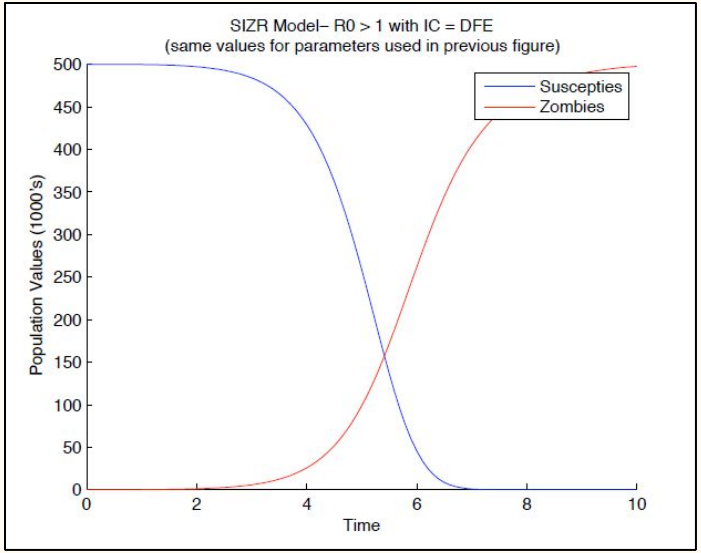
***Zeta:** Resurrection rate from R*

***Rho:** Zombification parameter*

Note: Infected people can still die a natural death (Delta parameter); Notice how the Beta parameter does not exist in Z' anymore.

The Latent Infection (SIZR) Model (2/2)

The Numerical Solution(s) (obtained through Euler's method): Alpha = 0.005; Beta = 0.0095; Zeta = 0.0001; Delta = 0.0001; and Rho = 0.005.



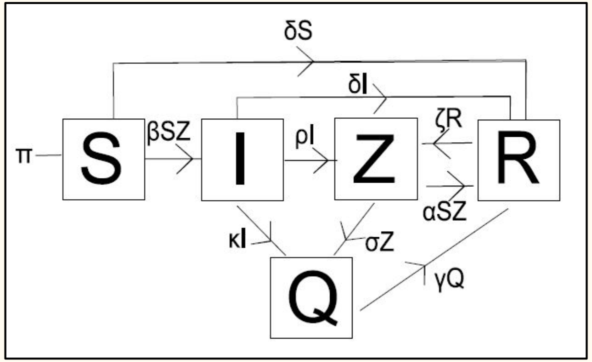
With R0 > 1, Zombies still take over.

- By having to move through the I class before becoming a Zombie, Zombies take over but it takes twice as long.
- All eigenvalues are nonpositive, therefore equilibrium is stable.
- That means that the Doomsday Scenario - namely Zombies taking over - is inevitable over time.

The Quarantine (SIZRQ) Model (1/2)

Q: <u>Population of Quarantined (Consists of Z and I population)</u>

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S \\
I' &= \beta SZ - \rho I - \delta I - \kappa I \\
Z' &= \rho I + \zeta R - \alpha SZ - \sigma Z \\
R' &= \delta S + \delta I + \alpha SZ - \zeta R + \gamma Q \\
Q' &= \kappa I + \sigma Z - \gamma Q\,.
\end{aligned}
$$



Sigma: *Zombies coming into Q*

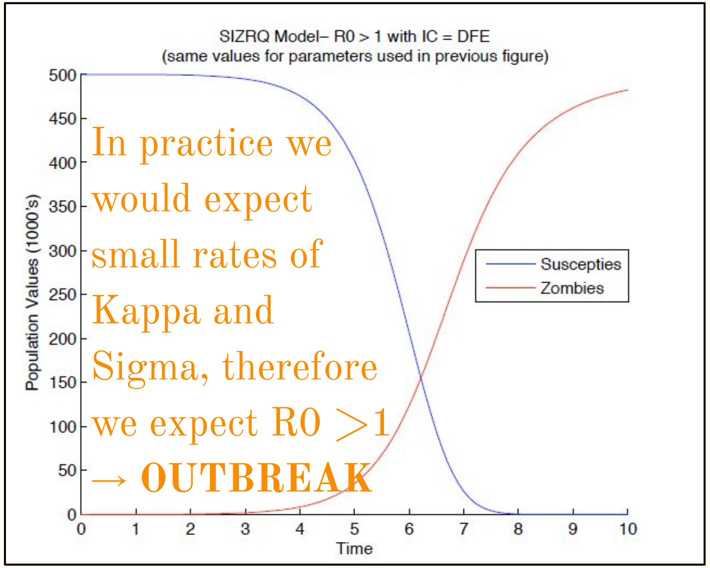
Kappa: *Infected coming into Q*

Gamma: *People breaking out of quarantine ('freedom parameter').*

Note: Quarantined people cannot infect anyone; People that die in quarantine join the R class

The Quarantine (SIZRQ) Model (2/2)

They determined the **basic reproductive ratio (RO)** using the **next-generation method** (per van den Driessche and Watmough, 2002); Numerical solution through Euler's Mtd.



In practice we would expect small rates of Kappa and Sigma, therefore we expect R0 >1 → **OUTBREAK**

With R0 > 1, Zombies take over.

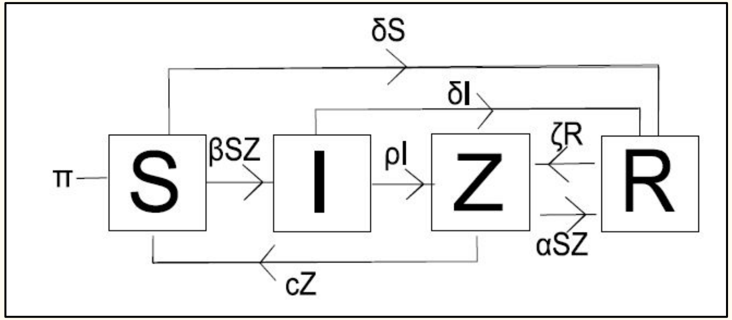
$$R_0 = \frac{\beta N \rho}{(\rho+\kappa)(\alpha N+\sigma)}. \Rightarrow R_0 \approx \frac{\beta\rho}{(\rho+\kappa)\alpha}.$$

- R0 > 1 → Outbreak persists **vs.** R0 < 1 → Outbreak is eradicated
- This is a better alternative to evaluating a degree 5 polynomial
- Stability (R0 >1) can be achieved only by increasing the infective rates, i.e. Kappa and Sigma; and vice versa.

The Treatment (SIZR*) Model (1/2)

Q: Population of Quarantined (Consists of Z and I population)

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S + cZ \\
I' &= \beta SZ - \rho I - \delta I \\
Z' &= \rho I + \zeta R - \alpha SZ - cZ \\
R' &= \delta S + \delta I + \alpha SZ - \zeta R .
\end{aligned}
$$

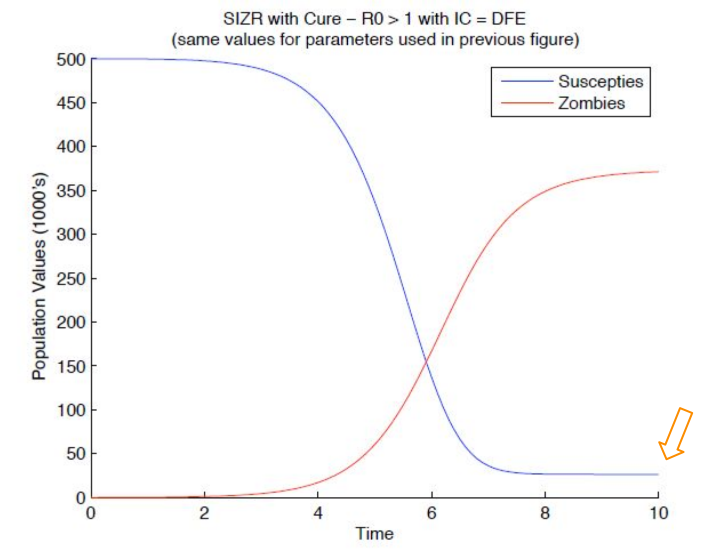


***c Parameter:** Zombies can become humans again!*

Note: With Treatment available, quarantine is no longer needed; Zombies can now become humans again!

The Treatment (SIZR*) Model (2/2)

The Numerical Solution(s) (obtained through Euler's method): Alpha = 0.005; Beta = 0.0095; Zeta = 0.0001; Delta = 0.0001; and Rho = 0.005.



With R0 > 1, Zombies take over.

- **Interesting:** Humans are not eradicated, they still exist in low numbers!
- This equilibrium is steady (all eigenvalues nonpositive) - speaks to the persistence of humans!
- This model is sensitive to parameter variation (more on that later)!

Impulsive Eradication (SZR*) Model (1/2)

Delta Z: Sudden eradication attempts from humans

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S & t &\neq t_n \\
Z' &= \beta SZ + \zeta R - \alpha SZ & t &\neq t_n \\
R' &= \delta S + \alpha SZ - \zeta R & t &\neq t_n \\
\Delta Z &= -knZ & t &= t_n\,,
\end{aligned}
$$

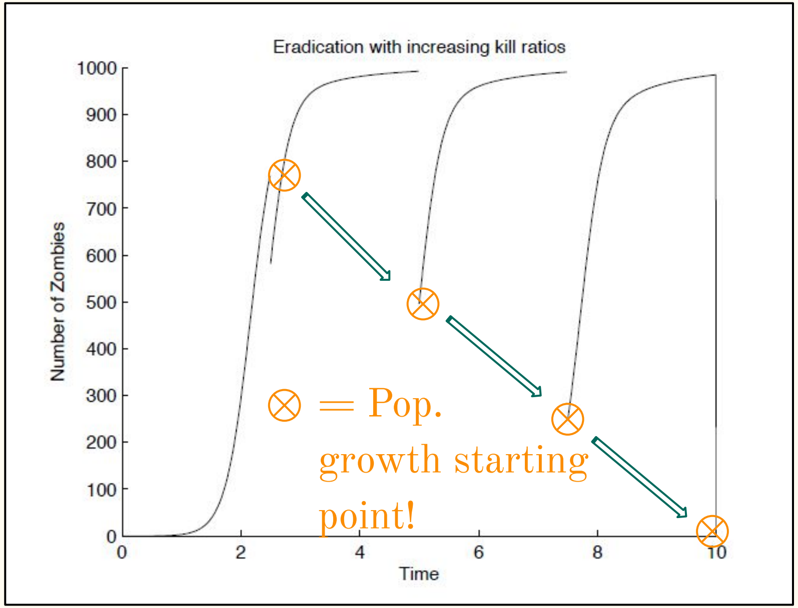
***k* Parameter:** *in [0,1] and refers to kill effectiveness, with 1 being a 100% kill ratio.*

***n* Parameter:** *number of attacks until kn>1 (where kn>1 corresponds to successful eradication.)*

Note: The delta Z term refers to a new time-step rather than a new population group.

Impulsive Eradication (SZR*) Model (2/2)

The Numerical Solution(s) (obtained through Euler's method): Alpha = 0.0075; Beta = 0.0055; Zeta = 0.009; Delta = 0.0001; and k = 0.25.



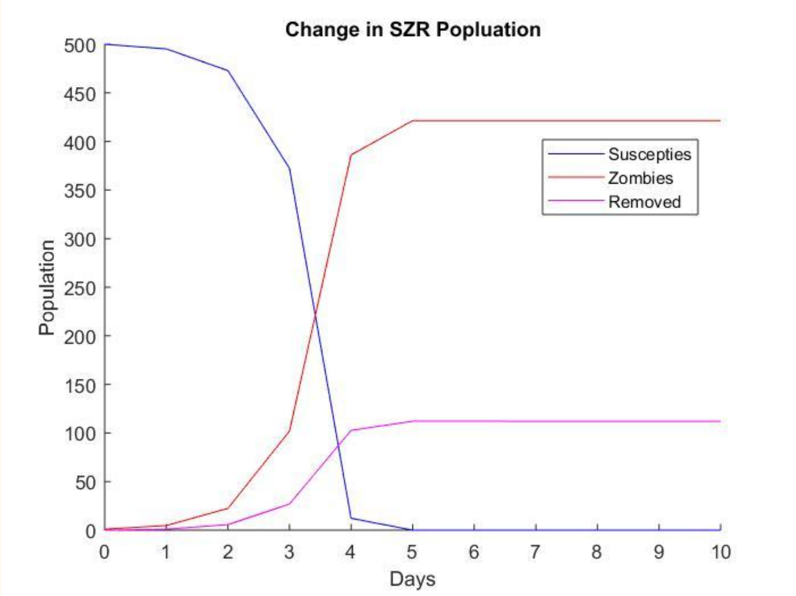
- After 2.5 days, 25% of zombies are destroyed; after 5 days, 50% of zombies are destroyed; after 7.5 days, 75% of remaining zombies are destroyed; after 10 days, 100% of zombies are destroyed.
- **Note:** Since k = 0.25, we need 4 attempts at destroying Zombies completely (25% effectiveness rate for attacks)
- **We can only win if we attack impulsively!**

3) Extending Munz et al. (2009): Varying Parameters

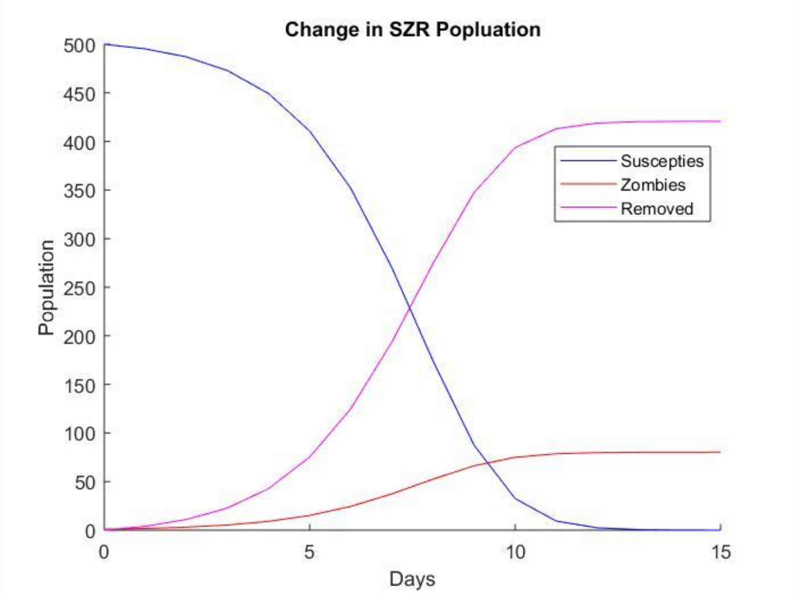
Extending the Base Model: SZR

- Vary alpha - defeated zombie parameter
  - Default: alpha = 0.005
  - Increase: alpha = 0.008 (Humans are better at defeating zombies)
  - Decrease: alpha = 0.002 (Humans are worst at defeating zombies)
- Vary beta - zombie encounter transmission parameter
  - Default: beta = 0.0095
  - Increase: beta = 0.0125 (More likely to run into zombies)
  - Decrease: beta = 0.0065 (Less likely to run into zombies)

Extending the Base Model: SZR



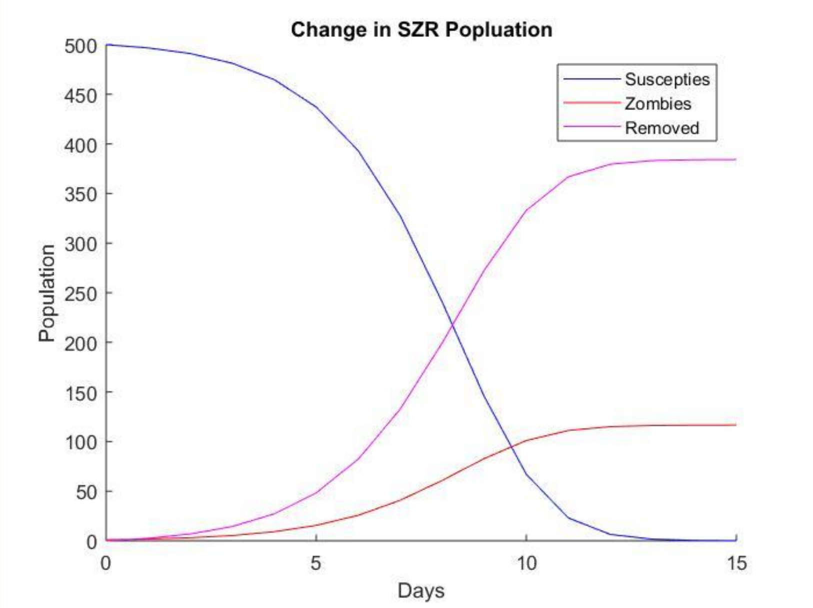
Alpha = 0.002



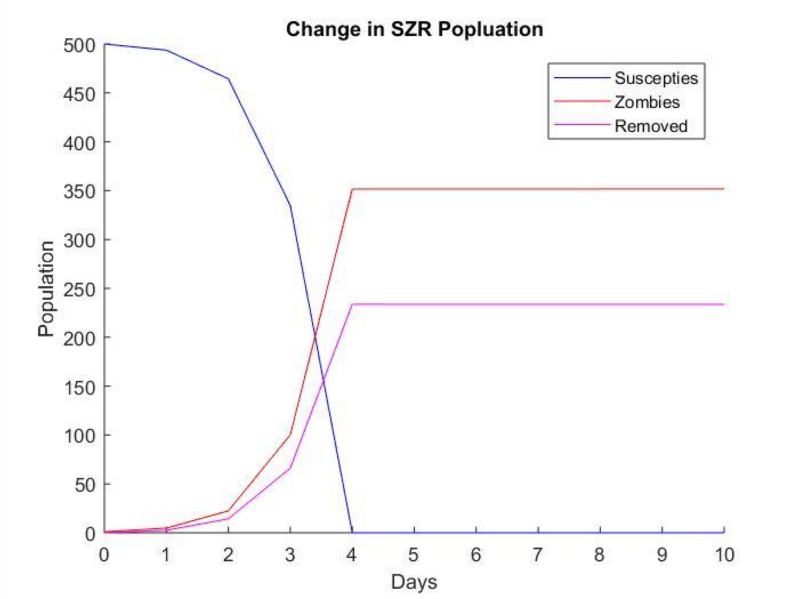
Alpha = 0.008

*Other parameters at their default value

Extending the Base Model: SZR



Beta = 0.0065



Beta = 0.0125

*Other parameters at their default value

Extending the Base Model: SZR

- As alpha increases, humans survive longer but still end up defeated
- As alpha decreases, humans are defeated more quickly

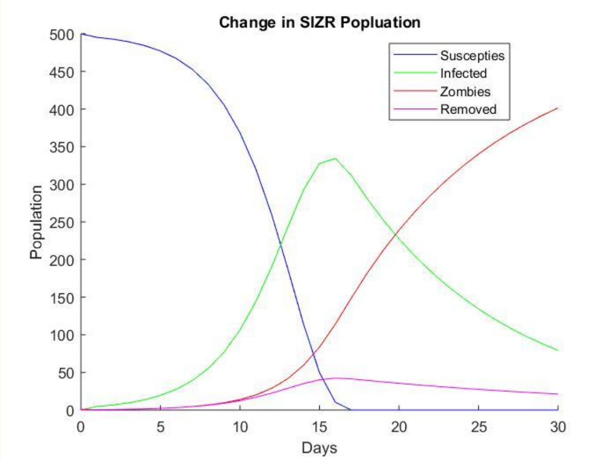
- As beta increases, humans encounter zombies more often and are defeated more quickly
- As beta decrease, humans encounter zombies less often and survive longer but still end up defeated

Extending the Base Model: SIZR

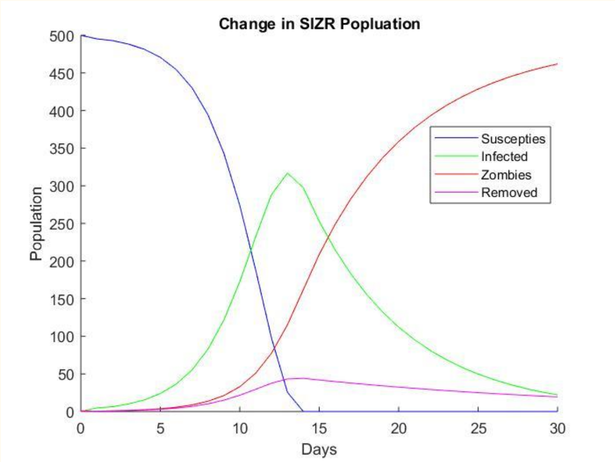
- Introduce “infected” class. Humans do not immediately turn into zombies.
- Vary rho - Infection parameter
  - Rho = 0.05 (Move from infected to zombies less frequently/slower)
  - Rho = 0.10
  - Rho = 0.15 (Move from infected to zombies more frequently/faster)

Extending the Base Model: SIZR

Rho = 0.05



Rho = 0.10



Rho = 0.15

*Other parameters at their default value

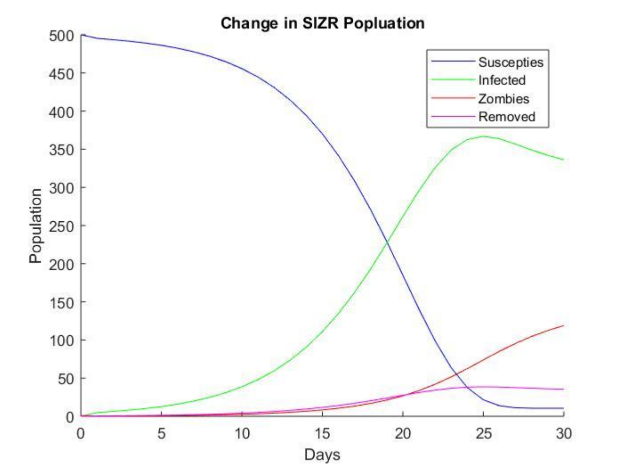
Extending the Base Model: SIZR

- As rho increases
  - Human survival decreases more quickly
  - The time someone stays in the infected class decreases
  - Infected turns into zombies more quickly
  - Human population still becomes defeated

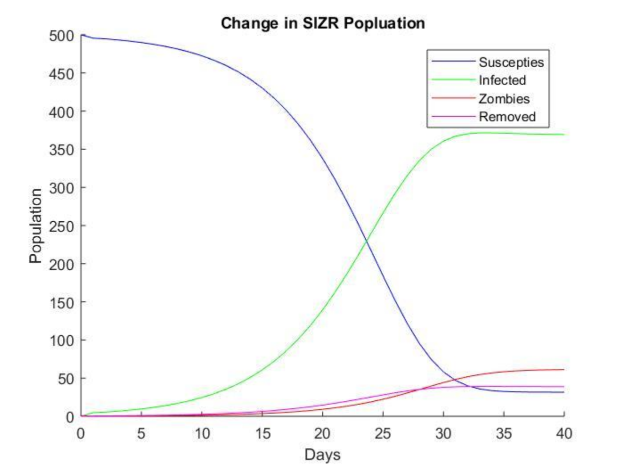
Extending the Base Model: SIZR with Treatment

- Take previous SIZR model, but now include a treatment parameter “c”
  - C = 0.10 (Least effective)
  - C = 0.30
  - C = 0.50 (Most effective)
- Treatment is not a cure. Recovered infected/zombies can become reinfected again
- Vary parameter for different levels of treatment “effectiveness”

Extending the Base Model: SIZR with Treatment



C = 0.10



C = 0.30

C = 0.50

*Rho = 0.05, other parameters at their default value

Extending the Base Model: SIZR with Treatment

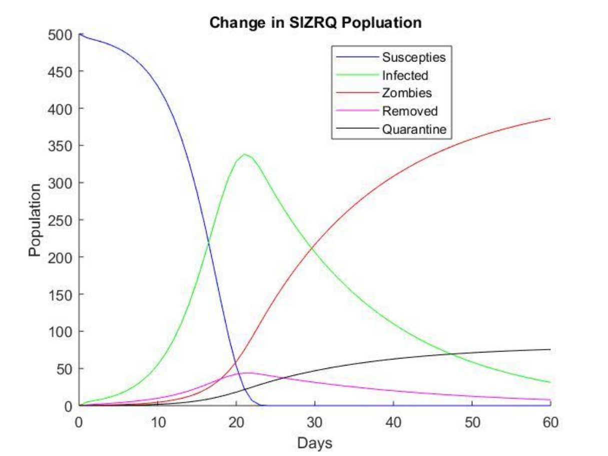
- Introducing a treatment parameter allows the human population to survive but in small numbers
- As c increases, the surviving population also increases
- Once c reaches a certain level of effectiveness (~0.43), human population will outnumber zombie population
- Zombies will not be defeated but will continue to survive in small numbers
    - Human/zombie coexistence

Extending the Base Model: SIZRQ

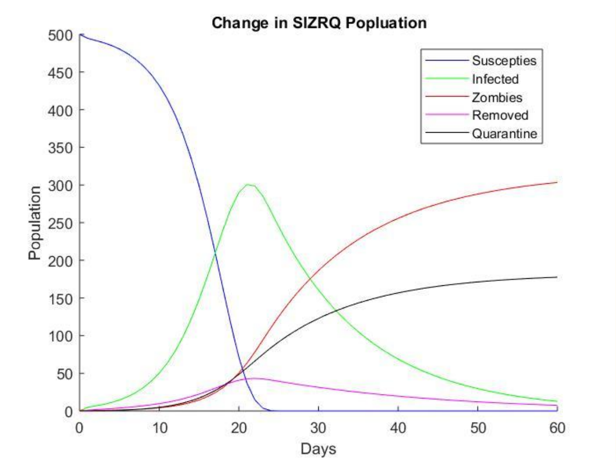
- Introduce quarantine class (remove treatment)
- Can either quarantine infected, zombies, or both
  - Parameter: k - quarantine of infected class
  - Parameter: sigma - quarantine of zombie class
    - K/Sigma = 0.01 (Least quarantine)
    - K/Sigma = 0.03
    - K/Sigma = 0.05 (Most quarantine)
- Can remove individuals from quarantine if necessary
  - Parameter: gamma
    - Gamma = 0.01 (Remove least from quarantine)
    - Gamma = 0.03
    - Gamma = 0.05 (Remove most from quarantine)

Extending the Base Model: SIZRQ

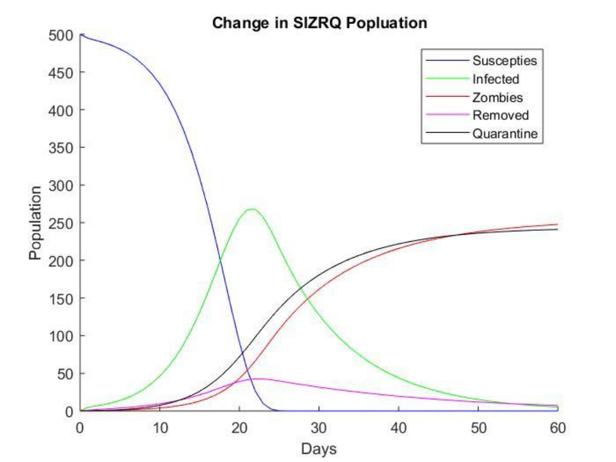
Quarantine only the infected class



K = 0.01



K = 0.03

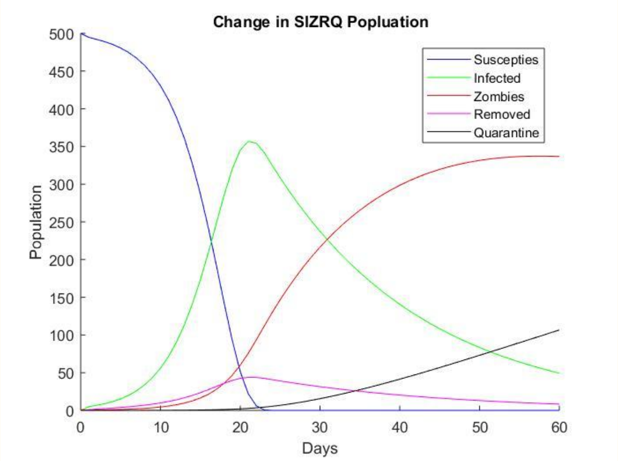


K = 0.05

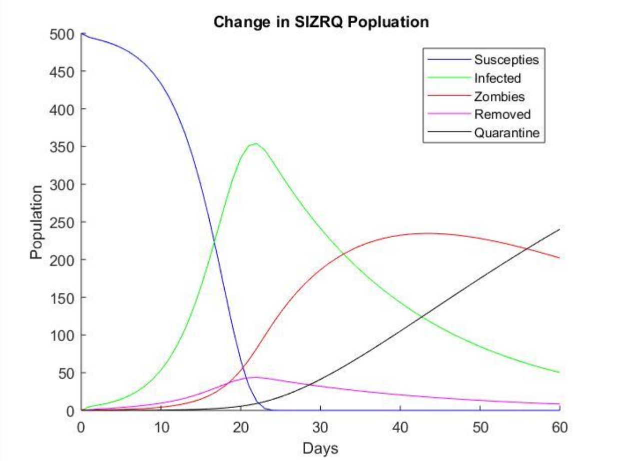
*Rho = 0.05, sigma = 0, gamma = 0, other parameters at their default value

Extending the Base Model: SIZRQ

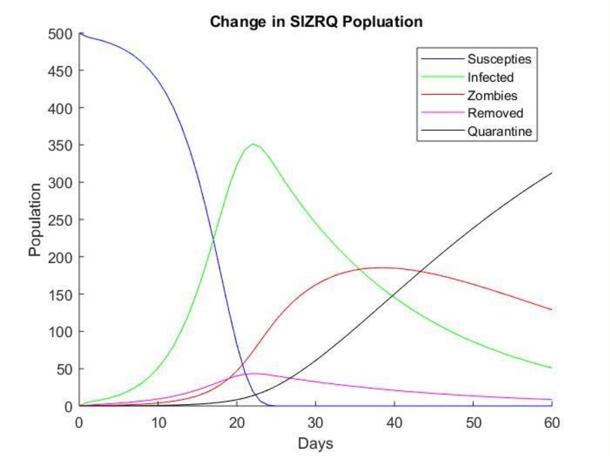
Quarantine only the zombie class



sigma = 0.01



sigma = 0.03

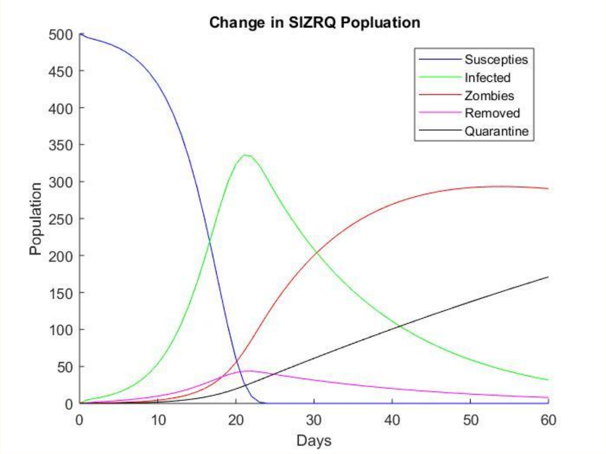


sigma = 0.05

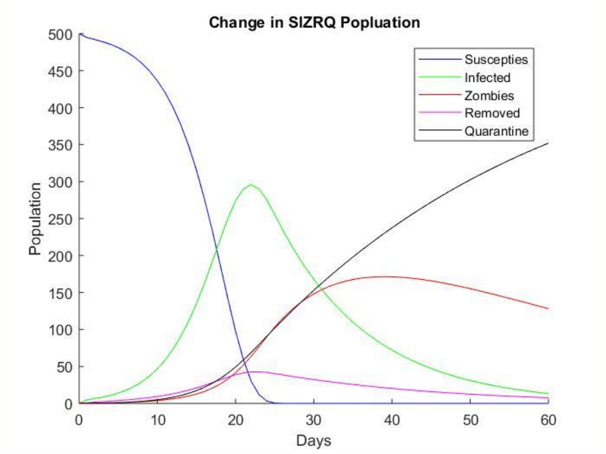
*Rho = 0.05, k= 0, gamma = 0, other parameters at their default value

Extending the Base Model: SIZRQ

Quarantine both the infected and zombie class



K, sigma = 0.01



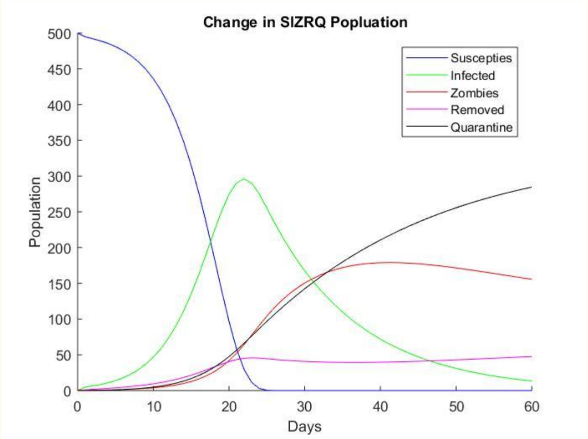
K, sigma = 0.03

K, sigma = 0.05

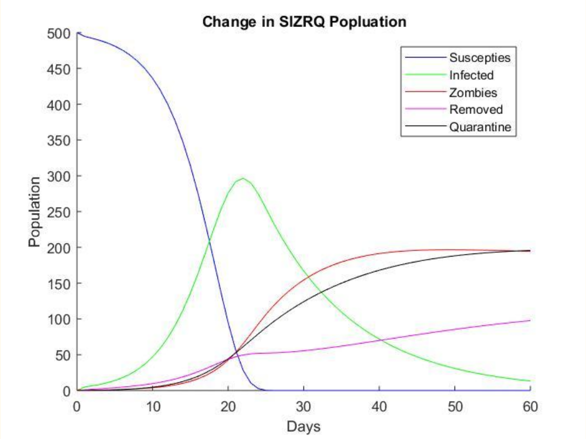
*Rho = 0.05, gamma = 0, other parameters at their default value

Extending the Base Model: SIZRQ

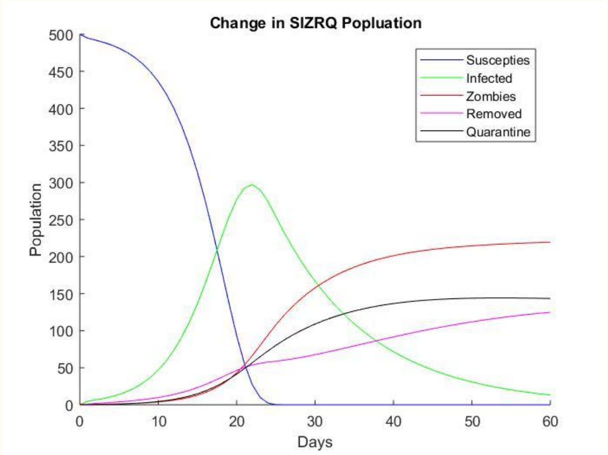
Quarantine both the infected and zombie class, move to removed if necessary



gamma = 0.01



gamma = 0.03



gamma = 0.05

*Rho = 0.05, sigma = 0.03, k = 0.03, other parameters at their default value

Extending the Base Model: SIZRQ

- When compared to the SIZR model:
  - Quarantine will slow down the spread of zombies
  - Allow the the human population to survive longer
  - Humans will still be defeated
- Difference in quarantine techniques does little to help humans survive longer
- Removing quarantined individuals has negligible effect of human population
  - Only adds to the “removed” population

Extending the Base Model: Conclusion

- Treatment is the only effective way for humans to survive
    - Zombies will not be completely defeated
- Other attempts such as quarantine will only delay humans being defeated, not prevent it

Appendix

Appendix A: The determinant of an NxN matrix

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = |A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Equation 6: Determinant of a 4x4 matrix

This process can be repeated until the NxN matrix is decomposed into a multiple of 2x2 matrices. Then det(A) = ab-cd.

Source:
https://www.studypug.com/algebra-help/the-determinant-of-a-3-x-3-matrix-general-and-shortcut-method