Linear Algebra Study Guide

May 8, 2020

1 Matrices

A is an $m \times n$, with $m$rows and $n$columns, $A = (A(j,k))_{j=1:m,k=1:n}$ or in the column form $A = col(A_k)_{k=1:n}$.

2 Matrix Vector Multiplication

$A = col(a_k)_{k=1:n}$, $v = (v_i)_{i=1:n}$ is a column vector of size n then $Av = a_1v_1 + a_2v_2 + ... + a_nv_n = \sum_{i=1}^{n} v_k a_k$. Av is a linear combination of the columns of A with coefficients the entries of v.

3 Matrix Matrix Multiplication

A is $m \times n$, B is $n \times p$ with $B = col(b_k)_{k=1:p}$ then AB is $m \times p$ and $AB = col(Ab_k)_{k=1:p}$.

4 Row-Vector Column-Vector Multiplication

$w^t = (w_1, w_2, ..., w_n)$, $v = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$ then $w^t v = \sum_{i=1}^{n} w_i v_i$

5 Matrix Matrix Matrix Multiplication

A is $m \times n$ with $A = row(r_j)_{j=1:m}$, B is $n \times p$, C is $p \times l$ with $C = col(c_k)_{k=1:p}$ then ABC is $m \times l$ with $(ABC)(j,k) = r_j B c_k$

6 Transpose of a Matrix

A is $m \times n$, then A^t is $n \times m$ and $A^t(j,k) = A(k,j)$

- $(AB)^t = B^t A^t$
- $(Av)^t = v^t A^t$

7 Identity Matrix

$I = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ | & ... & ... & | \\ 0 & ... & ... & 1 \end{bmatrix}$ column form $I = col(c_k)_{k=1:n}$ where $c_k = \begin{Bmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \\ 0 \\ ... \\ 0 \end{Bmatrix}$ with the 1 being in the kth position.
$AI = IA = A$
A is symmetric iff $A^t = A$

8 Matrix Rank, Nullspace, Range of a Matrix

- Definition: Vectors $w_1, w_2, ..., w_n$ (of the same size) are linearly independent iff $\sum c_i w_i = 0 <=> c_i = 0, \forall_i = 1 : p$
- Definition: Let A be an $m \times n$ matrix then the rank of A is equal to the largest number of linearly independent columns of A (which is equal to row rank).

- Definition: The nullspace of A is $Null(A) = v \in \mathbb{R}^n : Av = 0$
- Definition: The range of A is $Range(A) = u \in \mathbb{R}^m : \, there\, exsists\, v \in \mathbb{R}^n \, with\, Av = u$
- Recall: If $A = col(a_k)_{k=1:n}$ and $v = (v_i)_{i=1:n}$ then $Av = \sum_{k=1}^{n} v_k a_k$
- Definition: The space generated by vectors $w_1, w_2, ..., w_n$ then $< w_1, w_2, ..., w_n >= \{\sum_{i=1}^{p} \, with\, c_i \in \mathbb{R}\}$. The range of a matrix is equal to the space generated by the columns of the matrix.
- Definition: The dimension of a vector space in $\mathbb{R}^n$ is equal to the smallest number of linearly independent vectors from this space such that the space generated by their linearly independent vectors is equal to the vector space. These linearly independent vectors form a basis for the vector space.

9 Non-Singular Matrices

- Definition: Let A be a square matrix of size n. A is nonsingular iff there exists a square matrix A_{-1} of size n such that $AA_{-1} = A^{-1}A = I$
- Theorem: A is nonsingular if $Null(A) = \{0\}, Range(A) = \mathbb{R}^n, Rank(A) = n, det(A) \neq 0$

10 Determinants

The determinant can be viewed as the scaling factor of the transformation described by the matrix.

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$det(AB) = det(A)det(B)$$

$$det \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{bmatrix} = \prod_{i=1}^{n} d_i$$

Lemma: If $AB = I$, theN A is nonsingular, $BA = I$ then $B = A^{-1}$.

11 Diagonal Matrices