Modeling and control of manipulators

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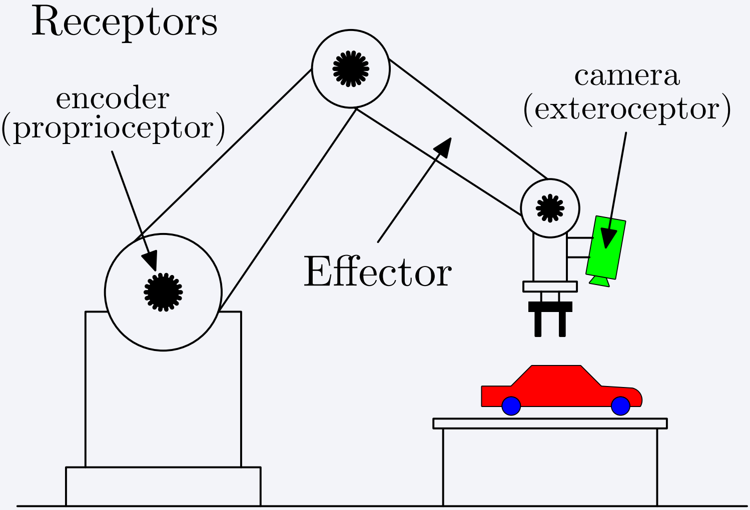
Warsaw, October 2013

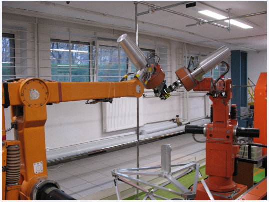

Receptors
encoder
(proprioceptor)
camera
(exteroceptor)
Effector

RNT robot

Polycrank robot

Mobile robot

Two IRp–6 robots



The robots were designed at the following faculties of Warsaw University of Technology:

RNT, Polycrank
- Power and Aeronautical Engineering

Mobile robot
- Mechatronics

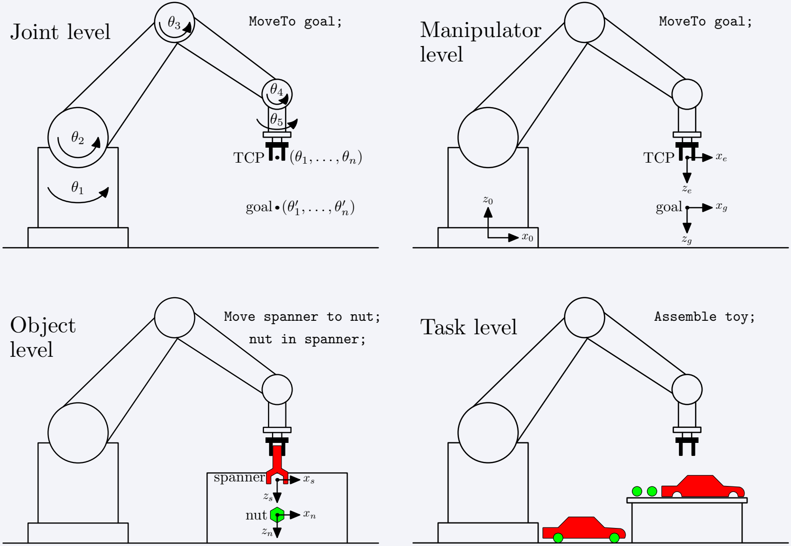
Control of all robots
- Electronics and Information

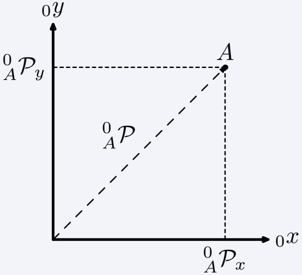
Receptors – general classification

Classification criterion – source of stimulus:

- Exteroceprors – external environment
  (stimulus external to the body)
  Sometimes exteroceptors (source in contact with the body) and telereceptors (remote source) are distinguished
- Interoceptors – internal organs
  (stimulus from internal organs)
- Proprioceptors – motion organs (e.g., limbs)
  (monitor movement and tension in muscles, tendons and joints)

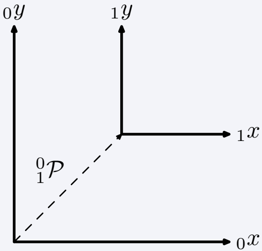
Robot ontologies





Coordinates of point A (or vector) in relation to frame 0

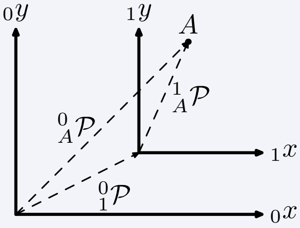
$$^0_A\mathcal{P} = \begin{bmatrix} ^0_A\mathcal{P}_x \\ ^0_A\mathcal{P}_y \end{bmatrix} \tag{1}$$



Translation of coordinate frame 1 in relation to coordinate frame 0 or location of coordinate frame 1 in relation to coordinate frame 0

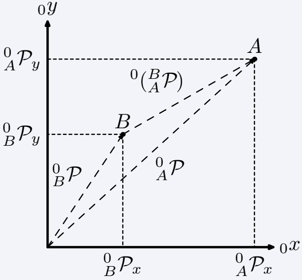
$$^0_1\mathcal{P} = \begin{bmatrix} ^0_1\mathcal{P}_x \\ ^0_1\mathcal{P}_y \end{bmatrix} \tag{2}$$

Coordinate transformation and translation



Location of a point expressed in relation to two translated frames

$$
\begin{aligned}
{}^0_A\mathcal{P} &= \begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \end{bmatrix} = {}^0_1\mathcal{P} + {}^0({}^1_A\mathcal{P}) \\
&= \begin{bmatrix} {}^0_1\mathcal{P}_x \\ {}^0_1\mathcal{P}_y \end{bmatrix} + \begin{bmatrix} {}^0({}^1_A\mathcal{P})_x \\ {}^0({}^1_A\mathcal{P})_y \end{bmatrix}
\end{aligned}
\tag{3}
$$

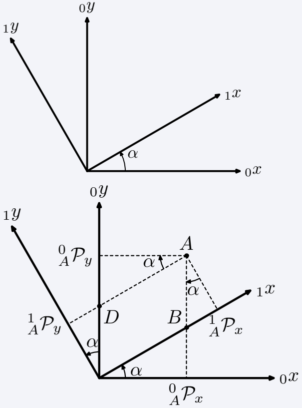


Translation of a point

$$
\begin{aligned}
{}^0_A\mathcal{P} &= \begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \end{bmatrix} = {}^0_B\mathcal{P} + {}^0\mathcal{P} = {}^0_B\mathcal{P} + {}^0({}^B_A\mathcal{P}) \\
&= \begin{bmatrix} {}^0_B\mathcal{P}_x \\ {}^0_B\mathcal{P}_y \end{bmatrix} + \begin{bmatrix} {}^0({}^B_A\mathcal{P})_x \\ {}^0({}^B_A\mathcal{P})_y \end{bmatrix}
\end{aligned}
\tag{4}
$$

If B is substituted for 1 the formulas (3) and (4) are the same.

Rotation transformation



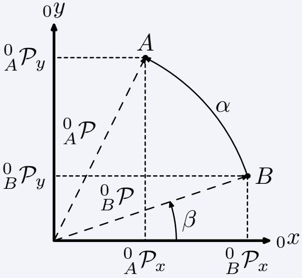
Frame 1 rotated in relation to frame 0 by an angle α

Coordinates of point A in relation to frames 0 and 1 rotated in relation to each other

$$
\begin{aligned}
{}^0_A\mathcal{P}_x &= \overline{0B}\cos\alpha = ({}^1_A\mathcal{P}_x - {}^1_A\mathcal{P}_y\tan\alpha)\cos\alpha \\
&= {}^1_A\mathcal{P}_x\cos\alpha - {}^1_A\mathcal{P}_y\sin\alpha
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
{}^0_A\mathcal{P}_y &= \overline{0D} + {}^0_A\mathcal{P}_x\tan\alpha = {}^1_A\mathcal{P}_y\cos^{-1}\alpha + ({}^1_A\mathcal{P}_x\cos\alpha - {}^1_A\mathcal{P}_y\sin\alpha)\tan\alpha \\
&= {}^1_A\mathcal{P}_y(1-\sin^2\alpha)\cos^{-1}\alpha + {}^1_A\mathcal{P}_x\sin\alpha = {}^1_A\mathcal{P}_x\sin\alpha + {}^1_A\mathcal{P}_y\cos\alpha
\end{aligned} \tag{6}
$$

$$
\begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} {}^1_A\mathcal{P}_x \\ {}^1_A\mathcal{P}_y \end{bmatrix} \tag{7}
$$



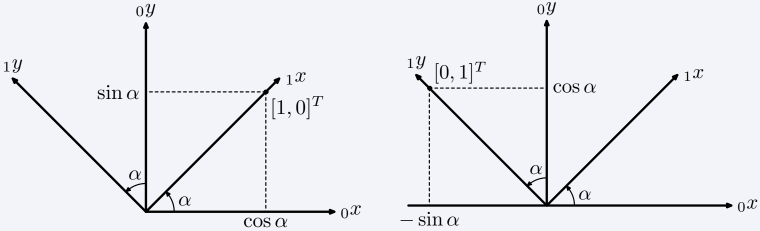
Point B rotated from A by an angle α around the origin of frame 0

Let $r = \|{}^0_A\mathcal{P}\| = \|{}^0_B\mathcal{P}\|$

$$
\begin{aligned}
{}^0_A\mathcal{P}_x &= r\cos(\alpha+\beta) = r(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = r\left(\cos\alpha\frac{{}^0_B\mathcal{P}_x}{r} - \sin\alpha\frac{{}^0_B\mathcal{P}_y}{r}\right)\\
{}^0_A\mathcal{P}_y &= r\sin(\alpha+\beta) = r(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = r\left(\sin\alpha\frac{{}^0_B\mathcal{P}_x}{r} + \cos\alpha\frac{{}^0_B\mathcal{P}_y}{r}\right)
\end{aligned} \tag{8}
$$

$$
\begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} {}^0_B\mathcal{P}_x \\ {}^0_B\mathcal{P}_y \end{bmatrix} \tag{9}
$$

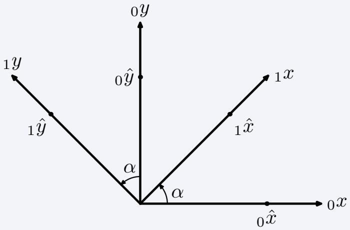
If ${}^0_B\mathcal{P}_x = {}^1_A\mathcal{P}_x$ and ${}^0_B\mathcal{P}_y = {}^1_A\mathcal{P}_y$ the formulas (7) and (9) are the same.



$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

$$\det {}^0_1\mathcal{R} = 1 \qquad \|{}^0_1\mathcal{R}_{[\bullet,1]}\| = 1 \qquad \|{}^0_1\mathcal{R}_{[\bullet,2]}\| = 1 \tag{12}$$



Dot product of versors:

$$\begin{array}{rclcl} {}_0\hat{x} \cdot {}_1\hat{x} & = & \cos\alpha & & \\ {}_0\hat{x} \cdot {}_1\hat{y} & = & \cos(90+\alpha) & = & -\sin\alpha \\ {}_0\hat{y} \cdot {}_1\hat{x} & = & \cos(90-\alpha) & = & \sin\alpha \\ {}_0\hat{y} \cdot {}_1\hat{y} & = & \cos\alpha & & \end{array} \tag{13}$$

$${}^0_1\mathcal{R} = \begin{bmatrix} {}_0\hat{x} \cdot {}_1\hat{x} & {}_0\hat{x} \cdot {}_1\hat{y} \\ {}_0\hat{y} \cdot {}_1\hat{x} & {}_0\hat{y} \cdot {}_1\hat{y} \end{bmatrix} \tag{14}$$

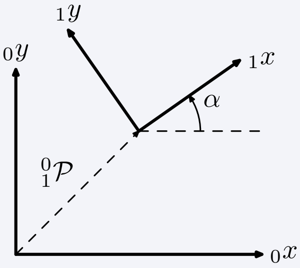
$$ {}^0_1\mathcal{R} = \begin{bmatrix} {}_0\hat{x}\cdot{}_1\hat{x} & {}_0\hat{x}\cdot{}_1\hat{y} \\ {}_0\hat{y}\cdot{}_1\hat{x} & {}_0\hat{y}\cdot{}_1\hat{y} \end{bmatrix} \qquad {}^1_0\mathcal{R} = \begin{bmatrix} {}_1\hat{x}\cdot{}_0\hat{x} & {}_1\hat{x}\cdot{}_0\hat{y} \\ {}_1\hat{y}\cdot{}_0\hat{x} & {}_1\hat{y}\cdot{}_0\hat{y} \end{bmatrix} \tag{15} $$

$$ {}^1_0\mathcal{R} = {}^0_1\mathcal{R}^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{16} $$

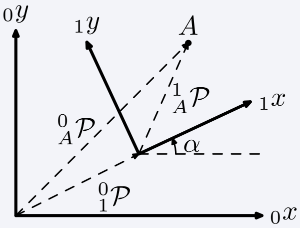
$$ {}^0_1\mathcal{R} = \mathrm{Rot}(z,\alpha) \qquad {}^1_0\mathcal{R} = \mathrm{Rot}(z,-\alpha) \tag{17} $$

$$ \mathrm{Rot}(z,-\alpha) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = {}^0_1\mathcal{R}^T \tag{18} $$

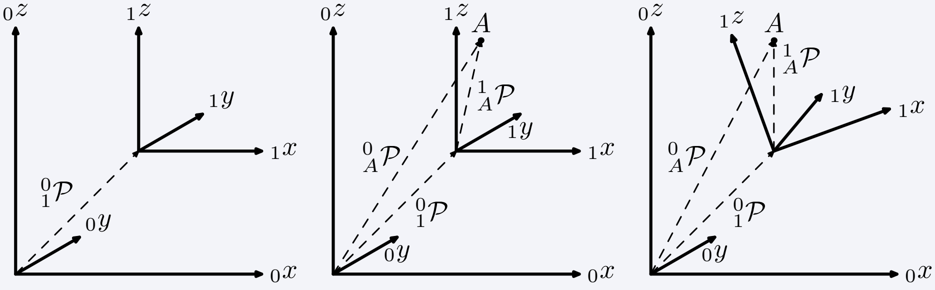
$$ {}^1_0\mathcal{R} = {}^0_1\mathcal{R}^{-1} = {}^0_1\mathcal{R}^T \tag{19} $$



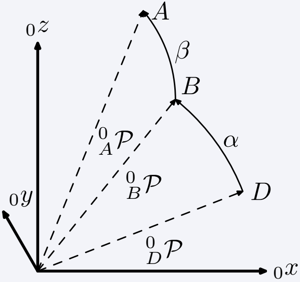
Operator creating the transformed frame: first rotates by α then translates the already rotated frame by ${}^0_1\mathcal{P}$. The same formula is used for transforming the coordinates of a point expressed in the transformed frame into the coordinates expressed in the original frame.



$$
{}^0_A\mathcal{P} = {}^0_1\mathcal{P} + {}^0({}^1_A\mathcal{P}) = {}^0_1\mathcal{P} + {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} =
\begin{bmatrix} {}^0_1\mathcal{P}_x \\ {}^0_1\mathcal{P}_y \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} {}^1_A\mathcal{P}_x \\ {}^1_A\mathcal{P}_y \end{bmatrix} \tag{20}
$$



$$
{}^0_A\mathcal{P} = {}^0_1\mathcal{P} + {}^0({}^1_A\mathcal{P}) = \begin{bmatrix} {}^0_1\mathcal{P}_x \\ {}^0_1\mathcal{P}_y \\ {}^0_1\mathcal{P}_z \end{bmatrix} + {}^0_1\mathcal{R} \begin{bmatrix} {}^1_A\mathcal{P}_x \\ {}^1_A\mathcal{P}_y \\ {}^1_A\mathcal{P}_z \end{bmatrix} \tag{21}
$$

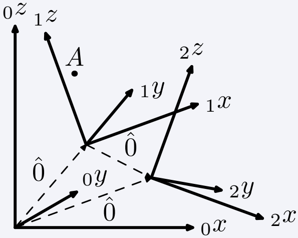


$$ {}^0_B\mathcal{P} = {}^0\mathcal{R}_\alpha \, {}^0_D\mathcal{P} \tag{22} $$

$$ {}^0_A\mathcal{P} = {}^0\mathcal{R}_\beta \, {}^0_B\mathcal{P} \tag{23} $$

$$ {}^0_A\mathcal{P} = {}^0\mathcal{R}_\beta \, {}^0\mathcal{R}_\alpha \, {}^0_D\mathcal{P} \tag{24} $$

First operator ${}^0\mathcal{R}_\alpha$ is applied.
Then operator ${}^0\mathcal{R}_\beta$ is applied.
Both operations are related to the same frame.
The composition is right–to–left.



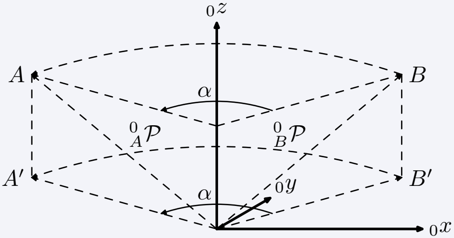
$$\hat{0} = [0,0,0]^T \tag{25}$$

$$ {}^1_A\mathcal{P} = {}^1_2\mathcal{R}\,{}^2_A\mathcal{P} \tag{26}$$

$$ {}^0_A\mathcal{P} = {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} \tag{27}$$

$$ {}^0_A\mathcal{P} = {}^0_1\mathcal{R}\,{}^1_2\mathcal{R}\,{}^2_A\mathcal{P} \tag{28}$$

First frame 0 is rotated by ${}^0_1\mathcal{R}$ to obtain frame 1.
Then frame 1 is rotated by ${}^1_2\mathcal{R}$ to obtain frame 2.
Those transformations are related to a moving frame
The composition is left–to–right.



$$ {}^0_B\mathcal{P}_z = {}^0_A\mathcal{P}_z $$

$$ \begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \\ {}^0_A\mathcal{P}_z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^0_B\mathcal{P}_x \\ {}^0_B\mathcal{P}_y \\ {}^0_B\mathcal{P}_z \end{bmatrix} \tag{29} $$

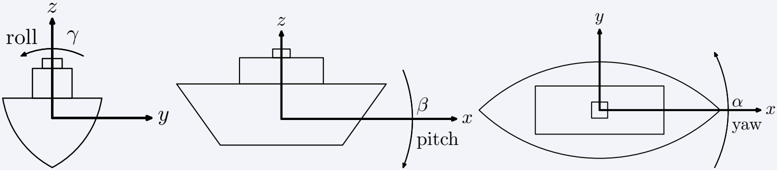
$$ \mathrm{Rot}(z,\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30} $$

Rotation about principal axes

$$\mathrm{Rot}(x,\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \tag{31}$$

$$\mathrm{Rot}(y,\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{32}$$

$$\mathrm{Rot}(z,\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$



$$
{}^0_1\mathcal{R} = \mathrm{Rot}_{RPY}(\gamma,\beta,\alpha) = \mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Rot}(x,\gamma) =
\begin{bmatrix} \mathrm{c}\alpha & -\mathrm{s}\alpha & 0 \\ \mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \mathrm{c}\beta & 0 & \mathrm{s}\beta \\ 0 & 1 & 0 \\ -\mathrm{s}\beta & 0 & \mathrm{c}\beta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\gamma & -\mathrm{s}\gamma \\ 0 & \mathrm{s}\gamma & \mathrm{c}\gamma \end{bmatrix} =
\begin{bmatrix} \mathrm{c}\alpha\,\mathrm{c}\beta & \mathrm{c}\alpha\,\mathrm{s}\beta\,\mathrm{s}\gamma - \mathrm{s}\alpha\,\mathrm{c}\gamma & \mathrm{c}\alpha\,\mathrm{s}\beta\,\mathrm{c}\gamma + \mathrm{s}\alpha\,\mathrm{s}\gamma \\ \mathrm{s}\alpha\,\mathrm{c}\beta & \mathrm{s}\alpha\,\mathrm{s}\beta\,\mathrm{s}\gamma + \mathrm{c}\alpha\,\mathrm{c}\gamma & \mathrm{s}\alpha\,\mathrm{s}\beta\,\mathrm{c}\gamma - \mathrm{c}\alpha\,\mathrm{s}\gamma \\ -\mathrm{s}\beta & \mathrm{c}\beta\,\mathrm{s}\gamma & \mathrm{c}\beta\,\mathrm{c}\gamma \end{bmatrix}
\tag{34}
$$

where $\mathrm{s}\alpha = \sin\alpha$ and $\mathrm{c}\alpha = \cos\alpha$ etc.

$$
{}^{0}_{1}\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\alpha\,\mathrm{c}\beta & \mathrm{c}\alpha\,\mathrm{s}\beta\,\mathrm{s}\gamma - \mathrm{s}\alpha\,\mathrm{c}\gamma & \mathrm{c}\alpha\,\mathrm{s}\beta\,\mathrm{c}\gamma + \mathrm{s}\alpha\,\mathrm{s}\gamma \\ \mathrm{s}\alpha\,\mathrm{c}\beta & \mathrm{s}\alpha\,\mathrm{s}\beta\,\mathrm{s}\gamma + \mathrm{c}\alpha\,\mathrm{c}\gamma & \mathrm{s}\alpha\,\mathrm{s}\beta\,\mathrm{c}\gamma - \mathrm{c}\alpha\,\mathrm{s}\gamma \\ -\mathrm{s}\beta & \mathrm{c}\beta\,\mathrm{s}\gamma & \mathrm{c}\beta\,\mathrm{c}\gamma \end{bmatrix} \tag{35}
$$

If $\mathrm{c}\beta \neq 0$

$$
\begin{aligned}
\tfrac{r_{32}}{r_{33}} &= \tan\gamma & \Rightarrow \quad \gamma &= \mathrm{Atan2}(r_{32}, r_{33}) \\
\tfrac{r_{21}}{r_{11}} &= \tan\alpha & \Rightarrow \quad \alpha &= \mathrm{Atan2}(r_{21}, r_{11}) \\
\mathrm{c}^2\beta &= r_{32}^2 + r_{33}^2 & \Rightarrow \quad \beta &= \mathrm{Atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})
\end{aligned} \tag{36}
$$

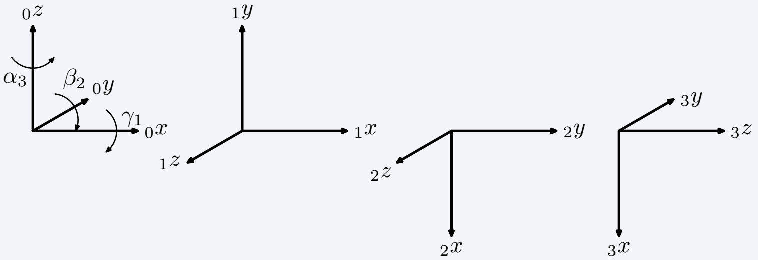
If $\beta = 90$ then $r_{12} = \sin(\gamma - \alpha)$ and $r_{13} = \cos(\gamma - \alpha)$
$\Rightarrow \alpha = 0$ and $\gamma = \mathrm{Atan2}(r_{12}, r_{13})$

If $\beta = -90$ then $r_{12} = -\sin(\gamma + \alpha)$ and $r_{22} = \cos(\gamma + \alpha)$
$\Rightarrow \alpha = 0$ and $\gamma = -\mathrm{Atan2}(r_{12}, r_{22})$

Rotations about fixed and moving frame axes

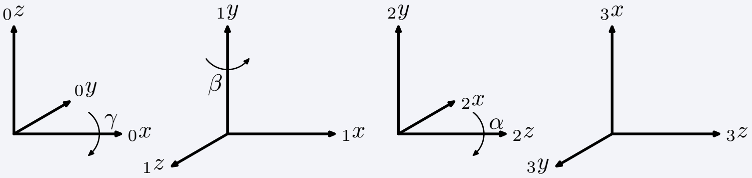
RPY – rotation about the axes of a fixed frame of reference:

$\mathrm{Rot}({}_0x, \gamma) : \ \gamma = 90, \quad \mathrm{Rot}({}_0y, \beta) : \ \beta = 90, \quad \mathrm{Rot}({}_0z, \alpha) : \ \alpha = 90$



Rotation about the axes of a moving frame of reference:

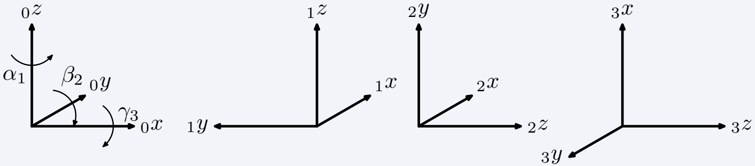
$\mathrm{Rot}({}_0x, \gamma) : \ \gamma = 90, \quad \mathrm{Rot}({}_1y, \beta) : \ \beta = 90, \quad \mathrm{Rot}({}_2z, \alpha) : \ \alpha = 90$



Rotations about fixed and moving frame axes in reverse order

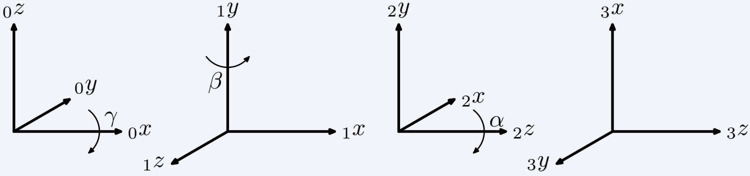
RPY – rotation about the axes of a fixed frame of reference:

$\mathrm{Rot}({}_0z, \alpha):\ \alpha = 90, \quad \mathrm{Rot}({}_0y, \beta):\ \beta = 90, \quad \mathrm{Rot}({}_0x, \gamma):\ \gamma = 90$

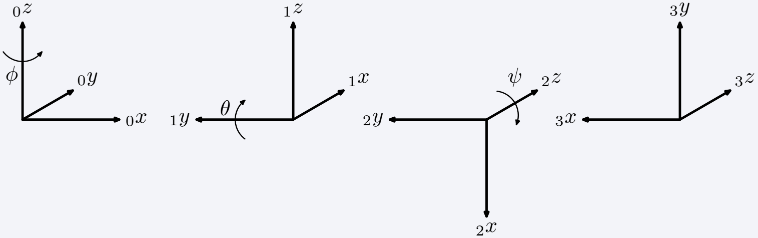


Rotation about the axes of a moving frame of reference:

$\mathrm{Rot}({}_0x, \gamma):\ \gamma = 90, \quad \mathrm{Rot}({}_1y, \beta):\ \beta = 90, \quad \mathrm{Rot}({}_2z, \alpha):\ \alpha = 90$



ZYZ Euler angles – rotation about moving frame axes



$$
{}^0_3\mathcal{R} = \mathrm{Rot}_{zyz}(\phi,\theta,\psi) = \mathrm{Rot}(z,\phi)\mathrm{Rot}(y,\theta)\mathrm{Rot}(z,\psi) =
\begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \mathrm{c}\theta & 0 & \mathrm{s}\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix}
\begin{bmatrix} \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} =
\begin{bmatrix} \mathrm{c}\phi\,\mathrm{c}\theta\,\mathrm{c}\psi - \mathrm{s}\phi\,\mathrm{s}\psi & -\mathrm{c}\phi\,\mathrm{c}\theta\,\mathrm{s}\psi - \mathrm{s}\phi\,\mathrm{c}\psi & \mathrm{c}\phi\,\mathrm{s}\theta \\ \mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\psi & -\mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{c}\psi & \mathrm{s}\phi\,\mathrm{s}\theta \\ -\mathrm{s}\theta\,\mathrm{c}\psi & \mathrm{s}\theta\,\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix} \tag{37}
$$

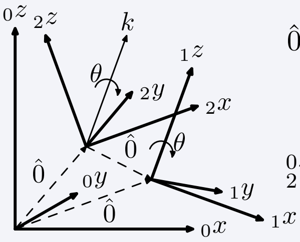
$$
{}^0_3\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\phi\,\mathrm{c}\theta\,\mathrm{c}\psi - \mathrm{s}\phi\,\mathrm{s}\psi & -\mathrm{c}\phi\,\mathrm{c}\theta\,\mathrm{s}\psi - \mathrm{s}\phi\,\mathrm{c}\psi & \mathrm{c}\phi\,\mathrm{s}\theta \\ \mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\psi & -\mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{c}\psi & \mathrm{s}\phi\,\mathrm{s}\theta \\ -\mathrm{s}\theta\,\mathrm{c}\psi & \mathrm{s}\theta\,\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix} \tag{38}
$$

If $\mathrm{s}\theta \neq 0$

$$
\begin{array}{lcl} \frac{r_{32}}{-r_{31}} = \tan\psi & \Rightarrow & \psi = \mathrm{Atan2}(r_{32}, -r_{31}) \\ \frac{r_{23}}{r_{13}} = \tan\phi & \Rightarrow & \phi = \mathrm{Atan2}(r_{23}, r_{13}) \\ \mathrm{s}^2\theta = r_{13}^2 + r_{23}^2 & \Rightarrow & \theta = \mathrm{Atan2}(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}) \end{array} \tag{39}
$$

If $\theta = 0$ then $r_{11} = \cos(\phi + \psi)$ and $r_{21} = \sin(\phi + \psi)$
$\Rightarrow \psi = 0$ and $\phi = \mathrm{Atan2}(r_{21}, r_{11})$

If $\theta = 180$ then $r_{12} = \sin(\psi - \phi)$ and $r_{22} = \cos(\psi - \phi)$
$\Rightarrow \phi = 0$ and $\psi = \mathrm{Atan2}(r_{12}, r_{22})$



$$\hat{0} = [0,0,0]^T, \quad {}^0k = {}^0_1z, \quad \|{}^0k\| = 1 \tag{40}$$

Prior to rotation by θ:

$${}^0_2\mathcal{R} = {}^0_1\mathcal{R}\,{}^1_2\mathcal{R} \quad \Rightarrow \quad {}^1_2\mathcal{R} = {}^0_1\mathcal{R}^{-1}\,{}^0_2\mathcal{R} \tag{41}$$

After rotation by θ:

$$\begin{aligned}
\underbrace{\mathrm{Rot}({}^0k,\theta)\,{}^0_2\mathcal{R}}_{\text{fixed frame}} &= \underbrace{{}^0_1\mathcal{R}\,\mathrm{Rot}({}_1z,\theta)\,{}^1_2\mathcal{R}}_{\text{moving frame}} \\
\mathrm{Rot}({}^0k,\theta)\,{}^0_2\mathcal{R} &= {}^0_1\mathcal{R}\,\mathrm{Rot}({}_1z,\theta)\,{}^0_1\mathcal{R}^{-1}\,{}^0_2\mathcal{R} \\
\mathrm{Rot}({}^0k,\theta) &= {}^0_1\mathcal{R}\,\mathrm{Rot}({}_1z,\theta)\,{}^0_1\mathcal{R}^{-1}
\end{aligned} \tag{42}$$

$${}^0_1\mathcal{R} = \begin{bmatrix} n_x & o_x & k_x \\ n_y & o_y & k_y \\ n_z & o_z & k_z \end{bmatrix} \Rightarrow {}^0_1\mathcal{R}^{-1} = {}^0_1\mathcal{R}^T = \begin{bmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ k_x & k_y & k_z \end{bmatrix} \tag{43}$$

$$\mathrm{Rot}({}_1z,\theta) = \begin{bmatrix} \mathrm{c}\theta & -\mathrm{s}\theta & 0 \\ \mathrm{s}\theta & \mathrm{c}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}$$

$$k = n \times o = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} n_y o_z - n_z o_y \\ n_z o_x - n_x o_z \\ n_x o_y - n_y o_x \end{bmatrix} \tag{45}$$

Dot products of versors of ${}^0_1\mathcal{R}^{-1}$:

$$n_x n_y + o_x o_y + k_x k_y = 0 \quad \Rightarrow \quad k_x k_y = -n_x n_y - o_x o_y \tag{46}$$

After multiplication (42) and using (44), (45) and (46) we get:

$$\mathrm{Rot}({}^0k,\theta) = \begin{bmatrix} k_x k_x(1-\mathrm{c}\theta)+\mathrm{c}\theta & k_y k_x(1-\mathrm{c}\theta)-k_z\mathrm{s}\theta & k_z k_x(1-\mathrm{c}\theta)+k_y\mathrm{s}\theta \\ k_x k_y(1-\mathrm{c}\theta)+k_z\mathrm{s}\theta & k_y k_y(1-\mathrm{c}\theta)+\mathrm{c}\theta & k_z k_y(1-\mathrm{c}\theta)-k_x\mathrm{s}\theta \\ k_x k_z(1-\mathrm{c}\theta)-k_y\mathrm{s}\theta & k_y k_z(1-\mathrm{c}\theta)+k_x\mathrm{s}\theta & k_z k_z(1-\mathrm{c}\theta)+\mathrm{c}\theta \end{bmatrix} \tag{47}$$

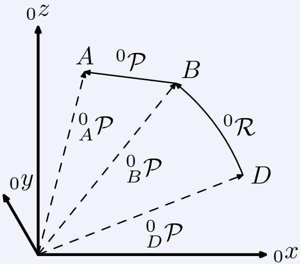
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} =$$
$$\begin{bmatrix} k_x k_x(1-\mathrm{c}\theta)+\mathrm{c}\theta & k_y k_x(1-\mathrm{c}\theta)-k_z\mathrm{s}\theta & k_z k_x(1-\mathrm{c}\theta)+k_y\mathrm{s}\theta \\ k_x k_y(1-\mathrm{c}\theta)+k_z\mathrm{s}\theta & k_y k_y(1-\mathrm{c}\theta)+\mathrm{c}\theta & k_z k_y(1-\mathrm{c}\theta)-k_x\mathrm{s}\theta \\ k_x k_z(1-\mathrm{c}\theta)-k_y\mathrm{s}\theta & k_y k_z(1-\mathrm{c}\theta)+k_x\mathrm{s}\theta & k_z k_z(1-\mathrm{c}\theta)+\mathrm{c}\theta \end{bmatrix} \tag{48}$$

$$r_{11}+r_{22}+r_{33} = k_x^2(1-\mathrm{c}\theta)+\mathrm{c}\theta+k_y^2(1-\mathrm{c}\theta)+\mathrm{c}\theta+k_z^2(1-\mathrm{c}\theta)+\mathrm{c}\theta = (k_x^2+k_y^2+k_z^2)(1-\mathrm{c}\theta)+3\mathrm{c}\theta = (1-\mathrm{c}\theta)+3\mathrm{c}\theta = 1+2\mathrm{c}\theta \tag{49}$$

$$\theta = \arccos\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \tag{50}$$

If $\mathrm{s}\theta \neq 0$

$$k_x = \frac{r_{32}-r_{23}}{2\mathrm{s}\theta}, \qquad k_y = \frac{r_{13}-r_{31}}{2\mathrm{s}\theta}, \qquad k_z = \frac{r_{21}-r_{12}}{2\mathrm{s}\theta} \tag{51}$$



Rotation and translation operators in relation to a fixed frame:

$$
{}^{0}\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad {}^{0}\mathcal{P} = \begin{bmatrix} {}^{0}\mathcal{P}_x \\ {}^{0}\mathcal{P}_y \\ {}^{0}\mathcal{P}_z \end{bmatrix} \tag{52}
$$

First rotate point D in relation to frame 0 and then translate the resulting point B in relation to frame 0

$$
{}^{0}_{A}\mathcal{P} = {}^{0}\mathcal{R}\,{}^{0}_{D}\mathcal{P} + {}^{0}\mathcal{P} \tag{53}
$$

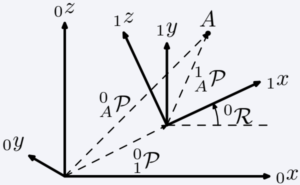
$$
\begin{bmatrix} {}^{0}_{A}\mathcal{P}_x \\ {}^{0}_{A}\mathcal{P}_y \\ {}^{0}_{A}\mathcal{P}_z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} {}^{0}_{D}\mathcal{P}_x \\ {}^{0}_{D}\mathcal{P}_y \\ {}^{0}_{D}\mathcal{P}_z \end{bmatrix} + \begin{bmatrix} {}^{0}\mathcal{P}_x \\ {}^{0}\mathcal{P}_y \\ {}^{0}\mathcal{P}_z \end{bmatrix} \tag{54}
$$

$$
\begin{aligned}
{}^{0}_{A}\mathcal{P}_x &= r_{11}\,{}^{0}_{D}\mathcal{P}_x + r_{12}\,{}^{0}_{D}\mathcal{P}_y + r_{13}\,{}^{0}_{D}\mathcal{P}_z + {}^{0}\mathcal{P}_x \\
{}^{0}_{A}\mathcal{P}_y &= r_{21}\,{}^{0}_{D}\mathcal{P}_x + r_{22}\,{}^{0}_{D}\mathcal{P}_y + r_{23}\,{}^{0}_{D}\mathcal{P}_z + {}^{0}\mathcal{P}_y \\
{}^{0}_{A}\mathcal{P}_z &= r_{31}\,{}^{0}_{D}\mathcal{P}_x + r_{32}\,{}^{0}_{D}\mathcal{P}_y + r_{33}\,{}^{0}_{D}\mathcal{P}_z + {}^{0}\mathcal{P}_z
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
{}^0_A\mathcal{P}_x &= r_{11}\,{}^0_D\mathcal{P}_x + r_{12}\,{}^0_D\mathcal{P}_y + r_{13}\,{}^0_D\mathcal{P}_z + {}^0\mathcal{P}_x \\
{}^0_A\mathcal{P}_y &= r_{21}\,{}^0_D\mathcal{P}_x + r_{22}\,{}^0_D\mathcal{P}_y + r_{23}\,{}^0_D\mathcal{P}_z + {}^0\mathcal{P}_y \\
{}^0_A\mathcal{P}_z &= r_{31}\,{}^0_D\mathcal{P}_x + r_{32}\,{}^0_D\mathcal{P}_y + r_{33}\,{}^0_D\mathcal{P}_z + {}^0\mathcal{P}_z \\
1 &= 0\,{}^0_D\mathcal{P}_x + 0\,{}^0_D\mathcal{P}_y + 0\,{}^0_D\mathcal{P}_z + 1
\end{aligned}
\tag{56}
$$

$$
\begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \\ {}^0_A\mathcal{P}_z \\ 1 \end{bmatrix}
=
\begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^0\mathcal{P}_x \\ r_{21} & r_{22} & r_{23} & {}^0\mathcal{P}_y \\ r_{31} & r_{32} & r_{33} & {}^0\mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} {}^0_D\mathcal{P}_x \\ {}^0_D\mathcal{P}_y \\ {}^0_D\mathcal{P}_z \\ 1 \end{bmatrix}
\tag{57}
$$

$$
{}^0\mathcal{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^0\mathcal{P}_x \\ r_{21} & r_{22} & r_{23} & {}^0\mathcal{P}_y \\ r_{31} & r_{32} & r_{33} & {}^0\mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad \Rightarrow \quad
{}^0\mathcal{T} = \left[\begin{array}{c|c} {}^0\mathcal{R}_{3\times 3} & {}^0\mathcal{P}_{3\times 1} \\ \hline 0_{1\times 3} & 1 \end{array}\right]
\tag{58}
$$



$$
{}^0_1\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad {}^0_1\mathcal{P} = \begin{bmatrix} {}^0_1\mathcal{P}_x \\ {}^0_1\mathcal{P}_y \\ {}^0_1\mathcal{P}_z \end{bmatrix} \tag{59}
$$

The coordinates expressed in frame 1 are transformed into a coordinate frame with the origin coincident with frame 1, but the orientation of frame 0, and then the translation between the two frames is taken into account

$$
{}^0_A\mathcal{P} = {}^0({}^1_A\mathcal{P}) + {}^0_1\mathcal{P} = {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{P} \tag{60}
$$

$$
\begin{bmatrix} {}^0_A\mathcal{P}_x \\ {}^0_A\mathcal{P}_y \\ {}^0_A\mathcal{P}_z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} {}^1_A\mathcal{P}_x \\ {}^1_A\mathcal{P}_y \\ {}^1_A\mathcal{P}_z \end{bmatrix} + \begin{bmatrix} {}^0_1\mathcal{P}_x \\ {}^0_1\mathcal{P}_y \\ {}^0_1\mathcal{P}_z \end{bmatrix} \tag{61}
$$

$$
\begin{aligned}
{}^0_A\mathcal{P}_x &= r_{11}\,{}^1_A\mathcal{P}_x + r_{12}\,{}^1_A\mathcal{P}_y + r_{13}\,{}^1_A\mathcal{P}_z + {}^0_1\mathcal{P}_x \\
{}^0_A\mathcal{P}_y &= r_{21}\,{}^1_A\mathcal{P}_x + r_{22}\,{}^1_A\mathcal{P}_y + r_{23}\,{}^1_A\mathcal{P}_z + {}^0_1\mathcal{P}_y \\
{}^0_A\mathcal{P}_z &= r_{31}\,{}^1_A\mathcal{P}_x + r_{32}\,{}^1_A\mathcal{P}_y + r_{33}\,{}^1_A\mathcal{P}_z + {}^0_1\mathcal{P}_z
\end{aligned} \tag{62}
$$

$$\begin{array}{ccccccccc}
{}^{0}_{A}\mathcal{P}_x & = & r_{11}\,{}^{1}_{A}\mathcal{P}_x & + & r_{12}\,{}^{1}_{A}\mathcal{P}_y & + & r_{13}\,{}^{1}_{A}\mathcal{P}_z & + & {}^{0}_{1}\mathcal{P}_x \\
{}^{0}_{A}\mathcal{P}_y & = & r_{21}\,{}^{1}_{A}\mathcal{P}_x & + & r_{22}\,{}^{1}_{A}\mathcal{P}_y & + & r_{23}\,{}^{1}_{A}\mathcal{P}_z & + & {}^{0}_{1}\mathcal{P}_y \\
{}^{0}_{A}\mathcal{P}_z & = & r_{31}\,{}^{1}_{A}\mathcal{P}_x & + & r_{32}\,{}^{1}_{A}\mathcal{P}_y & + & r_{33}\,{}^{1}_{A}\mathcal{P}_z & + & {}^{0}_{1}\mathcal{P}_z \\
1 & = & 0\,{}^{1}_{A}\mathcal{P}_x & + & 0\,{}^{1}_{A}\mathcal{P}_y & + & 0\,{}^{1}_{A}\mathcal{P}_z & + & 1
\end{array} \tag{63}$$

$$\begin{bmatrix} {}^{0}_{A}\mathcal{P}_x \\ {}^{0}_{A}\mathcal{P}_y \\ {}^{0}_{A}\mathcal{P}_z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{0}_{1}\mathcal{P}_x \\ r_{21} & r_{22} & r_{23} & {}^{0}_{1}\mathcal{P}_y \\ r_{31} & r_{32} & r_{33} & {}^{0}_{1}\mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{1}_{A}\mathcal{P}_x \\ {}^{1}_{A}\mathcal{P}_y \\ {}^{1}_{A}\mathcal{P}_z \\ 1 \end{bmatrix} \tag{64}$$

$${}^{0}_{1}\mathcal{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{0}_{1}\mathcal{P}_x \\ r_{21} & r_{22} & r_{23} & {}^{0}_{1}\mathcal{P}_y \\ r_{31} & r_{32} & r_{33} & {}^{0}_{1}\mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad {}^{0}_{1}\mathcal{T} = \left[\begin{array}{c|c} {}^{0}_{1}\mathcal{R}_{3\times 3} & {}^{0}_{1}\mathcal{P}_{3\times 1} \\ \hline 0_{1\times 3} & 1 \end{array}\right] \tag{65}$$

Homogeneous transformation operator

$$ {}^{0}_{A}\mathcal{P}_{4\times 1} = {}^{0}\mathcal{T}\,{}^{0}_{D}\mathcal{P}_{4\times 1} \tag{66} $$

Homogeneous transformation of coordinates

$$ {}^{0}_{A}\mathcal{P}_{4\times 1} = {}^{0}_{1}\mathcal{T}\,{}^{1}_{A}\mathcal{P}_{4\times 1} \tag{67} $$

If ${}^{0}_{D}\mathcal{P} = {}^{1}_{A}\mathcal{P}$ then ${}^{0}\mathcal{T} = {}^{0}_{1}\mathcal{T}$.

$$ {}^{0}_{1}\mathcal{T} = {}^{0}\mathcal{T} = \mathrm{Trans}(\mathcal{P})\mathrm{Rot}(k,\theta) \tag{68} $$

In general $\mathcal{P}_{4\times 1} = \begin{bmatrix} \mathcal{P}_x \\ \mathcal{P}_y \\ \mathcal{P}_z \\ 1 \end{bmatrix}, \quad \mathcal{R}_{4\times 4} = \mathrm{Rot}(k,\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 & \mathcal{P}_x \\ 0 & 1 & 0 & \mathcal{P}_y \\ 0 & 0 & 1 & \mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \mathcal{P}_x \\ r_{21} & r_{22} & r_{23} & \mathcal{P}_y \\ r_{31} & r_{32} & r_{33} & \mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{69}$$

$$\mathcal{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{T}^{-1} = \begin{bmatrix} n_x & n_y & n_z & -n \cdot p \\ o_x & o_y & o_z & -o \cdot p \\ a_x & a_y & a_z & -a \cdot p \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{70}$$

Proof

$$\mathcal{T}^{-1}\mathcal{T} = \begin{bmatrix} n_x & n_y & n_z & -n \cdot p \\ o_x & o_y & o_z & -o \cdot p \\ a_x & a_y & a_z & -a \cdot p \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{71}$$

Proof, cnt.

$$\mathcal{T}^{-1}\mathcal{T} = \begin{bmatrix} n \cdot n & n \cdot o & n \cdot a & n \cdot p - n \cdot p \\ o \cdot n & o \cdot o & o \cdot a & o \cdot p - o \cdot p \\ a \cdot n & a \cdot o & a \cdot a & a \cdot a - a \cdot p \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{72}$$

Length of versors is 1 $\Rightarrow$

$$\|n\| = \|o\| = \|a\| = 1 \quad \Rightarrow \quad n \cdot n = 1,\ o \cdot o = 1,\ a \cdot a = 1 \tag{73}$$

Versors are at right angles $\Rightarrow$

$$n \cdot o = o \cdot n = 0,\ n \cdot a = a \cdot n = 0,\ o \cdot a = a \cdot o = 0 \tag{74}$$

$$\mathcal{T}^{-1}\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathcal{I} \tag{75}$$

Position vectors

$$\begin{bmatrix} n_x + p_x \\ n_y + p_y \\ n_x + p_z \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{76}$$

$$\begin{bmatrix} o_x + p_x \\ o_y + p_y \\ o_z + p_z \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{77}$$

$$\begin{bmatrix} a_x + p_x \\ a_y + p_y \\ a_z + p_z \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{78}$$

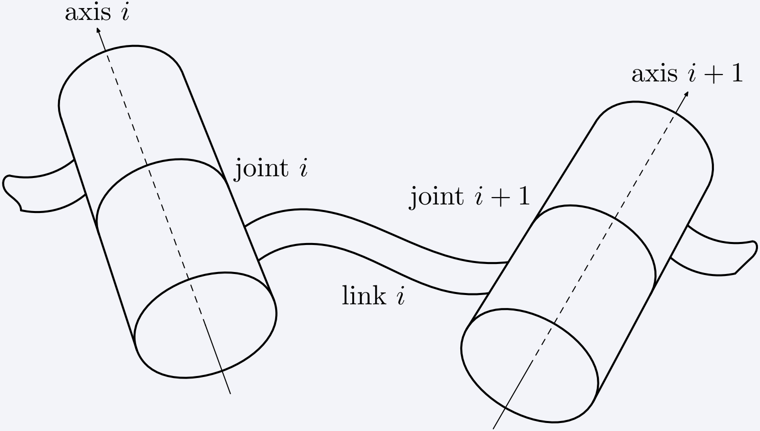
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{79}$$

$$\begin{bmatrix} n_x \\ n_y \\ n_x \\ 0 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{80}$$

$$\begin{bmatrix} o_x \\ o_y \\ o_z \\ 0 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{81}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \\ 0 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{82}$$

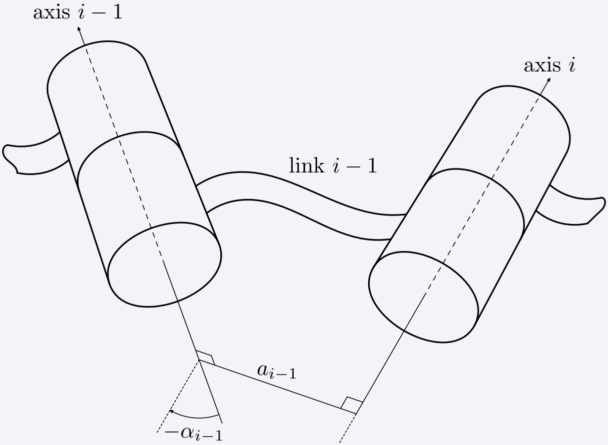
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_x & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{83}$$



Prismatic joint – translation along the joint axis

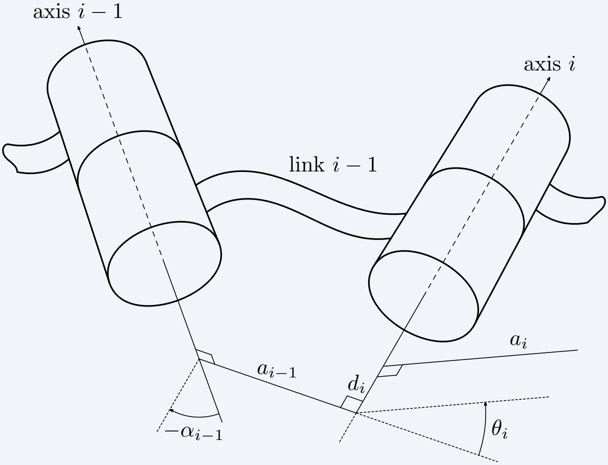
Revolute joint – rotation about the joint axis

Joint i connects link $i-1$ with link i



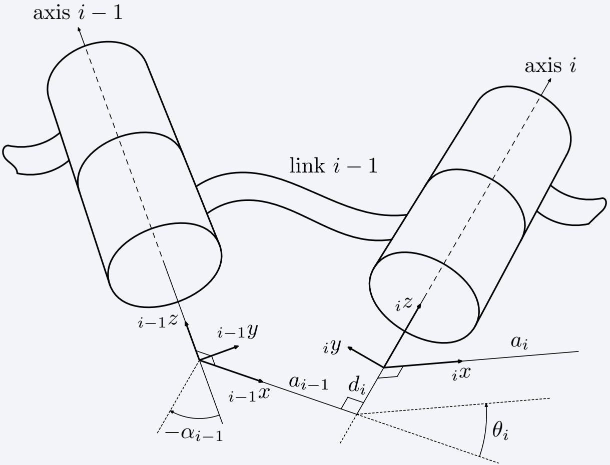
Link length a_{i-1} – length of link $i-1$ measured as a distance between joint $i-1$ axis and joint i axis

Link twist α_{i-1} – angle of rotation about the axis defined by the line along the distance between axes of joints $i-1$ and i



Joint offset d_i – distance between link $i-1$ and i measured along joint i axis

Joint angle θ_i – angle of rotation about the i axis between link $i-1$ and i



$$ {}^{i-1}_{i}\mathcal{T} = \mathrm{Rot}({}_{i-1}x, \alpha_{i-1})\mathrm{Trans}({}_{i-1}x, a_{i-1})\mathrm{Rot}({}_{i}z, \theta_i)\mathrm{Trans}({}_{i}z, d_i) \tag{84} $$

$$ {}^{i-1}_{i}\mathcal{T} = \mathrm{Rot}({}_{i-1}x, \alpha_{i-1})\mathrm{Trans}({}_{i-1}x, a_{i-1})\mathrm{Rot}({}_{i}z, \theta_i)\mathrm{Trans}({}_{i}z, d_i) \tag{85} $$

$$ \begin{aligned} {}^{i-1}_{i}\mathcal{T} \quad = \quad & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\alpha_{i-1} & -\mathrm{s}\alpha_{i-1} & 0 \\ 0 & \mathrm{s}\alpha_{i-1} & \mathrm{c}\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} \mathrm{c}\theta_i & -\mathrm{s}\theta_i & 0 & 0 \\ \mathrm{s}\theta_i & \mathrm{c}\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{86} $$

$$ {}^{i-1}_{i}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_i & -\mathrm{s}\theta_i & 0 & a_{i-1} \\ \mathrm{s}\theta_i\mathrm{c}\alpha_{i-1} & \mathrm{c}\theta_i\mathrm{c}\alpha_{i-1} & -\mathrm{s}\alpha_{i-1} & -d_i\mathrm{s}\alpha_{i-1} \\ \mathrm{s}\theta_i\mathrm{s}\alpha_{i-1} & \mathrm{c}\theta_i\mathrm{s}\alpha_{i-1} & \mathrm{c}\alpha_{i-1} & d_i\mathrm{c}\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{87} $$

Kinematic model of a manipulator is the mathematical relationship between the end-effector pose (its position and orientation) related to the base Cartesian reference coordinate frame and the joint variables.

Direct kinematic problem (DKP): knowing the values of joint variables θ_i compute the homogeneous matrix ${}^0_n\mathcal{T}$, where n is the frame attached to the end-effector.

Inverse kinematic problem (IKP): knowing the homogeneous matrix ${}^0_n\mathcal{T}$ determine the joint variables θ_i.

Analytic solution to DKP for serial manipulators is always possible, while IKP is possible only in the case of adequate kinematic structure of the manipulator.

Analytic solution to IKP for parallel manipulators is usually possible, while DKP is possible only in the case of adequate kinematic structure of the manipulator.

Algorithm for serial manipulators:

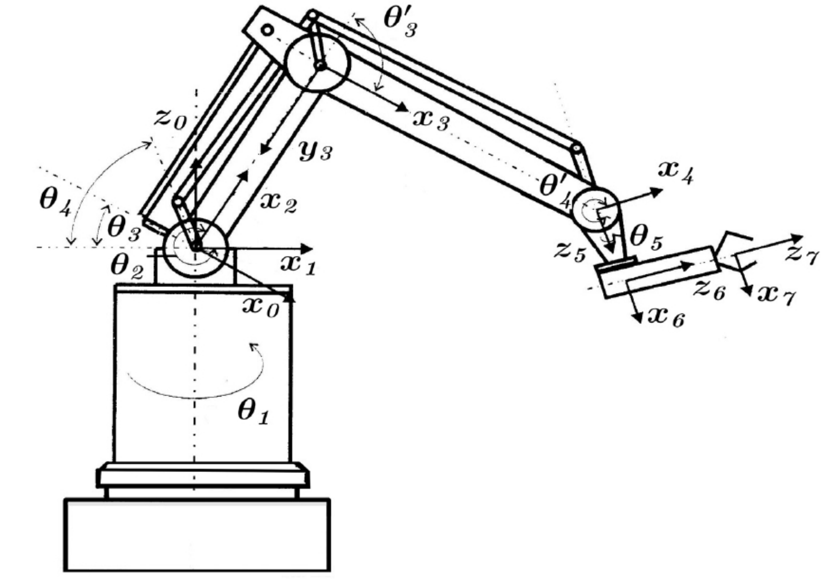
1. Draw the manipulator
2. Identify the axes of rotation or translation (for revolute and prismatic joints)
3. Assign the ${}_iz$ axis to the ith axis of rotation/traslation
4. Find the common perpendiculars between the ${}_{i-1}z$ and ${}_iz$ axes (if those axes intersect the common perpendicular is defined along ${}_{i-1}z \times {}_iz$)
5. Define the ${}_ix$ axes – the axis ${}_{i-1}x$ coincides with the common perpendicular between the ${}_{i-1}z$ and ${}_iz$ axes and points from ${}_{i-1}z$ to ${}_iz$
6. Determine the ${}_iy$ axes ${}_iy = {}_iz \times {}_ix$
7. The base coordinate frame 0 should be located in such a way that α_0, a_0 and θ_1 or d_1 are equal to 0 $\Rightarrow {}_0z = {}_1z$ and ${}^0_0\mathcal{P} = {}^0_1\mathcal{P}$

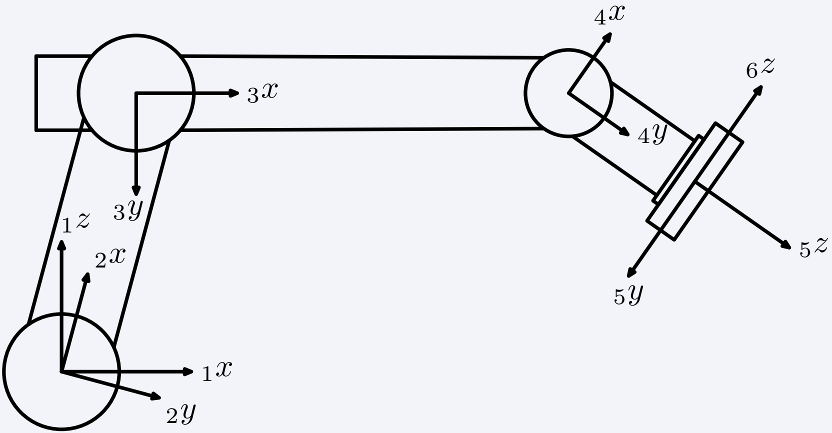
1. Determine the link and joint parameters
   - a_{i-1}: the distance between axes ${}_{i-1}z$ and ${}_{i}z$ measured along ${}_{i-1}x$
   - α_{i-1}: the angle between axes ${}_{i-1}z$ and ${}_{i}z$ measured around ${}_{i-1}x$
   - d_i: the distance between axes ${}_{i-1}x$ and ${}_{i}x$ measured along ${}_{i}z$
   - θ_i: the angle between axes ${}_{i-1}x$ and ${}_{i}x$ measured around ${}_{i}z$
2. Determine the homogeneous matrices ${}^{i-1}_{i}\mathcal{T}$ using (87)
3. Solve the direct kinematics problem

$$ {}^{0}_{n}\mathcal{T} = {}^{0}_{1}\mathcal{T}\,{}^{1}_{2}\mathcal{T}\ldots{}^{n-1}_{n}\mathcal{T} = \prod_{i=1}^{n} {}^{i-1}_{i}\mathcal{T} \tag{88} $$

   where n – the end-effector frame
4. Solve the inverse kinematics problem

Modified IRp-6 robot



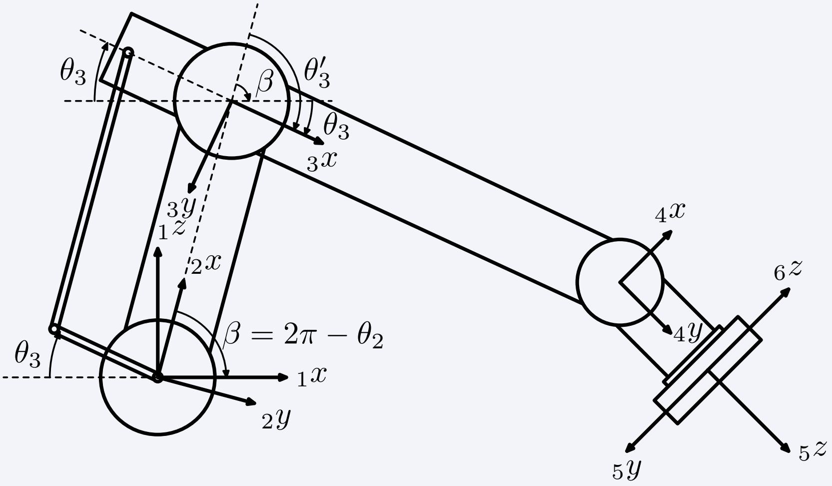


$$a_{i-1} = {}_{i-1}z \xrightarrow{{}_{i-1}x} {}_iz \qquad \alpha_{i-1} = {}_{i-1}z \xrightarrow{{}_{i-1}x}\circlearrowleft {}_iz$$

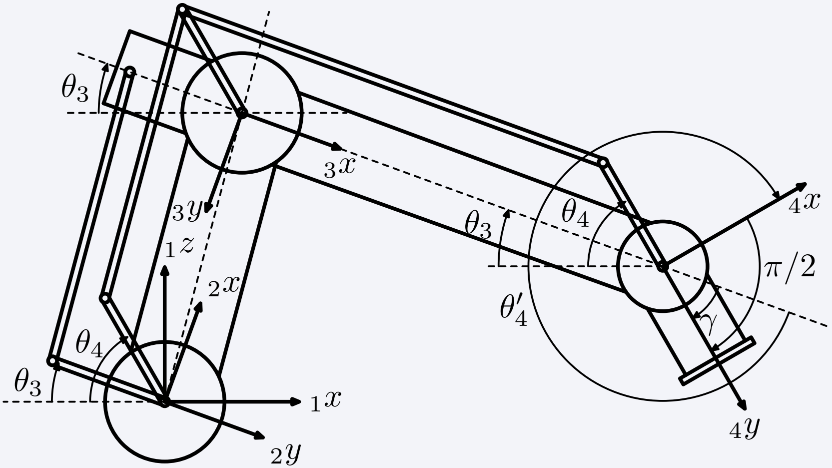
$$d_i = {}_{i-1}x \xrightarrow{{}_iz} {}_ix \qquad \theta_i = {}_{i-1}x \xrightarrow{{}_iz}\circlearrowleft {}_ix$$

Denavit-Hartenberg parameters for the modified IRp-6 robot

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	$-\frac{\pi}{2}$	0	θ_2
3	a_2	0	0	$\theta_3' = \theta_3 - \theta_2$
4	a_3	0	0	$\theta_4' = \theta_4 - \theta_3 - \frac{\pi}{2}$
5	0	$-\frac{\pi}{2}$	d_5	θ_5
6	0	$\frac{\pi}{2}$	0	θ_6



$$\theta_3' = \beta + \theta_3 = 2\pi - \theta_2 + \theta_3 \sim \theta_3 - \theta_2$$



$$\gamma = \theta_4 - \theta_3$$

$$\frac{\pi}{2} - \gamma = 2\pi - \theta_4' \quad \Rightarrow \quad \theta_4' = 2\pi - \frac{\pi}{2} + (\theta_4 - \theta_3) \ \sim \ \theta_4 - \theta_3 - \frac{\pi}{2}$$

$$ {}^{0}_{1}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 & 0 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{89} $$

$$ {}^{1}_{2}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{s}\theta_2 & -\mathrm{c}\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{90} $$

$$ {}^{2}_{3}\mathcal{T} = \begin{bmatrix} \mathrm{c}(\theta_3 - \theta_2) & -\mathrm{s}(\theta_3 - \theta_2) & 0 & a_2 \\ \mathrm{s}(\theta_3 - \theta_2) & \mathrm{c}(\theta_3 - \theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{91} $$

$\theta_4' = \theta_4 - \theta_3 - \frac{\pi}{2} \Rightarrow$

$$\cos\left(-\left(\frac{\pi}{2} + \theta_3 - \theta_4\right)\right) = \cos\left(\frac{\pi}{2} + \theta_3 - \theta_4\right) = -\sin\left(\theta_3 - \theta_4\right)$$

$$\sin\left(-\left(\frac{\pi}{2} + \theta_3 - \theta_4\right)\right) = -\sin\left(\frac{\pi}{2} + \theta_3 - \theta_4\right) = -\cos\left(\theta_3 - \theta_4\right)$$

$$ {}^{3}_{4}\mathcal{T} = \begin{bmatrix} -\mathrm{s}\left(\theta_3 - \theta_4\right) & \mathrm{c}\left(\theta_3 - \theta_4\right) & 0 & a_3 \\ -\mathrm{c}\left(\theta_3 - \theta_4\right) & -\mathrm{s}\left(\theta_3 - \theta_4\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{92}$$

$$ {}^{4}_{5}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_5 & -\mathrm{s}\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ -\mathrm{s}\theta_5 & -\mathrm{c}\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{93}$$

$$
{}^{5}_{6}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_6 & -\mathrm{s}\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathrm{s}\theta_6 & \mathrm{c}\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{94}
$$

Abbreviations:
$\mathrm{c}_i = \mathrm{c}\theta_i$, $\mathrm{s}_i = \mathrm{s}\theta_i$, $\mathrm{c}_{32} = \mathrm{c}\left(\theta_3 - \theta_2\right)$, $\mathrm{s}_{32} = \mathrm{s}\left(\theta_3 - \theta_2\right)$.

Composition of matrices:

$$
{}^{0}_{2}\mathcal{T} = \begin{bmatrix} \mathrm{c}_1\mathrm{c}_2 & -\mathrm{c}_1\mathrm{s}_2 & -\mathrm{s}_1 & 0 \\ \mathrm{s}_1\mathrm{c}_2 & -\mathrm{s}_1\mathrm{s}_2 & \mathrm{c}_1 & 0 \\ -\mathrm{s}_2 & -\mathrm{c}_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{95}
$$

$$
{}^{0}_{3}\mathcal{T} = \begin{bmatrix} \mathrm{c}_1\left(\mathrm{c}_2\mathrm{c}_{32} - \mathrm{s}_2\mathrm{s}_{32}\right) & -\mathrm{c}_1\left(\mathrm{c}_2\mathrm{s}_{32} + \mathrm{s}_2\mathrm{c}_{32}\right) & -\mathrm{s}_1 & \mathrm{c}_1\mathrm{c}_2\, a_2 \\ \mathrm{s}_1\left(\mathrm{c}_2\mathrm{c}_{32} - \mathrm{s}_2\mathrm{s}_{32}\right) & -\mathrm{s}_1\left(\mathrm{c}_2\mathrm{s}_{32} + \mathrm{s}_2\mathrm{c}_{32}\right) & \mathrm{c}_1 & \mathrm{s}_1\mathrm{c}_2\, a_2 \\ -\mathrm{c}_2\mathrm{s}_{32} - \mathrm{s}_2\mathrm{c}_{32} & \mathrm{s}_2\mathrm{s}_{32} - \mathrm{c}_2\mathrm{c}_{32} & 0 & -\mathrm{s}_2\, a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{96}
$$

$$
{}^{0}_{4}\mathcal{T} = \begin{bmatrix} \mathrm{c}_1\mathrm{s}_4 & \mathrm{c}_1\mathrm{c}_4 & -\mathrm{s}_1 & \mathrm{c}_1\left(a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right) \\ \mathrm{s}_1\mathrm{s}_4 & \mathrm{s}_1\mathrm{c}_4 & \mathrm{c}_1 & \mathrm{s}_1\left(a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right) \\ \mathrm{c}_4 & -\mathrm{s}_4 & 0 & -a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{97}
$$

$$
{}^{0}_{5}\mathcal{T} = \begin{bmatrix} \mathrm{c}_1\mathrm{s}_4\mathrm{c}_5 + \mathrm{s}_1\mathrm{s}_5 & -\mathrm{c}_1\mathrm{s}_4\mathrm{s}_5 + \mathrm{s}_1\mathrm{c}_5 & \mathrm{c}_1\mathrm{c}_4 & \mathrm{c}_1\left(\mathrm{c}_4\, d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right) \\ \mathrm{s}_1\mathrm{s}_4\mathrm{c}_5 - \mathrm{c}_1\mathrm{s}_5 & -\mathrm{s}_1\mathrm{s}_4\mathrm{s}_5 - c1\,c5 & \mathrm{s}_1\mathrm{c}_4 & \mathrm{s}_1\left(\mathrm{c}_4\, d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right) \\ \mathrm{c}_4\mathrm{c}_5 & -\mathrm{c}_4\mathrm{s}_5 & -\mathrm{s}_4 & -\mathrm{s}_4\, d_5 - a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{98}
$$

$$
{}^0_6\mathcal{T} = \begin{bmatrix} (c_1 s_4 c_5 + s_1 s_5)\, c_6 + c_1 c_4 s_6 & -(c_1 s_4 c_5 + s_1 s_5)\, s_6 + c_1 c_4 c_6 & c_1 s_4 s_5 - s_1 c_5 & c_1\,(c_4\, d_5 + a_3 c_3 + a_2 c_2) \\ (s_1 s_4 c_5 - c_1 s_5)\, c_6 + s_1 c_4 s_6 & -(s_1 s_4 c_5 - c_1 s_5)\, s_6 + s_1 c_4 c_6 & s_1 s_4 s_5 + c_1 c_5 & s_1\,(c_4\, d_5 + a_3 c_3 + a_2 c_2) \\ c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & -s_4\, d_5 - a_3 s_3 - a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{99}
$$

n – number of degrees of freedom (usually $n = 6$)

$$\underbrace{{}^0_n\mathcal{T}}_{\text{direct kinematic problem}} = \underbrace{{}^0_n\mathcal{T}_d}_{\text{constant}} \tag{100}$$

$$\underbrace{{}^0_1\mathcal{T}^{-1}\,{}^0_n\mathcal{T}}_{\mathcal{T}(\theta_2,\ldots,\theta_n)} = \underbrace{{}^0_1\mathcal{T}^{-1}\,{}^0_n\mathcal{T}_d}_{\mathcal{T}(\theta_1)} \tag{101}$$

$$\underbrace{{}^0_n\mathcal{T}\,{}^{n-1}_n\mathcal{T}^{-1}}_{\mathcal{T}(\theta_1,\ldots,\theta_{n-1})} = \underbrace{{}^0_n\mathcal{T}_d\,{}^{n-1}_n\mathcal{T}^{-1}}_{\mathcal{T}(\theta_n)} \tag{102}$$

We are looking for elements producing $f(\theta_i) = \text{const}$.
Formulas (101) and (102) can be repeated several times with next ${}^{i-1}_i\mathcal{T}^{-1}$.

$$
{}^{2}_{4}\mathcal{T} = {}^{2}_{3}\mathcal{T}\,{}^{3}_{4}\mathcal{T} = \begin{bmatrix} -\mathrm{c}_{32}\mathrm{s}_{34} + \mathrm{s}_{32}\mathrm{c}_{34} & \mathrm{c}_{32}\mathrm{c}_{34} + \mathrm{s}_{32}\mathrm{s}_{34} & 0 & a_3\mathrm{c}_{32} + a_2 \\ -\mathrm{c}_{32}\mathrm{c}_{34} - \mathrm{s}_{32}\mathrm{s}_{34} & -\mathrm{c}_{32}\mathrm{s}_{34} + \mathrm{s}_{32}\mathrm{c}_{34} & 0 & a_3\mathrm{s}_{32} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{103}
$$

$$
{}^{2}_{4}\mathcal{T} = \begin{bmatrix} \mathrm{s}(\theta_3 - \theta_2 - \theta_3 + \theta_4) & \mathrm{c}(\theta_3 - \theta_2 - \theta_3 + \theta_4) & 0 & a_3\mathrm{c}_{32} + a_2 \\ -\mathrm{c}(\theta_3 - \theta_2 - \theta_3 + \theta_4) & \mathrm{s}(\theta_3 - \theta_2 - \theta_3 + \theta_4) & 0 & \mathrm{s}_{32} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{104}
$$

$$
{}^{2}_{4}\mathcal{T} = \begin{bmatrix} \mathrm{s}_{42} & \mathrm{c}_{42} & 0 & a_3\mathrm{c}_{32} + a_2 \\ -\mathrm{c}_{42} & \mathrm{s}_{42} & 0 & \mathrm{s}_{32} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \theta_{42} = \theta_4 - \theta_2 \tag{105}
$$

$$ {}^{1}_{4}\mathcal{T} = {}^{1}_{2}\mathcal{T}\,{}^{2}_{4}\mathcal{T} = $$

$$ \begin{bmatrix} c_2 s_{42} + s_2 c_{42} & c_2 c_{42} - s_2 s_{42} & 0 & c_2(a_3 c_{32} + a_2) - s_2 a_3 s_{32} \\ 0 & 0 & 1 & 0 \\ c_2 c_{42} - s_2 s_{42} & -c_2 s_{42} - s_2 c_{42} & 0 & -s_2(a_3 c_{32} + a_2) - c_2 a_3 s_{32} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{106} $$

$$ {}^{1}_{4}\mathcal{T} = \begin{bmatrix} s(\theta_2 + \theta_4 - \theta_2) & c(\theta_2 + \theta_4 - \theta_2) & 0 & a_3(c_2 c_{32} - s_2 s_{32}) + a_2 c_2 \\ 0 & 0 & 1 & 0 \\ c(\theta_2 + \theta_4 - \theta_2) & -s(\theta_2 + \theta_4 - \theta_2) & 0 & -a_3(s_2 c_{32} + c_2 s_{32}) - a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{107} $$

$$
{}^{1}_{4}\mathcal{T} = \begin{bmatrix} \mathrm{s}_4 & \mathrm{c}_4 & 0 & a_3\mathrm{c}(\theta_2+\theta_3-\theta_2)+a_2\mathrm{c}_2 \\ 0 & 0 & 1 & 0 \\ \mathrm{c}_4 & -\mathrm{s}_4 & 0 & -a_3\mathrm{s}(\theta_2+\theta_3-\theta_2)-a_2\mathrm{s}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{108}
$$

$$
{}^{1}_{4}\mathcal{T} = \begin{bmatrix} \mathrm{s}_4 & \mathrm{c}_4 & 0 & a_3\mathrm{c}_3+a_2\mathrm{c}_2 \\ 0 & 0 & 1 & 0 \\ \mathrm{c}_4 & -\mathrm{s}_4 & 0 & -a_3\mathrm{s}_3-a_2\mathrm{s}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{109}
$$

$$
{}^{4}_{6}\mathcal{T} = {}^{4}_{5}\mathcal{T}\,{}^{5}_{6}\mathcal{T} = \begin{bmatrix} \mathrm{c}_5\mathrm{c}_6 & -\mathrm{c}_5\mathrm{s}_6 & \mathrm{s}_5 & 0 \\ \mathrm{s}_6 & \mathrm{c}_6 & 0 & d_5 \\ -\mathrm{s}_5\mathrm{c}_6 & \mathrm{s}_5\mathrm{s}_6 & \mathrm{c}_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{110}
$$

$$ {}^{1}_{6}\mathcal{T} = {}^{1}_{4}\mathcal{T}\,{}^{4}_{6}\mathcal{T} \tag{111} $$

$$ {}^{1}_{6}\mathcal{T} = \begin{bmatrix} \mathrm{s}_4\mathrm{c}_5\mathrm{c}_6 + \mathrm{c}_4\mathrm{s}_6 & -\mathrm{s}_4\mathrm{c}_5\mathrm{s}_6 + \mathrm{c}_4\mathrm{c}_6 & \mathrm{s}_4\mathrm{s}_5 & \mathrm{c}_4\,d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2 \\ -\mathrm{s}_5\mathrm{c}_6 & \mathrm{s}_5\mathrm{s}_6 & \mathrm{c}_5 & 0 \\ \mathrm{c}_4\mathrm{c}_5\mathrm{c}_6 - \mathrm{s}_4\mathrm{s}_6 & -\mathrm{c}_4\mathrm{c}_5\mathrm{s}_6 - \mathrm{s}_4\mathrm{c}_6 & \mathrm{c}_4\mathrm{s}_5 & -\mathrm{s}_4\,d_5 - a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{112} $$

Those intermediate matrices are more useful for the derivation of the inverse kinematics problem solution than those obtained by systematic left-right matrix multiplication.

$$ {}^{0}_{1}\mathcal{T}^{-1}\,{}^{0}_{n}\mathcal{T}_d = {}^{0}_{1}\mathcal{T}^{-1}\,{}^{0}_{n}\mathcal{T} \tag{113} $$

$$ {}^{0}_{n}\mathcal{T}_d = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{114} $$

$$ {}^{0}_{1}\mathcal{T}^{-1}\,{}^{0}_{n}\mathcal{T}_d = \begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = $$

$$ \begin{bmatrix} \mathrm{c}_1 r_{11} + \mathrm{s}_1 r_{21} & \mathrm{c}_1 r_{12} + \mathrm{s}_1 r_{22} & \mathrm{c}_1 r_{13} + \mathrm{s}_1 r_{23} & \mathrm{c}_1 p_x + \mathrm{s}_1 p_y \\ \mathrm{c}_1 r_{21} - \mathrm{s}_1 r_{11} & \mathrm{c}_1 r_{22} - \mathrm{s}_1 r_{12} & \mathrm{c}_1 r_{23} - \mathrm{s}_1 r_{13} & \mathrm{c}_1 p_y - \mathrm{s}_1 p_x \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{115} $$

$$\begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} & c_1 p_x + s_1 p_y \\ c_1 r_{21} - s_1 r_{11} & c_1 r_{22} - s_1 r_{12} & c_1 r_{23} - s_1 r_{13} & c_1 p_y - s_1 p_x \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & c_4 d_5 + a_3 c_3 + a_2 c_2 \\ -s_5 c_6 & s_5 s_6 & c_5 & 0 \\ c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & -s_4 d_5 - a_3 s_3 - a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{116}$$

$${}^1_6\mathcal{T}_{24} \quad \Rightarrow \quad c_1 p_y - s_1 p_x = 0 \quad \Rightarrow \quad c_1 p_y = s_1 p_x \quad \Rightarrow \quad \frac{s_1}{c_1} = \frac{p_y}{p_x}$$

$$\theta_1 = \arctan\left(\frac{p_y}{p_x}\right) \tag{117}$$

$${}^1_6\mathcal{T}_{23} \quad \Rightarrow \quad c_1 r_{23} - s_1 r_{13} = c_5 \qquad s_5 = \pm\sqrt{1 - c_5^2} \quad \Rightarrow$$

$$\theta_5 = \arctan\left(\frac{\pm\sqrt{1-(c_1 r_{23} - s_1 r_{13})^2}}{c_1 r_{23} - s_1 r_{13}}\right) \tag{118}$$

$$\frac{{}^1_6\mathcal{T}_{22}}{{}^1_6\mathcal{T}_{21}} \quad\Rightarrow\quad \frac{c_1 r_{22} - s_1 r_{12}}{c_1 r_{21} - s_1 r_{11}} = \frac{s_5 s_6}{-s_5 c_6}$$

Assuming $s_5 \neq 0$

$$\theta_6 = \arctan\left(\frac{c_1 r_{22} - s_1 r_{12}}{-(c_1 r_{21} - s_1 r_{11})}\right) \tag{119}$$

$$\frac{{}^1_6\mathcal{T}_{13}}{{}^1_6\mathcal{T}_{33}} \quad\Rightarrow\quad \frac{c_1 r_{13} + s_1 r_{23}}{r_{33}} = \frac{s_4 s_5}{c_4 s_5}$$

Assuming $s_5 \neq 0$

$$\theta_4 = \arctan\left(\frac{c_1 r_{13} + s_1 r_{23}}{r_{33}}\right) \tag{120}$$

When $s_5 = 0$ either $\theta_5 = 0$ or $\theta_5 = \pi \quad \Rightarrow$

$$
{}^1_6\mathcal{T} = \begin{bmatrix} \pm s_4c_6 + c_4s_6 & \pm s_4s_6 + c_4c_6 & 0 & a_2c_2 + a_3c_3 \\ 0 & 0 & \pm 1 & 0 \\ \pm c_4c_6 - s_4s_6 & \pm c_4s_6 - s_4c_6 & 0 & -a_2s_2 - a_3s_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{121}
$$

$$
{}^1_6\mathcal{T} = \begin{bmatrix} s\,(\theta_6 \pm \theta_4) & c\,(\theta_6 \pm \theta_4) & 0 & a_2c_2 + a_3c_3 \\ 0 & 0 & \pm 1 & 0 \\ \pm c\,(\theta_6 \pm \theta_4) & \pm s\,(\theta_6 \pm \theta_4) & 0 & -a_2s_2 - a_3s_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{122}
$$

$$
\frac{{}^1_6\mathcal{T}_{11}}{{}^1_6\mathcal{T}_{12}} \quad \Rightarrow \quad \frac{c_1 r_{11} + s_1 r_{21}}{c_1 r_{12} + s_1 r_{22}} = \frac{s\,(\theta_6 \pm \theta_4)}{c\,(\theta_6 \pm \theta_4)} \quad \Rightarrow
$$

$$
(\theta_6 \pm \theta_4) = \arctan\left(\frac{c_1 r_{11} + s_1 r_{21}}{c_1 r_{12} + s_1 r_{22}}\right) \qquad \theta_4 = \theta_4^{\text{previous}} \tag{123}
$$

${}^{1}_{6}\mathcal{T}_{14}$ and ${}^{1}_{6}\mathcal{T}_{34}$ $\Rightarrow$

$$\left\{\begin{array}{lcl} c_4 d_5 + a_3 c_3 + a_2 c_2 & = & c_1 p_x + s_1 p_y \\ -s_4 d_5 - a_3 s_3 - a_2 s_2 & = & p_z \end{array}\right. \tag{124}$$

$$\left\{\begin{array}{lcl} a_3 c_3 + a_2 c_2 & = & c_1 p_x + s_1 p_y - c_4 d_5 \\ a_3 s_3 + a_2 s_2 & = & -\left(p_z + s_4 d_5\right) \end{array}\right. \tag{125}$$

Introducing the following substitutions:

$$\begin{array}{l} E = c_1 p_x + s_1 p_y - c_4 d_5 \\ F = -\left(p_z + s_4 d_5\right) \end{array} \tag{126}$$

The following standard system of equations is obtained:

$$\left\{\begin{array}{lcl} a_3 c_3 & = & E - a_2 c_2 \\ a_3 s_3 & = & F - a_2 s_2 \end{array}\right. \tag{127}$$

Solution of the equations (127)

$$\begin{cases} (a_3 \mathrm{c}_3)^2 & = & E^2 + (a_2 \mathrm{c}_2)^2 - 2Ea_2 \mathrm{c}_2 \\ (a_3 \mathrm{s}_3)^2 & = & F^2 + (a_2 \mathrm{s}_2)^2 - 2Fa_2 \mathrm{s}_2 \end{cases} \tag{128}$$

Addition of those equations results in:

$$a_3^2 = E^2 + F^2 + a_2^2 - 2Ea_2 \mathrm{c}_2 - 2Fa_2 \mathrm{s}_2 \tag{129}$$

$$E^2 + F^2 + a_2^2 - a_3^2 = 2Ea_2 \mathrm{c}_2 + 2Fa_2 \mathrm{s}_2 \tag{130}$$

$$K = E^2 + F^2 + a_2^2 - a_3^2 \tag{131}$$

Introduction of polar coordinates:

$$\begin{cases} G = 2Ea_2 = \zeta \mathrm{s}(\varphi) \\ H = 2Fa_2 = \zeta \mathrm{c}(\varphi) \end{cases} \tag{132}$$

$$\begin{cases} \zeta = \sqrt{G^2 + H^2} \\ \varphi = \arctan\left(\frac{G}{H}\right) \end{cases} \tag{133}$$

Equation (130) assumes the following form:

$$K = \zeta \mathrm{s}\varphi \mathrm{c}\theta_2 + \zeta \mathrm{c}\varphi \mathrm{s}\theta_2 \tag{134}$$

$$\begin{cases} \mathrm{s}\left(\theta_2 + \varphi\right) = \frac{K}{\zeta} \\ \mathrm{c}\left(\theta_2 + \varphi\right) = \pm\sqrt{1 - \left(\frac{K}{\zeta}\right)^2} \end{cases} \tag{135}$$

$$\theta_2 + \varphi = \arctan\left(\frac{\frac{K}{\zeta}}{\pm\sqrt{1 - \left(\frac{K}{\zeta}\right)^2}}\right)$$

$$\theta_2 = \arctan\left(\frac{\frac{K}{\zeta}}{\pm\sqrt{1 - \left(\frac{K}{\zeta}\right)^2}}\right) - \arctan\left(\frac{G}{H}\right) \tag{136}$$

Using (127), i.e.:

$$\left\{ \begin{array}{rcl} a_3 \mathrm{c}_3 & = & E - a_2 \mathrm{c}_2 \\ a_3 \mathrm{s}_3 & = & F - a_2 \mathrm{s}_2 \end{array} \right. \tag{137}$$

the following is obtained

$$\frac{F - a_2 \mathrm{s}_2}{E - a_2 \mathrm{c}_2} = \frac{\mathrm{s}_3}{\mathrm{c}_3} \tag{138}$$

thus

$$\theta_3 = \arctan\left(\frac{F - a_2 \mathrm{s}_2}{E - a_2 \mathrm{c}_2}\right) \tag{139}$$

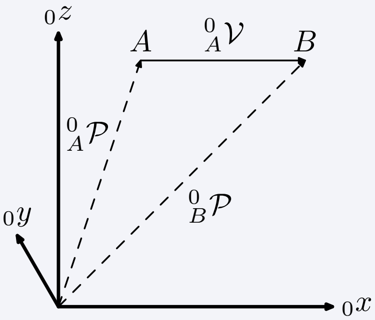
Selection of the correct solution out of the many obtained is performed using the trajectory continuity assumption. The correct solution is the one nearest to the current pose of the arm.

The case of a revolute joint:
θ_i – joint angle
ϕ_i – motor position

- direct drive $\Rightarrow \theta_i = \phi_i$
- geared motor $\Rightarrow \theta_i = g\phi_i$, where g is the gear ratio
- screw and nut $\Rightarrow \theta_i = f_g(\phi_i)$ – nonlinear relationship

Actually ϕ_i is servo controlled.

Point A is moving towards point B along a straight line

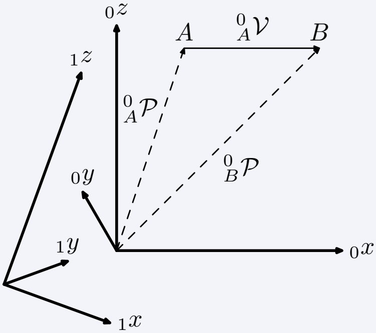


$$ {}^0\mathcal{P}(t) = {}^0_A\mathcal{P} \tag{140} $$

$$ {}^0\mathcal{P}(t+\Delta t) = {}^0_B\mathcal{P} \tag{141} $$

$$ \Delta t \to 0 \quad \Rightarrow \quad B \to A \tag{142} $$

$$ {}^0\mathcal{V} = \frac{\mathrm{d}\,{}^0\mathcal{P}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{{}^0\mathcal{P}(t+\Delta t) - {}^0\mathcal{P}(t)}{\Delta t} = {}^0_A\mathcal{V} \tag{143} $$



If frames 0 and 1 are not moving with relation to each other

$$^1_A\mathcal{V} = {}^1({}^0_A\mathcal{V}) = {}^1_0\mathcal{R}\,{}^0_A\mathcal{V} \quad (144)$$

If frames 0 and 1 are moving with relation to each other, but their relative orientation does not change

$$^1_A\mathcal{V} = {}^1_0\mathcal{V} + {}^1({}^0_A\mathcal{V}) = {}^1_0\mathcal{V} + {}^1_0\mathcal{R}\,{}^0_A\mathcal{V} \quad (145)$$

$$ {}^{1}_{A}\mathcal{P} = {}^{1}_{0}\mathcal{P} + {}^{1}_{0}\mathcal{R}\,{}^{0}_{A}\mathcal{P} \tag{146} $$

$$ \frac{\mathrm{d}\,{}^{1}_{A}\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^{1}_{0}\mathcal{P}}{\mathrm{d}t} + \frac{\mathrm{d}\left({}^{1}_{0}\mathcal{R}\,{}^{0}_{A}\mathcal{P}\right)}{\mathrm{d}t} \tag{147} $$

If ${}^{1}_{0}\mathcal{R} = \text{const}$

$$ \frac{\mathrm{d}\,{}^{1}_{A}\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^{1}_{0}\mathcal{P}}{\mathrm{d}t} + {}^{1}_{0}\mathcal{R}\frac{\mathrm{d}\,{}^{0}_{A}\mathcal{P}}{\mathrm{d}t} \tag{148} $$

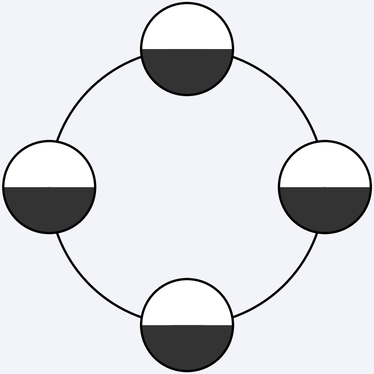
$$ {}^{1}_{A}\mathcal{V} = {}^{1}_{0}\mathcal{V} + {}^{1}_{0}\mathcal{R}\,{}^{0}_{A}\mathcal{V} \tag{149} $$

$\Rightarrow$ (149) and (145) are the same.
If moreover ${}^{1}_{0}\mathcal{V} = 0$

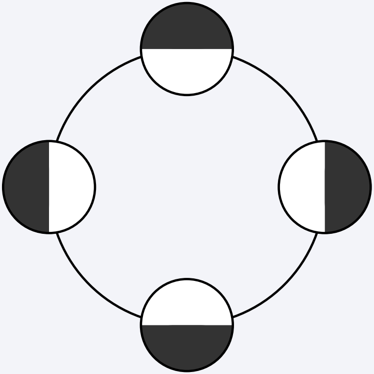
$$ {}^{1}_{A}\mathcal{V} = {}^{1}_{0}\mathcal{R}\,{}^{0}_{A}\mathcal{V} \tag{150} $$

$\Rightarrow$ (150) and (144)

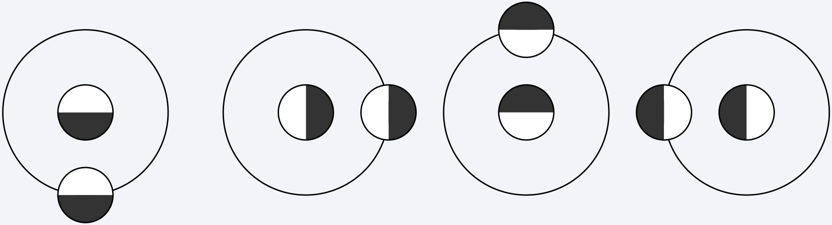
Translation without rotation



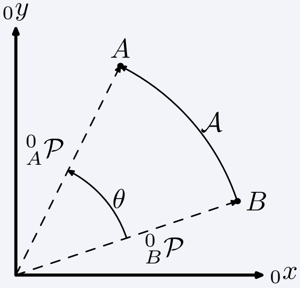
Translation with rotation



An object can be translated along any curve without changing its orientation (pure translation) or with a change in its orientation (extra rotation is induced).



All points of a rigid body rotate with the same angular velocity.



$$\|{}^0_B\mathcal{P}\| = \|{}^0_A\mathcal{P}\| = \rho = \text{const} \quad (151)$$

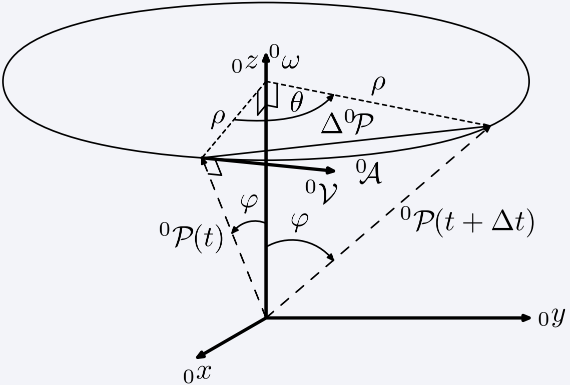
Measure of an angle:

$$\theta = \frac{\mathcal{A}}{\rho} \quad [\text{radian}] \quad (152)$$

e.g., for the full angle: $\theta = \frac{2\pi\rho}{\rho} = 2\pi$

Linear velocity along the arc:

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \frac{\mathrm{d}(\theta\rho)}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\rho = \|\omega\|\rho \;\Rightarrow\; \mathcal{V} = \omega \times \rho \quad (153)$$



$\rho = \|{}^0\mathcal{P}\| \sin\varphi$
$\varphi = \text{const}$

$\theta \to 0 \;\Rightarrow\; {}^0\mathcal{A} \to \Delta\,{}^0\mathcal{P}$

${}^0\mathcal{A}$ – length of the directed arc traversed by ${}^0\mathcal{P}$ when this vector is rotated by θ about axis ${}_0z$, i.e., by a directed angle $\theta_{{}_0z}$

$$\Delta\,{}^0\mathcal{P} = {}^0\mathcal{P}(t+\Delta t) - {}^0\mathcal{P} \approx {}^0\mathcal{A} = \theta_{{}_0z}\rho = \theta_{{}_0z}\|{}^0\mathcal{P}\| \sin\varphi \tag{154}$$

$${}^0\mathcal{V} = \frac{\mathrm{d}{}^0\mathcal{A}}{\mathrm{d}t} = \frac{\mathrm{d}\theta_{{}_0z}}{\mathrm{d}t}\|{}^0\mathcal{P}\| \sin\varphi = {}^0\omega\|{}^0\mathcal{P}\| \sin\varphi \tag{155}$$

As ${}^0\mathcal{V} \perp {}^0\mathcal{P}$ and ${}^0\mathcal{V} \perp {}_0z$ and hence ${}^0\mathcal{V} \perp \theta_{{}_0z}$

$$\Rightarrow\;\; {}^0\mathcal{V} = {}^0\omega \times {}^0\mathcal{P} \tag{156}$$

An $n \times n$ matrix $\mathcal{S}$ is skew symmetric if

$\mathcal{S} + \mathcal{S}^T = \mathcal{O} \quad \Rightarrow \quad s_{kl} = -s_{lk} \quad \Rightarrow \quad s_{ll} = 0.$

$\mathcal{S}$ can be treated as an operator, e.g., for $\mathcal{P} \in \mathbb{R}^3$, i.e.,

$\mathcal{P} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T$

$$\mathcal{S}(\mathcal{P}) = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \tag{157}$$

$$\begin{aligned} \mathcal{S}({}^0_A\mathcal{P})\,{}^0_B\mathcal{P} &= \begin{bmatrix} 0 & -{}^0_Ap_z & {}^0_Ap_y \\ {}^0_Ap_z & 0 & -{}^0_Ap_x \\ -{}^0_Ap_y & {}^0_Ap_x & 0 \end{bmatrix} \begin{bmatrix} {}^0_Bp_x \\ {}^0_Bp_y \\ {}^0_Bp_z \end{bmatrix} = \\ &= \begin{bmatrix} {}^0_Ap_y\,{}^0_Bp_z - {}^0_Ap_z\,{}^0_Bp_y \\ {}^0_Ap_z\,{}^0_Bp_x - {}^0_Ap_x\,{}^0_Bp_z \\ {}^0_Ap_x\,{}^0_Bp_y - {}^0_Ap_y\,{}^0_Bp_x \end{bmatrix} = {}^0_A\mathcal{P} \times {}^0_B\mathcal{P} \end{aligned} \tag{158}$$

Operator $\mathcal{S}$ is linear

$$\mathcal{S}(a\,{}_A^0\mathcal{P} + \; b\,{}_B^0\mathcal{P}) = a\mathcal{S}({}_A^0\mathcal{P}) + b\mathcal{S}({}_B^0\mathcal{P}), \quad a, b \in \mathbb{R} \tag{159}$$

For rotation matrices ${}_0^1\mathcal{R}$, where frames 1 and 0 have a common origin

$${}_0^1\mathcal{R}\,{}_A^0\mathcal{P} \times \;{}_0^1\mathcal{R}\,{}_B^0\mathcal{P} = \;{}_A^1\mathcal{P} \times {}_B^1\mathcal{P} = \;{}_0^1\mathcal{R}({}_A^0\mathcal{P} \times \;{}_B^0\mathcal{P}) \tag{160}$$

Similarity transformation

$${}_0^1\mathcal{R}\mathcal{S}({}_A^0\mathcal{P})\,{}_0^1\mathcal{R}^T\,{}_B^1\mathcal{P} = \;{}_0^1\mathcal{R}\mathcal{S}({}_A^0\mathcal{P})\,{}_1^0\mathcal{R}\,{}_B^1\mathcal{P} = \;{}_0^1\mathcal{R}\mathcal{S}({}_A^0\mathcal{P})\,{}_B^0\mathcal{P} =$$

$${}_0^1\mathcal{R}({}_A^0\mathcal{P} \times \;{}_B^0\mathcal{P}) = \;{}_0^1\mathcal{R}\,{}_A^0\mathcal{P} \times \;{}_0^1\mathcal{R}\,{}_B^0\mathcal{P} = ({}_0^1\mathcal{R}\,{}_A^0\mathcal{P}) \times \;{}_B^1\mathcal{P} = \mathcal{S}({}_0^1\mathcal{R}\,{}_A^0\mathcal{P})\,{}_B^1\mathcal{P}$$

$$\Rightarrow \quad {}_0^1\mathcal{R}\mathcal{S}({}_A^0\mathcal{P})\,{}_0^1\mathcal{R}^T = \mathcal{S}({}_0^1\mathcal{R}\,{}_A^0\mathcal{P}) = \mathcal{S}({}_A^1\mathcal{P}) \tag{161}$$

$$\text{For any skew symmetric matrix } \mathcal{S}_{n\times n}: \;\; {}_A^0\mathcal{P}^T\,\mathcal{S}\,{}_A^0\mathcal{P} = \mathcal{O} \tag{162}$$

Let ${}^0\mathcal{P} = \text{const}$, $\text{Rot}({}^0\mathcal{P}, \theta) = {}^0\mathcal{R}(\theta)$

$$ {}^0\mathcal{R}(\theta){}^0\mathcal{R}^T(\theta) = \mathcal{I} \tag{163} $$

$$ \frac{\mathrm{d}\big({}^0\mathcal{R}(\theta)\ {}^0\mathcal{R}^T(\theta)\big)}{\mathrm{d}\theta} = \mathcal{O} \tag{164} $$

$$ \frac{\mathrm{d}\big({}^0\mathcal{R}(\theta)\big)}{\mathrm{d}\theta}\ {}^0\mathcal{R}^T(\theta) + {}^0\mathcal{R}(\theta)\ \frac{\mathrm{d}\big({}^0\mathcal{R}^T(\theta)\big)}{\mathrm{d}\theta} = \mathcal{O} \tag{165} $$

$$ \mathcal{S} \triangleq \frac{\mathrm{d}\big({}^0\mathcal{R}(\theta)\big)}{\mathrm{d}\theta}\ {}^0\mathcal{R}^T(\theta) \tag{166} $$

$$ \Rightarrow\ \mathcal{S}^T = \left(\frac{\mathrm{d}\big({}^0\mathcal{R}(\theta)\big)}{\mathrm{d}\theta}\ {}^0\mathcal{R}^T(\theta)\right)^T = {}^0\mathcal{R}(\theta)\left(\frac{\mathrm{d}\big({}^0\mathcal{R}^T(\theta)\big)}{\mathrm{d}\theta}\right) \tag{167} $$

$$ \Rightarrow\ \mathcal{S} + \mathcal{S}^T = \mathcal{O} \tag{168} $$

$$ \Rightarrow\ \frac{\mathrm{d}\big({}^0\mathcal{R}(\theta)\big)}{\mathrm{d}\theta} = \mathcal{S}\,{}^0\mathcal{R}(\theta), \qquad \text{How to compute } \mathcal{S}? \tag{169} $$

$$\mathrm{Rot}(\mathcal{P},\theta) = \begin{bmatrix} \mathcal{P}_x^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & \mathcal{P}_y\mathcal{P}_x(1-\mathrm{c}\theta)-\mathcal{P}_z\mathrm{s}\theta & \mathcal{P}_z\mathcal{P}_x(1-\mathrm{c}\theta)+\mathcal{P}_y\mathrm{s}\theta \\ \mathcal{P}_x\mathcal{P}_y(1-\mathrm{c}\theta)+\mathcal{P}_z\mathrm{s}\theta & \mathcal{P}_y^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & \mathcal{P}_z\mathcal{P}_y(1-\mathrm{c}\theta)-\mathcal{P}_y\mathrm{s}\theta \\ \mathcal{P}_x\mathcal{P}_z(1-\mathrm{c}\theta)-\mathcal{P}_y\mathrm{s}\theta & \mathcal{P}_y\mathcal{P}_z(1-\mathrm{c}\theta)+\mathcal{P}_y\mathrm{s}\theta & \mathcal{P}_z^2(1-\mathrm{c}\theta)+\mathrm{c}\theta \end{bmatrix} \tag{170}$$

For $\mathcal{P} = \text{const}$, $\quad \dfrac{\mathrm{d}\,\mathrm{Rot}(\mathcal{P},\theta)}{\mathrm{d}\theta} =$

$$\begin{bmatrix} \mathcal{P}_x^2\mathrm{s}\theta-\mathrm{s}\theta & \mathcal{P}_y\mathcal{P}_x\mathrm{s}\theta-\mathcal{P}_z\mathrm{c}\theta & \mathcal{P}_z\mathcal{P}_x\mathrm{s}\theta+\mathcal{P}_y\mathrm{c}\theta \\ \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta+\mathcal{P}_z\mathrm{c}\theta & \mathcal{P}_y^2\mathrm{s}\theta-\mathrm{s}\theta & \mathcal{P}_z\mathcal{P}_y\mathrm{s}\theta-\mathcal{P}_x\mathrm{c}\theta \\ \mathcal{P}_x\mathcal{P}_z\mathrm{s}\theta-\mathcal{P}_y\mathrm{c}\theta & \mathcal{P}_y\mathcal{P}_z\mathrm{s}\theta+\mathcal{P}_x\mathrm{c}\theta & \mathcal{P}_z^2\mathrm{s}\theta-\mathrm{s}\theta \end{bmatrix} \tag{171}$$

$$\mathcal{S}(\mathcal{P}) = \begin{bmatrix} 0 & -\mathcal{P}_z & \mathcal{P}_y \\ \mathcal{P}_z & 0 & -\mathcal{P}_x \\ -\mathcal{P}_y & \mathcal{P}_x & 0 \end{bmatrix} \tag{172}$$

Assume $\|\mathcal{P}\| = 1$ and prove that $\dfrac{\mathrm{d}\,\mathrm{Rot}(\mathcal{P},\theta)}{\mathrm{d}\theta}\mathrm{Rot}^T(\mathcal{P},\theta) = \mathcal{S}(\mathcal{P})$

$$\mathcal{S}(\mathcal{P})_{[1,1]} = \left(\frac{\mathrm{d}\,\mathrm{Rot}(\mathcal{P},\theta)}{\mathrm{d}\theta} \;\; \mathrm{Rot}^{T}(\,\mathcal{P},\theta) \right)_{[1,1]} =$$

$$\begin{aligned}
&\big(\mathcal{P}_x^2 \mathrm{s}\theta - \mathrm{s}\theta\big)\big(\mathcal{P}_x^2(1-\mathrm{c}\theta) + \mathrm{c}\theta\big) + \\
&\big(\mathcal{P}_y\mathcal{P}_x \mathrm{s}\theta - \mathcal{P}_z \mathrm{c}\theta\big)\big(\mathcal{P}_y\mathcal{P}_x(1-\mathrm{c}\theta) - \mathcal{P}_z \mathrm{s}\theta\big) + \\
&\big(\mathcal{P}_z\mathcal{P}_x \mathrm{s}\theta + \mathcal{P}_y \mathrm{c}\theta\big)\big(\mathcal{P}_z\mathcal{P}_x(1-\mathrm{c}\theta) + \mathcal{P}_y \mathrm{s}\theta\big) =
\end{aligned}$$

$$\begin{aligned}
&\big(\mathcal{P}_x^2 \mathrm{s}\theta - \mathrm{s}\theta\big)\big(\mathcal{P}_x^2 - \mathcal{P}_x^2\mathrm{c}\theta + \mathrm{c}\theta\big) + \\
&\big(\mathcal{P}_y\mathcal{P}_x \mathrm{s}\theta - \mathcal{P}_z \mathrm{c}\theta\big)\big(\mathcal{P}_y\mathcal{P}_x - \mathcal{P}_y\mathcal{P}_x\mathrm{c}\theta - \mathcal{P}_z \mathrm{s}\theta\big) + \\
&\big(\mathcal{P}_z\mathcal{P}_x \mathrm{s}\theta + \mathcal{P}_y \mathrm{c}\theta\big)\big(\mathcal{P}_z\mathcal{P}_x - \mathcal{P}_z\mathcal{P}_x\mathrm{c}\theta + \mathcal{P}_y \mathrm{s}\theta\big) =
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_x^4\mathrm{s}\theta - \mathcal{P}_x^2\mathrm{s}\theta - \mathcal{P}_x^4\mathrm{c}\theta\mathrm{s}\theta + \mathcal{P}_x^2\mathrm{c}\theta\mathrm{s}\theta + \mathcal{P}_x^2\mathrm{s}\theta\mathrm{c}\theta - \mathrm{s}\theta\mathrm{c}\theta + \\
&\mathcal{P}_y^2\mathcal{P}_x^2\mathrm{s}\theta - \mathcal{P}_y\mathcal{P}_x\mathcal{P}_z\mathrm{c}\theta - \mathcal{P}_y^2\mathcal{P}_x^2\mathrm{c}\theta\mathrm{s}\theta + \mathcal{P}_y\mathcal{P}_x\mathcal{P}_z\mathrm{c}^2\theta - \mathcal{P}_z\mathcal{P}_y\mathcal{P}_x\mathrm{s}^2\theta + \mathcal{P}_z^2\mathrm{c}\theta\mathrm{s}\theta + \\
&\mathcal{P}_z^2\mathcal{P}_x^2\mathrm{s}\theta + \mathcal{P}_z\mathcal{P}_x\mathcal{P}_y\mathrm{c}\theta - \mathcal{P}_z^2\mathcal{P}_x^2\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_z\mathcal{P}_x\mathcal{P}_y\mathrm{c}^2\theta + \mathcal{P}_y\mathcal{P}_z\mathcal{P}_x\mathrm{s}^2\theta + \mathcal{P}_y^2\mathrm{s}\theta\mathrm{c}\theta =
\end{aligned} \tag{173}$$

$$
\begin{aligned}
&\mathcal{P}_x^4 \mathrm{s}\theta + \mathcal{P}_x^2\mathcal{P}_y^2 \mathrm{s}\theta + \mathcal{P}_x^2\mathcal{P}_z^2 \mathrm{s}\theta - \mathcal{P}_x^2 \mathrm{s}\theta + \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{c}\theta + \\
&\mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{c}^2\theta - \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{c}^2\theta + \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{s}^2\theta - \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z \mathrm{s}^2\theta - \mathcal{P}_x^4 \mathrm{c}\theta\mathrm{s}\theta + \\
&\mathcal{P}_x^2 \mathrm{c}\theta\mathrm{s}\theta + \mathcal{P}_y^2 \mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_z^2 \mathrm{c}\theta\mathrm{s}\theta - \mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_x^2 \mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_x^2\mathcal{P}_y^2 \mathrm{c}\theta\mathrm{s}\theta - \mathcal{P}_x^2\mathcal{P}_z^2 \mathrm{s}\theta\mathrm{c}\theta = \\
\\
&\mathcal{P}_x^2 \mathrm{s}\theta\big(\mathcal{P}_x^2 + \mathcal{P}_y^2 + \mathcal{P}_z^2 - 1\big) + \\
&\mathrm{c}\theta\mathrm{s}\theta\big(\mathcal{P}_x^2 + \mathcal{P}_y^2 + \mathcal{P}_z^2 - 1 - \mathcal{P}_x^4 + \mathcal{P}_x^2 - \mathcal{P}_x^2\mathcal{P}_y^2 - \mathcal{P}_x^2\mathcal{P}_z^2\big) = \\
\\
&0 + \mathcal{P}_x^2 \mathrm{c}\theta\mathrm{s}\theta\big(1 - \mathcal{P}_x^2 - \mathcal{P}_y^2 - \mathcal{P}_z^2\big) = 0 \\
\\
&\Rightarrow \quad \mathcal{S}_{[1,1]}(\mathcal{P}) = 0
\end{aligned}
\tag{174}
$$

$$\mathcal{S}(\mathcal{P})_{[1,2]} = \left(\frac{\mathrm{d}\,\mathrm{Rot}(\mathcal{P},\theta)}{\mathrm{d}\theta}\, \mathrm{Rot}^T(\mathcal{P},\theta) \right)_{[1,2]} =$$

$$\begin{aligned}
&\left(\mathcal{P}_x^2 \mathrm{s}\theta - \mathrm{s}\theta\right)\left(\mathcal{P}_x\mathcal{P}_y(1-\mathrm{c}\theta) + \mathcal{P}_z\mathrm{s}\theta\right) + \\
&\left(\mathcal{P}_y\mathcal{P}_x\mathrm{s}\theta - \mathcal{P}_z\mathrm{c}\theta\right)\left(\mathcal{P}_y^2(1-\mathrm{c}\theta) + \mathrm{c}\theta\right) + \\
&\left(\mathcal{P}_z\mathcal{P}_x\mathrm{s}\theta + \mathcal{P}_y\mathrm{c}\theta\right)\left(\mathcal{P}_z\mathcal{P}_y(1-\mathrm{c}\theta) - \mathcal{P}_y\mathrm{s}\theta\right) =
\end{aligned}$$

$$\begin{aligned}
&\left(\mathcal{P}_x^2 \mathrm{s}\theta - \mathrm{s}\theta\right)\left(\mathcal{P}_x\mathcal{P}_y - \mathcal{P}_x\mathcal{P}_y\mathrm{c}\theta + \mathcal{P}_z\mathrm{s}\theta\right) + \\
&\left(\mathcal{P}_y\mathcal{P}_x\mathrm{s}\theta - \mathcal{P}_z\mathrm{c}\theta\right)\left(\mathcal{P}_y^2 - \mathcal{P}_y^2\mathrm{c}\theta + \mathrm{c}\theta\right) + \\
&\left(\mathcal{P}_z\mathcal{P}_x\mathrm{s}\theta + \mathcal{P}_y\mathrm{c}\theta\right)\left(\mathcal{P}_z\mathcal{P}_y - \mathcal{P}_z\mathcal{P}_y\mathrm{c}\theta - \mathcal{P}_y\mathrm{s}\theta\right) =
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_x^3\mathcal{P}_y\mathrm{s}\theta - \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta - \mathcal{P}_x^3\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_x^2\mathcal{P}_z\mathrm{s}^2\theta - \mathcal{P}_z\mathrm{s}^2\theta + \\
&\mathcal{P}_x\mathcal{P}_y^3\mathrm{s}\theta - \mathcal{P}_y^2\mathcal{P}_z\mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_y^3\mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_y^2\mathcal{P}_z\mathrm{c}^2\theta + \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_z\mathrm{c}^2\theta + \\
&\mathcal{P}_x\mathcal{P}_y\mathcal{P}_z^2\mathrm{s}\theta + \mathcal{P}_y^2\mathcal{P}_z\mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_z^2\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_z\mathcal{P}_y^2\mathrm{c}^2\theta - \mathcal{P}_x^2\mathcal{P}_z\mathrm{s}^2\theta - \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta =
\end{aligned} \tag{175}$$

$$
\begin{aligned}
&\mathcal{P}_x^3\mathcal{P}_y\mathrm{s}\theta - \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta + \mathcal{P}_x\mathcal{P}_y^3\mathrm{s}\theta + \mathcal{P}_x\mathcal{P}_y\mathcal{P}_z^2\mathrm{s}\theta +\\
&\mathcal{P}_y^2\mathcal{P}_z\mathrm{c}\theta - \mathcal{P}_y^2\mathcal{P}_z\mathrm{c}\theta - \mathcal{P}_x^2\mathcal{P}_z\mathrm{s}^2\theta + \mathcal{P}_x^2\mathcal{P}_z\mathrm{s}^2\theta - \mathcal{P}_z\mathrm{s}^2\theta - \mathcal{P}_z\mathrm{c}^2\theta +\\
&\mathcal{P}_y^2\mathcal{P}_z\mathrm{c}^2\theta - \mathcal{P}_y^2\mathcal{P}_z\mathrm{c}^2\theta - \mathcal{P}_x\mathcal{P}_y^3\mathrm{s}\theta\mathrm{c}\theta + \mathcal{P}_x^3\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta +\\
&\mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_z^2\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta - \mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\mathrm{c}\theta =\\
\\
&\mathcal{P}_x\mathcal{P}_y\mathrm{s}\theta\big(\mathcal{P}_x^2 + \mathcal{P}_y^2 + \mathcal{P}_z^2 - 1\big) - \mathcal{P}_z +\\
&\big(-\mathcal{P}_x\mathcal{P}_y\mathcal{P}_z^2 - \mathcal{P}_x^3\mathcal{P}_y - \mathcal{P}_x\mathcal{P}_y^3 + \mathcal{P}_x\mathcal{P}_y\big)\mathrm{s}\theta\mathrm{c}\theta =\\
\\
&-\mathcal{P}_z + \mathcal{P}_x\mathcal{P}_y\big(-\mathcal{P}_z^2 - \mathcal{P}_x^2 - \mathcal{P}_y^2 + 1\big)\mathrm{s}\theta\mathrm{c}\theta = -\mathcal{P}_z\\
\\
&\Rightarrow \quad \mathcal{S}_{[1,2]}(\mathcal{P}) = -\mathcal{P}_z
\end{aligned}
\tag{176}
$$

Other elements of the matrix $\mathcal{S}$ can be computed in a similar way.

Assuming $\mathcal{P} = \text{const}$, as links rotate around fixed axis

$$\begin{aligned}
&\frac{\mathrm{d}\,\mathcal{R}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{Rot}(\mathcal{P},\theta)}{\mathrm{d}\theta}\,\frac{\mathrm{d}\,\theta}{\mathrm{d}t} = \\
&\mathcal{S}(\mathcal{P})\mathrm{Rot}(\mathcal{P},\theta)\frac{\mathrm{d}\,\theta}{\mathrm{d}t} = \mathcal{S}(\mathcal{P}\frac{\mathrm{d}\,\theta}{\mathrm{d}t})\mathrm{Rot}(\mathcal{P},\theta) = \mathcal{S}(\omega)\mathcal{R}(t) \\
&\Rightarrow \quad \frac{\mathrm{d}\,\mathcal{R}(t)}{\mathrm{d}t} = \mathcal{S}(\omega)\mathcal{R}(t)
\end{aligned} \tag{177}$$

Assuming that frame 1 rotates with angular velocity ${}^0_1\omega$ in relation to frame 0 and that origins of both frames coincide calculate the linear velocity in relation to frame 0 of a fixed point ${}^1_A\mathcal{P}$ (fixed in relation to frame 1).

$$\begin{aligned}
&\frac{\mathrm{d}\,{}^0_A\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{R}\,{}^1_A\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\frac{\mathrm{d}\,{}^1_A\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\cdot\mathcal{O} = \\
&\mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1_A\mathcal{P} = {}^0_1\omega \times ({}^0_1\mathcal{R}\,{}^1_A\mathcal{P}) = {}^0_1\omega \times {}^0({}^1_A\mathcal{P}) = {}^0_1\omega \times {}^0_A\mathcal{P}
\end{aligned} \tag{178}$$

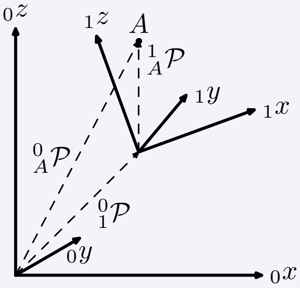
$$
\begin{aligned}
&{}^0_2\mathcal{R} = {}^0_1\mathcal{R}\,{}^1_2\mathcal{R} \\
&\frac{\mathrm{d}\,{}^0_2\mathcal{R}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_2\mathcal{R} + {}^0_1\mathcal{R}\frac{\mathrm{d}\,{}^1_2\mathcal{R}}{\mathrm{d}t} = \\
&\mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1_2\mathcal{R} + {}^0_1\mathcal{R}\mathcal{S}({}^1_2\omega)\,{}^1_2\mathcal{R} = \\
&\mathcal{S}({}^0_1\omega)\,{}^0_2\mathcal{R} + {}^0_1\mathcal{R}\mathcal{S}({}^1_2\omega)\,{}^0_1\mathcal{R}^T\,{}^0_1\mathcal{R}\,{}^1_2\mathcal{R} = \\
&\mathcal{S}({}^0_1\omega)\,{}^0_2\mathcal{R} + \mathcal{S}(\,{}^0_1\mathcal{R}\,{}^1_2\omega)\,{}^0_2\mathcal{R} = \\
&\left[\mathcal{S}({}^0_1\omega) + \mathcal{S}(\,{}^0_1\mathcal{R}\,{}^1_2\omega)\right]\,{}^0_2\mathcal{R} = \\
&\left[\mathcal{S}({}^0_1\omega) + \mathcal{S}\left(\,{}^0(\,{}^1_2\omega)\right)\right]\,{}^0_2\mathcal{R} = \\
&\mathcal{S}\left(\,{}^0_1\omega + {}^0(\,{}^1_2\omega)\right)\,{}^0_2\mathcal{R}
\end{aligned}
\tag{179}
$$

$$
\frac{\mathrm{d}\,{}^0_2\mathcal{R}}{\mathrm{d}t} = \mathcal{S}({}^0_2\omega)\,{}^0_2\mathcal{R}
$$

$$
\Rightarrow \quad {}^0_2\omega = {}^0_1\omega + {}^0(\,{}^1_2\omega)
$$

Addition of angular velocities for an n-joint manipulator

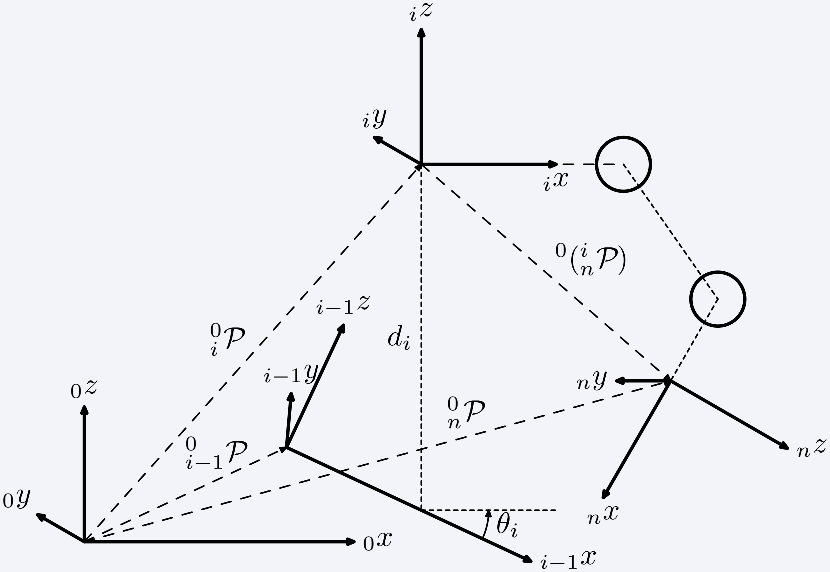
$$
\begin{aligned}
&{}^{0}_{n}\mathcal{R} = {}^{0}_{1}\mathcal{R}\,{}^{1}_{2}\mathcal{R} \ldots {}^{n-1}_{n}\mathcal{R} \\
&\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}t} \triangleq \dot{\mathcal{R}} \\
&{}^{0}_{n}\dot{\mathcal{R}} = \mathcal{S}({}^{0}_{n}\omega)\,{}^{0}_{n}\mathcal{R} \\
&{}^{0}_{n}\omega = {}^{0}_{1}\omega + {}^{0}_{1}\mathcal{R}\,{}^{1}_{2}\omega + \ldots + {}^{0}_{n-1}\mathcal{R}\,{}^{n-1}_{n}\omega \\
&{}^{0}_{n}\omega = {}^{0}_{1}\omega + {}^{0}({}^{1}_{2}\omega) + \ldots + {}^{0}({}^{n-1}_{n}\omega)
\end{aligned}
\tag{180}
$$



$$
\begin{aligned}
&{}^1_A\mathcal{P} = \text{const} \\
&{}^0_1\mathcal{P} \neq \text{const} \\
&{}^0_1\omega \neq \mathcal{O} \\
&{}^0_A\mathcal{P} = {}^0_1\mathcal{P} + {}^0_1\mathcal{R}\,{}^1_A\mathcal{P}
\end{aligned}
\tag{181}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}\,{}^0_A\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{P}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1_A\mathcal{P}}{\mathrm{d}t} = \\
&\frac{\mathrm{d}\,{}^0_1\mathcal{P}}{\mathrm{d}t} + \mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,\mathcal{O} = \\
&{}^0_1\mathcal{V} + {}^0_1\omega \times {}^0({}^1_A\mathcal{P})
\end{aligned}
\tag{182}
$$

Assuming that only the i-th joint is moving:



$$
\begin{aligned}
{}^0_1\omega &= {}^0_1\mathcal{R}\,{}_1z\dot{\theta}_1, \quad {}^1_2\omega = {}^1_2\mathcal{R}\,{}_2z\dot{\theta}_2, \quad \ldots, \quad {}^{n-1}_{n}\omega = {}^{n-1}_{n}\mathcal{R}\,{}_nz\dot{\theta}_n \\
{}^0_n\omega &= {}^0_1\omega + {}^0_1\mathcal{R}\,{}^1_2\omega + \ \ldots \ + {}^{0}_{n-1}\mathcal{R}\,{}^{n-1}_{n}\omega \\
{}^0_n\omega &= {}^0_1\mathcal{R}\,{}_1z\dot{\theta}_1 + {}^0_1\mathcal{R}\,{}^1_2\mathcal{R}\,{}_2z\dot{\theta}_2 + \ \ldots \ + {}^{0}_{n-1}\mathcal{R}\,{}^{n-1}_{n}\mathcal{R}\,{}_nz\dot{\theta}_n \\
{}^0_n\omega &= {}^0_1\mathcal{R}\,{}_1z\dot{\theta}_1 + {}^0_2\mathcal{R}\,{}_2z\dot{\theta}_2 + \ \ldots \ + {}^0_n\mathcal{R}\,{}_nz\dot{\theta}_n
\end{aligned} \tag{183}
$$

$$
{}^0_n\omega = \sum_{i=1}^{n} {}^0_i\mathcal{R}\,{}_iz\dot{\theta}_i = \sum_{i=1}^{n} {}^0_iz\dot{\theta}_i = \sum_{i=1}^{n} {}^0_iz\zeta_i\dot{q}_i \tag{184}
$$

$$
q_i = \begin{cases} d_i & - \quad \text{prismatic joint} \\ \theta_i & - \quad \text{revolute joint} \end{cases} \qquad \zeta_i = \begin{cases} 0 & - \quad \text{prismatic joint} \\ 1 & - \quad \text{revolute joint} \end{cases}
$$

$$
\begin{aligned}
q &= [q_1, q_2, \ldots, q_n]^T \\
{}^0_n\omega &= \mathcal{J}_\omega \dot{q} \\
\mathcal{J}_\omega &= [\,{}^0_1z\zeta_1,\ {}^0_2z\zeta_2,\ \ldots,\ {}^0_nz\zeta_n]
\end{aligned} \tag{185}
$$

$$ {}^{0}_{n}\mathcal{V} = {}^{0}_{n}\dot{\mathcal{P}} = \frac{\mathrm{d}\,{}^{0}_{n}\mathcal{P}}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial\,{}^{0}_{n}\mathcal{P}}{\partial q_i}\dot{q}_i \tag{186} $$

$$ {}^{0}_{n}\mathcal{V} = \mathcal{J}_{\mathcal{V}}\dot{q} \tag{187} $$

$$ \mathcal{J}_{\mathcal{V}} = \begin{bmatrix} \frac{\partial\,{}^{0}_{n}\mathcal{P}}{\partial q_1} & \frac{\partial\,{}^{0}_{n}\mathcal{P}}{\partial q_2} & \dots & \frac{\partial\,{}^{0}_{n}\mathcal{P}}{\partial q_n} \end{bmatrix} \tag{188} $$

If joint i is prismatic ($q_i = d_i$) and all other joints are frozen:

$$ {}^{0}_{n}\dot{\mathcal{P}} = {}^{0}_{i}z\,\dot{d}_i = {}^{0}_{i}z\,\dot{q}_i \tag{189} $$

$$ \Rightarrow \quad \frac{\partial\,{}^{0}_{n}\mathcal{P}}{\partial q_i} = {}^{0}_{i}z \tag{190} $$

If joint i is revolute ($q_i = \theta_i$) and all other joints are frozen:

$$ {}^{0}_{n}\omega = {}^{0}_{i}\omega = {}^{0}_{i}z\,\dot{q}_i = {}^{0}_{i}z\,\dot{\theta}_i \tag{191} $$

$$ {}^{0}({}^{i}_{n}\mathcal{P}) = {}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \tag{192} $$

$$ {}^{0}_{n}\mathcal{V} = {}^{0}_{i}\omega \times {}^{0}({}^{i}_{n}\mathcal{P}) = \dot{\theta}_i \, {}^{0}_{i}z \times \left({}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \right) = {}^{0}_{i}z \times \left({}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \right) \dot{q}_i \tag{193} $$

$$ \Rightarrow \quad \frac{\partial \, {}^{0}_{n}\mathcal{P}}{\partial q_i} = {}^{0}_{i}z \times \left({}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \right) \tag{194} $$

$$ \mathcal{J}_{\mathcal{V}} = [\mathcal{J}_{\mathcal{V}_1} \;\; \mathcal{J}_{\mathcal{V}_2} \;\; \ldots \;\; \mathcal{J}_{\mathcal{V}_n}] \tag{195} $$

$$ \mathcal{J}_{\mathcal{V}_i} = \begin{cases} {}^{0}_{i}z & - \quad \text{prismatic joint} \\ {}^{0}_{i}z \times \left({}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \right) & - \quad \text{revolute joint} \end{cases} \tag{196} $$

$$ \Rightarrow \quad \mathcal{J}_{\mathcal{V}_i} = (1 - \zeta_i) \, {}^{0}_{i}z + \zeta_i \left[{}^{0}_{i}z \times \left({}^{0}_{n}\mathcal{P} - {}^{0}_{i}\mathcal{P} \right) \right] \tag{197} $$

$$ {}^{0}_{n}\mathcal{U} = \begin{bmatrix} {}^{0}_{n}\mathcal{V} \\ {}^{0}_{n}\omega \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{\mathcal{V}} \\ \mathcal{J}_{\omega} \end{bmatrix} \dot{q} = \mathcal{J} \, \dot{q} = {}^{0,n}_{q}\mathcal{J} \, \dot{q} \tag{198} $$

$$
{}^0_n\mathcal{U} = \begin{bmatrix} {}^0_n\mathcal{U}_x(q_1, q_2, \ldots, q_n) \\ {}^0_n\mathcal{U}_y(q_1, q_2, \ldots, q_n) \\ \vdots \\ {}^0_n\mathcal{U}_\psi(q_1, q_2, \ldots, q_n) \end{bmatrix} = \begin{bmatrix} \frac{\partial\, {}^0_n\mathcal{U}_x}{\partial q_1} & \frac{\partial\, {}^0_n\mathcal{U}_x}{\partial q_2} & \ldots & \frac{\partial\, {}^0_n\mathcal{U}_x}{\partial q_n} \\ \frac{\partial\, {}^0_n\mathcal{U}_y}{\partial q_1} & \frac{\partial\, {}^0_n\mathcal{U}_y}{\partial q_2} & \ldots & \frac{\partial\, {}^0_n\mathcal{U}_y}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\, {}^0_n\mathcal{U}_\psi}{\partial q_1} & \frac{\partial\, {}^0_n\mathcal{U}_\psi}{\partial q_2} & \ldots & \frac{\partial\, {}^0_n\mathcal{U}_\psi}{\partial q_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{199}
$$

Analytic Jacobian:

- depends on the representation of orientation within ${}^0_n\mathcal{U}$,
- is difficult to calculate directly by differentiation – that is why we prefer to calculate the geometric Jacobian,
- does not have to be a square matrix,
- is a linear transformation for fixed $[q_1, q_2, \ldots, q_n]^T$.

$$ {}^0_i\mathcal{T} = \begin{bmatrix} {}^0_i r_{11} & {}^0_i r_{12} & {}^0_i r_{13} & {}^0_i \mathcal{P}_x \\ {}^0_i r_{21} & {}^0_i r_{22} & {}^0_i r_{23} & {}^0_i \mathcal{P}_y \\ {}^0_i r_{31} & {}^0_i r_{32} & {}^0_i r_{33} & {}^0_i \mathcal{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{200} $$

$$ \Rightarrow \quad {}^0_i z = \begin{bmatrix} {}^0_i r_{13} \\ {}^0_i r_{23} \\ {}^0_i r_{33} \end{bmatrix} \qquad {}^0_i\mathcal{P} = \begin{bmatrix} {}^0_i \mathcal{P}_x \\ {}^0_i \mathcal{P}_y \\ {}^0_i \mathcal{P}_z \end{bmatrix} \tag{201} $$

$\Rightarrow$ All data necessary to compute the geometric Jacobian can be extracted from the matrices derived while solving the Direct Kinematics Problem.

All joints of the modified IRp-6 robot are revolute.

$$\mathcal{J}_\omega = \begin{bmatrix} {}^0_1z & {}^0_2z & {}^0_3z & {}^0_4z & {}^0_5z & {}^0_6z \end{bmatrix} \tag{202}$$

Each versor ${}^0_iz$ is extracted out of its respective matrix ${}^0_i\mathcal{T}$, $i = 1, \ldots, 6$, i.e., (89), (95), (96), (97), (98), (99)

$$\begin{aligned} {}^0_1z &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad {}^0_2z = \begin{bmatrix} -\mathrm{s}_1 \\ \mathrm{c}_1 \\ 0 \end{bmatrix} \quad {}^0_3z = \begin{bmatrix} -\mathrm{s}_1 \\ \mathrm{c}_1 \\ 0 \end{bmatrix} \\ {}^0_4z &= \begin{bmatrix} -\mathrm{s}_1 \\ \mathrm{c}_1 \\ 0 \end{bmatrix} \quad {}^0_5z = \begin{bmatrix} \mathrm{c}_1\mathrm{c}_4 \\ \mathrm{s}_1\mathrm{c}_4 \\ -\mathrm{s}_4 \end{bmatrix} \quad {}^0_6z = \begin{bmatrix} \mathrm{c}_1\mathrm{s}_4\mathrm{s}_5 - \mathrm{s}_1\mathrm{c}_5 \\ \mathrm{s}_1\mathrm{s}_4\mathrm{s}_5 + \mathrm{c}_1\mathrm{c}_5 \\ \mathrm{c}_4\mathrm{s}_5 \end{bmatrix} \end{aligned} \tag{203}$$

Computation of $\mathcal{J}_{\mathcal{V}}$ requires the extraction of ${}^0_i\mathcal{P}$ out of ${}^0_i\mathcal{T}$, $i = 1, \ldots, 6$, i.e., (89), (95), (96), (97), (98), (99)

$$
{}^0_1\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^0_2\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^0_3\mathcal{P} = \begin{bmatrix} \mathrm{c}_1\mathrm{c}_2 a_2 \\ \mathrm{s}_1\mathrm{c}_2 a_2 \\ -\mathrm{s}_2 a_2 \end{bmatrix} \qquad {}^0_4\mathcal{P} = \begin{bmatrix} \mathrm{c}_1\left[a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ \mathrm{s}_1\left[a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ -a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \end{bmatrix}
$$

$$
{}^0_5\mathcal{P} = \begin{bmatrix} \mathrm{c}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ \mathrm{s}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ -\mathrm{s}_4 d_5 - a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \end{bmatrix} \quad {}^0_6\mathcal{P} = \begin{bmatrix} \mathrm{c}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ \mathrm{s}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ -\mathrm{s}_4 d_5 - a_3\mathrm{s}_3 - a_2\mathrm{s}_2 \end{bmatrix} \tag{204}
$$

Computation of ${}^0_6\mathcal{P} - {}^0_i\mathcal{P}$, $i = 1, \ldots, 6$

$$
{}^0({}^1_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_1\mathcal{P} = \begin{bmatrix} c_1 \left[c_4 d_5 + a_3 c_3 + a_2 c_2\right] \\ s_1 \left[c_4 d_5 + a_3 c_3 + a_2 c_2\right] \\ -s_4 d_5 - a_3 s_3 - a_2 s_2 \end{bmatrix}
$$

$$
{}^0({}^2_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_2\mathcal{P} = \begin{bmatrix} c_1 \left[c_4 d_5 + a_3 c_3 + a_2 c_2\right] \\ s_1 \left[c_4 d_5 + a_3 c_3 + a_2 c_2\right] \\ -s_4 d_5 - a_3 s_3 - a_2 s_2 \end{bmatrix} \tag{205}
$$

$$
{}^0({}^3_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_3\mathcal{P} = \begin{bmatrix} c_1 \left[c_4 d_5 + a_3 c_3\right] \\ s_1 \left[c_4 d_5 + a_3 c_3\right] \\ -s_4 d_5 - a_3 s_3 \end{bmatrix} \quad {}^0({}^4_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_4\mathcal{P} = \begin{bmatrix} c_1 c_4 d_5 \\ s_1 c_4 d_5 \\ -s_4 d_5 \end{bmatrix}
$$

$$
{}^0({}^5_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_5\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^0({}^6_6\mathcal{P}) = {}^0_6\mathcal{P} - {}^0_6\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{206}
$$

$$\mathcal{J}_{\mathcal{V}} = \left[{}^0_1z \times {}^0({}^1_6\mathcal{P}) \quad {}^0_2z \times {}^0({}^2_6\mathcal{P}) \quad {}^0_3z \times {}^0({}^3_6\mathcal{P}) \right. \\ \left. {}^0_4z \times {}^0({}^4_6\mathcal{P}) \quad {}^0_5z \times {}^0({}^5_6\mathcal{P}) \quad {}^0_6z \times {}^0({}^6_6\mathcal{P}) \right] \tag{207}$$

$${}^0_1z \times {}^0({}^1_6\mathcal{P}) = \begin{bmatrix} -\mathrm{s}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ \mathrm{c}_1\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \\ 0 \end{bmatrix} \tag{208}$$

$${}^0_2z \times {}^0({}^2_6\mathcal{P}) = \begin{bmatrix} -\mathrm{c}_1\left[\mathrm{s}_4 d_5 + a_3\mathrm{s}_3 + a_2\mathrm{s}_2\right] \\ -\mathrm{s}_1\left[\mathrm{s}_4 d_5 + a_3\mathrm{s}_3 + a_2\mathrm{s}_2\right] \\ -\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2\right] \end{bmatrix} \tag{209}$$

$${}^0_3z \times {}^0({}^3_6\mathcal{P}) = \begin{bmatrix} -\mathrm{c}_1\left[\mathrm{s}_4 d_5 + a_3\mathrm{s}_3\right] \\ -\mathrm{s}_1\left[\mathrm{s}_4 d_5 + a_3\mathrm{s}_3\right] \\ -\left[\mathrm{c}_4 d_5 + a_3\mathrm{c}_3\right] \end{bmatrix} \qquad {}^0_4z \times {}^0({}^4_6\mathcal{P}) = \begin{bmatrix} -\mathrm{c}_1\mathrm{s}_4 d_5 \\ -\mathrm{s}_1\mathrm{s}_4 d_5 \\ -\mathrm{c}_4 d_5 \end{bmatrix}$$

$${}^0_5z \times {}^0({}^5_6\mathcal{P}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^0_6z \times {}^0({}^6_6\mathcal{P}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{210}$$

Rotational velocity of the end-effector

E – end-effector (tool) frame

$$ {}^{6}_{E}\mathcal{T} = \left[\begin{array}{c|c} {}^{6}_{E}\mathcal{R}_{3\times 3} & {}^{6}_{E}\mathcal{P}_{3\times 1} \\ \hline 0_{1\times 3} & 1 \end{array}\right] = \text{const} \quad \Rightarrow \quad {}^{6}_{E}\mathcal{R} = \text{const} \tag{211} $$

$$ {}^{0}_{6}\mathcal{T}\,{}^{6}_{E}\mathcal{T} = {}^{0}_{E}\mathcal{T} \tag{212} $$

$$ \Rightarrow \quad {}^{0}_{6}\mathcal{R}\,{}^{6}_{E}\mathcal{R} = {}^{0}_{E}\mathcal{R} \tag{213} $$

$$ \Rightarrow \quad {}^{0}_{6}\dot{\mathcal{R}}\,{}^{6}_{E}\mathcal{R} + {}^{0}_{6}\mathcal{R}\,{}^{6}_{E}\dot{\mathcal{R}} = {}^{0}_{E}\dot{\mathcal{R}} \tag{214} $$

$$ \Rightarrow \quad {}^{0}_{6}\dot{\mathcal{R}}\,{}^{6}_{E}\mathcal{R} = {}^{0}_{E}\dot{\mathcal{R}} \tag{215} $$

$$ \Rightarrow \quad \mathcal{S}({}^{0}_{6}\omega)\,{}^{0}_{6}\mathcal{R}\,{}^{6}_{E}\mathcal{R} = \mathcal{S}({}^{0}_{E}\omega)\,{}^{0}_{E}\mathcal{R} \tag{216} $$

$$ \Rightarrow \quad \mathcal{S}({}^{0}_{6}\omega)\,{}^{0}_{E}\mathcal{R} = \mathcal{S}({}^{0}_{E}\omega)\,{}^{0}_{E}\mathcal{R} \tag{217} $$

$$ \Rightarrow \quad \mathcal{S}({}^{0}_{6}\omega) = \mathcal{S}({}^{0}_{E}\omega) \quad \Rightarrow \quad {}^{0}_{6}\omega = {}^{0}_{E}\omega \tag{218} $$

$$ \Rightarrow \ {}^{E}\left({}^{0}_{E}\omega\right) = {}^{E}\left({}^{0}_{6}\omega\right) = {}^{E}_{0}\mathcal{R}\,{}^{0}_{6}\omega = {}^{E}_{0}\mathcal{R}\,{}^{0}_{6}\mathcal{R}\,{}^{6}\left({}^{0}_{6}\omega\right) = {}^{E}_{6}\mathcal{R}\,{}^{6}\left({}^{0}_{6}\omega\right) \tag{219} $$

$$ \Rightarrow \ {}^{E}\left({}^{0}_{E}\omega\right) = {}^{6}_{E}\mathcal{R}^{T}\,{}^{6}\left({}^{0}_{6}\omega\right) \tag{220} $$

$$ {}^{E}\left({}^{6}_{E}\mathcal{P}\right) = {}^{E}_{6}\mathcal{R}\,{}^{6}_{E}\mathcal{P} = {}^{6}_{E}\mathcal{R}^{T}\,{}^{6}_{E}\mathcal{P} \tag{221} $$

Linear velocity of the end-effector E due to the rotation of frame 6 and the displacement of the origin of E wrt 6:

$$ \begin{aligned} {}^{E}({}^{6}_{E}\mathcal{V}) = {}^{E}({}^{0}_{6}\omega) \times {}^{E}({}^{6}_{E}\mathcal{P}) &= \left({}^{E}_{0}\mathcal{R}\,{}^{0}_{6}\omega\right) \times {}^{E}({}^{6}_{E}\mathcal{P}) \\ &= \left({}^{E}_{6}\mathcal{R}\,{}^{6}_{0}\mathcal{R}\,{}^{0}_{6}\omega\right) \times {}^{E}({}^{6}_{E}\mathcal{P}) \end{aligned} \tag{222} $$

$$ \begin{aligned} \Rightarrow \quad {}^{E}({}^{6}_{E}\mathcal{V}) &= \left({}^{E}_{6}\mathcal{R}\,{}^{6}_{0}\mathcal{R}\,{}^{0}_{6}\omega\right) \times \left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}_{E}\mathcal{P}\right) \\ &= -\left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}_{E}\mathcal{P}\right) \times \left({}^{E}_{6}\mathcal{R}\,{}^{6}_{0}\mathcal{R}\,{}^{0}_{6}\omega\right) \\ &= -\left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}_{E}\mathcal{P}\right) \times \left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}({}^{0}_{6}\omega)\right) \\ &= -\mathcal{S}\left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}_{E}\mathcal{P}\right)\left({}^{6}_{E}\mathcal{R}^{T}\,{}^{6}({}^{0}_{6}\omega)\right) \\ &= -{}^{6}_{E}\mathcal{R}^{T}\,\mathcal{S}\,({}^{6}_{E}\mathcal{P})\;{}^{6}_{E}\mathcal{R}\,{}^{6}_{E}\mathcal{R}^{T}\,{}^{6}({}^{0}_{6}\omega) \end{aligned} \tag{223} $$

$$ \Rightarrow \; {}^{E}\left({}^{6}_{E}\mathcal{V}\right) = -\,{}^{6}_{E}\mathcal{R}^{T}\,\mathcal{S}\left({}^{6}_{E}\mathcal{P}\right)\,{}^{6}\left({}^{0}_{6}\omega\right) \tag{224} $$

Velocity of the last link expressed in the end-effector frame:

$$ {}^{E}\left({}^{0}_{6}\mathcal{V}\right) = {}^{E}_{6}\mathcal{R}\ {}^{6}\left({}^{0}_{6}\mathcal{V}\right) = {}^{6}_{E}\mathcal{R}^{T}\ {}^{6}\left({}^{0}_{6}\mathcal{V}\right) \tag{225} $$

Velocity of the end-effector in relation to the base coordinate frame expressed in the end-effector frame

$$ {}^{E}\left({}^{0}_{E}\mathcal{V}\right) = {}^{E}\left({}^{0}_{6}\mathcal{V}\right) + {}^{E}\left({}^{6}_{E}\mathcal{V}\right) = {}^{6}_{E}\mathcal{R}^{T}\ {}^{6}\left({}^{0}_{6}\mathcal{V}\right) - {}^{6}_{E}\mathcal{R}^{T}\,\mathcal{S}\left({}^{6}_{E}\mathcal{P}\right)\,{}^{6}\left({}^{0}_{6}\omega\right) \tag{226} $$

Rotational velocity of the end-effector

$$ {}^{E}\left({}^{0}_{E}\omega\right) = {}^{E}_{6}\mathcal{R}\ {}^{6}\left({}^{0}_{6}\omega\right) = {}^{6}_{E}\mathcal{R}^{T}\ {}^{6}\left({}^{0}_{6}\omega\right) \tag{227} $$

General velocity of the last link of the manipulator

$$ {}^{6}\left({}^{0}_{6}\mathcal{U}\right) = \begin{bmatrix} {}^{6}\left({}^{0}_{6}\mathcal{V}\right) \\ {}^{6}\left({}^{0}_{6}\omega\right) \end{bmatrix} \tag{228} $$

$$ {}^{E}\left({}^{0}_{E}\mathcal{U}\right) = \begin{bmatrix} {}^{E}\left({}^{0}_{E}\mathcal{V}\right) \\ {}^{E}\left({}^{0}_{E}\omega\right) \end{bmatrix} = {}^{E}_{6}\mathcal{H} \begin{bmatrix} {}^{6}\left({}^{0}_{6}\mathcal{V}\right) \\ {}^{6}\left({}^{0}_{6}\omega\right) \end{bmatrix} = {}^{E}_{6}\mathcal{H}\ {}^{6}\left({}^{0}_{6}\mathcal{U}\right) \tag{229} $$

$$ {}^{E}_{6}\mathcal{H} = \left[\begin{array}{c|c} {}^{6}_{E}\mathcal{R}^{T} & -{}^{6}_{E}\mathcal{R}^{T}\mathcal{S}\left({}^{6}_{E}\mathcal{P}\right) \\ \hline \mathcal{O} & {}^{6}_{E}\mathcal{R}^{T} \end{array}\right] \tag{230} $$

$$ {}^{0}_{6}\mathcal{U} = {}^{0,6}_{q}\mathcal{J}\,\dot{q} \tag{231} $$

$$ {}^{6}\left({}^{0}_{6}\mathcal{U}\right) = {}^{6}_{0}\mathcal{H}_{*}\,{}^{0,6}_{q}\mathcal{J}\,\dot{q} \tag{232} $$

$$ {}^{6}_{0}\mathcal{H}_{*} = \left[\begin{array}{c|c} {}^{0}_{6}\mathcal{R}^{T} & \mathcal{O} \\ \hline \mathcal{O} & {}^{0}_{6}\mathcal{R}^{T} \end{array}\right] \tag{233} $$

General velocity of the end-effector in relation to frame 0 expressed in the end-effector frame

$$ {}^{E}\left({}^{0}_{E}\mathcal{U}\right) = {}^{E}_{6}\mathcal{H}\ {}^{6}_{0}\mathcal{H}_{*}\,{}^{0,6}_{q}\mathcal{J}\,\dot{q} \tag{234} $$

Explanation:

- (230) (i.e., ${}^{A}_{B}\mathcal{H}$) is used when two frames A and B are fixed to the same rigid body that is moving in relation to a static reference frame, e.g., 0, and we want to find the compound velocity of frame A wrt frame 0, when we know the compound velocity of frame B wrt 0.
- (233) (i.e., ${}^{A}_{B}\mathcal{H}_{*}$) is used when we want to express the components of a compound velocity wrt to frames that are treated as static wrt each other (i.e., the motion of those frames is frozen for the considered instant or they are static).

Inverse of matrix $\mathcal{H}$

$$ {}^{6}_{E}\mathcal{H} = {}^{E}_{6}\mathcal{H}^{-1} = \left[\begin{array}{c|c} {}^{6}_{E}\mathcal{R}^{T} & -{}^{6}_{E}\mathcal{R}^{T}\,\mathcal{S}\left({}^{6}_{E}\mathcal{P}\right) \\ \hline \mathcal{O} & {}^{6}_{E}\mathcal{R}^{T} \end{array}\right]^{-1} = \left[\begin{array}{c|c} {}^{6}_{E}\mathcal{R} & \mathcal{S}\left({}^{6}_{E}\mathcal{P}\right)\,{}^{6}_{E}\mathcal{R} \\ \hline \mathcal{O} & {}^{6}_{E}\mathcal{R} \end{array}\right] \tag{235} $$

The inverse of equation (229):

$$ {}^{6}\left({}^{0}_{6}\mathcal{U}\right) = {}^{6}_{E}\mathcal{H}\;{}^{E}\left({}^{0}_{6}\mathcal{U}\right) \tag{236} $$

$$ {}^{0}_{n}\mathcal{U} = {}^{0,n}_{q}\mathcal{J}\,\dot{q} \tag{237} $$

When ${}^{0,n}_{q}\mathcal{J}$ loses rank the Jacobian is said to be singular.

- Workspace boundary singularities – manipulator fully stretched or folded
- Internal workspace singularities – two or more joint axes aligned

In a singular configuration the manipulator loses one or more degrees of freedom.

The determinant of the Jacobian of the modified IRp-6 manipulator defines singular configurations

$$ \det {}^{0,6}_{q}\mathcal{J} = \mathrm{s}_5\, a_2\, a_3 (\mathrm{s}_2\mathrm{c}_3 - \mathrm{c}_2\mathrm{s}_3)(\mathrm{c}_4\, d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2) = 0 \tag{238} $$

$$ \begin{array}{lcll} \mathrm{s}_5 = 0 & \Rightarrow & \theta_5 = k\pi, \quad k \in \mathbb{Z} & \surd \\ \mathrm{s}_2\mathrm{c}_3 - \mathrm{c}_2\mathrm{s}_3 = 0 & \Rightarrow & \tan\theta_2 = \tan\theta_3 \Rightarrow \theta_2 = \theta_3 + k\pi & \times \\ \mathrm{c}_4\, d_5 + a_3\mathrm{c}_3 + a_2\mathrm{c}_2 = 0 & \Rightarrow & \forall_{i\in\{2,3,4\}}\, \theta_i = k\pi + \dfrac{\pi}{2} & \times \end{array} \tag{239} $$

$$ {}^{0}\mathcal{F} = \begin{bmatrix} {}^{0}F \\ {}^{0}N \end{bmatrix} \quad {}^{0}\mathcal{X} = \begin{bmatrix} {}^{0}\mathcal{P}_1 \\ \phi\, {}^{0}\mathcal{P}_2 \end{bmatrix} \quad q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \tag{240} $$

Principle of virtual work. In equilibrium:

$$ {}^{0}\mathcal{F}^{T}\, \delta\, {}^{0}\mathcal{X} = \tau^{T}\, \delta q \tag{241} $$

Virtual displacements: $\delta \mathcal{X}$ and δq,
i.e., infinitesimal displacements not violating the constraints

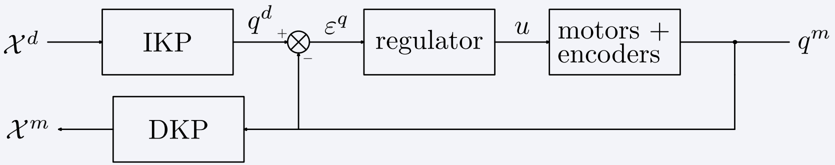
$$ \delta\, {}^{0}\mathcal{X} = {}^{0,n}_{\;q}\mathcal{J}\, \delta q \tag{242} $$

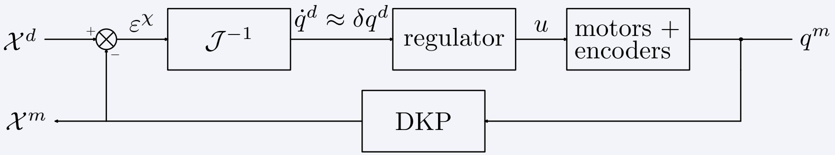
$$ \Rightarrow \quad {}^{0}\mathcal{F}^{T}\; {}^{0,n}_{\;q}\mathcal{J}\, \delta q = \tau^{T}\, \delta q \tag{243} $$

$$ \Rightarrow \quad {}^{0}\mathcal{F}^{T}\; {}^{0,n}_{\;q}\mathcal{J} = \tau^{T} \tag{244} $$

$$ \Rightarrow \quad \tau = {}^{0,n}_{\;q}\mathcal{J}^{T}\, {}^{0}\mathcal{F} \tag{245} $$

Singular Jacobian $\Rightarrow$ manipulator cannot exert forces in some directions
Example: If a two-revolute-joint planar manipulator is fully outstretched in a certain direction a very large external force can be applied along that direction and negligible joint torques are required.


$\mathcal{X}^d$
IKP
q^d
+
−
ε^q
regulator
u
motors + encoders
q^m
$\mathcal{X}^m$
DKP


$\mathcal{X}^d$
+
−
$\varepsilon^{\mathcal{X}}$
$\mathcal{J}^{-1}$
$\dot{q}^d \approx \delta q^d$
regulator
u
motors + encoders
q^m
$\mathcal{X}^m$
DKP

Mechanics:

- Kinematics – description of motion without the study of its causes (no mass, no force)
- Dynamics – study of effects that forces produce on object motion (forces acting on masses produce motion)
- Statics – deals with forces acting on bodies at rest (equilibrium of forces)

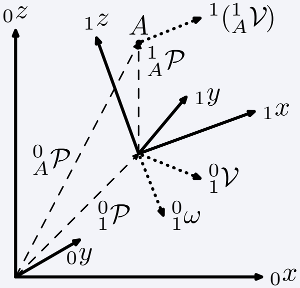
(torque – pair of forces, moment of inertia – mass distribution)

Direct dynamics model (important for simulation):

$$\tau \rightarrow q, \dot{q}, \ddot{q} \tag{246}$$

Inverse dynamics model (important for control):

$$q, \dot{q}, \ddot{q} \rightarrow \tau \tag{247}$$



$$^0_1\mathcal{P} \neq \text{const} \qquad {}^1_A\mathcal{P} \neq \text{const}$$

$$^0_1\mathcal{V} \neq \mathcal{O} \qquad {}^1({}^1_A\mathcal{V}) \neq \mathcal{O}$$

$$^0_1\omega \neq \mathcal{O}$$

$$^0_A\mathcal{P} = {}^0_1\mathcal{P} + {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} \tag{248}$$

$$\frac{\mathrm{d}\,{}^0_A\mathcal{P}}{\mathrm{d}t} = \frac{\mathrm{d}\,{}^0_1\mathcal{P}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1_A\mathcal{P}}{\mathrm{d}t} =$$
$$\frac{\mathrm{d}\,{}^0_1\mathcal{P}}{\mathrm{d}t} + \mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) = \tag{249}$$
$$^0_1\mathcal{V} + {}^0_1\omega \times {}^0({}^1_A\mathcal{P}) + {}^0({}^1_A\mathcal{V})$$

$$
\begin{aligned}
&\frac{\mathrm{d}^2\,{}^0_A\mathcal{P}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}\,{}^0_A\mathcal{P}}{\mathrm{d}t}\Big) = \\
&\frac{\mathrm{d}}{\mathrm{d}t}\Big({}^0_1\mathcal{V} + {}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V})\Big) = \\
&\frac{\mathrm{d}\,{}^0_1\mathcal{V}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\omega}{\mathrm{d}t} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \frac{\mathrm{d}({}^0_1\mathcal{R}\,{}^1_A\mathcal{P})}{\mathrm{d}t} + \\
&+\frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1({}^1_A\mathcal{V})}{\mathrm{d}t} = \\
&\frac{\mathrm{d}\,{}^0_1\mathcal{V}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\omega}{\mathrm{d}t} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \Big(\frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1_A\mathcal{P}}{\mathrm{d}t}\Big) + \\
&+\frac{\mathrm{d}\,{}^0_1\mathcal{R}}{\mathrm{d}t}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1({}^1_A\mathcal{V})}{\mathrm{d}t} = \\
&\frac{\mathrm{d}\,{}^0_1\mathcal{V}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\omega}{\mathrm{d}t} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \Big(\mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V})\Big) + \\
&+\mathcal{S}({}^0_1\omega)\,{}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1({}^1_A\mathcal{V})}{\mathrm{d}t} =
\end{aligned}
\tag{250}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}\,{}^0_1\mathcal{V}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\omega}{\mathrm{d}t} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \left({}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V})\right) + \\
&+ {}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1({}^1_A\mathcal{V})}{\mathrm{d}t} = \\
&\frac{\mathrm{d}\,{}^0_1\mathcal{V}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^0_1\omega}{\mathrm{d}t} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \left({}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P}\right) + \\
&+ {}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,\frac{\mathrm{d}\,{}^1({}^1_A\mathcal{V})}{\mathrm{d}t} = \\
&{}^0_1\mathcal{A} + {}^0_1\mathcal{E} \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P} + {}^0_1\omega \times \left({}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1_A\mathcal{P}\right) + \\
&+ 2\,{}^0_1\omega \times {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{V}) + {}^0_1\mathcal{R}\,{}^1({}^1_A\mathcal{A}) = \\
&{}^0_1\mathcal{A} + {}^0_1\mathcal{E} \times {}^0({}^1_A\mathcal{P}) + {}^0_1\omega \times \left({}^0_1\omega \times {}^0({}^1_A\mathcal{P})\right) + \\
&+ 2\,{}^0_1\omega \times {}^0({}^1_A\mathcal{V}) + {}^0({}^1_A\mathcal{A})
\end{aligned}
\tag{251}
$$

$$\frac{\mathrm{d}^2\,{}^0_A\mathcal{P}}{\mathrm{d}t^2} = \underbrace{{}^0_1\mathcal{A}}_{\substack{\text{linear}\\ \text{acceleration}\\ \text{of frame 1}\\ \text{wrt. frame 0}}} + \underbrace{{}^0_1\mathcal{E} \times {}^0({}^1_A\mathcal{P})}_{\substack{\text{tangent}\\ \text{acceleration}\\ \text{due to angular}\\ \text{acceleration of 1 wrt 0}}} + \underbrace{{}^0_1\omega \times \left({}^0_1\omega \times {}^0({}^1_A\mathcal{P})\right)}_{\substack{\text{centripetal}\\ \text{acceleration}}} + \underbrace{2\,{}^0_1\omega \times {}^0({}^1_A\mathcal{V})}_{\substack{\text{Coriolis}\\ \text{acceleration}}} + \underbrace{{}^0({}^1_A\mathcal{A})}_{\substack{\text{acceleration}\\ \text{of A wrt 1}}} \tag{252}$$

All of the above free vectors are expressed wrt frame 0 (i.e., wrt a frame with an orientation of frame 0). From the point of view of the observer located in frame 1 only the last term is real. All other terms are fictious, arising due to the motion of frame 1 wrt 0. Frame 0 is inertial, while frame 1 is a non-inertial frame of reference.

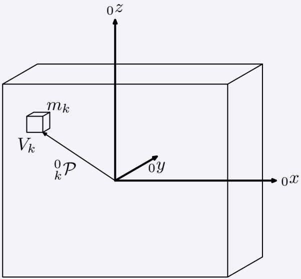
Angular velocity of frame 2 rotating wrt frame 1 while that is rotating wrt frame 0:

$$ {}_2^0\omega = {}_1^0\omega + {}^0({}_2^1\omega) = {}_1^0\omega + {}_1^0\mathcal{R}\,{}_2^1\omega \tag{253} $$

Angular acceleration:

$$ \begin{aligned} \frac{\mathrm{d}\,{}_2^0\omega}{\mathrm{d}t} &= \frac{\mathrm{d}\,{}_1^0\omega}{\mathrm{d}t} + \frac{\mathrm{d}\,{}_1^0\mathcal{R}}{\mathrm{d}t}\,{}_2^1\omega + {}_1^0\mathcal{R}\frac{\mathrm{d}\,{}_2^1\omega}{\mathrm{d}t} \\ &= {}_1^0\mathcal{E} + \mathcal{S}({}_1^0\omega){}_1^0\mathcal{R}\,{}_2^1\omega + {}_1^0\mathcal{R}\,{}_2^1\mathcal{E} \\ &= {}_1^0\mathcal{E} + {}_1^0\omega \times {}^0({}_2^1\omega) + {}_1^0\mathcal{R}\,{}_2^1\mathcal{E} \\ &= {}_1^0\mathcal{E} + {}_1^0\omega \times {}^0({}_2^1\omega) + {}^0({}_2^1\mathcal{E}) \end{aligned} \tag{254} $$

c.o.m. – centre of mass



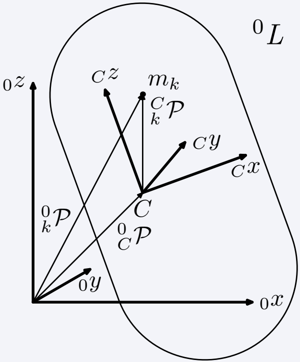
$$\begin{aligned} M &= \textstyle\sum_k m_k \\ m_k &\to \mathrm{d}\,m \quad V_k \to \mathrm{d}\,V \\ M &= \textstyle\int \mathrm{d}\,m = \int_V \varrho\, \mathrm{d}\,V \\ \varrho &- \text{density} \end{aligned} \tag{255}$$

$$ {}^0_C\mathcal{P} = \frac{1}{M}\sum_k m_k\, {}^0_k\mathcal{P} \tag{256}$$

$$ {}^C_C\mathcal{P} = \mathcal{O} \;\Rightarrow\; \frac{1}{M}\sum_k m_k\, {}^C_k\mathcal{P} = \mathcal{O} \;\Rightarrow\; \frac{1}{M}\sum_k m_k\, {}^0(\,{}^C_k\mathcal{P}) = \mathcal{O} \tag{257}$$

$$ {}^0_C\mathcal{V} = \frac{\mathrm{d}\,{}^0_C\mathcal{P}}{\mathrm{d}t} = \frac{1}{M}\sum_k m_k \frac{\mathrm{d}\,{}^0_k\mathcal{P}}{\mathrm{d}t} = \frac{1}{M}\sum_k m_k\, {}^0_k\mathcal{V} \tag{258}$$

$$ {}^C_C\mathcal{V} = \mathcal{O} \;\Rightarrow\; \frac{1}{M}\sum_k m_k\, {}^C_k\mathcal{V} = \mathcal{O} \;\Rightarrow\; \frac{1}{M}\sum_k m_k\, {}^0(\,{}^C_k\mathcal{V}) = \mathcal{O} \tag{259}$$



$$\begin{aligned}
{}^0L &= \textstyle\sum_k {}^0_kL \\
&= \textstyle\sum_k {}^0_k\mathcal{P} \times m_k\, {}^0_k\mathcal{V} \\
&= \textstyle\sum_k m_k\, {}^0_k\mathcal{P} \times {}^0_k\mathcal{V} \\
&= \textstyle\sum_k m_k \left({}^0_C\mathcal{P} + {}^0({}^C_k\mathcal{P}) \right) \times \left({}^0_C\mathcal{V} + {}^0({}^C_k\mathcal{V}) \right) \\
&= \textstyle\sum_k m_k \Big[{}^0_C\mathcal{P} \times {}^0_C\mathcal{V} + {}^0_C\mathcal{P} \times {}^0({}^C_k\mathcal{V}) \\
&\quad + {}^0({}^C_k\mathcal{P}) \times {}^0_C\mathcal{V} + {}^0({}^C_k\mathcal{P}) \times {}^0({}^C_k\mathcal{V}) \Big] =
\end{aligned} \tag{260}$$

$$\begin{aligned}
&= \left({}^0_C\mathcal{P} \times {}^0_C\mathcal{V} \right) \textstyle\sum_k m_k + \sum_k m_k \left({}^0({}^C_k\mathcal{P}) \times {}^0({}^C_k\mathcal{V}) \right) \\
&\quad + {}^0_C\mathcal{P} \times \underbrace{\sum_k m_k\, {}^0({}^C_k\mathcal{V})}_{=\mathcal{O}} + \underbrace{\left(\sum_k m_k\, {}^0({}^C_k\mathcal{P}) \right)}_{=\mathcal{O}} \times {}^0_C\mathcal{V} =
\end{aligned}$$

$$
\begin{aligned}
&= \left({}^0_C\mathcal{P} \times {}^0_C\mathcal{V}\right) \textstyle\sum_k m_k + \sum_k m_k \left({}^0({}^C_k\mathcal{P}) \times {}^0({}^C_k\mathcal{V})\right) \\
&= M\left({}^0_C\mathcal{P} \times {}^0_C\mathcal{V}\right) + \textstyle\sum_k \left({}^0({}^C_k\mathcal{P}) \times m_k\, {}^0({}^C_k\mathcal{V})\right) \\
&= \underbrace{\left({}^0_C\mathcal{P} \times M\, {}^0_C\mathcal{V}\right)}_{{}^0_C L \text{ – angular momentum of c.o.m.}} + \underbrace{\sum_k {}^0({}^C_k L)}_{{}^0({}^C L) \text{ – angular momentum due to the motion of particles relative to c.o.m.}}
\end{aligned}
$$

If all particles k belong to a single rigid body (i.e., frame C is affixed to the body):

$$
\begin{aligned}
{}^0({}^C_k\mathcal{V}) &= {}^0_C\omega \times {}^0({}^C_k\mathcal{P}) \\
{}^C_0\mathcal{R}\, {}^0({}^C_k\mathcal{V}) &= {}^C_0\mathcal{R}\, {}^0_C\omega \times {}^C_0\mathcal{R}\, {}^0({}^C_k\mathcal{P}) \qquad (261) \\
{}^C_k\mathcal{V} &= {}^C({}^0_C\omega) \times {}^C_k\mathcal{P}
\end{aligned}
$$

Taking ${}^{C}_{k}\mathcal{P} = \begin{bmatrix} {}^{C}_{k}\mathcal{P}_x \\ {}^{C}_{k}\mathcal{P}_y \\ {}^{C}_{k}\mathcal{P}_z \end{bmatrix}$ and ${}^{C}({}^{0}_{C}\omega) = \begin{bmatrix} {}_{C}\omega_x \\ {}_{C}\omega_y \\ {}_{C}\omega_z \end{bmatrix}$ we get:

$$
\begin{aligned}
&{}^{C}L = \textstyle\sum_k \left[{}^{C}_{k}\mathcal{P} \times m_k \, {}^{C}_{k}\mathcal{V} \right] = \\
&= \textstyle\sum_k \left[{}^{C}_{k}\mathcal{P} \times m_k \left({}^{C}({}^{0}_{C}\omega) \times {}^{C}_{k}\mathcal{P} \right) \right] = \\
&= \textstyle\sum_k m_k \begin{bmatrix} 0 & -{}^{C}_{k}\mathcal{P}_z & {}^{C}_{k}\mathcal{P}_y \\ {}^{C}_{k}\mathcal{P}_z & 0 & -{}^{C}_{k}\mathcal{P}_x \\ -{}^{C}_{k}\mathcal{P}_y & {}^{C}_{k}\mathcal{P}_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -{}_{C}\omega_z & {}_{C}\omega_y \\ {}_{C}\omega_z & 0 & -{}_{C}\omega_x \\ -{}_{C}\omega_y & {}_{C}\omega_x & 0 \end{bmatrix} \begin{bmatrix} {}^{C}_{k}\mathcal{P}_x \\ {}^{C}_{k}\mathcal{P}_y \\ {}^{C}_{k}\mathcal{P}_z \end{bmatrix}
\end{aligned}
\tag{262}
$$

$$
{}^{C}L = \sum_k m_k \begin{bmatrix} {}_{C}\omega_x({}^{C}_{k}\mathcal{P}_y^2 + {}^{C}_{k}\mathcal{P}_z^2) - {}_{C}\omega_y\, {}^{C}_{k}\mathcal{P}_x\, {}^{C}_{k}\mathcal{P}_y - {}_{C}\omega_z\, {}^{C}_{k}\mathcal{P}_x\, {}^{C}_{k}\mathcal{P}_z \\ -{}_{C}\omega_x\, {}^{C}_{k}\mathcal{P}_y\, {}^{C}_{k}\mathcal{P}_x + {}_{C}\omega_y({}^{C}_{k}\mathcal{P}_x^2 + {}^{C}_{k}\mathcal{P}_z^2) - {}_{C}\omega_z\, {}^{C}_{k}\mathcal{P}_y\, {}^{C}_{k}\mathcal{P}_z \\ -{}_{C}\omega_x\, {}^{C}_{k}\mathcal{P}_z\, {}^{C}_{k}\mathcal{P}_x - {}_{C}\omega_y\, {}^{C}_{k}\mathcal{P}_z\, {}^{C}_{k}\mathcal{P}_y + {}_{C}\omega_z({}^{C}_{k}\mathcal{P}_x^2 + {}^{C}_{k}\mathcal{P}_y^2) \end{bmatrix} =
$$

$$
\begin{bmatrix} \sum_k m_k({}^{C}_{k}\mathcal{P}_y^2 + {}^{C}_{k}\mathcal{P}_z^2) & -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_x\, {}^{C}_{k}\mathcal{P}_y & -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_x\, {}^{C}_{k}\mathcal{P}_z \\ -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_y\, {}^{C}_{k}\mathcal{P}_x & \sum_k m_k({}^{C}_{k}\mathcal{P}_x^2 + {}^{C}_{k}\mathcal{P}_z^2) & -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_y\, {}^{C}_{k}\mathcal{P}_z \\ -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_z\, {}^{C}_{k}\mathcal{P}_x & -\sum_k m_k\, {}^{C}_{k}\mathcal{P}_z\, {}^{C}_{k}\mathcal{P}_y & \sum_k m_k({}^{C}_{k}\mathcal{P}_x^2 + {}^{C}_{k}\mathcal{P}_y^2) \end{bmatrix}
$$

$$
\begin{bmatrix} {}_{C}\omega_x \\ {}_{C}\omega_y \\ {}_{C}\omega_z \end{bmatrix} = {}^{C}I\, {}^{C}({}^{0}_{C}\omega) \tag{263}
$$

$$
\begin{aligned}
{}^{C}L &= {}^{C}I\,{}^{C}({}^{0}_{C}\omega) \\
{}^{0}({}^{C}L) &= {}^{0}_{C}\mathcal{R}\,{}^{C}L \\
&= {}^{0}_{C}\mathcal{R}\,{}^{C}I\,{}^{C}({}^{0}_{C}\omega) \qquad (264) \\
&= {}^{0}_{C}\mathcal{R}\,{}^{C}I\,{}^{0}_{C}\mathcal{R}^{-1}\,{}^{0}_{C}\mathcal{R}\,{}^{C}({}^{0}_{C}\omega) \\
&= {}^{0}({}^{C}I)\,{}^{0}_{C}\omega
\end{aligned}
$$

$$
\Rightarrow \quad {}^{0}({}^{C}I) = {}^{0}_{C}\mathcal{R}\,{}^{C}I\,{}^{0}_{C}\mathcal{R}^{-1} \qquad (265)
$$

– this is a similarity transformation

$$
{}^{0}({}^{C}I) = {}^{0}_{C}\mathcal{R}\,{}^{C}I\,{}^{0}_{C}\mathcal{R}^{T} \qquad (266)
$$

– moment of inertia about c.o.m. expressed in frame 0

${}^0L$ – total angular momentum; ${}^0_CL$ – angular momentum of c.o.m.;
${}^0({}^CL)$ – angular momentum about c.o.m.

$$\begin{aligned} {}^0L &= {}^0_CL + {}^0({}^CL) \\ &= {}^0_C\mathcal{P} \times M\,{}^0_C\mathcal{V} + {}^0({}^CI)\,{}^0_C\omega \\ &= M\,{}^0_C\mathcal{P} \times \left({}^0_C\omega \times {}^0_C\mathcal{P}\right) + {}^0({}^CI)\,{}^0_C\omega \end{aligned} \tag{267}$$

Taking ${}^0_C\mathcal{P} = \begin{bmatrix} {}^0_C\mathcal{P}_x \\ {}^0_C\mathcal{P}_y \\ {}^0_C\mathcal{P}_z \end{bmatrix}$ and ${}^0_C\omega = \begin{bmatrix} {}^0_C\omega_x \\ {}^0_C\omega_y \\ {}^0_C\omega_z \end{bmatrix}$ we get:

$$ {}^0_CL = M \begin{bmatrix} 0 & -{}^0_C\mathcal{P}_z & {}^0_C\mathcal{P}_y \\ {}^0_C\mathcal{P}_z & 0 & -{}^0_C\mathcal{P}_x \\ -{}^0_C\mathcal{P}_y & {}^0_C\mathcal{P}_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -{}^0_C\omega_z & {}^0_C\omega_y \\ {}^0_C\omega_z & 0 & -{}^0_C\omega_x \\ -{}^0_C\omega_y & {}^0_C\omega_x & 0 \end{bmatrix} \begin{bmatrix} {}^0_C\mathcal{P}_x \\ {}^0_C\mathcal{P}_y \\ {}^0_C\mathcal{P}_z \end{bmatrix} \tag{268}$$

$$
{}^0_C L = M \begin{bmatrix} {}^0_C\omega_x({}^0_C\mathcal{P}_y^2 + {}^0_C\mathcal{P}_z^2) - {}^0_C\omega_y\,{}^0_C\mathcal{P}_x\,{}^0_C\mathcal{P}_y - {}^0_C\omega_z\,{}^0_C\mathcal{P}_x\,{}^0_C\mathcal{P}_z \\ -{}^0_C\omega_x\,{}^0_C\mathcal{P}_y\,{}^0_C\mathcal{P}_x + {}^0_C\omega_y({}^0_C\mathcal{P}_x^2 + {}^0_C\mathcal{P}_z^2) - {}^0_C\omega_z\,{}^0_C\mathcal{P}_y\,{}^0_C\mathcal{P}_z \\ -{}^0_C\omega_x\,{}^0_C\mathcal{P}_z\,{}^0_C\mathcal{P}_x - {}^0_C\omega_y\,{}^0_C\mathcal{P}_z\,{}^0_C\mathcal{P}_y + {}^0_C\omega_z({}^0_C\mathcal{P}_x^2 + {}^0_C\mathcal{P}_y^2) \end{bmatrix} =
$$
$$
\begin{bmatrix} M({}^0_C\mathcal{P}_y^2 + {}^0_C\mathcal{P}_z^2) & -M\,{}^0_C\mathcal{P}_x\,{}^0_C\mathcal{P}_y & -M\,{}^0_C\mathcal{P}_x\,{}^0_C\mathcal{P}_z \\ -M\,{}^0_C\mathcal{P}_y\,{}^0_C\mathcal{P}_x & M({}^0_C\mathcal{P}_x^2 + {}^0_C\mathcal{P}_z^2) & -M\,{}^0_C\mathcal{P}_y\,{}^0_C\mathcal{P}_z \\ -M\,{}^0_C\mathcal{P}_z\,{}^0_C\mathcal{P}_x & -M\,{}^0_C\mathcal{P}_z\,{}^0_C\mathcal{P}_y & M({}^0_C\mathcal{P}_x^2 + {}^0_C\mathcal{P}_y^2) \end{bmatrix} \begin{bmatrix} {}^0_C\omega_x \\ {}^0_C\omega_y \\ {}^0_C\omega_z \end{bmatrix}
$$
$$
= {}^0_C I\,{}^0_C\omega \tag{269}
$$

Total angular momentum:

$$
{}^0 L = {}^0_C I\,{}^0_C\omega + {}^0({}^C I)\,{}^0_C\omega = \left[{}^0_C I + {}^0({}^C I)\right]{}^0_C\omega \tag{270}
$$

- Newton's First Law of Motion
  Every body will remain at rest, or in a state of uniform motion unless acted upon by a force.
- Newton's Second Law of Motion
  A not equilibrated force acting upon a body produces an acceleration proportional to the force and inversely proportional to the mass of the body. This acceleration is in the direction of the force.
- Newton's Third Law of Motion
  Every action has an equal and opposite reaction.

Newton's laws of dynamics are formulated wrt an inertial frame of reference.

Newton's second law for a rigid body i:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(M_i\,{}^0_{C_i}\mathcal{V}\right)}_{\substack{\text{change of}\\ \text{linear}\\ \text{momentum}}} = \underbrace{\sum_j {}^0_iF_j}_{\substack{\text{sum of external}\\ \text{forces acting}\\ \text{on body } i}} \tag{271}$$

Euler's equation for a rigid body i:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left({}^0({}^{C_i}I_i)\,{}^0_{C_i}\omega\right)}_{\substack{\text{change of}\\ \text{angular}\\ \text{momentum}}} = \underbrace{\sum_j {}^0({}^{C_i}_{\ i}N_j)}_{\substack{\text{sum of external}\\ \text{torques acting}\\ \text{on body } i}} \tag{272}$$

Both equations are formulated wrt an inertial frame 0, although (272) concerns angular momentum and torque about c.o.m.

The centre of mass of body i is represented by a coordinate frame C_i affixed to that body (its origin is located at c.o.m.). Frame 0 is an inertial frame, while frame C_i usually is not.

$$M_i \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{0}_{C_i}\mathcal{V} = \sum_j {}^{0}_{i}F_j \tag{273}$$

$$M_i\,{}^{C_i}_{0}\mathcal{R}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{0}_{C_i}\mathcal{V} = \sum_j {}^{C_i}_{0}\mathcal{R}\,{}^{0}_{i}F_j \tag{274}$$

$$M_i\,{}^{C_i}_{0}\mathcal{R}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{0}_{C_i}\mathcal{V} = \sum_j {}^{C_i}({}^{0}_{i}F_j) \tag{275}$$

$$M_i\,{}^{C_i}_{0}\mathcal{R}\,{}^{0}_{C_i}\mathcal{A} = \sum_j {}^{C_i}({}^{0}_{i}F_j) \tag{276}$$

Dynamics of rotation about the c.o.m. observed w.r.t. an inertial frame:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}({}^{C_i}I_i)\,{}^{0}_{C_i}\omega\right) = \sum_j {}^{0}({}^{C_i}_{i}N_j) \tag{277}$$

${}^{0}({}^{C_i}I_i)$ depends on time, but ${}^{C_i}I_i = \text{const}$

$${}^{0}({}^{C_i}I_i)\,{}^{0}_{C_i}\omega = {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{0}_{C_i}\mathcal{R}^{-1}\,{}^{0}_{C_i}\omega = {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}_{0}\mathcal{R}\,{}^{0}_{C_i}\omega = {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega) \tag{278}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}({}^{C_i}I_i)\,{}^{0}_{C_i}\omega\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)\right) \tag{279}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}({}^{C_i}I_i)\,{}^{0}_{C_i}\omega\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}_{C_i}\mathcal{R}\right){}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega) + {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right) \tag{280}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{0}({}^{C_i}I_i)\,{}^{0}_{C_i}\omega\right) = \mathcal{S}({}^{0}_{C_i}\omega)\,{}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega) + {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right) \tag{281}$$

$$\mathcal{S}({}^{0}_{C_i}\omega)\,{}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega) + {}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right) = \sum_j {}^{0}({}^{C_i}_{i}N_j) \tag{282}$$

$$ {}^{C_i}_{0}\mathcal{R}\mathcal{S}({}^{0}_{C_i}\omega)\,{}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)+{}^{C_i}_{0}\mathcal{R}\,{}^{0}_{C_i}\mathcal{R}\,{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right)=\sum_j {}^{C_i}_{0}\mathcal{R}\,{}^{0}({}^{C_i}_{i}N_j) \tag{283} $$

$$ \mathcal{S}({}^{C_i}_{0}\mathcal{R}\,{}^{0}_{C_i}\omega)\,{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)+{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right)=\sum_j {}^{C_i}_{i}N_j \tag{284} $$

$$ \mathcal{S}\left({}^{C_i}({}^{0}_{C_i}\omega)\right){}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)+{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right)=\sum_j {}^{C_i}_{i}N_j \tag{285} $$

$$ {}^{C_i}({}^{0}_{C_i}\omega)\times\left({}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)\right)+{}^{C_i}I_i\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{C_i}({}^{0}_{C_i}\omega)\right)=\sum_j {}^{C_i}_{i}N_j \tag{286} $$

$$ \underbrace{{}^{C_i}({}^{0}_{C_i}\omega)\times\left({}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\omega)\right)}_{\text{gyroscopic term}}+\underbrace{{}^{C_i}I_i\,{}^{C_i}({}^{0}_{C_i}\mathcal{E})}_{\substack{\text{rate of change of}\\ \text{angular momentum}\\ \text{w.r.t. non-inertial}\\ \text{reference frame}}}=\sum_j {}^{C_i}_{i}N_j \tag{287} $$

Angular velocity of link $i+1$ connected to link i by a rotational joint. Using (253) we get:

$$ {}^{0}_{i+1}\omega = {}^{0}_{i}\omega + {}^{0}({}^{i}_{i+1}\omega) = {}^{0}_{i}\omega + {}^{0}_{i}\mathcal{R}\,{}^{i}_{i+1}\omega \tag{288} $$

As ${}^{i}_{i+1}\omega = {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}z\,\dot{\theta}_{i+1}$

$$ {}^{0}_{i+1}\omega = {}^{0}_{i}\omega + {}^{0}_{i}\mathcal{R}\;{}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}z\,\dot{\theta}_{i+1} = {}^{0}_{i}\omega + {}^{0}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}z\dot{\theta}_{i+1} \tag{289} $$

Multiplying equation (289) by ${}^{i+1}_{0}\mathcal{R}$ we get:

$$ {}^{i+1}({}^{0}_{i+1}\omega) = {}^{i+1}_{i}\mathcal{R}\,{}^{i}({}^{0}_{i}\omega) + {}^{i+1}_{i+1}z\,\dot{\theta}_{i+1} \tag{290} $$

Angular velocity of link $i+1$ connected to link i by a prismatic joint:

$$ {}^{i+1}({}^{0}_{i+1}\omega) = {}^{i+1}({}^{0}_{i}\omega) \tag{291} $$

$$ {}^{i+1}({}^{0}_{i+1}\omega) = {}^{i+1}_{i}\mathcal{R}\,{}^{i}({}^{0}_{i}\omega) \tag{292} $$

Angular acceleration of link $i+1$ connected to link i by a rotational joint (from (254)):

$$ {}^{0}_{i+1}\mathcal{E} = {}^{0}_{i}\mathcal{E} + {}^{0}_{i}\omega \times {}^{0}({}^{i}_{i+1}\omega) + {}^{0}({}^{i}_{i+1}\mathcal{E}) \tag{293} $$

Multiplying equation (293) by ${}^{i+1}_{0}\mathcal{R}$ we get:

$$ {}^{i+1}({}^{0}_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{0}_{i}\mathcal{E}) + {}^{i+1}_{i}\mathcal{R}^{i}({}^{0}_{i}\omega) \times {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{i}_{i+1}\omega) + {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{i}_{i+1}\mathcal{E}) \tag{294} $$

As ${}^{i}_{i+1}\omega = {}^{i}_{i+1}\mathcal{R}\, {}^{i+1}_{i+1}z\, \dot{\theta}_{i+1}$ and ${}^{i}_{i+1}\mathcal{E} = {}^{i}_{i+1}\mathcal{R}\, {}^{i+1}_{i+1}z\, \ddot{\theta}_{i+1}$ we get:

$$ {}^{i+1}({}^{0}_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{0}_{i}\mathcal{E}) + {}^{i+1}_{i}\mathcal{R}^{i}({}^{0}_{i}\omega) \times {}^{i+1}_{i+1}z\, \dot{\theta}_{i+1} + {}^{i+1}_{i+1}z\, \ddot{\theta}_{i+1} \tag{295} $$

If the link is connected by a prismatic joint $\dot{\theta}_{i+1} = 0$ and $\ddot{\theta}_{i+1} = 0$, thus:

$$ {}^{i+1}({}^{0}_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{0}_{i}\mathcal{E}) \tag{296} $$

Linear acceleration of link $i+1$ connected to link i (from (251)):

$$ {}^{0}_{i+1}\mathcal{A} = {}^{0}_{i}\mathcal{A} + {}^{0}_{i}\mathcal{E} \times {}^{0}({}^{i}_{i+1}\mathcal{P}) + {}^{0}_{i}\omega \times \left({}^{0}_{i}\omega \times {}^{0}({}^{i}_{i+1}\mathcal{P}) \right) + 2\, {}^{0}_{i}\omega \times {}^{0}({}^{i}_{i+1}\mathcal{V}) + {}^{0}({}^{i}_{i+1}\mathcal{A}) \tag{297} $$

Multiplying equation (297) by ${}^{i+1}_{0}\mathcal{R}$ we get:

$$ \begin{aligned} {}^{i+1}({}^{0}_{i+1}\mathcal{A}) &= {}^{i+1}_{i}\mathcal{R}\Big[{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \left({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P} \right) \Big] \\ &\quad + 2\, {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{0}_{i}\omega) \times {}^{i+1}_{i}\mathcal{R}\, {}^{i}_{i+1}\mathcal{V} + {}^{i+1}_{i}\mathcal{R}\, {}^{i}_{i+1}\mathcal{A} \end{aligned} \tag{298} $$

As for a prismatic joint $i+1$ we have ${}^{i}_{i+1}\mathcal{V} = {}^{i}_{i+1}\mathcal{R}\, {}^{i+1}_{i+1}z\, \dot{d}_{i+1}$ and ${}^{i}_{i+1}\mathcal{A} = {}^{i}_{i+1}\mathcal{R}\, {}^{i+1}_{i+1}z\, \ddot{d}_{i+1}$ we get:

$$ \begin{aligned} {}^{i+1}({}^{0}_{i+1}\mathcal{A}) &= {}^{i+1}_{i}\mathcal{R}\Big[{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \left({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P} \right) \Big] \\ &\quad + 2\, {}^{i+1}_{i}\mathcal{R}\, {}^{i}({}^{0}_{i}\omega) \times {}^{i+1}_{i+1}z\, \dot{d}_{i+1} + {}^{i+1}_{i+1}z\, \ddot{d}_{i+1} \end{aligned} \tag{299} $$

As from (291) for a prismatic joint $i+1$ ${}^{0}_{i+1}\omega = {}^{0}_{i}\omega$ we get:

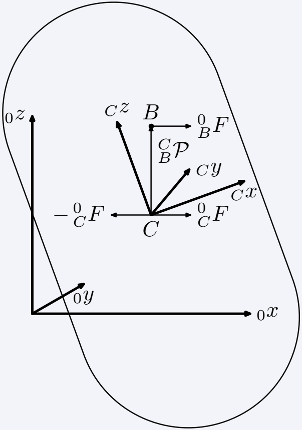
$$
\begin{aligned}
{}^{i+1}({}^{0}_{i+1}\mathcal{A}) &= {}^{i+1}_{i}\mathcal{R}\Big[{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \Big({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P}\Big)\Big] \\
&\quad + 2\,{}^{i+1}({}^{0}_{i+1}\omega) \times {}^{i+1}_{i+1}z\,\dot{d}_{i+1} + {}^{i+1}_{i+1}z\,\ddot{d}_{i+1}
\end{aligned}
\tag{300}
$$

Considering a revolute joint $i+1$ in (298) we have ${}^{i}_{i+1}\mathcal{V} = \mathcal{O}$ and ${}^{i}_{i+1}\mathcal{A} = \mathcal{O}$, thus:

$$
{}^{i+1}({}^{0}_{i+1}\mathcal{A}) = {}^{i+1}_{i}\mathcal{R}\Big[{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \Big({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P}\Big)\Big] \tag{301}
$$

Linear acceleration of centre of mass of link i is obtained similarly:

$$
{}^{i}({}^{0}_{C_i}\mathcal{A}) = {}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{C_i}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \Big({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{C_i}\mathcal{P}\Big) \tag{302}
$$



$$ {}^{0}_{C}F = {}^{0}_{B}F \tag{303} $$

$$ {}^{0}_{C}F - {}^{0}_{C}F = 0 \tag{304} $$

C – centre of mass (chosen as origin/reference)

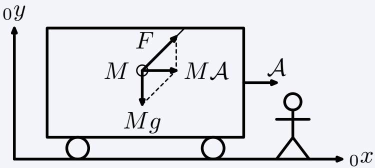
${}^{0}_{C}F$ – acts at c.o.m. and induces linear acceleration of the body

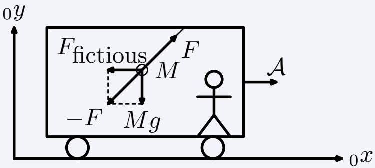
$-{}^{0}_{C}F$ and ${}^{0}_{B}F$ produce a torque around C and thus rotate the body

$$ {}^{0}_{C}\mathcal{R}({}^{C}_{B}\mathcal{P}) \times {}^{0}_{B}F = {}^{0}N \tag{305} $$

⇒ Unbalanced non-central force causes both linear acceleration of the body and its rotation

⇒ From the point of view of linear acceleration of the centre of mass all external forces acting on the body can be shifted in parallel to this centre





Inertial frame of reference

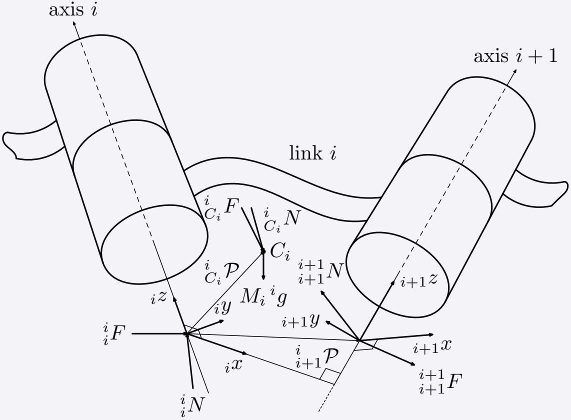
$$F + Mg = M\mathcal{A} \qquad (306)$$

Non-inertial frame of reference

$$-F = F_{\text{fictious}} + Mg \qquad (307)$$

$$\Rightarrow \quad F_{\text{fictious}} = -M\mathcal{A} \qquad (308)$$

Inclusion of fictious forces, in reality, transforms the considerations from a non-inertial frame of reference to an inertial one, thus magnitudes of forces are always related to an inertial frame, hence two indices suffice: one defining the point of force application and one defining the frame (its orientation) wrt which the force components are expressed.



All forces and torques are considered wrt an inertial frame 0 $\Rightarrow$ e.g.:

$${}^{i}_{C_i}F = {}^{i}({}^{0}_{C_i}F)$$

C_i	–	centre of mass of link i
${}_iF$	–	force exerted by link $i-1$ on link i
${}_{C_i}F$	–	inertial force acting at c.o.m. of link i
${}_iN$	–	torque exerted by link $i-1$ on link i
${}_{C_i}N$	–	inertial torque acting at c.o.m. of link i

Assumptions:

- Frame C_i has the same orientation as frame i, i.e.: ${}^{i}_{C_i}\mathcal{R} = \mathcal{I}$
  $\Rightarrow {}^{i}_{C_i}F = {}^{C_i}_{C_i}F$, ${}^{i}_{i}F = {}^{C_i}_{i}F$, ${}^{i}_{i+1}F = {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F = {}^{C_i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F$
- Forces acting on link i will be computed and expressed wrt frame i

Force balance equation:

$$ {}^{C_i}_{C_i}F = {}^{C_i}_{i}F - {}^{C_i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F + M_i\,{}^{C_i}g \tag{309} $$

LHS = inertial force
RHS = reaction forces from neighbouring links + gravity

$$ {}^{i}_{C_i}F = {}^{i}_{i}F - {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F + M_i\,{}^{i}g \tag{310} $$

$$ \Rightarrow \quad {}^{i}_{i}F = {}^{i}_{C_i}F + {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F - M_i\,{}^{i}g \tag{311} $$

From (276) we have:

$$ {}^{i}_{C_i}F = M_i\,{}^{i}({}^{0}_{C_i}\mathcal{A}) \tag{312} $$

Assumptions:

- Frame C_i has the same orientation as frame i, i.e.: ${}^{i}_{C_i}\mathcal{R} = \mathcal{I}$
  $\Rightarrow {}^{i}_{C_i}N = {}^{C_i}_{C_i}N$, ${}^{i}_{i}N = {}^{C_i}_{i}N$, ${}^{i}_{i+1}N = {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N = {}^{C_i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N$
- Forces and torques acting on the link will be computed and expressed wrt frame i
- Torques will be computed around the centre of mass
- Force of gravity acts at the centre of mass, thus arm of this force is 0

Torque balance equation:

$$ {}^{i}_{C_i}N = {}^{i}_{i}N - {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N + (-\,{}^{i}_{C_i}\mathcal{P}) \times {}^{i}_{i}F - ({}^{i}_{i+1}\mathcal{P} - {}^{i}_{C_i}\mathcal{P}) \times ({}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F) \quad (313) $$

$$ \Rightarrow \quad {}^{i}_{i}N = {}^{i}_{C_i}N + {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N + {}^{i}_{C_i}\mathcal{P} \times {}^{i}_{i}F + ({}^{i}_{i+1}\mathcal{P} - {}^{i}_{C_i}\mathcal{P}) \times ({}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F) \quad (314) $$

From (287) we have:

$$ {}^{i}_{C_i}N = {}^{i}({}^{0}_{C_i}\omega) \times \left({}^{C_i}I_i\,{}^{i}({}^{0}_{C_i}\omega)\right) + {}^{C_i}I_i\,{}^{i}({}^{0}_{C_i}\mathcal{E}) \quad (315) $$

Outward iterations ($i = 0 \to n-1$)

- ${}^0_0\omega = \mathcal{O}$, ${}^0_0\mathcal{E} = \mathcal{O}$
- Rotational velocity:
  For a rotational joint from (290) we have:

$$^{i+1}({}^0_{i+1}\omega) = {}^{i+1}_{i}\mathcal{R}\,{}^i({}^0_i\omega) + {}^{i+1}_{i+1}z\,\dot\theta_{i+1} \tag{316}$$

  For a prismatic joint from (292) we have:

$$^{i+1}({}^0_{i+1}\omega) = {}^{i+1}_{i}\mathcal{R}\,{}^i({}^0_i\omega) \tag{317}$$

- Rotational acceleration:
  For a rotational joint from (295) we have:

$$^{i+1}({}^0_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\,{}^i({}^0_i\mathcal{E}) + {}^{i+1}_{i}\mathcal{R}^i({}^0_i\omega) \times {}^{i+1}_{i+1}z\,\dot\theta_{i+1} + {}^{i+1}_{i+1}z\,\ddot\theta_{i+1} \tag{318}$$

  For a prismatic joint from (296) we have:

$$^{i+1}({}^0_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\,{}^i({}^0_i\mathcal{E}) \tag{319}$$

- <u>Linear acceleration</u>

  For a rotational joint from (301) we have:

$$ {}^{i+1}({}^{0}_{i+1}\mathcal{A}) = {}^{i+1}_{i}\mathcal{R}\Big[\,{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \Big({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P}\Big)\Big] \tag{320} $$

  For a prismatic joint from (300) we have:

$$ \begin{aligned} {}^{i+1}({}^{0}_{i+1}\mathcal{A}) &= {}^{i+1}_{i}\mathcal{R}\Big[\,{}^{i}({}^{0}_{i}\mathcal{A}) + {}^{i}({}^{0}_{i}\mathcal{E}) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}({}^{0}_{i}\omega) \times \Big({}^{i}({}^{0}_{i}\omega) \times {}^{i}_{i+1}\mathcal{P}\Big)\Big] \\ &\quad +2\,{}^{i+1}({}^{0}_{i+1}\omega) \times {}^{i+1}_{i+1}z\,\dot{d}_{i+1} + {}^{i+1}_{i+1}z\,\ddot{d}_{i+1} \end{aligned} \tag{321} $$

  Linear acceleration of the centre of mass results from (302)

$$ \begin{aligned} {}^{i+1}({}^{0}_{C_{i+1}}\mathcal{A}) &= {}^{i+1}({}^{0}_{i+1}\mathcal{A}) + {}^{i+1}({}^{0}_{i+1}\mathcal{E}) \times {}^{i+1}_{C_{i+1}}\mathcal{P} + \\ &\quad {}^{i+1}({}^{0}_{i+1}\omega) \times \Big({}^{i+1}({}^{0}_{i+1}\omega) \times {}^{i+1}_{C_{i+1}}\mathcal{P}\Big) \end{aligned} \tag{322} $$

- Inertial forces
  From (312) we obtain:

$$ {}^{i+1}_{C_{i+1}}F = M_{i+1}\,{}^{i+1}({}^{0}_{C_{i+1}}\mathcal{A}) \tag{323} $$

- Inertial torques
  From (315) we obtain:

$$ {}^{i+1}_{C_{i+1}}N = {}^{i+1}({}^{0}_{C_{i+1}}\omega) \times \left({}^{C_{i+1}}I_{i+1}\,{}^{i+1}({}^{0}_{C_{i+1}}\omega) \right) + {}^{C_{i+1}}I_{i+1}\,{}^{i+1}({}^{0}_{C_{i+1}}\mathcal{E}) \tag{324} $$

<u>Inward iterations</u> ($i = n \rightarrow 1$)

- For $i = n+1$: ${}^{n+1}_{n+1}F = \mathcal{O}$, ${}^{n+1}_{C_{n+1}}F = \mathcal{O}$
- Inter-link interaction forces resulting from (311):

$$
{}^{i}_{i}F = {}^{i}_{C_i}F + {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F - M_i\,{}^{i}g \tag{325}
$$

- Inter-link interaction torques resulting from (314):

$$
{}^{i}_{i}N = {}^{i}_{C_i}N + {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N + {}^{i}_{C_i}\mathcal{P} \times {}^{i}_{i}F + \left({}^{i}_{i+1}\mathcal{P} - {}^{i}_{C_i}\mathcal{P}\right) \times \left({}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F\right) \tag{326}
$$

- Force/torque to be developed in the joint:
  Revolute joint:

$$
\tau_i = {}^{i}_{i}N^{T}\,{}^{i}_{i}z \tag{327}
$$

  Prismatic joint:

$$
\tau_i = {}^{i}_{i}F^{T}\,{}^{i}_{i}z \tag{328}
$$

  Other than z components are canceled out by reaction forces/torques.

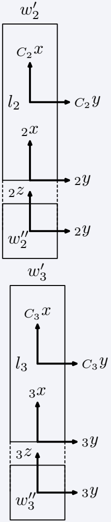
Moment of inertia of an object (with respect to a chosen axis of rotation χ) is:

$$^{\chi}I = \int_M {}^{\chi}r^2 \,\mathrm{d}m = \int_V \varrho \, {}^{\chi}r^2 \,\mathrm{d}\,V \tag{329}$$

m	–	elementary mass
M	–	mass of the object
V	–	volume of the object
ϱ	–	density of the object
r	–	perpendicular distance to the chosen axis of rotation χ

$\Rightarrow$ Moment of inertia is a property of the object, however it depends on the choice of axis of rotation

ϱ – uniform density of link 2



$$M_2 = \varrho l_2 w_2' w_2'' \tag{330}$$

$$\begin{aligned}
I_{2_{xx}} &= \int_V \varrho({}^{C_2}y^2 + {}^{C_2}z^2)\mathrm{d}\,V \\
&= \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \int_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \varrho({}^{C_2}y^2 + {}^{C_2}z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \varrho \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \int_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} ({}^{C_2}y^2 + {}^{C_2}z^2) x \Big|_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \mathrm{d}y\mathrm{d}z \\
&= \varrho l_2 \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \int_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} ({}^{C_2}y^2 + {}^{C_2}z^2)\mathrm{d}y\mathrm{d}z \\
&= \varrho l_2 \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} (\frac{{}^{C_2}y^3}{3} + {}^{C_2}z^2 y)\Big|_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} \mathrm{d}z \\
&= \varrho l_2 \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \Big(\frac{(\frac{w_2'}{2})^3}{3} - \frac{(-\frac{w_2'}{2})^3}{3} + {}^{C_2}z^2(\frac{w_2'}{2}) - {}^{C_2}z^2(-\frac{w_2'}{2})\Big)\mathrm{d}z
\end{aligned} \tag{331}$$

$$
\begin{aligned}
I_{2_{xx}} &= \varrho l_2 \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \left(\frac{(w_2')^3}{12} + \, C_2 z^2 w_2' \right) \mathrm{d}z \\
&= \varrho l_2 \left(\frac{(w_2')^3}{12} \, C_2 z + \frac{C_2 z^3}{3} w_2' \right) \Bigg|_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \\
&= \varrho l_2 \left(\frac{(w_2')^3}{12} \left(\frac{w_2''}{2}\right) - \frac{(w_2')^3}{12} \left(-\frac{w_2''}{2}\right) + \frac{(\frac{w_2''}{2})^3}{3} w_2' - \frac{(-\frac{w_2''}{2})^3}{3} w_2' \right) \\
&= \varrho l_2 \left(\frac{(w_2')^3}{12} (w_2'') + \frac{(w_2'')^3}{12} w_2' \right) \\
&= \varrho l_2 w_2' w_2'' \left(\frac{(w_2')^2}{12} + \frac{(w_2'')^2}{12} \right) \\
&= \frac{1}{12} \varrho l_2 w_2' w_2'' \left((w_2')^2 + (w_2'')^2 \right) \\
I_{2_{yy}} &= \frac{M_2}{12} \left((l_2)^2 + (w_2'')^2 \right) \\
I_{2_{zz}} &= \frac{M_2}{12} \left((w_2')^2 + (l_2)^2 \right)
\end{aligned}
\qquad (332)
$$

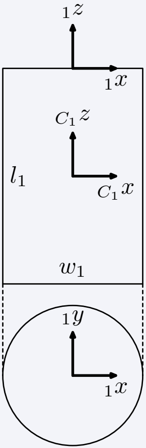
Moments of deviation:

$$\begin{aligned} I_{2_{xy}} &= -\int_V \varrho\, x\, y\, \mathrm{d}V \\ &= -\int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \int_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \varrho\, {}^{C_2}x\, {}^{C_2}y\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= -\varrho \int_{-\frac{w_2''}{2}}^{\frac{w_2''}{2}} \int_{-\frac{w_2'}{2}}^{\frac{w_2'}{2}} \frac{1}{2}\, {}^{C_2}x^2\, {}^{C_2}y \Big|_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \mathrm{d}y\mathrm{d}z = 0 \end{aligned} \tag{333}$$

By the same token:

$$\begin{aligned} I_{2_{xy}} &= I_{2_{yx}} = 0 \\ I_{2_{xz}} &= I_{2_{zx}} = 0 \\ I_{2_{yz}} &= I_{2_{zy}} = 0 \end{aligned} \tag{334}$$

$$ {}^{C_2}I_2 = \begin{bmatrix} \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right) & 0 & 0 \\ 0 & \frac{M_2}{12}\left((l_2)^2 + (w_2'')^2\right) & 0 \\ 0 & 0 & \frac{M_2}{12}\left((w_2')^2 + (l_2)^2\right) \end{bmatrix} \tag{335}$$



$$M_1 = \varrho\pi\left(\frac{w_1}{2}\right)^2 l_1 = \frac{\varrho\pi w_1^2 l_1}{4} \tag{336}$$

$$\begin{aligned} & {}^{C_1}x^2 + {}^{C_1}y^2 = \left(\tfrac{w_1}{2}\right)^2 \\ \Rightarrow \quad & {}^{C_1}x = \pm\sqrt{\frac{w_1^2}{4} - {}^{C_1}y^2} \end{aligned} \tag{337}$$

$$\begin{aligned} {}^{C_1}I_{1_{xx}} &= \int_V \varrho\left({}^{C_1}y^2 + {}^{C_1}z^2\right) \mathrm{d}\,V \\ &= \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{w_1}{2}}^{\frac{w_1}{2}} \int_{-\sqrt{\frac{w_1^2}{4} - {}^{C_2}y^2}}^{+\sqrt{\frac{w_1^2}{4} - {}^{C_2}y^2}} \varrho({}^{C_2}y^2 + {}^{C_2}z^2)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{w_1}{2}}^{\frac{w_1}{2}} \varrho\left({}^{C_2}y^2 + {}^{C_2}z^2\right) {}^{C_2}x\Big|_{-\sqrt{\frac{w_1^2}{4} - {}^{C_2}y^2}}^{+\sqrt{\frac{w_1^2}{4} - {}^{C_2}y^2}} \mathrm{d}y\mathrm{d}z \\ &= 2\varrho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{w_1}{2}}^{\frac{w_1}{2}} \left({}^{C_2}y^2 + {}^{C_2}z^2\right)\sqrt{\frac{w_1^2}{4} - {}^{C_2}y^2}\, \mathrm{d}y\mathrm{d}z \end{aligned} \tag{338}$$

– rather complex ⇒ cylindric coordinates

Elementary volume:

$$\mathrm{d}\,V = \mathrm{d}r\Big({}^{C_1}{}_z\,r\,\mathrm{d}\varphi\Big)\mathrm{d}z = {}^{C_1}{}_z\,r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \tag{339}$$

$$\begin{aligned}
{}^{C_1}I_{1_{zz}} &= \int_V \varrho\Big({}^{C_1}{}_z\,r\Big)^2\,\mathrm{d}\,V \\
&= \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\int_0^{\frac{w_1}{2}} \varrho\Big({}^{C_1}{}_z\,r\Big)^2\,{}^{C_1}{}_z\,r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \\
&= \varrho\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\int_0^{\frac{w_1}{2}}\Big({}^{C_1}{}_z\,r\Big)^3\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \\
&= \tfrac{1}{4}\varrho\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\Big({}^{C_1}{}_z\,r\Big)^4\Big|_0^{\frac{w_1}{2}}\,\mathrm{d}\varphi\,\mathrm{d}z \\
&= \tfrac{1}{64}\varrho w_1^4\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\,\mathrm{d}\varphi\,\mathrm{d}z \;=\; \tfrac{1}{64}\varrho w_1^4\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\;\varphi\Big|_0^{2\pi}\,\mathrm{d}z \\
&= \tfrac{\pi}{32}\varrho w_1^4\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\,\mathrm{d}z \;=\; \tfrac{\pi}{32}\varrho w_1^4\,{}^{C_1}z\Big|_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \;=\; \tfrac{\pi}{32}\varrho w_1^4 l_1 \;=\; \tfrac{1}{8}M_1 w_1^2
\end{aligned} \tag{340}$$

Transformation of Cartesian into cylindrical coordinates:

$$\begin{aligned} {}^{C_1}x &= {}^{C_{1z}}r\cos\varphi \\ {}^{C_1}y &= {}^{C_{1z}}r\sin\varphi \\ {}^{C_1}z &= {}^{C_1}z \end{aligned} \tag{341}$$

Using (339) and (341) we get:

$$\begin{aligned} {}^{C_1}I_{1_{xx}} &= \int_V \varrho\Big(\big({}^{C_1}y\big)^2 + \big({}^{C_1}z\big)^2\Big)\,\mathrm{d}V \\ &= \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\int_0^{\frac{w_1}{2}} \varrho\Big(\big({}^{C_{1z}}r\sin\varphi\big)^2 + \big({}^{C_1}z\big)^2\Big)\,{}^{C_{1z}}r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \\ &= \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\int_0^{\frac{w_1}{2}} \varrho\Big(\big({}^{C_{1z}}r\big)^3\sin^2\varphi + {}^{C_{1z}}r\big({}^{C_1}z\big)^2\Big)\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \\ &= \varrho\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}}\int_0^{2\pi}\left(\frac{1}{4}\big({}^{C_{1z}}r\big)^4\sin^2\varphi + \frac{1}{2}\big({}^{C_{1z}}r\big)^2\big({}^{C_1}z\big)^2\right)\Bigg|_0^{\frac{w_1}{2}}\,\mathrm{d}\varphi\,\mathrm{d}z \end{aligned} \tag{342}$$

$$
\begin{aligned}
{}^{C_1}I_{1_{xx}} &= \varrho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_0^{2\pi} \left(\frac{1}{64} w_1^4 \sin^2\varphi + \frac{1}{8} w_1^2 \left({}^{C_1}z\right)^2 \right) \mathrm{d}\varphi \, \mathrm{d}z \\
&= \varrho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_0^{2\pi} \frac{1}{64} w_1^4 \sin^2\varphi \, \mathrm{d}\varphi \, \mathrm{d}z + \varrho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_0^{2\pi} \frac{1}{8} w_1^2 \left({}^{C_1}z\right)^2 \mathrm{d}\varphi \, \mathrm{d}z \\
&= \frac{1}{64} \varrho w_1^4 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_0^{2\pi} \sin^2\varphi \, \mathrm{d}\varphi \, \mathrm{d}z + \frac{1}{8} \varrho w_1^2 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_0^{2\pi} \left({}^{C_1}z\right)^2 \mathrm{d}\varphi \, \mathrm{d}z
\end{aligned}
\tag{343}
$$

$$
\begin{aligned}
&\int_0^{2\pi} \sin^2\varphi \, \mathrm{d}\varphi = \int_0^{2\pi} \left(1 - \cos^2\varphi\right) \mathrm{d}\varphi = \int_0^{2\pi} \left(1 - \frac{1}{2}\left(1 + \cos 2\varphi\right)\right) \mathrm{d}\varphi \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2\varphi\right) \mathrm{d}\varphi = \frac{1}{2} \int_0^{2\pi} \mathrm{d}\varphi - \frac{1}{2} \int_0^{2\pi} \cos 2\varphi \, \mathrm{d}\varphi \\
&= \frac{1}{2} \varphi \Big|_0^{2\pi} - \frac{1}{2} \int_0^{4\pi} \cos\phi \, \frac{\mathrm{d}\phi}{2} = \pi - \frac{1}{4} \sin\phi \Big|_0^{4\pi} = \pi
\end{aligned}
\tag{344}
$$

$$
\begin{aligned}
{}^{C_1}I_{1_{xx}} &= \tfrac{1}{64}\varrho w_1^4 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \pi \,\mathrm{d}z + \tfrac{1}{8}\varrho w_1^2 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \left({}^{C_1}z\right)^2 \varphi\Big|_0^{2\pi} \mathrm{d}z \\
&= \tfrac{1}{64}\varrho\pi w_1^4 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \mathrm{d}z + \tfrac{1}{4}\varrho\pi w_1^2 \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \left({}^{C_1}z\right)^2 \mathrm{d}z \\
&= \tfrac{1}{64}\varrho\pi w_1^4 \, {}^{C_1}z\Big|_{-\frac{l_1}{2}}^{\frac{l_1}{2}} + \tfrac{1}{4}\varrho\pi w_1^2 \tfrac{1}{3}\left({}^{C_1}z\right)^3\Big|_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \\
&= \tfrac{1}{64}\varrho\pi w_1^4 l_1 + \tfrac{1}{12}\varrho\pi w_1^2 \tfrac{l_1^3}{4} \\
&= \tfrac{1}{64}\varrho\pi w_1^4 l_1 + \tfrac{1}{48}\varrho\pi w_1^2 l_1^3 \\
&= \tfrac{1}{16} M_1 w_1^2 + \tfrac{1}{12} M_1 l_1^2
\end{aligned}
\tag{345}
$$

Similar algebra produces:

$$
{}^{C_1}I_{1_{yy}} = \frac{1}{16} M_1 w_1^2 + \frac{1}{12} M_1 l_1^2 \tag{346}
$$

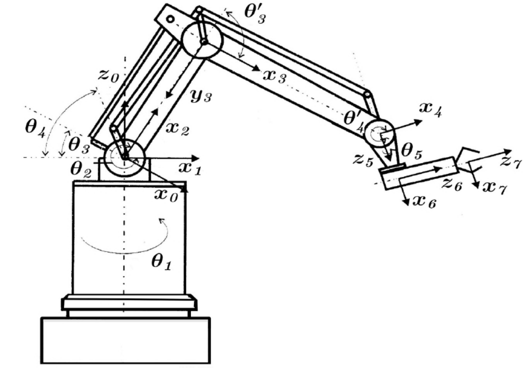
$$\begin{array}{ccccc} I_{1_{xy}} & = & I_{1_{yx}} & = & 0 \\ I_{1_{xz}} & = & I_{1_{zx}} & = & 0 \\ I_{1_{yz}} & = & I_{1_{zy}} & = & 0 \end{array} \tag{347}$$

$$^{C_1}I_1 = \begin{bmatrix} \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 & 0 \\ 0 & \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 \\ 0 & 0 & \frac{1}{8}M_1 w_1^2 \end{bmatrix} \tag{348}$$

Assumptions:

- 3 d.o.f. serial structure manipulator is considered
- All joints are revolute
- The vertical revolute column is followed by two revolute joints acting in one plane and at right angles to the column – i.e., articulated structure similar to the first 3 d.o.f. of the IRp-6 robot, however without the parallelogram mechanism ($\theta_1, \theta_2, \theta_3'$ are used)
- The column is treated as a cylinder with a uniformly distributed mass
- The other two links are treated as rectangular cross-section beams with uniformly distributed masses
- Friction is neglected
- Dynamics of drives is neglected

Example: kinematic parameters



i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	$-\frac{\pi}{2}$	0	θ_2
3	a_2	0	0	θ_3'

$$ {}^{0}_{1}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 & 0 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 & 0 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{349} $$

$$ {}^{1}_{2}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{s}\theta_2 & -\mathrm{c}\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{s}_2 & -\mathrm{c}_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{350} $$

$$ {}^{2}_{3}\mathcal{T} = \begin{bmatrix} \mathrm{c}\theta_3' & -\mathrm{s}\theta_3' & 0 & a_2 \\ \mathrm{s}\theta_3' & \mathrm{c}\theta_3' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 & a_2 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{351} $$

$$ {}^{0}_{1}\mathcal{R} = \begin{bmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}_{1}\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{352} $$

$$ {}^{1}_{2}\mathcal{R} = \begin{bmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 \\ 0 & 0 & 1 \\ -\mathrm{s}_2 & -\mathrm{c}_2 & 0 \end{bmatrix} \qquad {}^{1}_{2}\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{353} $$

$$ {}^{2}_{3}\mathcal{R} = \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{2}_{3}\mathcal{P} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \tag{354} $$

$$ {}^{3}_{4}\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{The tool (frame 4) is affixed to the end of link 3,} \\ \text{having the same orientation as frame 3.} \\ \text{The tool is weightless.} \end{array} \tag{355} $$

$$ {}^{1}_{0}\mathcal{R} = {}^{0}_{1}\mathcal{R}^{-1} = {}^{0}_{1}\mathcal{R}^{T} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{356} $$

$$ {}^{2}_{1}\mathcal{R} = {}^{1}_{2}\mathcal{R}^{-1} = {}^{1}_{2}\mathcal{R}^{T} = \begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} \tag{357} $$

$$ {}^{3}_{2}\mathcal{R} = {}^{2}_{3}\mathcal{R}^{-1} = {}^{2}_{3}\mathcal{R}^{T} = \begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{358} $$

$$ {}^{i}_{C_i}\mathcal{P}_x = \frac{\int x \mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_{V_i} \varrho x \, \mathrm{d}x\mathrm{d}y\mathrm{d}z}{\int_{V_i} \varrho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z} = \frac{\int_{V_i} \varrho x \, \mathrm{d}x\mathrm{d}y\mathrm{d}z}{M_i} \tag{359} $$

Taking into account the symmetry of the links and uniform distribution of their mass we get:

$$ {}^{1}_{C_1}\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ -\frac{l_1}{2} \end{bmatrix} \tag{360} $$

$$ {}^{2}_{C_2}\mathcal{P} = \begin{bmatrix} \frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \tag{361} $$

$$ {}^{3}_{C_3}\mathcal{P} = \begin{bmatrix} \frac{l_3}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \tag{362} $$

Position of the previous link's C.O.M. wrt next frame

$$
{}^{1}_{2}\mathcal{P} - {}^{1}_{C_1}\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -\frac{l_1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{l_1}{2} \end{bmatrix} \tag{363}
$$

$$
{}^{2}_{3}\mathcal{P} - {}^{2}_{C_2}\mathcal{P} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \tag{364}
$$

$$
{}^{3}_{4}\mathcal{P} - {}^{3}_{C_3}\mathcal{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \tag{365}
$$

All three joints are revolute thus we use (316). $n = 3$.

In general, for $i = 0 \to 2$ we use:

$$^{i+1}({}^{0}_{i+1}\omega) = {}^{i+1}_{i}\mathcal{R}\,{}^{i}({}^{0}_{i}\omega) + {}^{i+1}_{i+1}z\,\dot{\theta}_{i+1} \tag{366}$$

$$\text{For } i = 0: \quad {}^{1}({}^{0}_{1}\omega) = {}^{1}_{0}\mathcal{R}\,{}^{0}({}^{0}_{0}\omega) + {}^{1}_{1}z\,\dot{\theta}_{1} \tag{367}$$

$$\text{For } i = 1: \quad {}^{2}({}^{0}_{2}\omega) = {}^{2}_{1}\mathcal{R}\,{}^{1}({}^{0}_{1}\omega) + {}^{2}_{2}z\,\dot{\theta}_{2} \tag{368}$$

$$\text{For } i = 2: \quad {}^{3}({}^{0}_{3}\omega) = {}^{3}_{2}\mathcal{R}\,{}^{2}({}^{0}_{2}\omega) + {}^{3}_{3}z\,\dot{\theta}'_{3} \tag{369}$$

$${}^{0}_{0}\omega = \mathcal{O} \tag{370}$$

$i = 0$:

$$ {}^{1}({}^{0}_{1}\omega) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \tag{371} $$

$i = 1$

$$ {}^{2}({}^{0}_{2}\omega) = \begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_2 = \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{372} $$

$i = 2$

$$ {}^{3}({}^{0}_{3}\omega) = \begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}'_3 = \begin{bmatrix} -s_{2+3'}\dot{\theta}_1 \\ -c_{2+3'}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}'_3 \end{bmatrix} \tag{373} $$

All three joints are revolute thus we use (318). $n = 3$.

In general, for $i = 0 \rightarrow 2$ we use:

$$ {}^{i+1}({}^{0}_{i+1}\mathcal{E}) = {}^{i+1}_{i}\mathcal{R}\,{}^{i}({}^{0}_{i}\mathcal{E}) + {}^{i+1}_{i}\mathcal{R}^{i}({}^{0}_{i}\omega) \times {}^{i+1}_{i+1}z\,\dot{\theta}_{i+1} + {}^{i+1}_{i+1}z\,\ddot{\theta}_{i+1} \tag{374} $$

$i = 0$

$$ {}^{1}({}^{0}_{1}\mathcal{E}) = {}^{1}_{0}\mathcal{R}\,{}^{0}({}^{0}_{0}\mathcal{E}) + {}^{1}_{0}\mathcal{R}^{0}({}^{0}_{0}\omega) \times {}^{1}_{1}z\,\dot{\theta}_{1} + {}^{1}_{1}z\,\ddot{\theta}_{1} \tag{375} $$

$i = 1$

$$ {}^{2}({}^{0}_{2}\mathcal{E}) = {}^{2}_{1}\mathcal{R}\,{}^{1}({}^{0}_{1}\mathcal{E}) + {}^{2}_{1}\mathcal{R}^{1}({}^{0}_{1}\omega) \times {}^{2}_{2}z\,\dot{\theta}_{2} + {}^{2}_{2}z\,\ddot{\theta}_{2} \tag{376} $$

$i = 2$

$$ {}^{3}({}^{0}_{3}\mathcal{E}) = {}^{3}_{2}\mathcal{R}\,{}^{2}({}^{0}_{2}\mathcal{E}) + {}^{3}_{2}\mathcal{R}^{2}({}^{0}_{2}\omega) \times {}^{3}_{3}z\,\dot{\theta}'_{3} + {}^{3}_{3}z\,\ddot{\theta}'_{3} \tag{377} $$

$$ {}^{0}_{0}\mathcal{E} = \mathcal{O} \tag{378} $$

$$i = 0 \;\Rightarrow\; {}^{1}({}^{0}_{1}\mathcal{E}) =$$

$$\begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{\theta}_1 \tag{379}$$

$$\Rightarrow \quad {}^{1}({}^{0}_{1}\mathcal{E}) = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \tag{380}$$

Example: Rotational acceleration cnt.2

$i = 1 \;\Rightarrow\; {}^{2}({}^{0}_{2}\mathcal{E}) =$

$$\begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} -s_2\ddot{\theta}_1 \\ -c_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} -s_2\ddot{\theta}_1 \\ -c_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -c_2\dot{\theta}_1 \\ 0 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} -s_2\ddot{\theta}_1 \\ -c_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -c_2\dot{\theta}_1\dot{\theta}_2 \\ s_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2 \\ -c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \tag{381}$$

$$i = 2 \;\Rightarrow\; {}^{3}({}^{0}_{3}\mathcal{E}) =$$

$$\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2 \\ -c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} +$$
$$\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}'_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}'_3 \end{bmatrix} = \tag{382}$$

$$\begin{bmatrix} c_{3'}\Big(-s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2\Big) + s_{3'}\Big(-c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2\Big) \\ -s_{3'}\Big(-s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2\Big) + c_{3'}\Big(-c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2\Big) \\ \ddot{\theta}_2 \end{bmatrix} +$$
$$\begin{bmatrix} c_{3'}\Big(-s_2\dot{\theta}_1\Big) + s_{3'}\Big(-c_2\dot{\theta}_1\Big) \\ -s_{3'}\Big(-s_2\dot{\theta}_1\Big) + c_{3'}\Big(-c_2\dot{\theta}_1\Big) \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}'_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}'_3 \end{bmatrix} =$$

$$\Rightarrow\ {}^{3}({}^{0}_{3}\mathcal{E}) =$$

$$\begin{bmatrix} -c_{3'}s_2\ddot{\theta}_1 - c_{3'}c_2\dot{\theta}_1\dot{\theta}_2 - s_{3'}c_2\ddot{\theta}_1 + s_{3'}s_2\dot{\theta}_1\dot{\theta}_2 \\ s_{3'}s_2\ddot{\theta}_1 + s_{3'}c_2\dot{\theta}_1\dot{\theta}_2 - c_{3'}c_2\ddot{\theta}_1 + c_{3'}s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -c_{3'}s_2\dot{\theta}_1 - s_{3'}c_2\dot{\theta}_1 \\ s_{3'}s_2\dot{\theta}_1 - c_{3'}c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}'_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}'_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\Big(c_{3'}s_2 + s_{3'}c_2\Big)\ddot{\theta}_1 - \Big(c_{3'}c_2 - s_{3'}s_2\Big)\dot{\theta}_1\dot{\theta}_2 \\ -\Big(-s_{3'}s_2 + c_{3'}c_2\Big)\ddot{\theta}_1 + \Big(s_{3'}c_2 + c_{3'}s_2\Big)\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\Big(c_{3'}s_2 + s_{3'}c_2\Big)\dot{\theta}_1 \\ -\Big(-s_{3'}s_2 + c_{3'}c_2\Big)\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}'_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}'_3 \end{bmatrix} = \tag{383}$$

$$\Rightarrow {}^{3}({}^{0}_{3}\mathcal{E}) =$$

$$\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} -\dot\theta_1 s_{2+3'} \\ -\dot\theta_1 c_{2+3'} \\ \dot\theta_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot\theta'_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot\theta'_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} \\ \ddot\theta_2 + \ddot\theta'_3 \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta_2 & -\dot\theta_1 c_{2+3'} \\ \dot\theta_2 & 0 & \dot\theta_1 s_{2+3'} \\ \dot\theta_1 c_{2+3'} & -\dot\theta_1 s_{2+3'} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta'_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} \\ \ddot\theta_2 + \ddot\theta'_3 \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta_2 & -\dot\theta_1 c_{2+3'} \\ \dot\theta_2 & 0 & \dot\theta_1 s_{2+3'} \\ \dot\theta_1 c_{2+3'} & -\dot\theta_1 s_{2+3'} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta'_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} \\ \ddot\theta_2 + \ddot\theta'_3 \end{bmatrix} + \begin{bmatrix} -\dot\theta'_3\dot\theta_1 c_{2+3'} \\ \dot\theta'_3\dot\theta_1 s_{2+3'} \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} - \dot\theta'_3\dot\theta_1 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} + \dot\theta'_3\dot\theta_1 s_{2+3'} \\ \ddot\theta_2 + \ddot\theta'_3 \end{bmatrix} \tag{384}$$

All three joints are revolute thus we use (320). $n = 3$.

In general, for $i = 0 \rightarrow 2$ we use:

$$^{i+1}\left(^{0}_{i+1}\mathcal{A}\right) = {}^{i+1}_{i}\mathcal{R}\left[{}^{i}\left(^{0}_{i}\mathcal{A}\right) + {}^{i}\left(^{0}_{i}\mathcal{E}\right) \times {}^{i}_{i+1}\mathcal{P} + {}^{i}\left(^{0}_{i}\omega\right) \times \left({}^{i}\left(^{0}_{i}\omega\right) \times {}^{i}_{i+1}\mathcal{P}\right)\right] \quad (385)$$

$i = 0$

$$^{1}\left(^{0}_{1}\mathcal{A}\right) = {}^{1}_{0}\mathcal{R}\left[{}^{0}\left(^{0}_{0}\mathcal{A}\right) + {}^{0}\left(^{0}_{0}\mathcal{E}\right) \times {}^{0}_{1}\mathcal{P} + {}^{0}\left(^{0}_{0}\omega\right) \times \left({}^{0}\left(^{0}_{0}\omega\right) \times {}^{0}_{1}\mathcal{P}\right)\right] \quad (386)$$

$i = 1$

$$^{2}\left(^{0}_{2}\mathcal{A}\right) = {}^{2}_{1}\mathcal{R}\left[{}^{1}\left(^{0}_{1}\mathcal{A}\right) + {}^{1}\left(^{0}_{1}\mathcal{E}\right) \times {}^{1}_{2}\mathcal{P} + {}^{1}\left(^{0}_{1}\omega\right) \times \left({}^{1}\left(^{0}_{1}\omega\right) \times {}^{1}_{2}\mathcal{P}\right)\right] \quad (387)$$

$i = 2$

$$^{3}\left(^{0}_{3}\mathcal{A}\right) = {}^{3}_{2}\mathcal{R}\left[{}^{2}\left(^{0}_{2}\mathcal{A}\right) + {}^{2}\left(^{0}_{2}\mathcal{E}\right) \times {}^{2}_{3}\mathcal{P} + {}^{2}\left(^{0}_{2}\omega\right) \times \left({}^{2}\left(^{0}_{2}\omega\right) \times {}^{2}_{3}\mathcal{P}\right)\right] \quad (388)$$

$$^{0}_{0}\mathcal{A} = \mathcal{O} \quad (389)$$

$$i = 0 \quad \Rightarrow \quad {}^{1}\left({}^{0}_{1}\mathcal{A}\right) =$$

$$\begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \right] \tag{390}$$

$$\Rightarrow \quad {}^{1}\left({}^{0}_{1}\mathcal{A}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{391}$$

$$i = 1 \quad \Rightarrow \quad {}^{2}({}^{0}_{2}\mathcal{A}) =$$

$$\begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \right] \tag{392}$$

$$\Rightarrow \quad {}^{2}({}^{0}_{2}\mathcal{A}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{393}$$

$$i = 2 \quad \Rightarrow \quad {}^{3}({}^{0}_{3}\mathcal{A}) =$$

$$\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2 \\ -c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} +$$

$$\begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \left(\begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \right) \Bigg] =$$

$$\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg[\begin{bmatrix} 0 & -\ddot{\theta}_2 & -c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 & 0 & s_2\ddot{\theta}_1 + c_2\dot{\theta}_1\dot{\theta}_2 \\ c_2\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2 & -s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \right) \Bigg] \tag{394}$$

$$
{}^{3}\left({}^{0}_{3}\mathcal{A}\right) =
$$

$$
\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg[\begin{bmatrix} 0 & -\ddot{\theta}_2 & -c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 & 0 & s_2\ddot{\theta}_1 + c_2\dot{\theta}_1\dot{\theta}_2 \\ c_2\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2 & -s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \right) \Bigg] =
$$

$$
\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg[\begin{bmatrix} 0 \\ a_2\ddot{\theta}_2 \\ a_2\left(c_2\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ a_2\dot{\theta}_2 \\ a_2 c_2\dot{\theta}_1 \end{bmatrix} \Bigg] = \tag{395}
$$

$$
{}^{3}\left({}_{3}^{0}\mathcal{A}\right) =
$$

$$
\begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 \\ a_2\ddot{\theta}_2 \\ a_2\left(c_2\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) \end{bmatrix} + \begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2\dot{\theta}_1^2 c_2^2 \\ a_2\dot{\theta}_1^2 s_2 c_2 \\ -a_2\dot{\theta}_2\dot{\theta}_1 s_2 \end{bmatrix} \right] =
$$

$$
a_2 \begin{bmatrix} c_{3'} & s_{3'} & 0 \\ -s_{3'} & c_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{\theta}_2^2 - \dot{\theta}_1^2 c_2^2 \\ \ddot{\theta}_2 + \dot{\theta}_1^2 s_2 c_2 \\ c_2\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} =
$$

$$
a_2 \begin{bmatrix} c_{3'}\left(-\dot{\theta}_2^2 - \dot{\theta}_1^2 c_2^2\right) + s_{3'}\left(\ddot{\theta}_2 + \dot{\theta}_1^2 s_2 c_2\right) \\ s_{3'}\left(\dot{\theta}_2^2 + \dot{\theta}_1^2 c_2^2\right) + c_{3'}\left(\ddot{\theta}_2 + \dot{\theta}_1^2 s_2 c_2\right) \\ c_2\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} \tag{396}
$$

In general, for $i = 0 \rightarrow 2$ we use (322):

$$
\begin{aligned}
&{}^{i+1}({}^{0}_{C_{i+1}}\mathcal{A}) = \\
&{}^{i+1}({}^{0}_{i+1}\mathcal{A}) + {}^{i+1}({}^{0}_{i+1}\mathcal{E}) \times {}^{i+1}_{C_{i+1}}\mathcal{P} + {}^{i+1}({}^{0}_{i+1}\omega) \times \left({}^{i+1}({}^{0}_{i+1}\omega) \times {}^{i+1}_{C_{i+1}}\mathcal{P} \right)
\end{aligned} \tag{397}
$$

$i = 0$

$$
{}^{1}({}^{0}_{C_1}\mathcal{A}) = {}^{1}({}^{0}_{1}\mathcal{A}) + {}^{1}({}^{0}_{1}\mathcal{E}) \times {}^{1}_{C_1}\mathcal{P} + {}^{1}({}^{0}_{1}\omega) \times \left({}^{1}({}^{0}_{1}\omega) \times {}^{1}_{C_1}\mathcal{P} \right) \tag{398}
$$

$i = 1$

$$
{}^{2}({}^{0}_{C_2}\mathcal{A}) = {}^{2}({}^{0}_{2}\mathcal{A}) + {}^{2}({}^{0}_{2}\mathcal{E}) \times {}^{2}_{C_2}\mathcal{P} + {}^{2}({}^{0}_{2}\omega) \times \left({}^{2}({}^{0}_{2}\omega) \times {}^{2}_{C_2}\mathcal{P} \right) \tag{399}
$$

$i = 2$

$$
{}^{3}({}^{0}_{C_3}\mathcal{A}) = {}^{3}({}^{0}_{3}\mathcal{A}) + {}^{3}({}^{0}_{3}\mathcal{E}) \times {}^{3}_{C_3}\mathcal{P} + {}^{3}({}^{0}_{3}\omega) \times \left({}^{3}({}^{0}_{3}\omega) \times {}^{3}_{C_3}\mathcal{P} \right) \tag{400}
$$

$$ {}^{1}({}^{0}_{C_1}\mathcal{A}) = $$

$$ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -\frac{l_1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -\frac{l_1}{2} \end{bmatrix} \right) \tag{401} $$

$$ {}^{1}({}^{0}_{C_1}\mathcal{A}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{402} $$

$$
{}^{2}({}^{0}_{C_2}\mathcal{A}) =
$$

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\mathrm{s}_2\ddot{\theta}_1 - \mathrm{c}_2\dot{\theta}_1\dot{\theta}_2 \\ -\mathrm{c}_2\ddot{\theta}_1 + \mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \left(\begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} 0 & -\ddot{\theta}_2 & -\mathrm{c}_2\ddot{\theta}_1 + \mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 & 0 & \mathrm{s}_2\ddot{\theta}_1 + \mathrm{c}_2\dot{\theta}_1\dot{\theta}_2 \\ \mathrm{c}_2\ddot{\theta}_1 - \mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 & -\mathrm{s}_2\ddot{\theta}_1 - \mathrm{c}_2\dot{\theta}_1\dot{\theta}_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \left(\begin{bmatrix} 0 & -\dot{\theta}_2 & -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & \mathrm{s}_2\dot{\theta}_1 \\ \mathrm{c}_2\dot{\theta}_1 & -\mathrm{s}_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \right) \tag{403}
$$

$$
{}^{2}({}^{0}_{C_2}\mathcal{A}) =
$$

$$
\begin{bmatrix} 0 \\ \frac{a_2}{2}\ddot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - \frac{a_2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ \frac{a_2}{2}\dot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\dot{\theta}_1 \end{bmatrix} =
$$

$$
\begin{bmatrix} 0 \\ \frac{a_2}{2}\ddot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - \frac{a_2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta}_2 & -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & \mathrm{s}_2\dot{\theta}_1 \\ \mathrm{c}_2\dot{\theta}_1 & -\mathrm{s}_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{a_2}{2}\dot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\dot{\theta}_1 \end{bmatrix} =
$$

$$
\begin{bmatrix} 0 \\ \frac{a_2}{2}\ddot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - \frac{a_2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 \\ \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 \\ -\frac{a_2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} \tag{404}
$$

$$
{}^{3}\left({}^{0}_{C_3}\mathcal{A}\right) =
$$

$$
\begin{bmatrix} a_2 c_{3'}\left(-\dot\theta_2^2 - \dot\theta_1^2 c_2^2\right) + a_2 s_{3'}\left(\ddot\theta_2 + \dot\theta_1^2 s_2 c_2\right) \\ a_2 s_{3'}\left(\dot\theta_2^2 + \dot\theta_1^2 c_2^2\right) + a_2 c_{3'}\left(\ddot\theta_2 + \dot\theta_1^2 s_2 c_2\right) \\ a_2 c_2 \ddot\theta_1 - 2 a_2 s_2 \dot\theta_1 \dot\theta_2 \end{bmatrix} +
$$

$$
\begin{bmatrix} -\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} - \dot\theta_3'\dot\theta_1 c_{2+3'} \\ -\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} + \dot\theta_3'\dot\theta_1 s_{2+3'} \\ \ddot\theta_2 + \ddot\theta_3' \end{bmatrix} \times \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} + \tag{405}
$$

$$
\begin{bmatrix} -s_{2+3'}\dot\theta_1 \\ -c_{2+3'}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3' \end{bmatrix} \times \left(\begin{bmatrix} -s_{2+3'}\dot\theta_1 \\ -c_{2+3'}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3' \end{bmatrix} \times \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \right)
$$

$$
{}^{3}({}^{0}_{C_3}\mathcal{A}) = \begin{bmatrix} a_2 c_{3'}\left(-\dot{\theta}_2^2 - \dot{\theta}_1^2 c_2^2\right) + a_2 s_{3'}\left(\ddot{\theta}_2 + \dot{\theta}_1^2 s_2 c_2\right) \\ a_2 s_{3'}\left(\dot{\theta}_2^2 + \dot{\theta}_1^2 c_2^2\right) + a_2 c_{3'}\left(\ddot{\theta}_2 + \dot{\theta}_1^2 s_2 c_2\right) \\ a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & -\ddot{\theta}_2 - \ddot{\theta}'_3 & -\ddot{\theta}_1 c_{2+3'} + \dot{\theta}_1\dot{\theta}_2 s_{2+3'} + \dot{\theta}'_3\dot{\theta}_1 s_{2+3'} \\ \ddot{\theta}_2 + \ddot{\theta}'_3 & 0 & \ddot{\theta}_1 s_{2+3'} + \dot{\theta}_1\dot{\theta}_2 c_{2+3'} + \dot{\theta}'_3\dot{\theta}_1 c_{2+3'} \\ \ddot{\theta}_1 c_{2+3'} - \dot{\theta}_1\dot{\theta}_2 s_{2+3'} - \dot{\theta}'_3\dot{\theta}_1 s_{2+3'} & -\ddot{\theta}_1 s_{2+3'} - \dot{\theta}_1\dot{\theta}_2 c_{2+3'} - \dot{\theta}'_3\dot{\theta}_1 c_{2+3'} & 0 \end{bmatrix} \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} -s_{2+3'}\dot{\theta}_1 \\ -c_{2+3'}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}'_3 \end{bmatrix} \times \left(\begin{bmatrix} 0 & -\dot{\theta}_2 - \dot{\theta}'_3 & -c_{2+3'}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}'_3 & 0 & s_{2+3'}\dot{\theta}_1 \\ c_{2+3'}\dot{\theta}_1 & -s_{2+3'}\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \right) \tag{406}
$$

$$
{}^{3}({}^{0}_{C_3}\mathcal{A}) =
\begin{bmatrix}
-a_2 c_{3'}\dot{\theta}_2^2 - a_2 c_{3'}\dot{\theta}_1^2 c_2^2 + a_2 s_{3'}\ddot{\theta}_2 + a_2 s_{3'}\dot{\theta}_1^2 s_2 c_2 \\
a_2 s_{3'}\dot{\theta}_2^2 + a_2 s_{3'}\dot{\theta}_1^2 c_2^2 + a_2 c_{3'}\ddot{\theta}_2 + a_2 c_{3'}\dot{\theta}_1^2 s_2 c_2 \\
a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \\
\frac{a_3}{2}\left(\ddot{\theta}_1 c_{2+3'} - \dot{\theta}_1\dot{\theta}_2 s_{2+3'} - \dot{\theta}_3'\dot{\theta}_1 s_{2+3'}\right)
\end{bmatrix} +
\begin{bmatrix}
-s_{2+3'}\dot{\theta}_1 \\
-c_{2+3'}\dot{\theta}_1 \\
\dot{\theta}_2 + \dot{\theta}_3'
\end{bmatrix} \times
\begin{bmatrix}
0 \\
\frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \\
\frac{a_3}{2} c_{2+3'}\dot{\theta}_1
\end{bmatrix}
\tag{407}
$$

$$
{}^{3}\left({}^{0}_{C_3}\mathcal{A}\right) =
\begin{bmatrix}
-a_2 \mathrm{c}_{3'}\dot{\theta}_2^2 - a_2 \mathrm{c}_{3'}\dot{\theta}_1^2 \mathrm{c}_2^2 + a_2 \mathrm{s}_{3'}\ddot{\theta}_2 + a_2 \mathrm{s}_{3'}\dot{\theta}_1^2 \mathrm{s}_2 \mathrm{c}_2 \\
a_2 \mathrm{s}_{3'}\dot{\theta}_2^2 + a_2 \mathrm{s}_{3'}\dot{\theta}_1^2 \mathrm{c}_2^2 + a_2 \mathrm{c}_{3'}\ddot{\theta}_2 + a_2 \mathrm{c}_{3'}\dot{\theta}_1^2 \mathrm{s}_2 \mathrm{c}_2 \\
a_2 \mathrm{c}_2 \ddot{\theta}_1 - 2 a_2 \mathrm{s}_2 \dot{\theta}_1 \dot{\theta}_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \\
\frac{a_3}{2}\left(\ddot{\theta}_1 \mathrm{c}_{2+3'} - \dot{\theta}_1 \dot{\theta}_2 \mathrm{s}_{2+3'} - \dot{\theta}_3' \dot{\theta}_1 \mathrm{s}_{2+3'}\right)
\end{bmatrix} +
\begin{bmatrix}
0 & -\dot{\theta}_2 - \dot{\theta}_3' & -\mathrm{c}_{2+3'}\dot{\theta}_1 \\
\dot{\theta}_2 + \dot{\theta}_3' & 0 & \mathrm{s}_{2+3'}\dot{\theta}_1 \\
\mathrm{c}_{2+3'}\dot{\theta}_1 & -\mathrm{s}_{2+3'}\dot{\theta}_1 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \\
\frac{a_3}{2}\mathrm{c}_{2+3'}\dot{\theta}_1
\end{bmatrix}
\tag{408}
$$

$$
{}^{3}\left({}^{0}_{C_3}\mathcal{A}\right) =
\begin{bmatrix} -a_2 \mathrm{c}_{3'}\dot{\theta}_2^2 - a_2 \mathrm{c}_2 \mathrm{c}_{2+3'}\dot{\theta}_1^2 + a_2 \mathrm{s}_{3'}\ddot{\theta}_2 \\ a_2 \mathrm{s}_{3'}\dot{\theta}_2^2 + a_2 \mathrm{c}_2 \mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2 \mathrm{c}_{3'}\ddot{\theta}_2 \\ a_2 \mathrm{c}_2 \ddot{\theta}_1 - 2 a_2 \mathrm{s}_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix} +
\begin{bmatrix} 0 \\ \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \\ \frac{a_3}{2}\left(\mathrm{c}_{2+3'}\ddot{\theta}_1 - \mathrm{s}_{2+3'}\dot{\theta}_1\dot{\theta}_2 - \mathrm{s}_{2+3'}\dot{\theta}_1\dot{\theta}_3'\right) \end{bmatrix} +
\begin{bmatrix} -\frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 \\ \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 \\ -\frac{a_3}{2}\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix}
\tag{409}
$$

$$
{}^{3}({}^{0}_{C_3}\mathcal{A}) =
$$

$$
\begin{bmatrix}
-a_2 c_{3'}\dot{\theta}_2^2 - a_2 c_2 c_{2+3'}\dot{\theta}_1^2 + a_2 s_{3'}\ddot{\theta}_2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2} c_{2+3'}^2 \dot{\theta}_1^2 \\
a_2 s_{3'}\dot{\theta}_2^2 + a_2 c_2 s_{2+3'}\dot{\theta}_1^2 + a_2 c_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2} s_{2+3'} c_{2+3'}\dot{\theta}_1^2 \\
a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'}\ddot{\theta}_1 - a_3 s_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)
\end{bmatrix}
\tag{410}
$$

In general, for $i = 0 \to 2$ we use (323):

$$ {}^{i+1}_{C_{i+1}}F = M_{i+1}\, {}^{i+1}\left({}^{0}_{C_{i+1}}\mathcal{A}\right) \tag{411}$$

$$ i = 0 \quad \Rightarrow \quad {}^{1}_{C_1}F = M_1\, {}^{1}\left({}^{0}_{C_1}\mathcal{A}\right) \tag{412}$$

$$ i = 1 \quad \Rightarrow \quad {}^{2}_{C_2}F = M_2\, {}^{2}\left({}^{0}_{C_2}\mathcal{A}\right) \tag{413}$$

$$ i = 2 \quad \Rightarrow \quad {}^{3}_{C_3}F = M_3\, {}^{3}\left({}^{0}_{C_3}\mathcal{A}\right) \tag{414}$$

Example: Inertial forces cnt.

$$i = 0 \quad \Rightarrow \quad {}^{1}_{C_1}F = M_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{415}$$

$$i = 1 \quad \Rightarrow \quad {}^{2}_{C_2}F = M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} \tag{416}$$

$$i = 2 \quad \Rightarrow \quad {}^{3}_{C_3}F =$$

$$M_3 \begin{bmatrix} -a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{s}_{3'}\ddot{\theta}_2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}'_3\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}'_3\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}'_3\right) \end{bmatrix} \tag{417}$$

In general, for $i = 0 \rightarrow 2$ we use (324):

$$ {}^{i+1}_{C_{i+1}}N = {}^{i+1}({}^{0}_{C_{i+1}}\omega) \times \left({}^{C_{i+1}}I_{i+1}\, {}^{i+1}({}^{0}_{C_{i+1}}\omega) \right) + {}^{C_{i+1}}I_{i+1}\, {}^{i+1}({}^{0}_{C_{i+1}}\mathcal{E}) \tag{418} $$

$$ i = 0 \quad \Rightarrow \quad {}^{1}_{C_1}N = {}^{1}({}^{0}_{C_1}\omega) \times \left({}^{C_1}I_1\, {}^{1}({}^{0}_{C_1}\omega) \right) + {}^{C_1}I_1\, {}^{1}({}^{0}_{C_1}\mathcal{E}) \tag{419} $$

$$ i = 1 \quad \Rightarrow \quad {}^{2}_{C_2}N = {}^{2}({}^{0}_{C_2}\omega) \times \left({}^{C_2}I_2\, {}^{2}({}^{0}_{C_2}\omega) \right) + {}^{C_2}I_2\, {}^{2}({}^{0}_{C_2}\mathcal{E}) \tag{420} $$

$$ i = 2 \quad \Rightarrow \quad {}^{3}_{C_3}N = {}^{3}({}^{0}_{C_3}\omega) \times \left({}^{C_3}I_3\, {}^{3}({}^{0}_{C_3}\omega) \right) + {}^{C_3}I_3\, {}^{3}({}^{0}_{C_3}\mathcal{E}) \tag{421} $$

$$ {}^{1}({}^{0}_{C_1}\omega) = {}^{1}({}^{0}_{1}\omega), \quad {}^{2}({}^{0}_{C_2}\omega) = {}^{2}({}^{0}_{2}\omega), \quad {}^{3}({}^{0}_{C_3}\omega) = {}^{3}({}^{0}_{3}\omega) \tag{422} $$

$$ {}^{1}({}^{0}_{C_1}\mathcal{E}) = {}^{1}({}^{0}_{1}\mathcal{E}), \quad {}^{2}({}^{0}_{C_2}\mathcal{E}) = {}^{2}({}^{0}_{2}\mathcal{E}), \quad {}^{3}({}^{0}_{C_3}\mathcal{E}) = {}^{3}({}^{0}_{3}\mathcal{E}) \tag{423} $$

$$
\begin{aligned}
&{}^{1}_{C_1}N = {}^{1}({}^{0}_{1}\omega) \times \left({}^{C_1}I_1\,{}^{1}({}^{0}_{1}\omega)\right) + {}^{C_1}I_1\,{}^{1}({}^{0}_{1}\mathcal{E}) = \\
&\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \left(\begin{bmatrix} \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 & 0 \\ 0 & \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 \\ 0 & 0 & \frac{1}{8}M_1 w_1^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \right) + \\
&\begin{bmatrix} \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 & 0 \\ 0 & \frac{1}{16}M_1 w_1^2 + \frac{1}{12}M_1 l_1^2 & 0 \\ 0 & 0 & \frac{1}{8}M_1 w_1^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} = \\
&\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8}M_1 w_1^2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8}M_1 w_1^2 \ddot{\theta}_1 \end{bmatrix} = \\
&\begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8}M_1 w_1^2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8}M_1 w_1^2 \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8}M_1 w_1^2 \ddot{\theta}_1 \end{bmatrix}
\end{aligned}
\tag{424}
$$

$$
{}^{2}_{C_2}N = {}^{2}({}^{0}_{2}\omega) \times \left({}^{C_2}I_2\, {}^{2}({}^{0}_{2}\omega) \right) + {}^{C_2}I_2\, {}^{2}({}^{0}_{2}\mathcal{E}) = \begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times
$$

$$
\left(\begin{bmatrix} \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right) & 0 & 0 \\ 0 & \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right) & 0 \\ 0 & 0 & \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right) \end{bmatrix} \begin{bmatrix} -\mathrm{s}_2\dot{\theta}_1 \\ -\mathrm{c}_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \right)
$$

$$
+ \begin{bmatrix} \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right) & 0 & 0 \\ 0 & \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right) & 0 \\ 0 & 0 & \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right) \end{bmatrix}
$$

$$
\begin{bmatrix} -\mathrm{s}_2\ddot{\theta}_1 - \mathrm{c}_2\dot{\theta}_1\dot{\theta}_2 \\ -\mathrm{c}_2\ddot{\theta}_1 + \mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \tag{425}
$$

$$
\begin{aligned}
{}^{2}_{C_2}N = {} & \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} -s_2\dot{\theta}_1 \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right) \\ -c_2\dot{\theta}_1 \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right) \\ \dot{\theta}_2 \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right) \end{bmatrix} \\
& + \begin{bmatrix} \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right)\left(-s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2\right) \\ \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right)\left(-c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2\right) \\ \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right)\ddot{\theta}_2 \end{bmatrix} = \\
& \begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 & -s_2\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right) \\ -c_2\dot{\theta}_1 \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right) \\ \dot{\theta}_2 \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right) \end{bmatrix} \\
& + \begin{bmatrix} \frac{M_2}{12}\left((w_2')^2 + (w_2'')^2\right)\left(-s_2\ddot{\theta}_1 - c_2\dot{\theta}_1\dot{\theta}_2\right) \\ \frac{M_2}{12}\left((a_2)^2 + (w_2'')^2\right)\left(-c_2\ddot{\theta}_1 + s_2\dot{\theta}_1\dot{\theta}_2\right) \\ \frac{M_2}{12}\left((w_2')^2 + (a_2)^2\right)\ddot{\theta}_2 \end{bmatrix}
\end{aligned}
\tag{426}
$$

$$
\begin{aligned}
{}^{2}_{C_2}N = & \begin{bmatrix} \dot\theta_1\dot\theta_2 \mathrm{c}_2 \frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right) - \dot\theta_1\dot\theta_2 \mathrm{c}_2 \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right) \\ -\dot\theta_1\dot\theta_2 \mathrm{s}_2 \frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right) + \dot\theta_1\dot\theta_2 \mathrm{s}_2 \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right) \\ -\dot\theta_1^2 \mathrm{s}_2\mathrm{c}_2 \frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right) + \dot\theta_1^2 \mathrm{s}_2\mathrm{c}_2 \frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right) \end{bmatrix} \\
+ & \begin{bmatrix} \frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right)\left(-\mathrm{s}_2\ddot\theta_1 - \mathrm{c}_2\dot\theta_1\dot\theta_2\right) \\ \frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right)\left(-\mathrm{c}_2\ddot\theta_1 + \mathrm{s}_2\dot\theta_1\dot\theta_2\right) \\ \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right)\ddot\theta_2 \end{bmatrix} = \\
& \begin{bmatrix} \dot\theta_1\dot\theta_2 \mathrm{c}_2 \frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right) - \dot\theta_1\dot\theta_2 \mathrm{c}_2 \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right) \\ -\dot\theta_1\dot\theta_2 \mathrm{s}_2 \frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right) + \dot\theta_1\dot\theta_2 \mathrm{s}_2 \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right) \\ -\dot\theta_1^2 \mathrm{s}_2\mathrm{c}_2 \frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right) + \dot\theta_1^2 \mathrm{s}_2\mathrm{c}_2 \frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right) \end{bmatrix} \\
+ & \begin{bmatrix} -\frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right)\mathrm{s}_2\ddot\theta_1 - \mathrm{c}_2\dot\theta_1\dot\theta_2\frac{M_2}{12}\left((w_2')^2+(w_2'')^2\right) \\ -\frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right)\mathrm{c}_2\ddot\theta_1 + \mathrm{s}_2\dot\theta_1\dot\theta_2\frac{M_2}{12}\left((a_2)^2+(w_2'')^2\right) \\ \frac{M_2}{12}\left((w_2')^2+(a_2)^2\right)\ddot\theta_2 \end{bmatrix}
\end{aligned} \tag{427}
$$

$$
{}^{2}_{C_2}N =
\begin{bmatrix}
\dot\theta_1\dot\theta_2 \mathrm{c}_2 \frac{M_2}{12}\Big((w_2'')^2 - (w_2')^2\Big) \\
\dot\theta_1\dot\theta_2 \mathrm{s}_2 \frac{M_2}{12}\Big((a_2)^2 - (w_2'')^2\Big) \\
\dot\theta_1^2 \mathrm{s}_2\mathrm{c}_2 \frac{M_2}{12}\Big((a_2)^2 - (w_2')^2\Big)
\end{bmatrix} +
\begin{bmatrix}
-\frac{M_2}{12}\Big((w_2')^2 + (w_2'')^2\Big)\mathrm{s}_2\ddot\theta_1 - \mathrm{c}_2\dot\theta_1\dot\theta_2\frac{M_2}{12}\Big((w_2')^2 + (w_2'')^2\Big) \\
-\frac{M_2}{12}\Big((a_2)^2 + (w_2'')^2\Big)\mathrm{c}_2\ddot\theta_1 + \mathrm{s}_2\dot\theta_1\dot\theta_2\frac{M_2}{12}\Big((a_2)^2 + (w_2'')^2\Big) \\
\frac{M_2}{12}\Big((w_2')^2 + (a_2)^2\Big)\ddot\theta_2
\end{bmatrix} =
\frac{M_2}{12}
\begin{bmatrix}
-2\dot\theta_1\dot\theta_2\mathrm{c}_2(w_2')^2 - \ddot\theta_1\mathrm{s}_2\Big((w_2')^2 + (w_2'')^2\Big) \\
2\dot\theta_1\dot\theta_2\mathrm{s}_2 a_2^2 - \ddot\theta_1\mathrm{c}_2\Big(a_2^2 + (w_2'')^2\Big) \\
\dot\theta_1^2\mathrm{s}_2\mathrm{c}_2\Big(a_2^2 - (w_2')^2\Big) + \ddot\theta_2\Big((w_2')^2 + a_2^2\Big)
\end{bmatrix}
\tag{428}
$$

$$
{}^{3}_{C_3}N = {}^{3}({}^{0}_{3}\omega) \times \left({}^{C_3}I_3\,{}^{3}({}^{0}_{3}\omega) \right) + {}^{C_3}I_3\,{}^{3}({}^{0}_{3}\mathcal{E}) = \begin{bmatrix} -\mathrm{s}_{2+3'}\dot{\theta}_1 \\ -\mathrm{c}_{2+3'}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}'_3 \end{bmatrix} \times
$$

$$
\left(\begin{bmatrix} \frac{M_3}{12}\left((w'_3)^2+(w''_3)^2\right) & 0 & 0 \\ 0 & \frac{M_3}{12}\left((a_3)^2+(w''_3)^2\right) & 0 \\ 0 & 0 & \frac{M_3}{12}\left((w'_3)^2+(a_3)^2\right) \end{bmatrix} \begin{bmatrix} -\mathrm{s}_{2+3'}\dot{\theta}_1 \\ -\mathrm{c}_{2+3'}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}'_3 \end{bmatrix} \right)
$$

$$
+ \begin{bmatrix} \frac{M_3}{12}\left((w'_3)^2 + (w''_3)^2\right) & 0 & 0 \\ 0 & \frac{M_3}{12}\left((a_3)^2 + (w''_3)^2\right) & 0 \\ 0 & 0 & \frac{M_3}{12}\left((w'_3)^2 + (a_3)^2\right) \end{bmatrix}
$$

$$
\begin{bmatrix} -\ddot{\theta}_1\mathrm{s}_{2+3'} - \dot{\theta}_1\dot{\theta}_2\mathrm{c}_{2+3'} - \dot{\theta}'_3\dot{\theta}_1\mathrm{c}_{2+3'} \\ -\ddot{\theta}_1\mathrm{c}_{2+3'} + \dot{\theta}_1\dot{\theta}_2\mathrm{s}_{2+3'} + \dot{\theta}'_3\dot{\theta}_1\mathrm{s}_{2+3'} \\ \ddot{\theta}_2 + \ddot{\theta}'_3 \end{bmatrix} = \tag{429}
$$

$$
{}^{3}_{C_3}N = \begin{bmatrix} -s_{2+3'}\dot\theta_1 \\ -c_{2+3'}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3' \end{bmatrix} \times \begin{bmatrix} -s_{2+3'}\dot\theta_1 \frac{M_3}{12}\left((w_3')^2+(w_3'')^2\right) \\ -c_{2+3'}\dot\theta_1 \frac{M_3}{12}\left((a_3)^2+(w_3'')^2\right) \\ \left(\dot\theta_2+\dot\theta_3'\right)\frac{M_3}{12}\left((w_3')^2+(a_3)^2\right) \end{bmatrix} +
$$

$$
\begin{bmatrix} \left(-\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} - \dot\theta_3'\dot\theta_1 c_{2+3'}\right)\frac{M_3}{12}\left((w_3')^2+(w_3'')^2\right) \\ \left(-\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} + \dot\theta_3'\dot\theta_1 s_{2+3'}\right)\frac{M_3}{12}\left((a_3)^2+(w_3'')^2\right) \\ \left(\ddot\theta_2+\ddot\theta_3'\right)\frac{M_3}{12}\left((w_3')^2+(a_3)^2\right) \end{bmatrix} =
$$

$$
\begin{bmatrix} 0 & -\left(\dot\theta_2+\dot\theta_3'\right) & -c_{2+3'}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3' & 0 & s_{2+3'}\dot\theta_1 \\ c_{2+3'}\dot\theta_1 & -s_{2+3'}\dot\theta_1 & 0 \end{bmatrix} \begin{bmatrix} -s_{2+3'}\dot\theta_1 \frac{M_3}{12}\left((w_3')^2+(w_3'')^2\right) \\ -c_{2+3'}\dot\theta_1 \frac{M_3}{12}\left((a_3)^2+(w_3'')^2\right) \\ \left(\dot\theta_2+\dot\theta_3'\right)\frac{M_3}{12}\left((w_3')^2+(a_3)^2\right) \end{bmatrix} \tag{430}
$$

$$
+ \begin{bmatrix} \left(-\ddot\theta_1 s_{2+3'} - \dot\theta_1\dot\theta_2 c_{2+3'} - \dot\theta_3'\dot\theta_1 c_{2+3'}\right)\frac{M_3}{12}\left((w_3')^2+(w_3'')^2\right) \\ \left(-\ddot\theta_1 c_{2+3'} + \dot\theta_1\dot\theta_2 s_{2+3'} + \dot\theta_3'\dot\theta_1 s_{2+3'}\right)\frac{M_3}{12}\left((a_3)^2+(w_3'')^2\right) \\ \left(\ddot\theta_2+\ddot\theta_3'\right)\frac{M_3}{12}\left((w_3')^2+(a_3)^2\right) \end{bmatrix}
$$

Example: Inertial torques cnt.8

$$
\begin{aligned}
{}^{3}_{C_3}N = {} & \begin{bmatrix} \Big(-\ddot{\theta}_1 \mathrm{s}_{2+3'} - \dot{\theta}_1\dot{\theta}_2 \mathrm{c}_{2+3'} - \dot{\theta}'_3\dot{\theta}_1 \mathrm{c}_{2+3'} \Big) \frac{M_3}{12}\Big((w'_3)^2 + (w''_3)^2 \Big) \\ \Big(-\ddot{\theta}_1 \mathrm{c}_{2+3'} + \dot{\theta}_1\dot{\theta}_2 \mathrm{s}_{2+3'} + \dot{\theta}'_3\dot{\theta}_1 \mathrm{s}_{2+3'} \Big) \frac{M_3}{12}\Big((a_3)^2 + (w''_3)^2 \Big) \\ \Big(\ddot{\theta}_2 + \ddot{\theta}'_3 \Big) \frac{M_3}{12}\Big((w'_3)^2 + (a_3)^2 \Big) \end{bmatrix} + \\
& \begin{bmatrix} \mathrm{c}_{2+3'}\dot{\theta}_1 \frac{M_3}{12}\Big((a_3)^2 + (w''_3)^2 \Big)\Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big) - \Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big) \frac{M_3}{12}\Big((w'_3)^2 + (a_3)^2 \Big) \mathrm{c}_{2+3'}\dot{\theta}_1 \\ -\mathrm{s}_{2+3'}\dot{\theta}_1 \frac{M_3}{12}\Big((w'_3)^2 + (w''_3)^2 \Big)\Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big) + \Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big) \frac{M_3}{12}\Big((w'_3)^2 + (a_3)^2 \Big) \mathrm{s}_{2+3'}\dot{\theta}_1 \\ -\mathrm{s}_{2+3'}\dot{\theta}_1 \frac{M_3}{12}\Big((w'_3)^2 + (w''_3)^2 \Big) \mathrm{c}_{2+3'}\dot{\theta}_1 + \mathrm{c}_{2+3'}\dot{\theta}_1 \frac{M_3}{12}\Big((a_3)^2 + (w''_3)^2 \Big) \mathrm{s}_{2+3'}\dot{\theta}_1 \end{bmatrix} \\
= {} & \begin{bmatrix} -\frac{M_3}{12}\Big(\ddot{\theta}_1 \mathrm{s}_{2+3'} + \dot{\theta}_1\dot{\theta}_2 \mathrm{c}_{2+3'} + \dot{\theta}_1\dot{\theta}'_3 \mathrm{c}_{2+3'} \Big)\Big((w'_3)^2 + (w''_3)^2 \Big) \\ \frac{M_3}{12}\Big(-\ddot{\theta}_1 \mathrm{c}_{2+3'} + \dot{\theta}_1\dot{\theta}_2 \mathrm{s}_{2+3'} + \dot{\theta}_1\dot{\theta}'_3 \mathrm{s}_{2+3'} \Big)\Big((a_3)^2 + (w''_3)^2 \Big) \\ \frac{M_3}{12}\Big(\ddot{\theta}_2 + \ddot{\theta}'_3 \Big)\Big((w'_3)^2 + (a_3)^2 \Big) \end{bmatrix} + \\
& \begin{bmatrix} \frac{M_3}{12}\mathrm{c}_{2+3'}\dot{\theta}_1 \Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big)\Big((w''_3)^2 - \Big((w'_3)^2 \Big) \\ \frac{M_3}{12}\mathrm{s}_{2+3'}\dot{\theta}_1 \Big(\dot{\theta}_2 + \dot{\theta}'_3 \Big)\Big((a_3)^2 - (w''_3)^2 \Big) \\ \frac{M_3}{12}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 \Big((a_3)^2 - (w'_3)^2 \Big) \end{bmatrix}
\end{aligned} \tag{431}
$$

$$
{}^{3}_{C_3}N = \begin{bmatrix} -\frac{M_3}{12}\left(\ddot{\theta}_1 \mathrm{s}_{2+3'} + \dot{\theta}_1 \mathrm{c}_{2+3'}(\dot{\theta}_2 + \dot{\theta}_3')\right)\left((w_3')^2 + (w_3'')^2\right) \\ \frac{M_3}{12}\left(-\ddot{\theta}_1 \mathrm{c}_{2+3'} + \dot{\theta}_1 \mathrm{s}_{2+3'}(\dot{\theta}_2 + \dot{\theta}_3')\right)\left((a_3)^2 + (w_3'')^2\right) \\ \frac{M_3}{12}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{bmatrix} +
$$

$$
\begin{bmatrix} \frac{M_3}{12}\mathrm{c}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\left((w_3'')^2 - \left((w_3')^2\right)\right. \\ \frac{M_3}{12}\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\left((a_3)^2 - (w_3'')^2\right) \\ \frac{M_3}{12}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) \end{bmatrix} = \tag{432}
$$

$$
\frac{M_3}{12}\begin{bmatrix} -\ddot{\theta}_1 \mathrm{s}_{2+3'}\left((w_3')^2 + (w_3'')^2\right) - 2\dot{\theta}_1 \mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 \mathrm{c}_{2+3'}\left((a_3)^2 + (w_3'')^2\right) + 2\dot{\theta}_1 \mathrm{s}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{bmatrix}
$$

$$ {}^{0}g = \begin{bmatrix} 0 \\ 0 \\ -g_g \end{bmatrix} \tag{433} $$

$$ {}^{1}g = {}^{0}_{1}\mathcal{R}^{T}\,{}^{0}g = \begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g_g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g_g \end{bmatrix} \tag{434} $$

$$ {}^{2}g = {}^{0}_{2}\mathcal{R}^{T}\,{}^{0}g = {}^{1}_{2}\mathcal{R}^{T}\,{}^{0}_{1}\mathcal{R}^{T}\,{}^{0}g = \begin{bmatrix} \mathrm{c}_2 & 0 & -\mathrm{s}_2 \\ -\mathrm{s}_2 & 0 & -\mathrm{c}_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g_g \end{bmatrix} = \begin{bmatrix} \mathrm{s}_2 g_g \\ \mathrm{c}_2 g_g \\ 0 \end{bmatrix} \tag{435} $$

$$ {}^{3}g = {}^{0}_{3}\mathcal{R}^{T}\,{}^{0}g = {}^{2}_{3}\mathcal{R}^{T}\,{}^{1}_{2}\mathcal{R}^{T}\,{}^{0}_{1}\mathcal{R}^{T}\,{}^{0}g = {}^{2}_{3}\mathcal{R}^{T}\,{}^{0}_{2}\mathcal{R}^{T}\,{}^{0}g = $$
$$ \begin{bmatrix} \mathrm{c}_{3'} & \mathrm{s}_{3'} & 0 \\ -\mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{s}_2 g_g \\ \mathrm{c}_2 g_g \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\mathrm{s}_2\mathrm{c}_{3'} + \mathrm{c}_2\mathrm{s}_{3'}\right) g_g \\ \left(-\,\mathrm{s}_2\mathrm{s}_{3'} + \mathrm{c}_2\mathrm{c}_{3'}\right) g_g \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{s}_{2+3'}\, g_g \\ \mathrm{c}_{2+3'}\, g_g \\ 0 \end{bmatrix} \tag{436} $$

In general, for $i = n \to 0$ we use (325):

$$ {}_{i}^{i}F = {}_{C_i}^{i}F + {}_{i+1}^{i}\mathcal{R}\,{}_{i+1}^{i+1}F - M_i\,{}^{i}g \tag{437} $$

$n = 3 \Rightarrow n + 1 = 4 : \quad {}_{4}^{4}F = \mathcal{O} \quad - \quad$ no contact with the environment

$$ i = 3 \quad \Rightarrow \quad {}_{3}^{3}F = {}_{C_3}^{3}F + {}_{4}^{3}\mathcal{R}\,{}_{4}^{4}F - M_3\,{}^{3}g \tag{438} $$

$$ i = 2 \quad \Rightarrow \quad {}_{2}^{2}F = {}_{C_2}^{2}F + {}_{3}^{2}\mathcal{R}\,{}_{3}^{3}F - M_2\,{}^{2}g \tag{439} $$

$$ i = 1 \quad \Rightarrow \quad {}_{1}^{1}F = {}_{C_1}^{1}F + {}_{2}^{1}\mathcal{R}\,{}_{2}^{2}F - M_1\,{}^{1}g \tag{440} $$

$$
{}^3_3F = {}^3_{C_3}F + {}^3_4\mathcal{R}\,{}^4_4F - M_3\,{}^3g =
$$

$$
M_3\begin{bmatrix} -a_2\mathrm{c}_{3'}\dot\theta_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot\theta_1^2 + a_2\mathrm{s}_{3'}\ddot\theta_2 - \frac{a_3}{2}\left(\dot\theta_2+\dot\theta_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot\theta_1^2 \\ a_2\mathrm{s}_{3'}\dot\theta_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot\theta_1^2 + a_2\mathrm{c}_{3'}\ddot\theta_2 + \frac{a_3}{2}\left(\ddot\theta_2+\ddot\theta_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot\theta_1^2 \\ a_2\mathrm{c}_2\ddot\theta_1 - 2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1 - a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right) \end{bmatrix}
$$

$$
+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - M_3\begin{bmatrix} \mathrm{s}_{2+3'}g_g \\ \mathrm{c}_{2+3'}g_g \\ 0 \end{bmatrix} =
$$

$$
M_3\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot\theta_2 - a_2\mathrm{c}_{3'}\dot\theta_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}\left(\dot\theta_2+\dot\theta_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot\theta_1^2 - \mathrm{s}_{2+3'}g_g \\ a_2\mathrm{s}_{3'}\dot\theta_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot\theta_1^2 + a_2\mathrm{c}_{3'}\ddot\theta_2 + \frac{a_3}{2}\left(\ddot\theta_2+\ddot\theta_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot\theta_1^2 - \mathrm{c}_{2+3'}g_g \\ a_2\mathrm{c}_2\ddot\theta_1 - 2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1 - a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right) \end{bmatrix} \tag{441}
$$

$$
{}^{2}_{2}F = {}^{2}_{C_2}F - M_2\,{}^{2}g + {}^{2}_{3}\mathcal{R}\,{}^{3}_{3}F =
$$

$$
M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} - M_2 \begin{bmatrix} \mathrm{s}_2 g_g \\ \mathrm{c}_2 g_g \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
M_3 \begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \tag{442}
$$

$$
{}^{2}_{2}F = M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} \mathrm{c}_{3'}\left(a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}g_g\right) - \\ \mathrm{s}_{3'}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g\right) \\ \\ \mathrm{s}_{3'}\left(a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}g_g\right) + \\ \mathrm{c}_{3'}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g\right) \\ \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \tag{443}
$$

$$
{}^2_2F = M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}^2\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{3'}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 \\ -\mathrm{c}_{3'}\mathrm{s}_{2+3'}g_g - a_2\mathrm{s}_{3'}^2\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{s}_{3'}\mathrm{s}_{2+3'}\dot{\theta}_1^2 - a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\ddot{\theta}_2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \\ -\frac{a_3}{2}\mathrm{s}_{3'}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \mathrm{s}_{3'}\mathrm{c}_{2+3'}g_g \\ \\ a_2\mathrm{s}_{3'}^2\ddot{\theta}_2 - a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{s}_{3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{s}_{3'}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 \\ -\mathrm{s}_{3'}\mathrm{s}_{2+3'}g_g + a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{c}_{3'}\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}^2\ddot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \\ +\frac{a_3}{2}\mathrm{c}_{3'}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{3'}\mathrm{c}_{2+3'}g_g \\ \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \tag{444}
$$

$$
{}^2_2F = M_2 \begin{bmatrix} -\frac{a_2}{2}\dot\theta_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot\theta_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot\theta_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot\theta_1 - a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} -a_2\dot\theta_2^2 - a_2\mathrm{c}_2^2\dot\theta_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\left(\dot\theta_2 + \dot\theta_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\ddot\theta_2 + \ddot\theta_3'\right) - \mathrm{s}_2 g_g \\ a_2\ddot\theta_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot\theta_2 + \dot\theta_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot\theta_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot\theta_2 + \ddot\theta_3'\right) - \mathrm{c}_2 g_g \\ a_2\mathrm{c}_2\ddot\theta_1 - 2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1 - a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right) \end{bmatrix} \tag{445}
$$

$$
{}^1_1F = {}^1_{C_1}F + {}^1_2\mathcal{R}\,{}^2_2F - M_1\,{}^1g =
$$

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 \\ 0 & 0 & 1 \\ -\mathrm{s}_2 & -\mathrm{c}_2 & 0 \end{bmatrix} \Bigg(M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2\mathrm{c}_2^2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{s}_2 g_g \\ a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \Bigg)
$$

$$
-M_1 \begin{bmatrix} 0 \\ 0 \\ -g_g \end{bmatrix} \tag{446}
$$

$$
{}^1_1F = M_1 \begin{bmatrix} 0 \\ 0 \\ g_g \end{bmatrix} +
$$

$$
M_2 \begin{bmatrix} \mathrm{c}_2\Big(-\frac{a_2}{2}\dot\theta_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot\theta_1^2 - \mathrm{s}_2 g_g\Big) - \mathrm{s}_2\Big(\frac{a_2}{2}\ddot\theta_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 - \mathrm{c}_2 g_g\Big) \\ \frac{a_2}{2}\mathrm{c}_2\ddot\theta_1 - a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 \\ -\mathrm{s}_2\Big(-\frac{a_2}{2}\dot\theta_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot\theta_1^2 - \mathrm{s}_2 g_g\Big) - \mathrm{c}_2\Big(\frac{a_2}{2}\ddot\theta_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 - \mathrm{c}_2 g_g\Big) \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} \mathrm{c}_2\Big(-a_2\dot\theta_2^2 - a_2\mathrm{c}_2^2\dot\theta_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\Big(\dot\theta_2+\dot\theta_3'\Big)^2 - \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\Big(\ddot\theta_2+\ddot\theta_3'\Big) - \mathrm{s}_2 g_g\Big) \\ -\mathrm{s}_2\Big(a_2\ddot\theta_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\Big(\dot\theta_2+\dot\theta_3'\Big)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot\theta_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\Big(\ddot\theta_2+\ddot\theta_3'\Big) - \mathrm{c}_2 g_g\Big) \\ \\ a_2\mathrm{c}_2\ddot\theta_1 - 2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1 - a_3\mathrm{s}_{2+3'}\dot\theta_1\Big(\dot\theta_2 + \dot\theta_3'\Big) \\ \\ \mathrm{s}_2\Big(\mathrm{s}_2 g_g + a_2\dot\theta_2^2 + a_2\mathrm{c}_2^2\dot\theta_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\Big(\dot\theta_2+\dot\theta_3'\Big)^2 + \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot\theta_1^2 + \frac{a_3}{2}\mathrm{s}_{3'}\Big(\ddot\theta_2 + \ddot\theta_3'\Big)\Big) + \\ \mathrm{c}_2\Big(\mathrm{c}_2 g_g - a_2\ddot\theta_2 - a_2\mathrm{s}_2\mathrm{c}_2\dot\theta_1^2 + \frac{a_3}{2}\mathrm{s}_{3'}\Big(\dot\theta_2+\dot\theta_3'\Big)^2 - \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\Big(\ddot\theta_2+\ddot\theta_3'\Big)\Big) \end{bmatrix} \tag{447}
$$

$$
{}^1_1F = M_1 \begin{bmatrix} 0 \\ 0 \\ g_g \end{bmatrix} + M_2 \begin{bmatrix} s_2 c_2 g_g - \frac{a_2}{2} c_2 \dot{\theta}_2^2 - \frac{a_2}{2} c_2^3 \dot{\theta}_1^2 - s_2 c_2 g_g - \frac{a_2}{2} s_2 \ddot{\theta}_2 - \frac{a_2}{2} s_2^2 c_2 \dot{\theta}_1^2 \\ \frac{a_2}{2} c_2 \ddot{\theta}_1 - a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{a_2}{2} s_2 \dot{\theta}_2^2 + \frac{a_2}{2} s_2 c_2^2 \dot{\theta}_1^2 + s_2^2 g_g - \frac{a_2}{2} c_2 \ddot{\theta}_2 - \frac{a_2}{2} s_2 c_2^2 \dot{\theta}_1^2 + c_2^2 g_g \end{bmatrix} +
$$

$$
M_3 \begin{bmatrix} -a_2 c_2 \dot{\theta}_2^2 - a_2 c_2^3 \dot{\theta}_1^2 - \frac{a_3}{2} c_2 c_{3'} \left(\dot{\theta}_2 + \dot{\theta}'_3\right)^2 - \frac{a_3}{2} c_2^2 c_{2+3'} \dot{\theta}_1^2 - \frac{a_3}{2} c_2 s_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}'_3\right) \\ -s_2 c_2 g_g - a_2 s_2 \ddot{\theta}_2 - a_2 s_2^2 c_2 \dot{\theta}_1^2 + \frac{a_3}{2} s_2 s_{3'} \left(\dot{\theta}_2 + \dot{\theta}'_3\right)^2 - \frac{a_3}{2} s_2^2 c_{2+3'} \dot{\theta}_1^2 \\ -\frac{a_3}{2} s_2 c_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}'_3\right) + s_2 c_2 g_g \\ \\ a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}'_3\right) \\ \\ s_2^2 g_g + a_2 s_2 \dot{\theta}_2^2 + a_2 s_2 c_2^2 \dot{\theta}_1^2 + \frac{a_3}{2} s_2 c_{3'} \left(\dot{\theta}_2 + \dot{\theta}'_3\right)^2 + \frac{a_3}{2} s_2 c_2 c_{2+3'} \dot{\theta}_1^2 \\ + \frac{a_3}{2} s_2 s_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}'_3\right) + c_2^2 g_g - a_2 c_2 \ddot{\theta}_2 - a_2 s_2 c_2^2 \dot{\theta}_1^2 + \frac{a_3}{2} c_2 s_{3'} \left(\dot{\theta}_2 + \dot{\theta}'_3\right)^2 \\ -\frac{a_3}{2} s_2 c_2 c_{2+3'} \dot{\theta}_1^2 - \frac{a_3}{2} c_2 c_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}'_3\right) \end{bmatrix} \tag{448}
$$

$$
{}^1_1F = M_1 \begin{bmatrix} 0 \\ 0 \\ g_g \end{bmatrix} + M_2 \begin{bmatrix} -\frac{a_2}{2}\mathrm{c}_2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{a_2}{2}\mathrm{s}_2\ddot{\theta}_2 \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \\ \frac{a_2}{2}\mathrm{s}_2\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_2 + g_g \end{bmatrix} +
$$

$$
M_3 \begin{bmatrix} -a_2\mathrm{c}_2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{a_3}{2}\mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{2+3'}\left(\ddot{\theta}_2 + \ddot{\theta}_{3'}\right) \\ -a_2\mathrm{s}_2\ddot{\theta}_2 \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \\ g_g + a_2\mathrm{s}_2\dot{\theta}_2^2 - a_2\mathrm{c}_2\ddot{\theta}_2 + \frac{a_3}{2}\mathrm{s}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) \end{bmatrix} \tag{449}
$$

In general, for $i = n \to 0$ we use (326):

$$ {}^{i}_{i}N = {}^{i}_{C_i}N + {}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}N + {}^{i}_{C_i}\mathcal{P} \times {}^{i}_{i}F + ({}^{i}_{i+1}\mathcal{P} - {}^{i}_{C_i}\mathcal{P}) \times ({}^{i}_{i+1}\mathcal{R}\,{}^{i+1}_{i+1}F) \quad (450) $$

$n = 3 \Rightarrow n + 1 = 4$: ${}^{4}_{4}N = \mathcal{O}$ – no contact with the environment

$$ i = 3 \;\; \Rightarrow \;\; {}^{3}_{3}N = {}^{3}_{C_3}N + {}^{3}_{4}\mathcal{R}\,{}^{4}_{4}N + {}^{3}_{C_3}\mathcal{P} \times {}^{3}_{3}F + ({}^{3}_{4}\mathcal{P} - {}^{3}_{C_3}\mathcal{P}) \times ({}^{3}_{4}\mathcal{R}\,{}^{4}_{4}F) \quad (451) $$

$$ i = 2 \;\; \Rightarrow \;\; {}^{2}_{2}N = {}^{2}_{C_2}N + {}^{2}_{3}\mathcal{R}\,{}^{3}_{3}N + {}^{2}_{C_2}\mathcal{P} \times {}^{2}_{2}F + ({}^{2}_{3}\mathcal{P} - {}^{2}_{C_2}\mathcal{P}) \times ({}^{2}_{3}\mathcal{R}\,{}^{3}_{3}F) \quad (452) $$

$$ i = 1 \;\; \Rightarrow \;\; {}^{1}_{1}N = {}^{1}_{C_1}N + {}^{1}_{2}\mathcal{R}\,{}^{2}_{2}N + {}^{1}_{C_1}\mathcal{P} \times {}^{1}_{1}F + ({}^{1}_{2}\mathcal{P} - {}^{1}_{C_1}\mathcal{P}) \times ({}^{1}_{2}\mathcal{R}\,{}^{2}_{2}F) \quad (453) $$

$$
{}^3_3N = {}^{3}_{C_3}N + {}^3_4\mathcal{R}\,{}^4_4N + {}^{3}_{C_3}\mathcal{P} \times {}^3_3F + ({}^3_4\mathcal{P} - {}^{3}_{C_3}\mathcal{P}) \times ({}^3_4\mathcal{R}\,{}^4_4F) =
$$

$$
\frac{M_3}{12}\begin{bmatrix} -\ddot{\theta}_1 \mathrm{s}_{2+3'}\left((w_3')^2 + (w_3'')^2\right) - 2\dot{\theta}_1 \mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 \mathrm{c}_{2+3'}\left((a_3)^2 + (w_3'')^2\right) + 2\dot{\theta}_1 \mathrm{s}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{bmatrix}
$$

$$
+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \times M_3
$$

$$
\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix}
$$

$$
+ \begin{bmatrix} -\frac{a_3}{2} \\ 0 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \tag{454}
$$

$$
{}^3_3N = \frac{M_3}{12}\begin{bmatrix} -\ddot{\theta}_1 \mathrm{s}_{2+3'}\left((w_3')^2+(w_3'')^2\right) - 2\dot{\theta}_1 \mathrm{c}_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 \mathrm{c}_{2+3'}\left((a_3)^2+(w_3'')^2\right) + 2\dot{\theta}_1 \mathrm{s}_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(a_3)^2 \\ \mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2-(w_3')^2\right) + \left(\ddot{\theta}_2+\ddot{\theta}_3'\right)\left((w_3')^2+(a_3)^2\right) \end{bmatrix}
$$

$$
+M_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_3}{2} \\ 0 & \frac{a_3}{2} & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2+\dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2+\ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_3'\right) \end{bmatrix} \tag{455}
$$

$$
{}^3_3N = \frac{M_3}{12}\begin{bmatrix} -\ddot{\theta}_1 \mathrm{s}_{2+3'}\left((w_3')^2 + (w_3'')^2\right) - 2\dot{\theta}_1 \mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 \mathrm{c}_{2+3'}\left((a_3)^2 + (w_3'')^2\right) + 2\dot{\theta}_1 \mathrm{s}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} 0 \\ -\frac{a_3}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g\right) \end{bmatrix} \tag{456}
$$

$$
{}^{2}_{2}N = {}^{2}_{C_2}N + {}^{2}_{3}\mathcal{R}\,{}^{3}_{3}N + {}^{2}_{C_2}\mathcal{P} \times {}^{2}_{2}F + \left({}^{2}_{3}\mathcal{P} - {}^{2}_{C_2}\mathcal{P}\right) \times \left({}^{2}_{3}\mathcal{R}\,{}^{3}_{3}F\right) \tag{457}
$$

$$
{}^{2}_{3}\mathcal{R}\,{}^{3}_{3}N = \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\left(\frac{M_3}{12} \begin{bmatrix} -\ddot{\theta}_1 \mathrm{s}_{2+3'}\left((w_3')^2 + (w_3'')^2\right) - 2\dot{\theta}_1 \mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 \mathrm{c}_{2+3'}\left((a_3)^2 + (w_3'')^2\right) + 2\dot{\theta}_1 \mathrm{s}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{bmatrix} + M_3 \right.
$$

$$
\left. \begin{bmatrix} 0 \\ -\frac{a_3}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g\right) \end{bmatrix} \right) \tag{458}
$$

$$
{}^2_3\mathcal{R}\,{}^3_3N = \frac{M_3}{12}\begin{bmatrix}
c_{3'}\Big(-\ddot{\theta}_1 s_{2+3'}\Big((w_3')^2+(w_3'')^2\Big)-2\dot{\theta}_1 c_{2+3'}\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)(w_3')^2\Big) \\
-s_{3'}\Big(-\ddot{\theta}_1 c_{2+3'}\Big((a_3)^2+(w_3'')^2\Big)+2\dot{\theta}_1 s_{2+3'}\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)(a_3)^2\Big) \\
\\
s_{3'}\Big(-\ddot{\theta}_1 s_{2+3'}\Big((w_3')^2+(w_3'')^2\Big)-2\dot{\theta}_1 c_{2+3'}\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)(w_3')^2\Big) \\
+c_{3'}\Big(-\ddot{\theta}_1 c_{2+3'}\Big((a_3)^2+(w_3'')^2\Big)+2\dot{\theta}_1 s_{2+3'}\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)(a_3)^2\Big) \\
\\
s_{2+3'}c_{2+3'}\dot{\theta}_1^2\Big((a_3)^2-(w_3')^2\Big)+\Big(\ddot{\theta}_2+\ddot{\theta}_3'\Big)\Big((w_3')^2+(a_3)^2\Big)
\end{bmatrix}
$$

$$
+M_3
$$

$$
\begin{bmatrix}
\frac{a_3}{2}s_{3'}\Big(a_2c_2\ddot{\theta}_1-2a_2s_2\dot{\theta}_1\dot{\theta}_2+\frac{a_3}{2}c_{2+3'}\ddot{\theta}_1-a_3s_{2+3'}\dot{\theta}_1\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)\Big) \\
-\frac{a_3}{2}c_{3'}\Big(a_2c_2\ddot{\theta}_1-2a_2s_2\dot{\theta}_1\dot{\theta}_2+\frac{a_3}{2}c_{2+3'}\ddot{\theta}_1-a_3s_{2+3'}\dot{\theta}_1\Big(\dot{\theta}_2+\dot{\theta}_3'\Big)\Big) \\
\frac{a_3}{2}\Big(a_2s_{3'}\dot{\theta}_2^2+a_2c_2s_{2+3'}\dot{\theta}_1^2+a_2c_{3'}\ddot{\theta}_2+\frac{a_3}{2}\Big(\ddot{\theta}_2+\ddot{\theta}_3'\Big)+\frac{a_3}{2}s_{2+3'}c_{2+3'}\dot{\theta}_1^2-c_{2+3'}g_g\Big)
\end{bmatrix}
\tag{459}
$$

$$
{}^{2}_{3}\mathcal{R}\,{}^{3}_{3}N =
$$

$$
\frac{M_3}{12}\begin{bmatrix}
-\ddot{\theta}_1 c_{3'} s_{2+3'}\left((w_3')^2+(w_3'')^2\right)-2\dot{\theta}_1 c_{3'} c_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(w_3')^2 \\
+\ddot{\theta}_1 s_{3'} c_{2+3'}\left((a_3)^2+(w_3'')^2\right)-2\dot{\theta}_1 s_{3'} s_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(a_3)^2 \\
\\
-\ddot{\theta}_1 s_{3'} s_{2+3'}\left((w_3')^2+(w_3'')^2\right)-2\dot{\theta}_1 s_{3'} c_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(w_3')^2 \\
-\ddot{\theta}_1 c_{3'} c_{2+3'}\left((a_3)^2+(w_3'')^2\right)+2\dot{\theta}_1 c_{3'} s_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(a_3)^2 \\
\\
s_{2+3'} c_{2+3'}\dot{\theta}_1^2\left((a_3)^2-(w_3')^2\right)+\left(\ddot{\theta}_2+\ddot{\theta}_3'\right)\left((w_3')^2+(a_3)^2\right)
\end{bmatrix}
$$

$$
+M_3
$$

$$
\begin{bmatrix}
\frac{a_3}{2} s_{3'}\left(a_2 c_2\ddot{\theta}_1-2a_2 s_2\dot{\theta}_1\dot{\theta}_2+\frac{a_3}{2}c_{2+3'}\ddot{\theta}_1-a_3 s_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_3'\right)\right) \\
-\frac{a_3}{2} c_{3'}\left(a_2 c_2\ddot{\theta}_1-2a_2 s_2\dot{\theta}_1\dot{\theta}_2+\frac{a_3}{2}c_{2+3'}\ddot{\theta}_1-a_3 s_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_3'\right)\right) \\
\frac{a_3}{2}\left(a_2 s_{3'}\dot{\theta}_2^2+a_2 c_2 s_{2+3'}\dot{\theta}_1^2+a_2 c_{3'}\ddot{\theta}_2+\frac{a_3}{2}\left(\ddot{\theta}_2+\ddot{\theta}_3'\right)+\frac{a_3}{2}s_{2+3'}c_{2+3'}\dot{\theta}_1^2-c_{2+3'}\,g_g\right)
\end{bmatrix}
\tag{460}
$$

$$
{}^2_3\mathcal{R}\,{}^3_3N =
$$

$$
\frac{M_3}{12}\left[\begin{array}{l}
-\ddot{\theta}_1 s_2 (w_3'')^2 - \ddot{\theta}_1 c_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 c_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\
+\ddot{\theta}_1 s_{3'} c_{2+3'} (a_3)^2 - 2\dot{\theta}_1 s_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\
\\
-\ddot{\theta}_1 c_2 (w_3'')^2 - \ddot{\theta}_1 s_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 s_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\
-\ddot{\theta}_1 c_{3'} c_{2+3'} (a_3)^2 + 2\dot{\theta}_1 c_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\
\\
s_{2+3'} c_{2+3'} \dot{\theta}_1^2 \left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right)
\end{array}\right]
$$

$$
+M_3
$$

$$
\left[\begin{array}{c}
\frac{a_3}{2} s_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\
-\frac{a_3}{2} c_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2 a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\
\frac{a_3}{2} \left(a_2 s_{3'} \dot{\theta}_2^2 + a_2 c_2 s_{2+3'} \dot{\theta}_1^2 + a_2 c_{3'} \ddot{\theta}_2 + \frac{a_3}{2} \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2} s_{2+3'} c_{2+3'} \dot{\theta}_1^2 - c_{2+3'}\, g_g\right)
\end{array}\right]
\tag{461}
$$

$$
{}^{2}_{C_2}\mathcal{P} \times {}^{2}_{2}F = \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \times \Bigg(M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}c_2^2\dot{\theta}_1^2 - s_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}s_2c_2\dot{\theta}_1^2 - c_2 g_g \\ \frac{a_2}{2}c_2\ddot{\theta}_1 - a_2 s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + M_3
$$

$$
\begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2c_2^2\dot{\theta}_1^2 - \frac{a_3}{2}c_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}c_2c_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}s_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - s_2 g_g \\ a_2\ddot{\theta}_2 + a_2s_2c_2\dot{\theta}_1^2 - \frac{a_3}{2}s_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}s_2c_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}c_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - c_2 g_g \\ a_2c_2\ddot{\theta}_1 - 2a_2s_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}c_{2+3'}\ddot{\theta}_1 - a_3s_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \Bigg)
$$

$$
= M_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_2}{2} \\ 0 & \frac{a_2}{2} & 0 \end{bmatrix} \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}c_2^2\dot{\theta}_1^2 - s_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}s_2c_2\dot{\theta}_1^2 - c_2 g_g \\ \frac{a_2}{2}c_2\ddot{\theta}_1 - a_2 s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + M_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_2}{2} \\ 0 & \frac{a_2}{2} & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2c_2^2\dot{\theta}_1^2 - \frac{a_3}{2}c_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}c_2c_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}s_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - s_2 g_g \\ a_2\ddot{\theta}_2 + a_2s_2c_2\dot{\theta}_1^2 - \frac{a_3}{2}s_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}s_2c_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}c_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - c_2 g_g \\ a_2c_2\ddot{\theta}_1 - 2a_2s_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}c_{2+3'}\ddot{\theta}_1 - a_3s_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \tag{462}
$$

$$
{}^{2}_{C_2}\mathcal{P} \times {}^{2}_{2}F = M_2 \begin{bmatrix} 0 \\ -\frac{a_2^2}{4}\mathrm{c}_2\ddot{\theta}_1 + \frac{a_2^2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \\ \frac{a_2^2}{4}\ddot{\theta}_2 + \frac{a_2^2}{4}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \frac{a_2}{2}\mathrm{c}_2 g_g \end{bmatrix} +
$$

$$
M_3 \begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\left(a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 \right. \\ \left. + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g\right) \end{bmatrix} \tag{463}
$$

$$\left({}^{2}_{3}\mathcal{P} - {}^{2}_{C_2}\mathcal{P}\right) \times \left({}^{2}_{3}\mathcal{R}\,{}^{3}_{3}F\right) = \begin{bmatrix} \frac{a_2}{2} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} M_3$$

$$\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_2}{2} \\ 0 & \frac{a_2}{2} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{c}_{3'} & -\mathrm{s}_{3'} & 0 \\ \mathrm{s}_{3'} & \mathrm{c}_{3'} & 0 \\ 0 & 0 & 1 \end{bmatrix} M_3$$

$$\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \tag{464}$$

$$\left({}^2_3\mathcal{P} - {}^2_{C_2}\mathcal{P}\right) \times \left({}^2_3\mathcal{R}\,{}^3_3F\right) = M_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_2}{2} \\ \frac{a_2}{2}\mathrm{s}_{3'} & \frac{a_2}{2}\mathrm{c}_{3'} & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g \\ a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} = M_3$$

$$\begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\mathrm{s}_{3'}\left(a_2\mathrm{s}_{3'}\ddot{\theta}_2 - a_2\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{2+3'}\,g_g\right) + \\ \frac{a_2}{2}\mathrm{c}_{3'}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g\right) \end{bmatrix} \tag{465}$$

$$\left({}^2_3\mathcal{P} - {}^2_{C_2}\mathcal{P}\right) \times \left({}^2_3\mathcal{R}\,{}^3_3F\right) =$$

$$M_3\begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\left(a_2\mathrm{s}_{3'}^2\ddot{\theta}_2 - a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\dot{\theta}_2^2 - a_2\mathrm{c}_2\mathrm{s}_{3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2\right. \\ -\frac{a_3}{2}\mathrm{s}_{3'}\mathrm{c}_{2+3'}^2\dot{\theta}_1^2 - \mathrm{s}_{3'}\mathrm{s}_{2+3'}g_g + a_2\mathrm{s}_{3'}\mathrm{c}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{c}_{3'}\mathrm{s}_{2+3'}\dot{\theta}_1^2 \\ \left. + a_2\mathrm{c}_{3'}^2\ddot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{c}_{3'}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{3'}\mathrm{c}_{2+3'}g_g\right) \end{bmatrix} =$$

$$M_3\begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\left(a_2\ddot{\theta}_2 + a_2\mathrm{c}_2\mathrm{s}_2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g\right) \end{bmatrix} \tag{466}$$

Example: Inter-link interaction torques cnt.13

$$
\begin{aligned}
&{}^2_2N = {}^{\;2}_{C_2}N + {}^2_3\mathcal{R}\,{}^3_3N + {}^{\;2}_{C_2}\mathcal{P} \times {}^2_2F + \left({}^2_3\mathcal{P} - {}^{\;2}_{C_2}\mathcal{P}\right) \times \left({}^2_3\mathcal{R}\,{}^3_3F\right) = \\
&\frac{M_2}{12}\begin{bmatrix} -2\dot{\theta}_1\dot{\theta}_2 c_2 (w_2')^2 - \ddot{\theta}_1 s_2 \left((w_2')^2 + (w_2'')^2\right) \\ 2\dot{\theta}_1\dot{\theta}_2 s_2 a_2^2 - \ddot{\theta}_1 c_2 \left(a_2^2 + (w_2'')^2\right) \\ \dot{\theta}_1^2 s_2 c_2 \left(a_2^2 - (w_2')^2\right) + \ddot{\theta}_2 \left((w_2')^2 + a_2^2\right) \end{bmatrix} + M_2 \begin{bmatrix} 0 \\ -\frac{a_2^2}{4} c_2 \ddot{\theta}_1 + \frac{a_2^2}{2} s_2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{a_2^2}{4}\ddot{\theta}_2 + \frac{a_2^2}{4} s_2 c_2 \dot{\theta}_1^2 - \frac{a_2}{2} c_2 g_g \end{bmatrix} \\
&+\frac{M_3}{12}\left[\begin{array}{c} -\ddot{\theta}_1 s_2 (w_3'')^2 - \ddot{\theta}_1 c_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 c_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ +\ddot{\theta}_1 s_{3'} c_{2+3'} (a_3)^2 - 2\dot{\theta}_1 s_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \hline -\ddot{\theta}_1 c_2 (w_3'')^2 - \ddot{\theta}_1 s_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 s_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 c_{3'} c_{2+3'} (a_3)^2 + 2\dot{\theta}_1 c_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \hline s_{2+3'} c_{2+3'} \dot{\theta}_1^2 \left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{array}\right] \\
&+M_3\begin{bmatrix} \frac{a_3}{2} s_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ -\frac{a_3}{2} c_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_3}{2}\left(a_2 s_{3'} \dot{\theta}_2^2 + a_2 c_2 s_{2+3'} \dot{\theta}_1^2 + a_2 c_{3'} \ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2} s_{2+3'} c_{2+3'} \dot{\theta}_1^2 - c_{2+3'} g_g\right) \end{bmatrix} \\
&+M_3\begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\left(a_2 \ddot{\theta}_2 + a_2 s_2 c_2 \dot{\theta}_1^2 - \frac{a_3}{2} s_{3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2} s_2 c_{2+3'} \dot{\theta}_1^2 + \frac{a_3}{2} c_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - c_2 g_g\right) \end{bmatrix} \\
&+M_3\begin{bmatrix} 0 \\ -\frac{a_2}{2}\left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_2}{2}\left(a_2 \ddot{\theta}_2 + a_2 c_2 s_2 \dot{\theta}_1^2 - \frac{a_3}{2} s_{3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2} s_2 c_{2+3'} \dot{\theta}_1^2 + \frac{a_3}{2} c_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - c_2 g_g\right) \end{bmatrix}
\end{aligned} \tag{467}
$$

$$
\begin{aligned}
{}^2_2N &= {}^2_{C_2}N + {}^2_3\mathcal{R}\,{}^3_3N + {}^2_{C_2}\mathcal{P} \times {}^2_2F + ({}^2_3\mathcal{P} - {}^2_{C_2}\mathcal{P}) \times ({}^2_3\mathcal{R}\,{}^3_3F) = \\
&\frac{M_2}{12}\begin{bmatrix} -2\dot\theta_1\dot\theta_2 c_2 (w_2')^2 - \ddot\theta_1 s_2\left((w_2')^2 + (w_2'')^2\right) \\ 2\dot\theta_1\dot\theta_2 s_2 a_2^2 - \ddot\theta_1 c_2\left(a_2^2 + (w_2'')^2\right) \\ \dot\theta_1^2 s_2 c_2\left(a_2^2 - (w_2')^2\right) + \ddot\theta_2\left((w_2')^2 + a_2^2\right) \end{bmatrix} + M_2\begin{bmatrix} 0 \\ -\frac{a_2^2}{4}c_2\ddot\theta_1 + \frac{a_2^2}{2}s_2\dot\theta_1\dot\theta_2 \\ \frac{a_2^2}{4}\ddot\theta_2 + \frac{a_2^2}{4}s_2c_2\dot\theta_1^2 - \frac{a_2}{2}c_2 g_g \end{bmatrix} \\
&+\frac{M_3}{12}\left[\begin{array}{l} -\ddot\theta_1 s_2 (w_3'')^2 - \ddot\theta_1 c_{3'} s_{2+3'}(w_3')^2 - 2\dot\theta_1 c_{3'} c_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)(w_3')^2 \\ +\ddot\theta_1 s_{3'} c_{2+3'}(a_3)^2 - 2\dot\theta_1 s_{3'} s_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)(a_3)^2 \\ \hline -\ddot\theta_1 c_2 (w_3'')^2 - \ddot\theta_1 s_{3'} s_{2+3'}(w_3')^2 - 2\dot\theta_1 s_{3'} c_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)(w_3')^2 \\ -\ddot\theta_1 c_{3'} c_{2+3'}(a_3)^2 + 2\dot\theta_1 c_{3'} s_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)(a_3)^2 \\ \hline s_{2+3'} c_{2+3'}\dot\theta_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot\theta_2 + \ddot\theta_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{array}\right] \\
&+M_3\begin{bmatrix} \frac{a_3}{2}s_{3'}\left(a_2c_2\ddot\theta_1 - 2a_2s_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}c_{2+3'}\ddot\theta_1 - a_3s_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right)\right) \\ -\frac{a_3}{2}c_{3'}\left(a_2c_2\ddot\theta_1 - 2a_2s_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}c_{2+3'}\ddot\theta_1 - a_3s_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right)\right) \\ \frac{a_3}{2}\left(a_2s_{3'}\dot\theta_2^2 + a_2c_2s_{2+3'}\dot\theta_1^2 + a_2c_{3'}\ddot\theta_2 + \frac{a_3}{2}\left(\ddot\theta_2 + \ddot\theta_3'\right) + \frac{a_3}{2}s_{2+3'}c_{2+3'}\dot\theta_1^2 - c_{2+3'}g_g\right) \end{bmatrix} \\
&+M_3\begin{bmatrix} 0 \\ -a_2\left(a_2c_2\ddot\theta_1 - 2a_2s_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}c_{2+3'}\ddot\theta_1 - a_3s_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right)\right) \\ a_2\left(a_2\ddot\theta_2 + a_2s_2c_2\dot\theta_1^2 - \frac{a_3}{2}s_{3'}\left(\dot\theta_2 + \dot\theta_3'\right)^2 + \frac{a_3}{2}s_2c_{2+3'}\dot\theta_1^2 + \frac{a_3}{2}c_{3'}\left(\ddot\theta_2 + \ddot\theta_3'\right) - c_2g_g\right) \end{bmatrix}
\end{aligned}
\tag{468}
$$

$$
{}^1_1N = {}^1_{C_1}N + {}^1_2\mathcal{R}\,{}^2_2N + {}^1_{C_1}\mathcal{P} \times {}^1_1F + \left({}^1_2\mathcal{P} - {}^1_{C_1}\mathcal{P}\right) \times \left({}^1_2\mathcal{R}\,{}^2_2F\right) \tag{469}
$$

$$
\begin{aligned}
{}^1_2\mathcal{R}\,{}^2_2N = &\begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \\
&\left(\frac{M_2}{12} \begin{bmatrix} -2\dot{\theta}_1\dot{\theta}_2 c_2 (w_2')^2 - \ddot{\theta}_1 s_2 \left((w_2')^2 + (w_2'')^2\right) \\ 2\dot{\theta}_1\dot{\theta}_2 s_2 a_2^2 - \ddot{\theta}_1 c_2 \left(a_2^2 + (w_2'')^2\right) \\ \dot{\theta}_1^2 s_2 c_2 \left(a_2^2 - (w_2')^2\right) + \ddot{\theta}_2 \left((w_2')^2 + a_2^2\right) \end{bmatrix} + M_2 \begin{bmatrix} 0 \\ -\frac{a_2^2}{4} c_2 \ddot{\theta}_1 + \frac{a_2^2}{2} s_2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{a_2^2}{4}\ddot{\theta}_2 + \frac{a_2^2}{4} s_2 c_2 \dot{\theta}_1^2 - \frac{a_2}{2} c_2 g_g \end{bmatrix} \right. \\
&+ \frac{M_3}{12} \left[\begin{array}{l} -\ddot{\theta}_1 s_2 (w_3'')^2 - \ddot{\theta}_1 c_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 c_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ +\ddot{\theta}_1 s_{3'} c_{2+3'} (a_3)^2 - 2\dot{\theta}_1 s_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \hline -\ddot{\theta}_1 c_2 (w_3'')^2 - \ddot{\theta}_1 s_{3'} s_{2+3'} (w_3')^2 - 2\dot{\theta}_1 s_{3'} c_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(w_3')^2 \\ -\ddot{\theta}_1 c_{3'} c_{2+3'} (a_3)^2 + 2\dot{\theta}_1 c_{3'} s_{2+3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)(a_3)^2 \\ \hline s_{2+3'} c_{2+3'} \dot{\theta}_1^2 \left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right) \end{array} \right] \\
&+ M_3 \begin{bmatrix} \frac{a_3}{2} s_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ -\frac{a_3}{2} c_{3'} \left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ \frac{a_3}{2} \left(a_2 s_{3'} \dot{\theta}_2^2 + a_2 c_2 s_{2+3'} \dot{\theta}_1^2 + a_2 c_{3'} \ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2} s_{2+3'} c_{2+3'} \dot{\theta}_1^2 - c_{2+3'} g_g\right) \end{bmatrix} \\
&\left. + M_3 \begin{bmatrix} 0 \\ -a_2 \left(a_2 c_2 \ddot{\theta}_1 - 2a_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{a_3}{2} c_{2+3'} \ddot{\theta}_1 - a_3 s_{2+3'} \dot{\theta}_1 \left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \\ a_2 \left(a_2 \ddot{\theta}_2 + a_2 s_2 c_2 \dot{\theta}_1^2 - \frac{a_3}{2} s_{3'} \left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2} s_2 c_{2+3'} \dot{\theta}_1^2 + \frac{a_3}{2} c_{3'} \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - c_2 g_g\right) \end{bmatrix} \right)
\end{aligned} \tag{470}
$$

$$
\begin{aligned}
\left(\tfrac{1}{2}\mathcal{R}\,{}_2^2N\right)_z =& -\tfrac{M_2}{12}\mathrm{s}_2\left(-2\dot\theta_1\dot\theta_2\mathrm{c}_2(w_2')^2-\ddot\theta_1\mathrm{s}_2\left((w_2')^2+(w_2'')^2\right)\right)\\
&-\tfrac{M_2}{12}\mathrm{c}_2\left(2\dot\theta_1\dot\theta_2\mathrm{s}_2a_2^2-\ddot\theta_1\mathrm{c}_2\left(a_2^2+(w_2'')^2\right)\right)-M_2\mathrm{s}_2 0\\
&-M_2\mathrm{c}_2\left(-\tfrac{a_2^2}{4}\mathrm{c}_2\ddot\theta_1+\tfrac{a_2^2}{2}\mathrm{s}_2\dot\theta_1\dot\theta_2\right)\\
&-\tfrac{M_3}{12}\mathrm{s}_2\left(-\ddot\theta_1\mathrm{s}_2(w_3'')^2-\ddot\theta_1\mathrm{c}_{3'}\mathrm{s}_{2+3'}(w_3')^2-2\dot\theta_1\mathrm{c}_{3'}\mathrm{c}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(w_3')^2\right.\\
&\left.+\ddot\theta_1\mathrm{s}_{3'}\mathrm{c}_{2+3'}(a_3)^2-2\dot\theta_1\mathrm{s}_{3'}\mathrm{s}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(a_3)^2\right)\\
&-\tfrac{M_3}{12}\mathrm{c}_2\left(-\ddot\theta_1\mathrm{c}_2(w_3'')^2-\ddot\theta_1\mathrm{s}_{3'}\mathrm{s}_{2+3'}(w_3')^2-2\dot\theta_1\mathrm{s}_{3'}\mathrm{c}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(w_3')^2\right.\\
&\left.-\ddot\theta_1\mathrm{c}_{3'}\mathrm{c}_{2+3'}(a_3)^2+2\dot\theta_1\mathrm{c}_{3'}\mathrm{s}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(a_3)^2\right)\\
&-M_3\mathrm{s}_2\left(\tfrac{a_3}{2}\mathrm{s}_{3'}\left(a_2\mathrm{c}_2\ddot\theta_1-2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2+\tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1-a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right)\right)\right)\\
&-M_3\mathrm{c}_2\left(-\tfrac{a_3}{2}\mathrm{c}_{3'}\left(a_2\mathrm{c}_2\ddot\theta_1-2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2+\tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1-a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right)\right)\right)\\
&+M_3\mathrm{c}_2a_2\left(a_2\mathrm{c}_2\ddot\theta_1-2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2+\tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1-a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right)\right)
\end{aligned}
\tag{471}
$$

$$
\begin{aligned}
\left({}^{1}_{2}\mathcal{R}\,{}^{2}_{2}N\right)_z = &-\tfrac{M_2}{12}\left(2\dot\theta_1\dot\theta_2\mathrm{s}_2\mathrm{c}_2\left(a_2^2-(w_2')^2\right)-\ddot\theta_1\left((w_2'')^2+\mathrm{s}_2^2(w_2')^2+\mathrm{c}_2^2a_2^2\right)\right)\\
&-M_2\mathrm{c}_2\left(-\tfrac{a_2^2}{4}\mathrm{c}_2\ddot\theta_1+\tfrac{a_2^2}{2}\mathrm{s}_2\dot\theta_1\dot\theta_2\right)\\
&-\tfrac{M_3}{12}\left(-\ddot\theta_1(w_3'')^2-\ddot\theta_1\mathrm{s}_{2+3'}^2(w_3')^2-2\dot\theta_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(w_3')^2\right.\\
&\left.+\ddot\theta_1\mathrm{c}_{2+3'}^2(a_3)^2+2\dot\theta_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)(a_3)^2\right)\\
&+M_3\tfrac{a_3}{2}\mathrm{c}_{2+3'}\left(a_2\mathrm{c}_2\ddot\theta_1-2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2+\tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1-a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right)\right)\\
&+M_3\mathrm{c}_2a_2\left(a_2\mathrm{c}_2\ddot\theta_1-2a_2\mathrm{s}_2\dot\theta_1\dot\theta_2+\tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot\theta_1-a_3\mathrm{s}_{2+3'}\dot\theta_1\left(\dot\theta_2+\dot\theta_3'\right)\right)
\end{aligned}
\tag{472}
$$

$$\left({}^{1}_{2}\mathcal{R}\,{}^{2}_{2}N\right)_z = -\frac{M_2}{12}\left(2\dot{\theta}_1\dot{\theta}_2\mathrm{s}_2\mathrm{c}_2\left(a_2^2-(w_2')^2\right)-\ddot{\theta}_1\left((w_2'')^2+\mathrm{s}_2^2(w_2')^2+\mathrm{c}_2^2a_2^2\right)\right)$$
$$-M_2\mathrm{c}_2\left(-\frac{a_2^2}{4}\mathrm{c}_2\ddot{\theta}_1+\frac{a_2^2}{2}\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2\right)$$
$$-\frac{M_3}{12}\Bigg(-\ddot{\theta}_1(w_3'')^2-\ddot{\theta}_1\mathrm{s}_{2+3'}^2(w_3')^2-2\dot{\theta}_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(w_3')^2$$
$$+\ddot{\theta}_1\mathrm{c}_{2+3'}^2(a_3)^2+2\dot{\theta}_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot{\theta}_2+\dot{\theta}_3'\right)(a_3)^2\Bigg)$$
$$+M_3\left(\frac{a_3}{2}\mathrm{c}_{2+3'}+\mathrm{c}_2a_2\right)\left(a_2\mathrm{c}_2\ddot{\theta}_1-2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2+\frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1-a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_3'\right)\right) \tag{473}$$

$$
{}^{1}_{C_1}\mathcal{P} \times {}^{1}_{1}F = \begin{bmatrix} 0 \\ 0 \\ -\frac{l_1}{2} \end{bmatrix} \times \left(M_1 \begin{bmatrix} 0 \\ 0 \\ g_g \end{bmatrix} + M_2 \begin{bmatrix} -\frac{a_2}{2}c_2\left(\dot\theta_1^2 + \dot\theta_2^2\right) - \frac{a_2}{2}s_2\ddot\theta_2 \\ \frac{a_2}{2}c_2\ddot\theta_1 - a_2 s_2\dot\theta_1\dot\theta_2 \\ \frac{a_2}{2}s_2\dot\theta_2^2 - \frac{a_2}{2}c_2\ddot\theta_2 + g_g \end{bmatrix} + \right.
$$

$$
\left. M_3 \begin{bmatrix} -a_2c_2\left(\dot\theta_1^2+\dot\theta_2^2\right) - \frac{a_3}{2}c_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)^2 - \frac{a_3}{2}c_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}s_{2+3'}\left(\ddot\theta_2+\ddot\theta_{3'}\right) - a_2s_2\ddot\theta_2 \\ a_2c_2\ddot\theta_1 - 2a_2s_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}c_{2+3'}\ddot\theta_1 - a_3s_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right) \\ g_g + a_2s_2\dot\theta_2^2 - a_2c_2\ddot\theta_2 + \frac{a_3}{2}s_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)^2 - \frac{a_3}{2}c_{2+3'}\left(\ddot\theta_2 + \ddot\theta_3'\right) \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} 0 & \frac{l_1}{2} & 0 \\ -\frac{l_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(M_1 \begin{bmatrix} 0 \\ 0 \\ g_g \end{bmatrix} + M_2 \begin{bmatrix} -\frac{a_2}{2}c_2\left(\dot\theta_1^2 + \dot\theta_2^2\right) - \frac{a_2}{2}s_2\ddot\theta_2 \\ \frac{a_2}{2}c_2\ddot\theta_1 - a_2 s_2\dot\theta_1\dot\theta_2 \\ \frac{a_2}{2}s_2\dot\theta_2^2 - \frac{a_2}{2}c_2\ddot\theta_2 + g_g \end{bmatrix} + \right.
$$

$$
\left. M_3 \begin{bmatrix} -a_2c_2\left(\dot\theta_1^2+\dot\theta_2^2\right) - \frac{a_3}{2}c_{2+3'}\left(\dot\theta_2+\dot\theta_3'\right)^2 - \frac{a_3}{2}c_{2+3'}\dot\theta_1^2 - \frac{a_3}{2}s_{2+3'}\left(\ddot\theta_2+\ddot\theta_{3'}\right) - a_2s_2\ddot\theta_2 \\ a_2c_2\ddot\theta_1 - 2a_2s_2\dot\theta_1\dot\theta_2 + \frac{a_3}{2}c_{2+3'}\ddot\theta_1 - a_3s_{2+3'}\dot\theta_1\left(\dot\theta_2 + \dot\theta_3'\right) \\ g_g + a_2s_2\dot\theta_2^2 - a_2c_2\ddot\theta_2 + \frac{a_3}{2}s_{2+3'}\left(\dot\theta_2 + \dot\theta_3'\right)^2 - \frac{a_3}{2}c_{2+3'}\left(\ddot\theta_2 + \ddot\theta_3'\right) \end{bmatrix} \right)
$$

$$
\Rightarrow \quad \left({}^{1}_{C_1}\mathcal{P} \times {}^{1}_{1}F \right)_z = 0
$$

$$
\left({}^{1}_{2}\mathcal{P} - {}^{1}_{C_1}\mathcal{P}\right) \times \left({}^{1}_{2}\mathcal{R}\,{}^{2}_{2}F\right) = \begin{bmatrix} 0 \\ 0 \\ \frac{l_1}{2} \end{bmatrix} \times \left(M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \right.
$$

$$
\left. M_3 \begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2\mathrm{c}_2^2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{s}_2 g_g \\ a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} 0 & -\frac{l_1}{2} & 0 \\ \frac{l_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(M_2 \begin{bmatrix} -\frac{a_2}{2}\dot{\theta}_2^2 - \frac{a_2}{2}\mathrm{c}_2^2\dot{\theta}_1^2 - \mathrm{s}_2 g_g \\ \frac{a_2}{2}\ddot{\theta}_2 + \frac{a_2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \mathrm{c}_2 g_g \\ \frac{a_2}{2}\mathrm{c}_2\ddot{\theta}_1 - a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \right.
$$

$$
\left. M_3 \begin{bmatrix} -a_2\dot{\theta}_2^2 - a_2\mathrm{c}_2^2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{c}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 - \frac{a_3}{2}\mathrm{c}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{s}_2 g_g \\ a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \frac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \frac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \frac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g \\ a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \frac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right) \end{bmatrix} \right)
$$

$$
\Rightarrow \quad \left(\left({}^{1}_{2}\mathcal{P} - {}^{1}_{C_1}\mathcal{P}\right) \times \left({}^{1}_{2}\mathcal{R}\,{}^{2}_{2}F\right) \right)_z = 0 \tag{475}
$$

$$
\begin{aligned}
{}^1_1N_z &= \left({}^{1}_{C_1}N\right)_z + \left({}^1_2\mathcal{R}\,{}^2_2N\right)_z + \left({}^{1}_{C_1}\mathcal{P} \times {}^1_1F\right)_z + \left(({}^1_2\mathcal{P} - {}^{1}_{C_1}\mathcal{P}) \times ({}^1_2\mathcal{R}\,{}^2_2F)\right)_z \\
&= \tfrac{M_1}{8} w_1^2\ddot{\theta}_1 - \tfrac{M_2}{12}\Big(2\dot{\theta}_1\dot{\theta}_2 \mathrm{s}_2\mathrm{c}_2\left(a_2^2 - (w_2')^2\right) - \ddot{\theta}_1\left((w_2'')^2 + \mathrm{s}_2^2(w_2')^2 + \mathrm{c}_2^2 a_2^2\right)\Big) \\
&\quad - M_2\left(-\tfrac{a_2^2}{4}\mathrm{c}_2^2\ddot{\theta}_1 + \tfrac{a_2^2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1\dot{\theta}_2\right) - \tfrac{M_3}{12}\Big(-\ddot{\theta}_1(w_3'')^2 - \ddot{\theta}_1\mathrm{s}_{2+3'}^2(w_3')^2 \\
&\quad + \ddot{\theta}_1\mathrm{c}_{2+3'}^2(a_3)^2 + 2\dot{\theta}_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\left((a_3)^2 - (w_3')^2\right)\Big) \\
&\quad + M_3\tfrac{a_3}{2}\mathrm{c}_{2+3'}\Big(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\Big) + 0 + 0 \\
&= \tfrac{M_1}{8} w_1^2\ddot{\theta}_1 - \tfrac{M_2}{12}\Big(2\dot{\theta}_1\dot{\theta}_2 \mathrm{s}_2\mathrm{c}_2\left(a_2^2 - (w_2')^2\right) - \ddot{\theta}_1\left((w_2'')^2 + \mathrm{s}_2^2(w_2')^2 + \mathrm{c}_2^2 a_2^2\right)\Big) \\
&\quad - M_2\left(-\tfrac{a_2^2}{4}\mathrm{c}_2^2\ddot{\theta}_1 + \tfrac{a_2^2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1\dot{\theta}_2\right) - \tfrac{M_3}{12}\Big(-\ddot{\theta}_1(w_3'')^2 - \ddot{\theta}_1\mathrm{s}_{2+3'}^2(w_3')^2 \\
&\quad + \ddot{\theta}_1\mathrm{c}_{2+3'}^2(a_3)^2 + 2\dot{\theta}_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\left((a_3)^2 - (w_3')^2\right)\Big) \\
&\quad + M_3\tfrac{a_3}{2}\mathrm{c}_{2+3'}\Big(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\Big)
\end{aligned}
\tag{476}
$$

$$
\begin{aligned}
&{}^{2}_{2}N_z = \left({}^{2}_{C_2}N\right)_z + \left({}^{2}_{3}\mathcal{R}\,{}^{3}_{3}N\right)_z + \left({}^{2}_{C_2}\mathcal{P} \times {}^{2}_{2}F\right)_z + \left(({}^{2}_{3}\mathcal{P} - {}^{2}_{C_2}\mathcal{P}) \times ({}^{2}_{3}\mathcal{R}\,{}^{3}_{3}F)\right)_z \\
&= \tfrac{M_2}{12}\left(\dot{\theta}_1^2 \mathrm{s}_2\mathrm{c}_2\left(a_2^2 - (w_2')^2\right) + \ddot{\theta}_2\left((w_2')^2 + a_2^2\right)\right) \\
&+ M_2\left(\tfrac{a_2^2}{4}\ddot{\theta}_2 + \tfrac{a_2^2}{4}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \tfrac{a_2}{2}\mathrm{c}_2 g_g\right) \\
&+ \tfrac{M_3}{12}\left(\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right)\right) \\
&+ M_3\left(\tfrac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \tfrac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \right.\right. \\
&\left.\tfrac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g\right) + a_2\left(a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \tfrac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \right. \\
&\left.\left.\tfrac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \tfrac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g\right)\right)
\end{aligned}
\tag{477}
$$

$$
\begin{aligned}
{}^{3}_{3}N_z &= \left({}^{3}_{C_3}N\right)_z + \left({}^{3}_{4}\mathcal{R}\,{}^{4}_{4}N\right)_z + \left({}^{3}_{C_3}\mathcal{P} \times {}^{3}_{3}F\right)_z + \left(({}^{3}_{4}\mathcal{P} - {}^{3}_{C_3}\mathcal{P}) \times ({}^{3}_{4}\mathcal{R}\,{}^{4}_{4}F)\right)_z \\
&= \frac{M_3}{12}\left(\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right)\right) \\
&+ M_3\frac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \frac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \frac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\right. \\
&\left.- \mathrm{c}_{2+3'}g_g\right)
\end{aligned}
\tag{478}
$$

$$\tau_i = {}^i_iN^T\,{}^i_iz = \begin{bmatrix} {}^i_iN_x & {}^i_iN_y & {}^i_iN_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^i_iN_z \tag{479}$$

$$i = 1 \quad \Rightarrow \quad \tau_1 = {}^1_1N^T\,{}^1_1z = {}^1_1N_z \tag{480}$$

$$i = 2 \quad \Rightarrow \quad \tau_2 = {}^2_2N^T\,{}^2_2z = {}^2_2N_z \tag{481}$$

$$i = 3 \quad \Rightarrow \quad \tau_3 = {}^3_3N^T\,{}^3_3z = {}^3_3N_z \tag{482}$$

$$\tau_1 = \tfrac{M_1}{8} w_1^2\ddot{\theta}_1 - \tfrac{M_2}{12}\left(2\dot{\theta}_1\dot{\theta}_2 \mathrm{s}_2\mathrm{c}_2\left(a_2^2 - (w_2')^2\right) - \ddot{\theta}_1\left((w_2'')^2 + \mathrm{s}_2^2(w_2')^2 + \mathrm{c}_2^2 a_2^2\right)\right)$$
$$-M_2\left(-\tfrac{a_2^2}{4}\mathrm{c}_2^2\ddot{\theta}_1 + \tfrac{a_2^2}{2}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1\dot{\theta}_2\right) - \tfrac{M_3}{12}\left(-\ddot{\theta}_1(w_3'')^2 - \ddot{\theta}_1\mathrm{s}_{2+3'}^2(w_3')^2\right.$$
$$\left.+\ddot{\theta}_1\mathrm{c}_{2+3'}^2(a_3)^2 + 2\dot{\theta}_1\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\left((a_3)^2 - (w_3')^2\right)\right)$$
$$+M_3\tfrac{a_3}{2}\mathrm{c}_{2+3'}\left(a_2\mathrm{c}_2\ddot{\theta}_1 - 2a_2\mathrm{s}_2\dot{\theta}_1\dot{\theta}_2 + \tfrac{a_3}{2}\mathrm{c}_{2+3'}\ddot{\theta}_1 - a_3\mathrm{s}_{2+3'}\dot{\theta}_1\left(\dot{\theta}_2 + \dot{\theta}_3'\right)\right) \tag{483}$$

$$\tau_2 = \tfrac{M_2}{12}\left(\dot{\theta}_1^2\mathrm{s}_2\mathrm{c}_2\left(a_2^2 - (w_2')^2\right) + \ddot{\theta}_2\left((w_2')^2 + a_2^2\right)\right)$$
$$+M_2\left(\tfrac{a_2^2}{4}\ddot{\theta}_2 + \tfrac{a_2^2}{4}\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \tfrac{a_2}{2}\mathrm{c}_2 g_g\right)$$
$$+\tfrac{M_3}{12}\left(\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right)\right)$$
$$+M_3\left(\tfrac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \tfrac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \right.\right.$$
$$\left.\tfrac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}g_g\right) + a_2\left(a_2\ddot{\theta}_2 + a_2\mathrm{s}_2\mathrm{c}_2\dot{\theta}_1^2 - \tfrac{a_3}{2}\mathrm{s}_{3'}\left(\dot{\theta}_2 + \dot{\theta}_3'\right)^2 + \right.$$
$$\left.\left.\tfrac{a_3}{2}\mathrm{s}_2\mathrm{c}_{2+3'}\dot{\theta}_1^2 + \tfrac{a_3}{2}\mathrm{c}_{3'}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) - \mathrm{c}_2 g_g\right)\right) \tag{484}$$

$$\tau_3 = \tfrac{M_3}{12}\left(\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2\left((a_3)^2 - (w_3')^2\right) + \left(\ddot{\theta}_2 + \ddot{\theta}_3'\right)\left((w_3')^2 + (a_3)^2\right)\right)$$
$$+M_3\tfrac{a_3}{2}\left(a_2\mathrm{s}_{3'}\dot{\theta}_2^2 + a_2\mathrm{c}_2\mathrm{s}_{2+3'}\dot{\theta}_1^2 + a_2\mathrm{c}_{3'}\ddot{\theta}_2 + \tfrac{a_3}{2}\left(\ddot{\theta}_2 + \ddot{\theta}_3'\right) + \tfrac{a_3}{2}\mathrm{s}_{2+3'}\mathrm{c}_{2+3'}\dot{\theta}_1^2 - \mathrm{c}_{2+3'}\,g_g\right) \tag{485}$$

$$\tau = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{B}(\theta, \dot{\theta}) + \mathcal{G}(\theta) \tag{486}$$

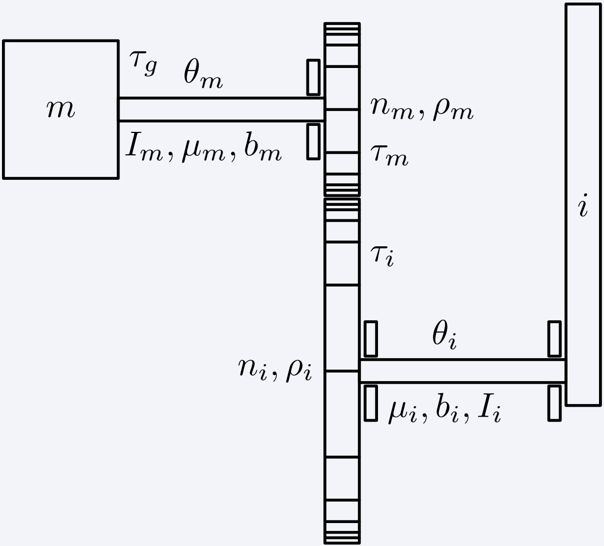
where:

$\mathcal{M}(\theta)$	–	manipulator inertia matrix
$\mathcal{B}(\theta, \dot{\theta})$	–	centrifugal and Coriolis force/torque vector
$\mathcal{G}(\theta)$	–	gravitational force/torque vector

$$\tau = \mathcal{M}(\theta)\ddot{\theta} + \mathcal{C}(\theta)\left[\dot{\theta}\dot{\theta}\right] + \mathcal{D}(\theta)\left[\dot{\theta}^2\right] + \mathcal{G}(\theta) \tag{487}$$

where:

$\mathcal{C}(\theta)$	–	Coriolis force/torque coefficient matrix
$\mathcal{D}(\theta)$	–	centrifugal force/torque coefficient matrix
$\left[\dot{\theta}\dot{\theta}\right]$	–	vector containing terms of the form: $\dot{\theta}_i\dot{\theta}_j$, where $i \neq j$
$\left[\dot{\theta}^2\right]$	–	vector containing terms of the form: $\dot{\theta}_i^2$



m
τ_g
θ_m
I_m, μ_m, b_m
n_m, ρ_m
τ_m
i
τ_i
θ_i
n_i, ρ_i
μ_i, b_i, I_i

m	–	motor side	i	–	link side
τ_m	–	transmitted torque	τ_i	–	received torque
τ_g	–	produced torque			
θ_m	–	rotor angle	θ_i	–	joint angle
ω_m	–	rotor angular velocity	ω_i	–	joint angular velocity
I_m	–	rotor+wheel inertia	I_i	–	wheel+load inertia
μ_m, μ_i	–	Coulomb friction coefficients			
b_m, b_i	–	viscous friction coefficients			
n_m, n_i	–	number of teeth			
ρ_m, ρ_i	–	gear radius			

Ideal gear $\Rightarrow$ no spring effect, no backlash, no friction

Number of teeth is proportional to radius of the wheel:

$$\frac{\rho_m}{\rho_i} = \frac{n_m}{n_i} \tag{488}$$

Usually $\frac{n_i}{n_m} > 100$

Displacement along the circumference of each wheel is the same:

$$\begin{aligned} \theta_m \rho_m &= \theta_i \rho_i \Rightarrow \frac{\theta_i}{\theta_m} = \frac{\rho_m}{\rho_i} \Rightarrow \frac{\theta_i}{\theta_m} = \frac{n_m}{n_i} \Rightarrow \theta_i = \theta_m \frac{n_m}{n_i} \\ \omega_m \rho_m &= \omega_i \rho_i \end{aligned} \tag{489}$$

Work done

$$\tau_m \theta_m = \tau_i \theta_i \;\Rightarrow\; \tau_m = \tau_i \frac{\theta_i}{\theta_m} = \tau_i \frac{n_m}{n_i} \tag{490}$$

Link side:

$$\tau_i = I_i \frac{\mathrm{d}^2\theta_i}{\mathrm{d}t^2} + b_i \frac{\mathrm{d}\theta_i}{\mathrm{d}t} + \mu_i \mathrm{sgn}\dot{\theta}_i \tag{491}$$

Motor side:

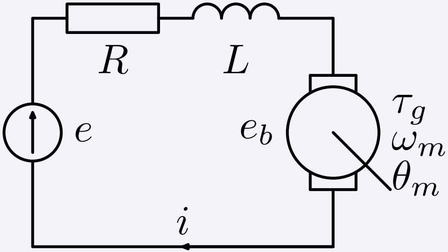
$$\tau_g = I_m \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_m \mathrm{sgn}\dot{\theta}_m + \tau_m \tag{492}$$

$$\tau_g = I_m \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_m \mathrm{sgn}\dot{\theta}_m + \tau_i \frac{n_m}{n_i} \tag{493}$$

$$\tau_g = I_m \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_m \mathrm{sgn}\dot{\theta}_m + \frac{n_m}{n_i}\left(I_i \frac{\mathrm{d}^2\theta_i}{\mathrm{d}t^2} + b_i \frac{\mathrm{d}\theta_i}{\mathrm{d}t} + \mu_i \mathrm{sgn}\dot{\theta}_i\right) \tag{494}$$

$$\tau_g = I_m \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_m \mathrm{sgn}\dot{\theta}_m + \frac{n_m}{n_i}\left(I_i \frac{n_m}{n_i} \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_i \frac{n_m}{n_i} \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_i \mathrm{sgn}\dot{\theta}_m\right) \tag{495}$$

$$\tau_g = \left(I_m + \frac{n_m^2}{n_i^2} I_i\right) \frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + \left(b_m + \frac{n_m^2}{n_i^2} b_i\right) \frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \left(\mu_m + \frac{n_m}{n_i} \mu_i\right) \mathrm{sgn}\dot{\theta}_m \tag{496}$$



e_b	–	back electromotoric force	e	–	applied voltage
R	–	rotor armature resistance	L	–	rotor armature inductance
K_i	–	torque constant	K_b	–	back-emf constant
i	–	armature current			

$$\begin{cases} e &= iR + L\dfrac{\mathrm{d}i}{\mathrm{d}t} + e_b \\ \tau_g &= K_i i \\ e_b &= K_b \dfrac{\mathrm{d}\theta_m}{\mathrm{d}t} = K_b \omega_m \\ \tau_g &= I_m \dfrac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m \dfrac{\mathrm{d}\theta_m}{\mathrm{d}t} + \mu_m \mathrm{sgn}\dot{\theta}_m + \tau_m \\ \mu_m &= 0 \quad \text{(simplifying assumption)} \end{cases} \tag{497}$$

$$\Rightarrow \begin{cases} \dfrac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{L}\Big(e - K_b\omega_m - iR\Big) \\ \dfrac{\mathrm{d}\theta_m}{\mathrm{d}t} &= \omega_m \\ \dfrac{\mathrm{d}\omega_m}{\mathrm{d}t} &= \frac{1}{I_m}\Big(K_i i - b_m\omega_m - \tau_m\Big) \end{cases} \tag{498}$$

$$\Rightarrow \begin{cases} \dfrac{\mathrm{d}i}{\mathrm{d}t} & = & -\frac{R}{L}i - \frac{K_b}{L}\omega_m + \frac{1}{L}e \\ \dfrac{\mathrm{d}\theta_m}{\mathrm{d}t} & = & \omega_m \\ \dfrac{\mathrm{d}\omega_m}{\mathrm{d}t} & = & \frac{K_i}{I_m}i - \frac{b_m}{I_m}\omega_m - \frac{1}{I_m}\tau_m \end{cases} \tag{499}$$

$$\begin{bmatrix} \dfrac{\mathrm{d}i}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\theta_m}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\omega_m}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} \\ 0 & 0 & 1 \\ \frac{K_i}{I_m} & 0 & -\frac{b_m}{I_m} \end{bmatrix} \begin{bmatrix} i \\ \theta_m \\ \omega_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} e - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_m} \end{bmatrix} \tau_m \tag{500}$$

Assuming: $\mu_m = 0$ (no dry friction) and $\tau_m = 0$ (no load; later treated as disturbance) we get:

$$\begin{cases} e &= iR + L\frac{\mathrm{d}i}{\mathrm{d}t} + K_b\frac{\mathrm{d}\theta_m}{\mathrm{d}t} \\ K_i i &= I_m\frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + b_m\frac{\mathrm{d}\theta_m}{\mathrm{d}t} \end{cases} \tag{501}$$

Applying the Laplace transform:

$$\begin{cases} e(s) &= i(s)R + Lsi(s) + K_b s\theta_m(s) \\ K_i i(s) &= I_m s^2\theta_m(s) + b_m s\theta_m \end{cases} \tag{502}$$

$$\begin{cases} e(s) &= \Big(R + Ls\Big)i(s) + K_b s\theta_m(s) \\ i(s) &= \Big(\frac{I_m}{K_i}s^2 + \frac{b_m}{K_i}s\Big)\theta_m \end{cases} \tag{503}$$

$$e(s) = \Big(R + Ls\Big)\Big(\frac{I_m}{K_i}s^2 + \frac{b_m}{K_i}s\Big)\theta_m + K_b s\theta_m(s) \tag{504}$$

$$e(s) = \frac{(R+Ls)(I_m s^2 + b_m s) + K_i K_b s}{K_i}\theta_m(s) \tag{505}$$

$$e(s) = \frac{(R+Ls)(I_m s^2+b_m s)+K_i K_b s}{K_i}\theta_m(s) \tag{506}$$

$$G(s) = \frac{\theta_m(s)}{e(s)} = \frac{K_i}{(R+Ls)(I_m s^2+b_m s)+K_i K_b s} \tag{507}$$

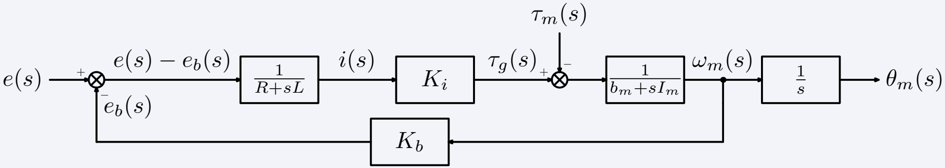
$$G(s) = \frac{K_i}{LI_m s^3+(RI_m+Lb_m)s^2+(Rb_m+K_i K_b)s} \tag{508}$$

$$G(s) = \frac{K_i}{s}\frac{1}{LI_m s^2+(RI_m+Lb_m)s+(Rb_m+K_i K_b)} \tag{509}$$

Applying the Laplace transform to (497) we get:

$$\begin{cases} e(s) &= Ri(s) + Lsi(s) + e_b(s) \\ \tau_g(s) &= K_i i(s) \\ e_b(s) &= K_b s\theta_m(s) = K_b\omega_m(s) \\ \tau_g(s) &= I_m s^2\omega_m(s) + b_m s\omega_m(s) + \theta_m(s) \\ \omega_m(s) &= s\theta_m(s) \end{cases} \tag{510}$$

$$\Rightarrow \begin{cases} i(s) &= \frac{1}{R+Ls}\Big(e(s) - e_b(s)\Big) \\ \tau_g(s) &= K_i i(s) \\ e_b(s) &= K_b\omega_m(s) \\ \omega_m(s) &= \frac{1}{I_m s+b_m}\Big(\tau_g(s) - \tau_m(s)\Big) \\ \theta_m(s) &= \frac{1}{s}\omega_m(s) \end{cases} \tag{511}$$



Power developed in the rotor winding (rotor armature) in an ideal case

$$\begin{cases} P = e_b i & - \quad \text{electric power} \\ P = \tau_g \omega_m & - \quad \text{mechanical power} \end{cases} \tag{512}$$

$$\begin{aligned} \tau_g \omega_m &= e_b i \\ K_i i \omega_m &= K_b \omega_m i \\ K_i &= K_b \end{aligned} \tag{513}$$

Nevertheless, functionally K_i and K_b are different.