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Sequent systems for compact bilinear logic

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Compact Bilinear Logic (CBL), introduced by Lambek [14], arises from the multiplicative fragment of Noncommutative Linear Logic of Abrusci [1] (also called Bilinear Logic in [13]) by identifying times with par and 0 with 1. In this paper, we present two sequent systems for CBL and prove the cut-elimination theorem for them. We also discuss a connection between cut-elimination for CBL and the Switching Lemma from [14].

1 Introduction and preliminaries

The most easy way of introducing Compact Bilinear Logic (CBL) is via its algebraic models. Let us begin with residuated monoids. A *residuated monoid* is a structure $(M, \leq, \cdot, \backslash, /, 1)$ such that $(M, \leq)$ is a poset, $(M, \cdot, 1)$ is a monoid, and $\backslash$, $/$ are binary operations on M, satisfying the equivalences: $ab \leq c$ iff $b \leq a\backslash c$ iff $a \leq c/b$, for all $a, b, c \in M$. Using the latter, one easily proves: $a \leq b$ entails $ca \leq cb$ and $ac \leq bc$, for all $a, b, c \in M$, which means that $(M, \leq, \cdot, 1)$ is a partially ordered monoid (p. o. monoid). The Lambek calculus with 1 (**L1**) is the logic of inequalities $a \leq b$ valid in residuated monoids [3].

Bilinear Logic (BL) corresponds to residuated monoids in which there exists an element 0, satisfying $0/(a\backslash 0) = a = (0/a)\backslash 0$, for all $a \in M$. One defines $a^{\mathrm{l}} = 0/a$ and $a^{\mathrm{r}} = a\backslash 0$. Then, $(b^{\mathrm{r}}a^{\mathrm{r}})^{\mathrm{l}} = (b^{\mathrm{l}}a^{\mathrm{l}})^{\mathrm{r}}$, for all $a, b \in M$; this element is denoted $a \oplus b$.

Algebraic models for CBL are residuated monoids with 0 (satisfying the equalities above) in which $0 = 1$ and $a \oplus b = ab$, for all $a, b \in M$. Lambek [14] calls them pregroups. A direct definition is given below.

A *pregroup* is a structure $(G, \leq, \cdot, ^{\mathrm{l}}, ^{\mathrm{r}}, 1)$ such that $(G, \leq, \cdot, 1)$ is a p. o. monoid and $^{\mathrm{l}}$, $^{\mathrm{r}}$ are unary operations on G, satisfying, for all $a \in G$,

(1) $$a^{\mathrm{l}}a \leq 1 \leq aa^{\mathrm{l}} \quad \text{and} \quad aa^{\mathrm{r}} \leq 1 \leq a^{\mathrm{r}}a.$$

The elements a^{l} and a^{r} are called *the left adjoint* and *the right adjoint*, respectively, of a. In every pregroup the following conditions are valid:

(2) $$1^{\mathrm{l}} = 1 = 1^{\mathrm{r}}, \quad (a^{\mathrm{l}})^{\mathrm{r}} = a = (a^{\mathrm{r}})^{\mathrm{l}},$$

(3) $$(ab)^{\mathrm{l}} = b^{\mathrm{l}}a^{\mathrm{l}}, \quad (ab)^{\mathrm{r}} = b^{\mathrm{r}}a^{\mathrm{r}},$$

(4) $$a \leq b \text{ iff } b^{\mathrm{l}} \leq a^{\mathrm{l}} \text{ iff } b^{\mathrm{r}} \leq a^{\mathrm{r}}.$$

CBL can be described as the logic of inequalities $a \leq b$ valid in pregroups. It naturally belongs to substructural logics, as they have been characterized in [15]. It is genuinely noncommutative: commutative pregroups must satisfy $a^{\mathrm{l}} = a^{\mathrm{r}}$, for all elements a, hence they amount to commutative p. o. groups [9].

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After Lambek [14], CBL is used as a formalism for natural language grammar, alternative to the Lambek calculus [2, 7, 8]. In [4], grammars based on CBL are proven to be equivalent to context-free grammars. Algebraic properties of pregroups are studied in [4, 6].

In the existing literature, CBL has been presented in the form of a rewriting system. The present paper provides the first 'logical' axiomatizations of CBL. Below we briefly recall the rewriting system from [14]. First, we define iterated adjoints in pregroups.

Let $\mathbb{Z}$ denote the set of integers. Let G be a pregroup. For $a \in G$ and $n \in \mathbb{Z}$, one defines an element $a^{(n)} \in G$, as follows:

(5) $$a^{(0)} = a, \quad a^{(n+1)} = (a^{(n)})^{\mathrm{r}}, \quad a^{(n-1)} = (a^{(n)})^{\mathrm{l}}.$$

By (2), this definition is correct. Also, the following conditions are valid in pregroups, for all elements a, b and $n \in \mathbb{Z}$:

(6) $$(ab)^{(2n)} = a^{(2n)}b^{(2n)}, \quad (ab)^{(2n+1)} = b^{(2n+1)}a^{(2n+1)},$$

(7) $$a \leq b \text{ iff } a^{(2n)} \leq b^{(2n)} \text{ iff } b^{(2n+1)} \leq a^{(2n+1)}.$$

Let $(P, \leq)$ be a nonempty poset. *Atoms* are expressions of the form $p^{(n)}$, for $p \in P$ and $n \in \mathbb{Z}$. We assume $p = p^{(0)}$. Finite strings of atoms are called *types* and are denoted by X, Y, Z, U, V. Basic rewriting rules are the following:

(CON) Contraction: $X,\, p^{(n)},\, p^{(n+1)}, Y \to X,\, Y$;

(EXP) Expansion: $X,\, Y \to X,\, p^{(n+1)},\, p^{(n)},\, Y$;

(IND) $X,\, p^{(2n)},\, Y \to X,\, q^{(2n)},\, Y$ and $X,\, q^{(2n+1)},\, Y \to X,\, p^{(2n+1)},\, Y$, if $p \leq q$ in $(P, \leq)$ (Induced Steps).

The relation $\Rightarrow$ is defined as the reflexive and transitive closure of the relation $\to$.

We define an equivalence relation $\sim$ by $X \sim Y$ iff both $X \Rightarrow Y$ and $Y \Rightarrow X$. It is nontrivial, since e. g. $p,\, p^{(1)},\, p \sim p$. (Notice that $aa^{\mathrm{l}}a = a$ and $aa^{\mathrm{r}}a = a$ are valid in pregroups.) Formal adjoints of types are defined as follows: $\varepsilon^{\mathrm{l}} = \varepsilon^{\mathrm{r}} = \varepsilon$ (ε denotes the empty string), and

(8) $$(p^{(n)})^{\mathrm{l}} = p^{(n-1)}, \quad (p^{(n)})^{\mathrm{r}} = p^{(n+1)},$$

(9) $$(a_1 \dots a_k)^{\mathrm{l}} = a_k^{\mathrm{l}} \dots a_1^{\mathrm{l}}, \quad (a_1 \dots a_k)^{\mathrm{r}} = a_k^{\mathrm{r}} \dots a_1^{\mathrm{r}},$$

for all $p \in P$, $n \in \mathbb{Z}$, $k \geq 1$, and atoms $a_1, \dots, a_k$. One easily shows that $X \sim Y$ entails $Z,\, X \sim Z,\, Y$, $X,\, Z \sim Y,\, Z$, $X^{\mathrm{l}} \sim Y^{\mathrm{l}}$ and $X^{\mathrm{r}} \sim Y^{\mathrm{r}}$. (First, show analogous conditions for $\Rightarrow$.) Then, on equivalence classes of $\sim$ one can define operations $\cdot$, $^{\mathrm{l}}$, $^{\mathrm{r}}$, an ordering $\leq$ and an element 1 as follows:

(10) $$[X] \cdot [Y] = [X, Y], \quad [X]^{\mathrm{l}} = [X^{\mathrm{l}}], \quad [X]^{\mathrm{r}} = [X^{\mathrm{r}}],$$

(11) $$[X] \leq [Y] \text{ iff } X \Rightarrow Y, \quad 1 = [\varepsilon].$$

The structure defined above is a pregroup. It is called *the free pregroup generated by the poset* $(P, \leq)$.

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets. A mapping $f : P_1 \longrightarrow P_2$ is said to be *order preserving*, if $a \leq_1 b$ entails $f(a) \leq_2 f(b)$, for all a, $b \in P_1$. It is easy to prove that for any pregroup G, every order preserving mapping $h : P \longrightarrow G$ is uniquely extendable to a homomorphism of the free pregroup generated by $(P, \leq)$ into G. Further, $[X] \leq [Y]$ is true in the free pregroup if and only if $h([X]) \leq h([Y])$ is true for all homomorphisms h, obtained in this way (all homomorphisms, indeed). Since $[X] \leq [Y]$ is equivalent to $X \Rightarrow Y$, then the rewriting system described above yields the inequalities valid in pregroups.

Lambek [14] proves an important combinatorial property of this system. One introduces new rewriting rules: Generalized Contraction (GCON) and Generalized Expansion (GEXP) as follows:

(GCON) $X,\ p^{(n)},\ q^{(n+1)},\ Y \to X,\ Y$ if either n is even and $p \leq q$ in P, or n is odd and $q \leq p$ in P;

(GEXP) $X,\ Y \to X,\ p^{(n+1)},\ q^{(n)},\ Y$ if either n is even and $p \leq q$ in P, or n is odd and $q \leq p$ in P.

Clearly, (GCON) amounts to (IND) followed by (CON), and (GEXP) amounts to (EXP) followed by (IND). The Lambek Switching Lemma is the following assertion: If $X \Rightarrow Y$, then there exist U, V such that $X \Rightarrow U$, $U \Rightarrow V$ and $V \Rightarrow Y$, and X can be transformed into U by (GCON) only, U into V by (IND) only, and V into Y by (GEXP) only. As a consequence, if $X \Rightarrow Y$ and Y is an atom or the empty string, then X can be transformed into Y by (CON) and (IND) only; also, if $X \Rightarrow Y$ and X is an atom or the empty string, then X can be transformed into Y by (EXP) and (IND) only. In [4], this fact is used to prove that grammars based on this system generate context-free languages; actually, they are weakly equivalent to context-free grammars (without ε-productions). Another important consequence is the polynomial time decidability of $X \Rightarrow Y$. For $X \Rightarrow Y$ iff $Y^1,\ X \Rightarrow \varepsilon$, and the latter must be derived by (CON) and (IND) only. These derivations can be simulated by a context-free grammar whose size is polynomially bounded by the size of the entry and the size of P, hence a standard parsing algorithm for context-free grammars yields a polynomial time decision procedure for this system.

In Section 2 we present a Gentzen style sequent system equivalent to this rewriting system and prove cut elimination. Actually, we make it clear that the cut elimination theorem for the new system is closely connected with the Switching Lemma for the rewriting system. In Section 3 we present a one-side sequent system for CBL, which is similar to systems for Classical Linear Logic [10, 12], and prove cut elimination.

2 A two-side sequent system for CBL

The rewriting system described in Section 1 can also be presented as a Gentzen style sequent calculus. Let $(P, \leq)$ be fixed. Atoms and types are defined as above. *Sequents* are of the form $X \Rightarrow Y$, where X, Y are types. The axioms and the inference rules of the calculus are the following:

(Id) $X \Rightarrow X$,

$$\text{(LA)}\ \frac{X, Y \Rightarrow Z}{X, p^{(n)}, p^{(n+1)}, Y \Rightarrow Z}, \qquad \text{(RA)}\ \frac{X \Rightarrow Y, Z}{X \Rightarrow Y, p^{(n+1)}, p^{(n)}, Z},$$

$$\text{(LIND)}\ \frac{X, p^{(n)}, Y \Rightarrow Z}{X, q^{(n)}, Y \Rightarrow Z}, \qquad \text{(RIND)}\ \frac{X \Rightarrow Y, p^{(n)}, Z}{X \Rightarrow Y, q^{(n)}, Z}.$$

(LIND) is limited by $q \leq p$ if n is even, and by $p \leq q$ if n is odd. (RIND) is limited by $p \leq q$ if n is even, and by $q \leq p$ if n is odd. X, Y, Z are arbitrary types, p, q are arbitrary elements of P, and n is an arbitrary integer.

The cut rule is the following:

$$\text{(CUT)}\ \frac{X \Rightarrow Y, \quad Y \Rightarrow Z}{X \Rightarrow Z}.$$

Let S denote the system axiomatized by (Id), (LA), (RA), (LIND) and (RIND), and let S' denote S enriched with (CUT).

Proposition 1 *For all types X, Y, $X \Rightarrow Y$ holds in the sense of the rewriting system if and only if $X \Rightarrow Y$ is provable in S'.*

Proof. Assume $X \Rightarrow Y$ hold in the sense of the rewriting system. Then, there exist types $Z_0, \ldots, Z_n$, $n \geq 0$, such that $Z_0 = X$, $Z_n = Y$, and $Z_{i-1} \to Z_i$, for all $1 \leq i \leq n$. We show that $Z_{i-1} \Rightarrow Z_i$ is provable in S', for all $1 \leq i \leq n$. If $Z_{i-1} \to Z_i$ is (CON), then we apply (LA) to axiom $Z_i \Rightarrow Z_i$. If $Z_{i-1} \to Z_i$ is (EXP), then we apply (RA) to axiom $Z_{i-1} \Rightarrow Z_{i-1}$. If $Z_{i-1} \to Z_i$ is (IND), then we apply (RIND) to axiom $Z_{i-1} \Rightarrow Z_{i-1}$; we can also apply (LIND) to axiom $Z_i \Rightarrow Z_i$. So, if $n = 0$, then $X \Rightarrow Y$ is axiom (Id), and if $n > 0$, then $X \Rightarrow Y$ is provable in S', using (CUT).

Assume $X \Rightarrow Y$ be provable in S'. We prove that $X \Rightarrow Y$ holds in the rewriting system by induction on axioms and rules of S'. If $X \Rightarrow Y$ is (Id), then the claim is true. For inference rules, we show that if the premise (premises) holds (hold) in the rewriting system, then the conclusion holds in this system. For (LA), the antecedent of the conclusion can be transformed into the antecedent of the premise by (CON). For (RA), the consequent of the premise can be transformed into the consequent of the conclusion by (EXP). For (LIND), the antecedent of the conclusion can be transformed into the antecedent of the premise by (IND). For (RIND), the consequent of the premise can be transformed into the consequent of the conclusion by (IND). For (CUT), if the premises hold in the rewriting system, then the conclusion also holds in this system, since $\Rightarrow$ is transitive. □

It follows from this proof that S' is equivalent to S' without (LIND) (or without (RIND)). We prove a form of cut elimination.

Theorem 1 *For all types X, Y, $X \Rightarrow Y$ is provable in S if and only if $X \Rightarrow Y$ is provable in S'.*

Proof. The 'only if' part is obvious. So, assume that $X \Rightarrow Y$ be provable in S'. By Proposition 1, $X \Rightarrow Y$ holds in the rewriting system. By the Switching Lemma, there exist types U, V such that $X \Rightarrow U$ holds by (GCON) only, $U \Rightarrow V$ holds by (IND) only, and $V \Rightarrow Y$ holds by (GEXP) only. Consequently, $X \Rightarrow U$ holds by (CON) and (IND) only, and $U \Rightarrow Y$ holds by (EXP) and (IND) only. Then, there exist types $Z_0, \dots, Z_m$, $m \geq 0$, such that $Z_0 = X$, $Z_m = U$, and, for all $1 \leq i \leq m$, $Z_{i-1} \rightarrow Z_i$ is either (CON) or (IND). We show that $Z_i \Rightarrow U$ is provable in S, for all $0 \leq i \leq m$. $Z_m \Rightarrow U$ is axiom (Id). Assume $Z_i \Rightarrow U$ be provable in S, $i > 0$. If $Z_{i-1} \rightarrow Z_i$ is (CON), then $Z_{i-1} \Rightarrow U$ is the result of applying (LA) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is (IND), then $Z_{i-1} \Rightarrow U$ is the result of applying (LIND) to $Z_i \Rightarrow U$. Consequently, $X \Rightarrow U$ is provable in S'.

Now, there exist types $V_0, \dots, V_n$, $n \geq 0$, such that $V_0 = U$, $V_n = Y$, and, for all $1 \leq i \leq n$, $V_{i-1} \rightarrow V_i$ is either (EXP) or (IND). We show that $X \Rightarrow V_i$ is provable in S, for all $0 \leq i \leq n$. $X \Rightarrow V_0$ is provable in S, by the first part of the proof. Assume $X \Rightarrow V_{i-1}$ be provable in S, $1 \leq i$. If $V_{i-1} \rightarrow V_i$ is (EXP), then $X \Rightarrow V_i$ is the result of applying (RA) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is (IND), then $X \Rightarrow V_i$ is the result of applying (RIND) to $X \Rightarrow V_{i-1}$. We have shown that $X \Rightarrow Y$ is provable in S. □

This proof suggests that both (LIND) and (RIND) are necessary in S. Indeed, if $p \leq q$ in P, then from axiom $\varepsilon \Rightarrow \varepsilon$ we infer $\varepsilon \Rightarrow p^{(1)}, p$, by (RA), hence $\varepsilon \Rightarrow p^{(1)}, q$, by (RIND). The latter sequent cannot be a conclusion of (LIND). It has exactly two proofs in S; the other one produces $\varepsilon \Rightarrow q^{(1)}, q$, by (RA), hence $\varepsilon \Rightarrow p^{(1)}, q$, by (RIND). Consequently, (RIND) is necessary in S. A dual argument shows that (LIND) is necessary. In S', this sequent can be proven, using (LIND): $p^{(1)}, p \Rightarrow p^{(1)}, q$ holds, by (LIND), and one uses (CUT) with the left premise $\varepsilon \Rightarrow p^{(1)}, p$.

Of course, Theorem 1 can also be proven directly, not using the Switching Lemma. One can show that (CUT) is admissible in S: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ are provable in S, then $X \Rightarrow Z$ is so. We introduce rules (GLA) and (GRA). (GLA) arises from (LA) by replacing $p^{(n)}$ with $q^{(n)}$ and requiring: $q \leq p$ if n is even, and $p \leq q$ if n is odd. (GRA) arises from (RA) in the same way, but now we require: $p \leq q$ if n is even, and $q \leq p$ if n is odd. Clearly, (GLA) amounts to (LA) followed by (LIND), and (GRA) amounts to (RA) followed by (RIND). Accordingly, (GLA) and (GRA) are derivable in S, hence the system S'' axiomatized by (Id), (GLA), (GRA), (LIND) and (RIND) is equivalent to S. We prove that (CUT) is admissible in S''. We proceed by induction on proofs of $X \Rightarrow Y$ and $Y \Rightarrow Z$ in S''. The interesting cases are the following.

(i) $X \Rightarrow Y$ follows from $X \Rightarrow Y', Y''$, by (GRA), with $Y = Y', p^{(n+1)}, q^{(n)}, Y''$, and $Y \Rightarrow Z$ is a result of applying (LIND) which touches one of the types $p^{(n+1)}, q^{(n)}$. If (LIND) touches $p^{(n+1)}$, then the premise of (LIND) is $Y', r^{(n+1)}, q^{(n)}, Y'' \Rightarrow Z$. Then, we change the instance of (GRA) to yield $X \Rightarrow Y', r^{(n+1)}, q^{(n)}, Y''$, drop the application of (LIND) and use the induction hypothesis. We only need to confirm the requirements to be fulfilled. If n is even, then $p \leq q$ and $r \leq p$, hence $r \leq q$, as required. If n is odd, then $q \leq p$ and $p \leq r$, hence $q \leq r$, as required. If (LIND) touches $q^{(n)}$, then we reason in a similar way.

(ii) $X \Rightarrow Y$ follows from $X \Rightarrow Y', p^{(n)}, Y''$, by (RIND), with $Y = Y', q^{(n)}, Y''$, and $Y \Rightarrow Z$ is the result of applying (GLA) which touches $q^{(n)}$. Again, we can change the instance of (GLA), drop the application of (RIND), and use the induction hypothesis.

(iii) $X \Rightarrow Y$ results from applying (RIND), and $Y \Rightarrow Z$ results from applying (LIND), and both applications of rules touch the same type in Y. Then, these two applications can be reduced to one application of (RIND).

(iv) $X \Rightarrow Y$ results from applying (GRA), and $Y \Rightarrow Z$ results from applying (GLA), and the types introduced by these rules overlap. Say, the premise of (GRA) is $X \Rightarrow Y', Y''$, the premise of (GLA) is $U, U' \Rightarrow Z$, $Y = Y', p^{(n+1)}, q^{(n)}, Y''$, $U = Y', p^{(n+1)}$, $Y'' = r^{(n+1)}, U'$. Then, (RIND) applied to $X \Rightarrow Y', Y''$ yields $X \Rightarrow U', U''$, and we use the induction hypothesis. The other case is treated in a similar way.

Conversely, from the cut-elimination theorem for S'' it is easy to derive the following weaker form of the Switching Lemma: if $X \Rightarrow Y$, then there exists type U such that $X \Rightarrow U, U \Rightarrow Y$, X can be transformed into U by (CON) and (IND) only, and U can be transformed into Y by (EXP) and (IND) only. Assume $X \Rightarrow Y$ holds. Then, $X \Rightarrow Y$ has a proof in S''. We proceed by induction on this proof. If $X \Rightarrow Y$ is (Id), there is nothing to prove. Let $X \Rightarrow Y$ arise by (GLA) from $X', X'' \Rightarrow Y$. Then, $X = X', q^{(n)}, p^{(n+1)}, X''$, and either n is even and $q \leq p$, or n is odd and $p \leq q$. By the induction hypothesis, there exists a type U such that $X'X''$ can be transformed into U by (CON) and (IND) only, and U can be transformed into Y by (EXP) and (IND) only. Since X can be transformed into $X'X''$ by (IND) and (CON), we get the thesis. The cases of (GRA), (LIND) and (RIND) are treated in a similar way. This weaker form of the Switching Lemma is sufficient for applications (polynomial time decidability, equivalence with context-free grammars).

3 A one-side sequent system for CBL

We present an axiomatization of CBL in the form of a one-side sequent system admitting cut elimination.

Atoms are 1 and types $p^{(n)}$ such that p is a variable and $n \in \mathbb{Z}$. Formulas are built from atoms by $\otimes$: if A, B are formulas, then $(A \otimes B)$ is a formula. Adjoints are defined in the metalanguage. For an atom A, A^{l} and A^{r} are defined by (8) with $1^{\mathrm{l}} = 1^{\mathrm{r}} = 1$. We also define:

(12) $$(A \otimes B)^{\mathrm{l}} = B^{\mathrm{l}} \otimes A^{\mathrm{l}}, \quad (A \otimes B)^{\mathrm{r}} = B^{\mathrm{r}} \otimes A^{\mathrm{r}}.$$

One proves $(A^{\mathrm{l}})^{\mathrm{r}} = A = (A^{\mathrm{r}})^{\mathrm{l}}$, by induction on A.

Sequents are finite sequences of formulas. Sequents are denoted by Γ, Δ, Φ. Given a pregroup G, any assignment of elements of G to variables can uniquely be extended to an assignment of elements of G to all formulas (we interpret $\otimes$ as $\cdot$). A formula A is *true under assignment* h if $1 \leq h(A)$ in G, and a sequent $A_1, \dots, A_n$ is *true* if the formula $A_1 \otimes \dots \otimes A_n$ is true. (The empty sequent is always true.) The system CBL described below yields all sequents true in all pregroups under all assignments.

The axiom is ε. The inference rules are the following:

$$(1\,\mathrm{I})\,\frac{\Gamma, \Delta}{\Gamma, 1, \Delta}, \quad (\otimes\,\mathrm{I})\,\frac{\Gamma, A, B, \Delta}{\Gamma, A \otimes B, \Delta}, \quad (\mathrm{l\,I})\,\frac{\Gamma, \Delta}{\Gamma, A, A^{\mathrm{l}}, \Delta}, \quad (\mathrm{r\,I})\,\frac{\Gamma, \Delta}{\Gamma, A^{\mathrm{r}}, A, \Delta}.$$

We restrict (l I) and (r I) to $A = p^{(n)}$. Then, (l I) and (r I) are the same rule. We write $\vdash \Gamma$ if Γ is provable in this system.

Lemma 1 *If* $\vdash \Gamma, \Delta$*, then* $\vdash \Gamma, A, A^{\mathrm{l}}, \Delta$ *and* $\vdash \Gamma, A^{\mathrm{r}}, A, \Delta$*, for every formula* A.

Proof. By induction on A. For $A = p^{(n)}$, it holds, by (l I) and (r I). For $A = 1$, it holds, by (1 I). Let $A = B \otimes C$. Then, $A^{\mathrm{l}} = C^{\mathrm{l}} \otimes B^{\mathrm{l}}$. Assume $\vdash \Gamma, \Delta$. Then, $\vdash \Gamma, B, B^{\mathrm{l}}, \Delta$, hence $\vdash \Gamma, B, C, C^{\mathrm{l}}, B^{\mathrm{l}}, \Delta$, by the induction hypothesis, and we apply ($\otimes$ I). For A^{r}, the argument is similar. □

Lemma 2 *If* $\vdash \Gamma, \Delta$ *and* $\vdash \Phi$*, then* $\vdash \Gamma, \Phi, \Delta$.

Proof. By induction on the proof of Φ. □

Lemma 3 *If* $\vdash \Gamma, 1, \Delta$*, then* $\vdash \Gamma, \Delta$.

Proof. By induction on the proof of $\Gamma, 1, \Delta$. □

Lemma 4 *If* $\vdash \Gamma, A \otimes B, \Delta$*, then* $\vdash \Gamma, A, B, \Delta$.

Proof. By induction on the proof of $\Gamma, A \otimes B, \Delta$. □

Notice that the next lemma is not a reversal of rules (l I) and (r I). It is crucial for cut elimination.

Lemma 5 *If* $\vdash \Gamma, A^{\mathrm{l}}, A, \Delta$, *then* $\vdash \Gamma, \Delta$. *If* $\vdash \Gamma, A, A^{\mathrm{r}}, \Delta$, *then* $\vdash \Gamma, \Delta$.

P r o o f. We proceed by induction on A. For $A = 1$, we use Lemma 3. Let $A = p^{(n)}$. We switch on induction on the proof of $\vdash \Gamma, A^{\mathrm{l}}, A, \Delta$. We only consider the nontrivial case: this sequent results by application of (l I) which introduces one of the formulas A^{l}, A.

(i) The rule introduces A^{l}. Then, $\Gamma = \Gamma', p^{(n)}$ and the premise is $\Gamma', p^{(n)}, \Delta$. So, the premise equals Γ, Δ.

(ii) The rule introduces A. Then, $\Delta = p^{(n-1)}, \Delta'$ and the premise is $\Gamma, p^{(n-1)}, \Delta'$. Again, the premise equals Γ, Δ.

Let $A = B \otimes C$. Our assumption is $\vdash \Gamma, C^{\mathrm{l}} \otimes B^{\mathrm{l}}, B \otimes C, \Delta$, so we can use Lemma 4 and the induction hypothesis. The second part of the lemma can be proven in a similar way. □

We prove the cut elimination theorem.

Theorem 2 *If* $\vdash \Gamma, A^{\mathrm{l}}$ *and* $\vdash A, \Delta$, *then* $\vdash \Gamma, \Delta$. *If* $\vdash \Gamma, A$ *and* $\vdash A^{\mathrm{r}}, \Delta$, *then* $\vdash \Gamma, \Delta$.

P r o o f. Assume $\vdash \Gamma, A^{\mathrm{l}}$ and $\vdash A, \Delta$. By Lemma 2, we infer $\vdash \Gamma, A^{\mathrm{l}}, A, \Delta$, and consequently, $\vdash \Gamma, \Delta$, by Lemma 5. The second part is analogous. □

We define Γ^{l} and Γ^{r} by setting $\varepsilon^{\mathrm{l}} = \varepsilon^{\mathrm{r}} = \varepsilon$ and

$$(A_1, \ldots, A_n)^{\mathrm{l}} = A_n^{\mathrm{l}}, \ldots, A_1^{\mathrm{l}}, \quad (A_1, \ldots, A_n)^{\mathrm{r}} = A_n^{\mathrm{r}}, \ldots, A_1^{\mathrm{r}}.$$

Lemma 6 $\vdash \Gamma^{\mathrm{r}}, \Delta$ *if and only if* $\vdash \Delta, \Gamma^{\mathrm{l}}$.

P r o o f. Assume $\vdash \Gamma^{\mathrm{r}}, \Delta$. By Lemma 1 we get $\vdash \Gamma, \Gamma^{\mathrm{l}}$, hence $\vdash \Gamma, \Gamma^{\mathrm{r}}, \Delta, \Gamma^{\mathrm{l}}$, by Lemma 2. Then, $\vdash \Delta, \Gamma^{\mathrm{l}}$, by Lemma 5. The converse conditional can be proven in a similar way. □

We write $\vdash \Gamma \Rightarrow \Delta$ iff $\vdash \Gamma^{\mathrm{r}}, \Delta$ (or equivalently: $\vdash \Delta, \Gamma^{\mathrm{l}}$). We prove another form of cut elimination.

Theorem 3 *If* $\vdash \Gamma \Rightarrow \Delta$ *and* $\vdash \Delta \Rightarrow \Phi$, *then* $\vdash \Gamma \Rightarrow \Phi$.

P r o o f. Assume $\vdash \Gamma \Rightarrow \Delta$ and $\vdash \Delta \Rightarrow \Phi$. Then, $\vdash \Gamma^{\mathrm{r}}, \Delta$ and $\vdash \Delta^{\mathrm{r}}, \Phi$. Theorem 2 yields $\vdash \Gamma^{\mathrm{r}}, \Phi$, hence $\vdash \Gamma \Rightarrow \Phi$. □

Now, it is easy to prove that this system is complete with respect to the semantics of pregroups. Evidently, it is sound. The completeness can be proven by a Lindenbaum style construction. We define $A \sim B$ iff $\vdash A \Rightarrow B$ and $\vdash B \Rightarrow A$. By Theorem 3, $\sim$ is an equivalence relation. Further, it is a congruence on the syntactic algebra of CBL. To show this fact assume $\vdash A \Rightarrow B$. Then, $\vdash A^{\mathrm{r}}, B$, hence we get $\vdash A^{\mathrm{r}}, C^{\mathrm{r}}, C, B$, by (r I), and consequently, $\vdash (C \otimes A)^{\mathrm{r}}, (C \otimes B)$, by ($\otimes$ I) and the definition of adjoints. Thus, $\vdash C \otimes A \Rightarrow C \otimes B$. In a similar way one shows $\vdash A \otimes C \Rightarrow B \otimes C$. Also, $\vdash A^{\mathrm{r}}, B$ yields $\vdash B, A^{\mathrm{l}}$, by Lemma 6, which is $\vdash (B^{\mathrm{l}})^{\mathrm{r}}, A^{\mathrm{l}}$, and consequently, $\vdash B^{\mathrm{l}} \Rightarrow A^{\mathrm{l}}$. In a similar way one shows $\vdash B^{\mathrm{r}} \Rightarrow A^{\mathrm{r}}$. On the equivalence classes of $\sim$ we define operations $\cdot$, $^{\mathrm{l}}$, $^{\mathrm{r}}$ and the ordering $\leq$ by setting:

$$[A] \cdot [B] = [A \otimes B], \quad [A]^{\mathrm{l}} = [A^{\mathrm{l}}], \quad [A]^{\mathrm{r}} = [A^{\mathrm{r}}], \quad [A] \leq [B] \text{ iff } \vdash A \Rightarrow B.$$

The quotient structure is a pregroup with [1] being the unit. We have $\vdash A, A^{\mathrm{l}}$, by (l I), hence $\vdash 1, A, A^{\mathrm{l}}$, by (1 I), and $\vdash 1, A \otimes A^{\mathrm{l}}$, by ($\otimes$ I), which yields $[1] \leq [A] \cdot [A]^{\mathrm{l}}$. The condition $[A]^{\mathrm{l}} \cdot [A] \leq [1]$ means $\vdash A^{\mathrm{l}} \otimes A \Rightarrow 1$, and the latter amounts to $\vdash A^{\mathrm{r}} \otimes A, 1$ which is provable, by (r I), (1 I) and ($\otimes$ I). In a similar way one proves the inequalities for right adjoints, associativity and monotonicity. Accordingly, if $\vdash A$ does not hold, then $\vdash 1, A$ does not hold either, by Lemma 3, hence $[1] \leq [A]$ is not true, and consequently, the formula A is not true in this pregroup under the assignment $h(p) = [p]$, for variables p.

This system can be modified to admit a fixed ordering on the set of variables. Now, the variables should be treated as propositional constants, being elements of a poset $(P, \leq)$. We add a new rule:

$$\text{(RO)} \; \frac{\Gamma, p^{(n)}, \Delta}{\Gamma, q^{(n)}, \Delta},$$

licensed by $p \le q$ if n is even, and $q \le p$ if n is odd. All results of this section can be proven for this extended system. Only the proof of Lemma 5 requires some change. As in Section 2, we generalize rule (l I) (for atoms A) to the following one:

$$(\mathrm{Gl\,I})\ \frac{\Gamma, \Delta}{\Gamma, p^{(n+1)}, q^{(n)}, \Delta},$$

licensed as for (RO). Clearly, (Gr I) would be the same rule. (Gl I) amounts to (l I) followed by (RO), so the system CBL′ which results from CBL by replacing (l I) (for atoms A) with (Gl I) is equivalent to CBL. Accordingly, it is sufficient to prove Lemma 5 for CBL′, which is not difficult.

The latter system is equivalent to the rewriting system from Section 1 in the following sense: $\vdash \Gamma \Rightarrow \Delta$ iff $\Gamma^\circ \Rightarrow \Delta^\circ$ in the rewriting system, where Γ° arises from Γ by replacing all occurrences of $\otimes$ by commas and dropping parentheses. This fact can be shown in a syntactic way (by induction on proofs) or in a semantic way (using the completeness theorem).

We leave it as an open problem whether a cut-free axiomatization can be found for CBL with additive conjunction $\wedge$ and additive disjunction $\vee$, to be interpreted as lattice operations in pregroups.

Let us comment more on this point. In lattice ordered pregroups the following equalities are true:

$$\begin{aligned} a(b \vee c) &= ab \vee ac, & (a \vee b)c &= ac \vee bc,\\ a(b \wedge c) &= ab \wedge ac, & (a \wedge b)c &= ac \wedge bc,\\ (a \vee b)^{\mathrm{l}} &= a^{\mathrm{l}} \wedge b^{\mathrm{l}}, & (a \vee b)^{\mathrm{r}} &= a^{\mathrm{r}} \wedge b^{\mathrm{r}},\\ (a \wedge b)^{\mathrm{l}} &= a^{\mathrm{l}} \vee b^{\mathrm{l}}, & (a \wedge b)^{\mathrm{r}} &= a^{\mathrm{r}} \vee b^{\mathrm{r}}, \end{aligned}$$

for all elements a, b, c. One might try to axiomatize CBL with $\vee$ and $\wedge$ by affixing to the above system for CBL the new rules:

$$(\vee\,\mathrm{I1})\ \frac{\Gamma, A, \Delta}{\Gamma, A \vee B, \Delta}, \quad (\vee\,\mathrm{I2})\ \frac{\Gamma, B, \Delta}{\Gamma, A \vee B, \Delta}, \quad (\wedge\,\mathrm{I})\ \frac{\Gamma, A, \Delta; \quad \Gamma, B, \Delta}{\Gamma, A \wedge B, \Delta}.$$

Now, formulas are built from atoms by $\otimes$, $\vee$, $\wedge$, and adjoints are defined in the metalanguage by (12) and

$$(A \vee B)^{\mathrm{a}} = A^{\mathrm{a}} \wedge B^{\mathrm{a}}, \quad (A \wedge B)^{\mathrm{a}} = A^{\mathrm{a}} \vee B^{\mathrm{a}},$$

for $\mathrm{a} \in \{\mathrm{l}, \mathrm{r}\}$.

Unfortunately, the so-obtained system does not admit cut elimination. The formula

$$((A^{\mathrm{r}} \wedge B^{\mathrm{r}}) \otimes A) \vee ((A^{\mathrm{r}} \wedge B^{\mathrm{r}}) \otimes B)$$

is not provable, since neither $(A^{\mathrm{r}} \wedge B^{\mathrm{r}}) \otimes A$ nor $(A^{\mathrm{r}} \wedge B^{\mathrm{r}}) \otimes B$ is valid. But, this formula is valid, and it is provable with the aid of cut, since $(A^{\mathrm{r}} \wedge B^{\mathrm{r}}) \otimes (A \vee B)$ is provable without cut, and, for any formulas A, B, C, the sequent $A \otimes (B \vee C) \Rightarrow (A \otimes B) \vee (A \otimes C)$ is provable without cut (in the sense explained above).

A pregroup is said to be *proper* if it is not a p. o. group, that means, $a^{\mathrm{l}} \neq a^{\mathrm{r}}$, for some element a. In [4, 6], it has been shown that no finite pregroups and no totally ordered pregroups are proper. As a consequence, CBL does not have the finite model property (in opposition to e. g. **L1**, MALL and Cyclic MALL; see [12, 5]). There exist proper lattice ordered pregroups. A natural example is the pregroup of all order preserving functions on $\mathbb{Z}$ which are downward and upward unbounded. We set

$$f \le g \text{ iff } f(x) \le g(x) \text{ for all } x, \quad f^{\mathrm{l}}(x) = \min\{y : x \le f(y)\}, \quad f^{\mathrm{r}}(x) = \max\{y : f(y) \le x\}.$$

Also, $fg = f \circ g$ and $1 = \mathrm{Id}$. For $f(x) = 2x$, one gets $f^{\mathrm{l}}(x) = [(x+1)/2]$ and $f^{\mathrm{r}}(x) = [x/2]$, hence $f^{\mathrm{l}} \neq f^{\mathrm{r}}$ [14]; here $[x]$ stands for the greatest integer $m \le x$, and / stands for rational division. We have found a nice formula for iterated adjoints: $f^{(2n)}(x) = f(x+n) - n$ holds for all f in this structure. Let $n = 1$. Then, $y \le f^{(2)}(x)$ iff $f^{(1)}(y) \le x$ iff $\neg(x < f^{(1)}(y))$ iff $\neg(x + 1 \le f^{(1)}(y))$ iff $\neg(f(x+1) \le y)$ iff $y < f(x+1)$ iff $y \le f(x+1) - 1$. Thus, $f^{(2)}(x) = f(x+1) - 1$. By substituting $f^{(-2)}$ for f, we get $f^{(-2)}(x) = f(x-1) + 1$. Then, the required equality is proven by induction on n.

Accordingly, this structure is a proper pregroup. It forms a distributive lattice with operations

$$(f \wedge g)(x) = \min(f(x), g(x)), \quad (f \vee g)(x) = \max(f(x), g(x)).$$

It is worth noticing that CBL with (CUT) and either $\top$ or $\bot$ is inconsistent (all sequents are provable), which follows from the fact that every pregroup containing either $\top$, or $\bot$ is a one-element structure [6].

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