Introduction to Probability and Decision Theory

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(Adapted from joint slides with Michael Franke)

Probability

Definition:

Given a set of worlds W, a probability distribution over W is a function $P : W \to [0, 1]$ that satisfies the following properties:

- $P(W') = \sum_{w \in W'} P(\{w\}) \text{ for all } W' \subset W$

  $P(W) = 1$

Probability - terminology

$$P(x) \propto f(x) \text{ means } P(x) = \frac{f(x)}{\sum_{x'} f(x')}$$

$$P(x) \text{ means } P(\{x\})$$

Conditional probability

Let $P \in \Delta(T)$. The conditional probability of event $X \subseteq T$ given event $Y \subseteq T$ is calculated by Bayesian update as follows:

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}. \qquad \text{[only defined if } P(Y) \neq 0\text{]}$$

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Example

Who will sit the Iron Throne in spring if it is not Targaryen?

	Baratheon	Lannister	Greyjoy	Targaryen
$P(x)$	0.1	0.2	0	0.7
$P(x \mid \text{not Targ.})$				

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$P(x \mid \text{not Targ.})$	$1/3$	$2/3$	0	0

Bayes' rule

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From this, we can derive a simple rule that lets us "swap direction of conditionality":

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Abductive reasoning

If the outcome Y probabilistically depends on X, described by $P(Y \mid X)$, Bayes' rule let's us reason from observed outcome Y to likely cause X.

we use this to model interpretation as reverse production

Learning from language

Example

How many students passed the test?

	0	1	2	3
$P(x)$	0.25	0.25	0.25	0.25

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Example

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	0	1	2	3
$P(x)$	0.25	0.25	0.25	0.25
$P(x \mid$ "Some students passed."$)$	0	$1/3$	$1/3$	$1/3$

Inferring causes

Example

Your house has an alarm system which is intended to detect burglars. However, the system can suffer from *false positives*: the sprinkler can sometimes set it off accidentally.

	Burglar	Sprinkler
$P(x)$	0.01	0.99

Inferring causes

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	Burglar	Sprinkler
$P(x)$	0.01	0.99
$P(Alarm \mid x)$	0.9	0.1

Inferring causes

Example

Your house has an alarm system which is intended to detect burglars. However, the system can suffer from *false positives*: the sprinkler can sometimes set it off accidentally.

	Burglar	Sprinkler
$P(x)$	0.01	0.99
$P(Alarm\|x)$	0.9	0.1
$P(x\|Alarm)$	$1/12$	$11/12$

Marginalization

Joint distribution

A joint distribution $P(x_1, x_2, \ldots, x_n)$ assigns a probability to all elements of $X_1 \times X_2 \times \ldots X_n$.

Marginalization

The marginal distribution of x_1 is obtained by "marginalizing out" the other variables:

$$P(x_1) = \sum_{x_2 \in X_2, \ldots, x_n \in X_n} P(x_1, x_2, \ldots, X_n)$$

Marginalization

Example

There may be traffic on your way to the airport. If there is, you have a higher chance of missing your flight.

$P(x, y)$	Miss flight	Make flight
Traffic	0.4	0.3
No traffic	0.1	0.2

Marginalization

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	$P(x)$
Traffic	0.7
No traffic	0.3

Marginalization

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There may be traffic on your way to the airport. If there is, you have a higher chance of missing your flight.

$P(x,y)$	Miss flight	Make flight
Traffic	0.4	0.3
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	$P(x)$
Traffic	0.7
No traffic	0.3

	Miss flight	Make flight
$P(y)$	0.5	0.5

Measuring information

Consider a lottery, in which two binary digits (0 or 1) will be selected at random. You win the lottery by guessing both of the digits correctly. Alice and Bob are each going to play. Bob doesn't know anything about which numbers will be selected. Alice has some inside information: the lottery machine is rigged so that the first number will be 0.

Alice has more information than Bob. How much more?

Measuring information

Alice and Bob's distributions over lottery outcomes.

	00	01	10	11
$P_{\text{Alice}}(x)$	0.5	0.5	0	0
$P_{\text{Bob}}(x)$	0.25	0.25	0.25	0.25

Entropy

Definition

If P is a probability distribution, then the entropy of the distribution is: $\sum_x P(x) \log \frac{1}{P(x)}$.

$\log \frac{1}{P(x)}$ measures the surprisal associated with outcome x under distribution P.

Entropy

Computing the entropy for $P_{\mathsf{Alice}}(x)$.

	00	01	10	11
$P_{\mathsf{Alice}}(x)$	0.5	0.5	0	0
$\log \frac{1}{P_{\mathsf{Alice}}(x)}$	1	1	0	0

Entropy

Computing the entropy for $P_{\text{Alice}}(x)$.

	00	01	10	11
$P_{\text{Alice}}(x)$	0.5	0.5	0	0
$\log \frac{1}{P_{\text{Alice}}(x)}$	1	1	0	0

The entropy is: $P_{\text{Alice}}(00) \log \frac{1}{P_{\text{Alice}}(00)} + P_{\text{Alice}}(01) \log \frac{1}{P_{\text{Alice}}(01)} = 0.5 \cdot 1 + 0.5 \cdot 1.$

Entropy

Computing the entropy for $P_{\text{Alice}}(x)$.

	00	01	10	11
$P_{\text{Bob}}(x)$	0.25	0.25	0.25	0.25
$\log \frac{1}{P_{\text{Alice}}(x)}$	2	2	2	2

The entropy is 2.

Decision theory

Bob the Baker is in doubt whether the eggs in his refrigerator are still good and so he is wondering whether to bake or buy a cake. If the eggs are fine, Bob would prefer baking a cake himself because he can bake better cakes than the local store offers; otherwise, with foul eggs, his cake would be indigestible and a bought cake would be much preferred.

What should Bob the *Rational* Baker do?

Decision theory

- set of state distinctions (eggs good or foul)
- set of action alternatives (buy or bake)
- uncertainty about the state of nature (probability distribution)
- preferences conditional on choice and state (utility function)

Utility functions

Definition

If X is a set of (future) contingencies or choice options, a utility function U is map $\mathrm{U} : X \to \mathbb{R}$.

We say that x is preferred over x' iff $\mathrm{U}(x) > \mathrm{U}(x')$.

Decision problems

A decision problem is a quadruple $\mathcal{D} = \langle T, A, P, \mathrm{U} \rangle$ where:

- T is a finite set of world states
- A is a finite set of actions
- P is a probability distribution on T, i.e. $P \in \Delta(T)$
- U is a utility function $\mathrm{U} : A \times T \to \mathbb{R}$.

Decision problems

- Bob the Baker is uncertain whether his eggs are still good:
  - $T = \{t_{\text{foul}}, t_{\text{good}}\}$
- he is in doubt whether to bake or buy a cake:
  - $A = \{a_{\text{bake}}, a_{\text{buy}}\}$
- beliefs and preferences are as follows:

	$P(t)$	a_{bake}	a_{buy}
t_{foul}	.3	-10	5
t_{good}	.7	10	5

Expected utility

Given a decision problem $\mathcal{D} = \langle T, A, P, \mathrm{U} \rangle$, the expected utility of an action $a \in A$ is:

$$\mathrm{EU}(a) = \sum_{t \in T} P(t) \times \mathrm{U}(a, t) \, .$$

Expected utility

$\mathcal{D}$	$P(t)$	a_{bake}	a_{buy}
t_{foul}	.3	-10	5
t_{good}	.7	10	5

$$
\begin{aligned}
\text{EU}_{\mathcal{D}}(a_{\text{buy}}) &= P(t_{\text{foul}}) \times \text{U}(a_{\text{buy}}, t_{\text{foul}}) + P(t_{\text{good}}) \times \text{U}(a_{\text{buy}}, t_{\text{good}}) \\
&= .3 \times 5 + .7 \times 5 \\
&= 5
\end{aligned}
$$

Expected utility

$\mathcal{D}$	$P(t)$	a_{bake}	a_{buy}
t_{foul}	.3	-10	5
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\mathrm{EU}_{\mathcal{D}}(a_{\text{buy}}) &= P(t_{\text{foul}}) \times \mathrm{U}(a_{\text{buy}}, t_{\text{foul}}) + P(t_{\text{good}}) \times \mathrm{U}(a_{\text{buy}}, t_{\text{good}}) \\
&= .3 \times 5 + .7 \times 5 \\
&= 5 \\
\mathrm{EU}_{\mathcal{D}}(a_{\text{bake}}) &= P(t_{\text{foul}}) \times \mathrm{U}(a_{\text{bake}}, t_{\text{foul}}) + P(t_{\text{good}}) \times \mathrm{U}(a_{\text{bake}}, t_{\text{good}}) \\
&= .3 \times -10 + .7 \times 10 \\
&= 4
\end{aligned}$$

Soft-max choice rule

Soft-max choices (a.k.a. quantal response, logit choice . . .)

$$P(a_i) = \frac{\exp(\lambda u_i)}{\sum_j \exp(\lambda u_j)}$$

$\lambda \to \infty$ perfect rationality

$\lambda \to 0$ random choice

Motivation

- real people are not perfect utility maximizers
- they make mistakes $\rightsquigarrow$ sub-optimal choices
- still, high utility choices are more likely than low-utility ones

Properties of soft-max

- $P(a_i) > 0$ for all i with $u_i > -\infty$
- if $u_i = v_i + k$ for all i, then $\frac{\exp(\lambda u_i)}{\sum_j \exp(\lambda u_j)} = \frac{\exp(\lambda v_i)}{\sum_j \exp(\lambda v_j)}$ for all λ
- if $u_i = k \times v_i$ for all i, $k > 0$, then $\frac{\exp(\lambda u_i)}{\sum_j \exp(\lambda u_j)} = \frac{\exp(\lambda' v_i)}{\sum_j \exp(\lambda' v_j)}$ for all $\lambda' = k\lambda$