Shannon Information and Kolmogorov Complexity

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Abstract

We compare the elementary theories of Shannon information and Kolmogorov complexity, the extent to which they have a common purpose, and where they are fundamentally different. We discuss and relate the basic notions of both theories: Shannon entropy versus Kolmogorov complexity, the relation of both to universal coding, Shannon mutual information versus Kolmogorov ('algorithmic') mutual information, probabilistic sufficient statistic versus algorithmic sufficient statistic (related to lossy compression in the Shannon theory versus meaningful information in the Kolmogorov theory), and rate distortion theory versus Kolmogorov's structure function. Part of the material has appeared in print before, scattered through various publications, but this is the first comprehensive systematic comparison. The last mentioned relations are new.

Contents

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1 Introduction

Shannon information theory, usually called just ‘information’ theory was introduced in 1948, [22], by C.E. Shannon (1916–2001). *Kolmogorov complexity* theory, also known as ‘algorithmic information’ theory, was introduced with different motivations (among which Shannon’s probabilistic notion of information), independently by R.J. Solomonoff (born 1926), A.N. Kolmogorov (1903–1987) and G. Chaitin (born 1943) in 1960/1964, [24], 1965, [10], and 1969 [3], respectively. Both theories aim at providing a means for measuring ‘information’. They use the same unit to do this: the *bit*. In both cases, the amount of information in an object may be interpreted as the length of a description of the object. In the Shannon approach, however, the method of encoding objects is based on the presupposition that the objects to be encoded are outcomes of a known random source—it is only the characteristics of that random source that determine the encoding, not the characteristics of the objects that are its outcomes. In the Kolmogorov complexity approach we consider the individual objects themselves, in isolation so-to-speak, and the encoding of an object is a short computer program (compressed version of the object) that generates it and then halts. In the Shannon approach we are interested in the minimum expected number of bits to transmit a message from a random source of known characteristics through an error-free channel. Says Shannon [22]:

> “The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.”

In Kolmogorov complexity we are interested in the minimum number of bits from which a particular message or file can effectively be reconstructed: the minimum number of bits that suffice to store the file in reproducible format. This is the basic question of the ultimate compression of given individual files. A little reflection reveals that this is a great difference: for *every* source emitting but two messages the Shannon information (entropy) is at most 1 bit, but we can choose both messages concerned of arbitrarily high Kolmogorov complexity. Shannon stresses in his founding article that his notion is only concerned with *communication*, while Kolmogorov stresses in his founding article that his notion aims at supplementing the gap left by Shannon theory concerning the information in individual objects. Kolmogorov [12]:

> "Our definition of the quantity of information has the advantage that it refers to individual objects and not to objects treated as members of a set of objects with a probability distribution given on it. The probabilistic definition can be convincingly applied to the information contained, for example, in a stream of congratulatory telegrams. But it would not be clear how to apply it, for example, to an estimate of the quantity of information contained in a novel or in the translation of a novel into another language relative to the original. I think that the new definition is capable of introducing in similar applications of the theory at least clarity of principle."

To be sure, both notions are natural: Shannon ignores the object itself but considers only the characteristics of the random source of which the object is one of the possible outcomes, while Kolmogorov considers only the object itself to determine the number of bits in the ultimate compressed version irrespective of the manner in which the object arose. In this paper, we introduce, compare and contrast the Shannon and Kolmogorov approaches. An early comparison between Shannon entropy and Kolmogorov complexity is [14].

How to read this paper: We switch back and forth between the two theories concerned according to the following pattern: we first discuss a concept of Shannon's theory, discuss its properties as well as some questions it leaves open. We then provide Kolmogorov's analogue of the concept and show how it answers the question left open by Shannon's theory. To ease understanding of the two theories and how they relate, we supplied the overview below and then Sections 1.3 and Section 2, which discuss preliminaries, fix notation and introduce the basic notions. The other sections are largely independent from one another. Throughout the text, we assume some basic familiarity with elementary notions of probability theory and computation, but we have kept the treatment elementary. This may provoke scorn in the information theorist, who sees an elementary treatment of basic matters in his discipline, and likewise from the computation theorist concerning the treatment of aspects of the elementary theory of computation. But experience has shown that what one expert views as child's play is an insurmountable mountain for his opposite number. Thus, we decided to ignore background knowledge and cover both areas from first principles onwards, so that the opposite expert can easily access the unknown discipline, possibly helped along by the familiar analogues in his own ken of knowledge.

1.1 Overview and Summary

A summary of the basic ideas is given below. In the paper, these notions are discussed in the same order.

1. **Coding: Prefix codes, Kraft inequality** (Section 1.3) Since descriptions or *encodings* of objects are fundamental to both theories, we first review some elementary facts about coding. The most important of these is the *Kraft inequality*. This inequality gives the fundamental relationship between *probability density functions and prefix codes*, which are the type of codes we are interested in. Prefix codes and the Kraft inequality underly most of Shannon's, and a large part of Kolmogorov's theory.

2. **Shannon's Fundamental Concept: Entropy** (Section 2.1) Entropy is defined by a functional that maps *probability distributions* or, equivalently, *random variables*, to *real numbers*. This notion is derived from first principles as the only 'reasonable' way to measure the 'average amount of information conveyed when an outcome of the random variable is observed'. The notion is then related to encoding and communicating messages by Shannon's famous 'coding theorem'.

3. **Kolmogorov's Fundamental Concept: Kolmogorov Complexity** (Section 2.2) Kolmogorov complexity is defined by a function that maps *objects* (to be thought of as natural numbers or sequences of symbols, for example outcomes of the random variables figuring in the Shannon theory) to the *natural numbers*. Intuitively, the Kolmogorov complexity of a sequence is the length (in bits) of the shortest computer program that prints the sequence and then halts.

4. **Relating entropy and Kolmogorov complexity** (Section 2.3 and Appendix A) Although their primary aim is quite different, and they are functions defined on different spaces, there are close relations between entropy and Kolmogorov complexity. The formal relation "entropy = expected Kolmogorov complexity" is discussed in Section 2.3. The relation is further illustrated by explaining 'universal coding' (also introduced by Kolmogorov in 1965) which combines elements from both Shannon's and Kolmogorov's theory, and which lies at the basis of most practical data compression methods. While

related to the main theme of this paper, universal coding plays no direct role in the later sections, and therefore we delegated it to Appendix A.

Entropy and Kolmogorov Complexity are the basic notions of the two theories. They serve as building blocks for all other important notions in the respective theories. Arguably the most important of these notions is *mutual information*:

5. Mutual Information—Shannon and Kolmogorov Style (Section 3) Entropy and Kolmogorov complexity are concerned with information in a single object: a random variable (Shannon) or an individual sequence (Kolmogorov). Both theories provide a (distinct) notion of *mutual information* that measures the information that *one object gives about another object.* In Shannon's theory, this is the information that one random variable carries about another; in Kolmogorov's theory ('algorithmic mutual information'), it is the information one sequence gives about another. In an appropriate setting the former notion can be shown to be the expectation of the latter notion.

6. Mutual Information Non-Increase (Section 4) In the probabilistic setting the mutual information between two random variables cannot be increased by processing the outcomes. That stands to reason, since the mutual information is expressed in probabilities of the random variables involved. But in the algorithmic setting, where we talk about mutual information between two strings this is not evident at all. Nonetheless, up to some precision, the same non-increase law holds. This result was used recently to refine and extend the celebrated Gödel's incompleteness theorem.

7. Sufficient Statistic (Section 5) Although its roots are in the statistical literature, the notion of probabilistic "sufficient statistic" has a natural formalization in terms of mutual Shannon information, and can thus also be considered a part of Shannon theory. The probabilistic sufficient statistic extracts the information in the data about a model class. In the algorithmic setting, a sufficient statistic extracts the meaningful information from the data, leaving the remainder as accidental random "noise". In a certain sense the probabilistic version of sufficient statistic is the expectation of the algorithmic version. These ideas are generalized significantly in the next item.

8. Rate Distortion Theory versus Structure Function (Section 6) Entropy, Kolmogorov complexity and mutual information are concerned with *lossless* description or compression: messages must be described in such a way that from the description, the original message can be completely reconstructed. Extending the theories to *lossy* description or compression leads to rate-distortion theory in the Shannon setting, and the Kolmogorov structure function in the Kolmogorov section. The basic ingredients of the lossless theory (entropy and Kolmogorov complexity) remain the building blocks for such extensions. The Kolmogorov structure function significantly extends the idea of "meaningful information" related to the algorithmic sufficient statistic, and can be used to provide a foundation for inductive inference principles such as Minimum Description Length (MDL). Once again, the Kolmogorov structure function can be related to Shannon's rate-distortion function by taking expectations in an appropriate manner.

1.2 Preliminaries

Strings: Let $\mathcal{B}$ be some finite or countable set. We use the notation $\mathcal{B}^*$ to denote the set of finite *strings* or *sequences* over $\underline{\mathcal{X}}$. For example,

$\mathcal{B}$ ↗

$$\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\},$$

with ϵ denoting the *empty word* '' with no letters. Let $\mathcal{N}$ denotes the natural numbers. We identify $\mathcal{N}$ and $\{0,1\}^*$ according to the correspondence

$$(0,\epsilon),(1,0),(2,1),(3,00),(4,01),\ldots \tag{1.1}$$

The *length* $l(x)$ of x is the number of bits in the binary string x. For example, $l(010) = 3$ and $l(\epsilon) = 0$. If x is interpreted as an integer, we get $l(x) = \lfloor \log(x+1) \rfloor$ and, for $x \geq 2$,

$$\lfloor \log x \rfloor \leq l(x) \leq \lceil \log x \rceil. \tag{1.2}$$

Here, as in the sequel, $\lceil x \rceil$ is the smallest integer larger than or equal to x, $\lfloor x \rfloor$ is the largest integer smaller than or equal to x and log denotes logarithm to base two. We shall typically be concerned with encoding finite-length binary strings by other finite-length binary strings. The emphasis is on binary strings only for convenience; observations in any alphabet can be so encoded in a way that is 'theory neutral'.

Precision and $\overset{+}{<}, \overset{+}{=}$ notation: It is customary in the area of Kolmogorov complexity to use "additive constant c" or equivalently "additive $O(1)$ term" to mean a constant, accounting for the length of a fixed binary program, independent from every variable or parameter in the expression in which it occurs. In this paper we use the prefix complexity variant of Kolmogorov complexity for convenience. Since (in)equalities in the Kolmogorov complexity setting typically hold up to an additive constant, we use a special notation.

We will denote by $\overset{+}{<}$ an inequality to within an additive constant. More precisely, let f, g be functions from $\{0,1\}^*$ to $\mathcal{R}$, the *real numbers*. Then by '$f(x) \overset{+}{<} g(x)$' we mean that there exists a c such that for all $x \in \{0,1\}^*$, $f(x) < g(x) + c$. We denote by $\overset{+}{=}$ the situation when both $\overset{+}{<}$ and $\overset{+}{>}$ hold.

Probabilistic Notions: Let $\mathcal{X}$ be a finite or countable set. A function $f : \mathcal{X} \rightarrow [0,1]$ is a *probability mass function* if $\sum_{x \in \mathcal{X}} f(x) = 1$. We call f a *sub-probability mass function* if $\sum_{x \in \mathcal{X}} f(x) \leq 1$. Such sub-probability mass functions will sometimes be used for technical convenience. We can think of them as ordinary probability mass functions by considering the surplus probability to be concentrated on an undefined element $u \notin \mathcal{X}$.

In the context of (sub-) probability mass functions, $\mathcal{X}$ is called the *sample space*. Associated with mass function f and sample space $\mathcal{X}$ is the *random variable* X and the probability distribution P such that X takes value $x \in \mathcal{X}$ with probability $P(X = x) = f(x)$. A subset of $\mathcal{X}$ is called an *event*. We extend the probability of individual outcomes to events. With this terminology, $P(X = x) = f(x)$ is the probability that the singleton event $\{x\}$ occurs, and $P(X \in \mathcal{A}) = \sum_{x \in \mathcal{A}} f(x)$. In some cases (where the use of $f(x)$ would be confusing) we write p_x as an abbreviation of $P(X = x)$. In the sequel, we often refer to probability distributions in terms of their mass functions, i.e. we freely employ phrases like 'Let X be distributed according to f'.

Whenever we refer to probability mass functions without explicitly mentioning the sample space $\mathcal{X}$ is assumed to be $\mathcal{N}$ or, equivalently, $\{0,1\}^*$.

For a given probability mass function $f(x, y)$ on sample space $\mathcal{X} \times \mathcal{Y}$ with random variable (X, Y), we define the *conditional probability mass function* $f(y \mid x)$ of outcome $Y = y$ given outcome $X = x$ as

$$f(y|x) := \frac{f(x,y)}{\sum_y f(x,y)}.$$

Note that X and Y are not necessarily independent.

In some cases (esp. Section 5.3 and Appendix A), the notion of *sequential information source* will be needed. This may be thought of as a probability distribution over arbitrarily long binary sequences, of which an observer gets to see longer and longer initial segments. Formally, a sequential information source P is a probability distribution on the set $\{0,1\}^\infty$ of one-way infinite sequences. It is characterized by a *sequence of probability mass functions* $(f^{(1)}, f^{(2)}, \ldots)$ where $f^{(n)}$ is a probability mass function on $\{0,1\}^n$ that denotes the *marginal* distribution of P on the first n-bit segments. By definition, the sequence $f \equiv (f^{(1)}, f^{(2)}, \ldots)$ represents a sequential information source if for all $n > 0$, $f^{(n)}$ is related to $f^{(n+1)}$ as follows: for all $x \in \{0,1\}^n$, $\sum_{y \in \{0,1\}} f^{(n+1)}(xy) = f^{(n)}(x)$ and $f^{(0)}(x) = 1$. This is also called Kolmogorov's *compatibility condition* [20].

Some (by no means all!) probability mass functions on $\{0,1\}^*$ can be thought of as information sources. Namely, given a probability mass function g on $\{0,1\}^*$, we can define $g^{(n)}$ as the conditional distribution of x given that the length of x is n, with domain restricted to x of length n. That is, $g^{(n)} : \{0,1\}^n \rightarrow [0,1]$ is defined, for $x \in \{0,1\}^n$, as $g^{(n)}(x) = g(x) / \sum_{y \in \{0,1\}^n} g(y)$. Then g can be thought of as an information source if and only if the sequence $(g^{(1)}, g^{(2)}, \ldots)$ represents an information source.

Computable Functions: Partial functions on the natural numbers $\mathcal{N}$ are functions f such that $f(x)$ can be 'undefined' for some x. We abbreviate 'undefined' to '$\uparrow$'. A central notion in the theory of computation is that of the *partial recursive functions*. Formally, a function $f : \mathcal{N} \rightarrow \mathcal{N} \cup \{\uparrow\}$ is called *partial recursive* or *computable* if there exists a Turing Machine T that implements f. This means that for all x

1. If $f(x) \in \mathcal{N}$, then T, when run with input x outputs $f(x)$ and then halts.
2. If $f(x) = \uparrow$ (‘$f(x)$ is undefined’), then T with input x never halts.

Readers not familiar with computation theory may think of a Turing Machine as a computer program written in a general-purpose language such as C or Java.

A function $f : \mathcal{N} \to \mathcal{N} \cup \{\uparrow\}$ is called *total* if it is defined for all x (i.e. for all x, $f(x) \in \mathcal{N}$). A *total recursive* function is thus a function that is implementable on a Turing Machine that halts on all inputs. These definitions are extended to several arguments as follows: we fix, once and for all, some standard invertible pairing function $\langle \cdot, \cdot \rangle : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ and we say that $f : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \cup \{\uparrow\}$ is computable if there exists a Turing Machine T such that for all x_1, x_2, T with input $\langle x_1, x_2 \rangle$ outputs $f(x_1, x_2)$ and halts if $f(x_1, x_2) \in \mathcal{N}$ and otherwise T does not halt. By repeating this construction, functions with arbitrarily many arguments can be considered.

function

Real-valued Functions: We call a distribution $f : \mathcal{N} \to \mathcal{R}$ *recursive* or *computable* if there exists a Turing machine that, when input $\langle x, y \rangle$ with $x \in \{0,1\}^*$ and $y \in \mathcal{N}$, outputs $f(x)$ to precision $1/y$; more precisely, it outputs a pair $\langle p, q \rangle$ such that $|p/q - |f(x)|| < 1/y$ and an additional bit to indicate whether $f(x)$ larger or smaller than 0. Here $\langle \cdot, \cdot \rangle$ is the standard pairing function. In this paper all real-valued functions we consider are by definition total. Therefore, in line with the above definitions, for a real-valued function ‘computable’ (equivalently, recursive), means that there is a Turing Machine which for *all* x, computes $f(x)$ to arbitrary accuracy; ‘partial’ recursive real-valued functions are not considered.

It is convenient to distinguish between *upper* and *lower semi-computability*. For this purpose we consider both the argument of an auxiliary function ϕ and the value of ϕ as a pair of natural numbers according to the standard pairing function $\langle \cdot \rangle$. We define a function from $\mathcal{N}$ to the reals $\mathcal{R}$ by a Turing machine T computing a function ϕ as follows. Interpret the computation $\phi(\langle x, t \rangle) = \langle p, q \rangle$ to mean that the quotient p/q is the rational valued tth approximation of $f(x)$.

Definition 1.1 A function $f : \mathcal{N} \to \mathcal{R}$ is *lower semi-computable* if there is a Turing machine T computing a total function ϕ such that $\phi(x, t+1) \geq \phi(x, t)$ and $\lim_{t \to \infty} \phi(x, t) = f(x)$. This means that f can be computably approximated from below. A function f is *upper semi-computable* if $-f$ is lower semi-computable, Note that, if f is both upper- and lower semi-computable, then f is computable.

(Sub-) Probability mass functions: Probability mass functions on $\{0,1\}^*$ may be thought of as real-valued functions on $\mathcal{N}$. Therefore, the definitions of ‘computable’ and ‘recursive’ carry over unchanged from the real-valued function case.

1.3 Codes

We repeatedly consider the following scenario: a *sender* (say, A) wants to communicate or transmit some information to a *receiver* (say, B). The information to be transmitted is an element from some set $\mathcal{X}$ (This set may or may not consist of binary strings). It will be communicated by sending a binary string, called the *message*. When B receives the message, he can decode it again and (hopefully) reconstruct the element of $\mathcal{X}$ that was sent. To achieve this, A and B need to agree on a *code* or *description method* before communicating. Intuitively, this is a binary relation between *source words* and associated *code words*. The relation is fully characterized by the *decoding function*. Such a decoding function D can be any function $D : \{0,1\}^* \to \mathcal{X}$. The domain of D is the set of *code words* and the range of D is the set of *source words*. $D(y) = x$ is interpreted as “y is a code word for the source word x”. The set of all code words for source word x is the set $D^{-1}(x) = \{y : D(y) = x\}$. Hence, $E = D^{-1}$ can be called the *encoding* substitution (E is not necessarily a function). With each code D we can associate a *length function* $L_D : \mathcal{X} \to \mathcal{N}$ such that, for each source word x, $L(x)$ is the length of the shortest encoding of x:

D can be partial,
but should be
a surjection, no?

$$L_D(x) = \min\{l(y) : D(y) = x\}.$$

We denote by x^* the shortest y such that $D(y) = x$; if there is more than one such y, then x^* is defined to be the first such y in some agreed-upon order—for example, the lexicographical order.

In coding theory attention is often restricted to the case where the source word set is finite, say $\mathcal{X} = \{1, 2, \ldots, N\}$. If there is a constant l_0 such that $l(y) = l_0$ for all code words y (which implies, $L(x) = l_0$ for all source words x), then we call D a *fixed-length* code. It is easy to see that $l_0 \geq \log N$. For instance, in teletype transmissions the source has an alphabet of $N = 32$ letters, consisting of the 26 letters in the Latin alphabet plus 6 special characters. Hence, we need $l_0 = 5$ binary digits per source letter. In electronic computers we often use the fixed-length ASCII code with $l_0 = 8$.

Prefix code: It is immediately clear that in general we cannot uniquely recover x and y from $E(xy)$. Let E be the identity mapping. Then we have $E(00)E(00) = 0000 = E(0)E(000)$. We now introduce *prefix codes*, which do not suffer from this defect. A binary string x is a *proper prefix* of a binary string y if we can write $y = xz$ for $z \neq \epsilon$. A set $\{x, y, \ldots\} \subseteq \{0,1\}^*$ is *prefix-free* if for any pair of distinct elements in the set neither is a proper prefix of the other. A function $D : \{0,1\}^* \to \mathcal{N}$ defines a *prefix-code* if its domain is prefix-free. In order to decode a code sequence of a prefix-code, we simply start at the beginning and decode one code word at a time. When we come to the end of a code word, we know it is the end, since no code word is the prefix of any other code word in a prefix-code.

Suppose we encode each binary string $x = x_1 x_2 \ldots x_n$ as

$$\bar{x} = \underbrace{11\ldots1}_{n \text{ times}} 0 x_1 x_2 \ldots x_n .$$

The resulting code is prefix because we can determine where the code word $\bar{x}$ ends by reading it from left to right without backing up. Note $l(\bar{x}) = 2n + 1$; thus, we have encoded strings in $\{0,1\}^*$ in a prefix manner at the price of doubling their length. We can get a much more efficient code by applying the construction above to the length $l(x)$ of x rather than x itself: define $x' = \overline{l(x)}x$, where $l(x)$ is interpreted as a binary string according to the correspondence (1.1). Then the code D' with $D'(x') = x$ is a prefix code satisfying, for all $x \in \{0,1\}^*$, $l(x') = n + 2\log n + 1$ (here we ignore the 'rounding error' in (1.2)). D' is used throughout this paper as a standard code to encode natural numbers in a prefix free-manner; we call it the *standard prefix-code for the natural numbers.* We use $L_{\mathcal{N}}(x)$ as notation for $l(x')$. When x is interpreted as an integer (using the correspondence (1.1) and (1.2)), we see that, up to rounding, $L_{\mathcal{N}}(x) = \log x + 2 \log\log x + 1$.

length of word
according to
corresp. (1.1)
is
n = log x

Prefix codes and the Kraft inequality: Let $\mathcal{X}$ be the set of natural numbers and consider the straightforward non-prefix representation (1.1). There are two elements of $\mathcal{X}$ with a description of length 1, four with a description of length 2 and so on. However, for a prefix code D for the natural numbers there are less binary prefix code words of each length: if x is a prefix code word then no $y = xz$ with $z \neq \epsilon$ is a prefix code word. Asymptotically there are less prefix code words of length n than the 2^n source words of length n. Quantification of this intuition for countable $\mathcal{X}$ and arbitrary prefix-codes leads to a precise constraint on the number of code-words of given lengths. This important relation is known as the *Kraft Inequality* and is due to L.G. Kraft [13].

Theorem 1.2 *Let $l_1, l_2, \ldots$ be a finite or infinite sequence of natural numbers. There is a prefix-code with this sequence as lengths of its binary code words iff*

$$\sum_n 2^{-l_n} \leq 1.$$

Uniquely Decodable Codes: We want to code elements of $\mathcal{X}$ in a way that they can be uniquely reconstructed from the encoding. Such codes are called 'uniquely decodable'. Every prefix-code is a uniquely decodable code. For example, if $E(1) = 0$, $E(2) = 10$, $E(3) = 110$, $E(4) = 111$ then 1421 is encoded as 0111100, which can be easily decoded from left to right in a unique way.

On the other hand, not every uniquely decodable code satisfies the prefix condition. Prefix-codes are distinguished from other uniquely decodable codes by the property that the end of a code word is always recognizable as such. This means that decoding can be accomplished without the delay of observing subsequent code words, which is why prefix-codes are also called instantaneous codes.

There is good reason for our emphasis on prefix-codes. Namely, it turns out that Theorem 1.2 stays valid if we replace "prefix-code" by "uniquely decodable code." This important fact means that every uniquely decodable code can be replaced by a prefix-code without changing the set of code-word lengths. In Shannon's and Kolmogorov's theories, we are only interested in code word *lengths* of uniquely decodable codes rather than actual encodings. By the previous argument, we may restrict the set of codes we work with to prefix codes, which are much easier to handle.

Probability distributions and complete prefix codes: A uniquely decodable code is *complete* if the addition of any new code word to its code word set results in a non-uniquely decodable code. It is easy to see that a code is complete iff equality holds in the associated Kraft Inequality. Let $l_1, l_2, \ldots$ be the code words of some complete uniquely decodable code. Let us define $q_x = 2^{-l_x}$. By definition of completeness, we have

$\sum_x q_x = 1$. Thus, the q_x can be thought of as *probability mass functions* corresponding to some probability distribution Q. We say Q is the distribution *corresponding* to $l_1, l_2, \ldots$. In this way, each complete uniquely decodable code is mapped to a unique probability distribution. Of course, this is nothing more than a formal correspondence: we may choose to encode outcomes of X using a code corresponding to a distribution q, whereas the outcomes are actually distributed according to some $p \neq q$. But, as we show below, if X is distributed according to p, then the code to which p corresponds is, in an average sense, the code that achieves optimal compression of X.

2 Shannon Entropy versus Kolmogorov Complexity

2.1 Shannon Entropy

It seldom happens that a detailed mathematical theory springs forth in essentially final form from a single publication. Such was the case with Shannon information theory, which properly started only with the appearance of C.E. Shannon's paper "The mathematical theory of communication" [22]. In this paper, Shannon proposed a measure of information in a distribution, which he called the 'entropy'. The entropy $H(P)$ of a distribution P measures the 'the inherent uncertainty in P', or (in fact equivalently), 'how much information is gained when an outcome of P is observed'. To make this a bit more precise, let us imagine an observer who knows that X is distributed according to P. The observer then observes $X = x$. The entropy of P stands for the 'uncertainty of the observer about the outcome x *before* he observes it'. Now think of the observer as a 'receiver' who receives the message conveying the value of X. From this dual point of view, the entropy stands for

> the average amount of information that the observer has gained *after* receiving a realized outcome x of the random variable X. $(*)$

Below, we first give Shannon's mathematical definition of entropy, and we then connect it to its intuitive meaning $(*)$.

Definition 2.1 Let $\mathcal{X}$ be a finite or countable set, let X be a random variable taking values in $\mathcal{X}$ with distribution $P(X = x) = p_x$. Then the (Shannon-) *entropy* of random variable X is given by

$$H(X) = \sum_{x \in \mathcal{X}} p_x \log 1/p_x, \tag{2.1}$$

Entropy is defined here as a functional mapping random variables to real numbers. In many texts, entropy is, essentially equivalently, defined as a map from *distributions* of random variables to the real numbers. Thus, by definition: $H(P) := H(X) = \sum_{x \in \mathcal{X}} p_x \log 1/p_x$.

Motivation: The entropy function (2.1) can be motivated in different ways. The two most important ones are the *axiomatic* approach and the *coding interpretation.* In this paper we concentrate on the latter, but we first briefly sketch the former. The idea of the axiomatic approach is to postulate a small set of self-evident axioms that any measure of information relative to a distribution should satisfy. One then shows that the only measure satisfying all the postulates is the Shannon entropy. We outline this approach for finite sources $\mathcal{X} = \{1, \ldots, N\}$. We look for a function H that maps probability distributions on $\mathcal{X}$ to real numbers. For given distribution P, $H(P)$ should measure 'how much information is gained on average when an outcome is made available'. We can write $H(P) = H(p_1, \ldots, p_N)$ where p_i stands for the probability of i. Suppose we require that

1. $H(p_1, \ldots, p_N)$ is continuous in $p_1, \ldots, p_N$.
2. If all the p_i are equal, $p_i = 1/N$, then H should be a monotonic increasing function of N. With equally likely events there is more choice, or uncertainty, when there are more possible events.
3. If a choice is broken down into two successive choices, the original H should be the weighted sum of the individual values of H. Rather than formalizing this condition, we will give a specific example. Suppose that $\mathcal{X} = \{1, 2, 3\}$, and $p_1 = \frac{1}{2}, p_2 = 1/3, p_3 = 1/6$. We can think of $x \in \mathcal{X}$ as being generated in a two-stage process. First, an outcome in $\mathcal{X}' = \{0, 1\}$ is generated according to a distribution P'

with $p'_0 = p'_1 = \frac{1}{2}$. If $x' = 1$, we set $x = 1$ and the process stops. If $x' = 0$, then outcome '2' is generated with probability 2/3 and outcome '3' with probability 1/3, and the process stops. The final results have the same probabilities as before. In this particular case we require that

$$H(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) = H(\frac{1}{2}, \frac{1}{2}) + \frac{1}{2}H(\frac{2}{3}, \frac{1}{3}) + \frac{1}{2}H(1).$$

Thus, the entropy of P must be equal to entropy of the first step in the generation process, plus the weighted sum (weighted according to the probabilities in the first step) of the entropies of the second step in the generation process.

As a special case, if $\mathcal{X}$ is the n-fold product space of another space $\mathcal{Y}$, $X = (Y_1, \ldots, Y_n)$ and the Y_i are all independently distributed according to P_Y, then $H(P_X) = nH(P_Y)$. For example, the total entropy of n independent tosses of a coin with bias p is $nH(p, 1-p)$.

Theorem 2.2 *The only H satisfying the three above assumptions is of the form*

$$H = K \sum_{i=1}^{N} p_i \log 1/p_i,$$

see sec 1.11 of Li and Vitányi (p65)

with K a constant.

Thus, requirements (1)–(3) lead us to the definition of entropy (2.1) given above up to an (unimportant) scaling factor. We shall give a concrete interpretation of this factor later on. Besides the defining characteristics (1)–(3), the function H has a few other properties that make it attractive as a measure of information. We mention:

4. $H(p_1, \ldots, p_N)$ is a concave function of the p_i.

5. For each N, H achieves its unique maximum for the uniform distribution $p_i = 1/N$.

6. $H(p_1, \ldots, p_N)$ is zero iff one of the p_i has value 1. Thus, H is zero if and only if we do not gain any information at all if we are told that the outcome is i (since we already knew i would take place with certainty).

The Coding Interpretation: Immediately after stating Theorem 2.2, Shannon [22] continues, "this theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to provide a certain plausibility to some of our later definitions. The *real justification* of these definitions, however, will reside in their implications". Following this injunction, we emphasize the main practical interpretation of entropy as the length (number of bits) needed to encode outcomes in $\mathcal{X}$. This provides much clearer intuitions, it lies at the root of the many practical applications of information theory, and, most importantly for us, it simplifies the comparison to Kolmogorov complexity.

Example 2.3 The entropy of a random variable X with equally likely outcomes in a finite sample space $\mathcal{X}$ is given by $H(X) = \log |\mathcal{X}|$. By choosing a particular message x from $\mathcal{X}$, we remove the entropy from X by the assignment $X := x$ and produce or transmit *information* $I = \log |\mathcal{X}|$ by our selection of x. We show below that $I = \log |\mathcal{X}|$ (or, to be more precise, the integer $I' = \lceil \log |\mathcal{X}| \rceil$) can be interpreted as the number of bits needed to be transmitted from an (imagined) sender to an (imagined) receiver. ◇

We now connect entropy to minimum average code lengths. These are defined as follows:

Definition 2.4 Let source words $x \in \{0,1\}^*$ be produced by a random variable X with probability $P(X = x) = p_x$ for the event $X = x$. The characteristics of X are fixed. Now consider prefix codes $D : \{0,1\}^* \to \mathcal{N}$ with one code word per source word, and denote the length of the code word for x by l_x. We want to minimize the expected number of bits we have to transmit for the given source X and choose a prefix code D that achieves this. In order to do so, we must minimize the *average code-word length* $\bar{L}_D = \sum_x p_x l_x$. We define the *minimal average code word length* as $\bar{L} = \min\{\bar{L}_D : D \text{ is a prefix-code}\}$. A prefix-code D such that $\bar{L}_D = \bar{L}$ is called an *optimal prefix-code* with respect to prior probability P of the source words.

The (minimal) average code length of an (optimal) code does not depend on the details of the set of code words, but only on the set of code-word lengths. It is just the expected code-word length with respect to the given distribution. Shannon discovered that the minimal average code word length is about equal to the entropy of the source word set. This is known as the *Noiseless Coding Theorem*. The adjective "noiseless" emphasizes that we ignore the possibility of errors.

Theorem 2.5 *Let $\bar{L}$ and P be as above. If $H(P) = \sum_x p_x \log 1/p_x$ is the entropy, then*

$$H(P) \leq \bar{L} \leq H(P) + 1. \qquad (2.2)$$

up to rounding, the minimum avg length is equal to the entropy

We are typically interested in encoding a binary string of length n with entropy proportional to n (Example A.1). The essence of (2.2) is that, for all but the smallest n, the difference between entropy and minimal expected code length is completely negligible.

It turns out that the optimum $\bar{L}$ in (2.2) is relatively easy to achieve, with the Shannon-Fano code. Let there be N symbols (also called basic messages or source words). Order these symbols according to decreasing probability, say $\mathcal{X} = \{1, 2, \ldots, N\}$ with probabilities $p_1, p_2, \ldots, p_N$. Let $P_r = \sum_{i=1}^{r-1} p_i$, for $r = 1, \ldots, N$. The binary code $E : \mathcal{X} \to \{0,1\}^*$ is obtained by coding r as a binary number $E(r)$, obtained by truncating the binary expansion of P_r at length $l(E(r))$ such that

$$\log 1/p_r \leq l(E(r)) < 1 + \log 1/p_r.$$

This code is the *Shannon-Fano code*. It has the property that highly probable symbols are mapped to short code words and symbols with low probability are mapped to longer code words (just like in a less optimal, non-prefix-free, setting is done in the Morse code). Moreover,

$$2^{-l(E(r))} \leq p_r < 2^{-l(E(r))+1}.$$

Note that the code for symbol r differs from all codes of symbols $r+1$ through N in one or more bit positions, since for all i with $r + 1 \leq i \leq N$,

$$P_i \geq P_r + 2^{-l(E(r))}.$$

Therefore the binary expansions of P_r and P_i differ in the first $l(E(r))$ positions. This means that E is one-to-one, and it has an inverse: the decoding mapping E^{-1}. Even better, since no value of E is a prefix of any other value of E, the set of code words is a prefix-code. This means we can recover the source message from the code message by scanning it from left to right without look-ahead. If H_1 is the average number of bits used per symbol of an original message, then $H_1 = \sum_r p_r l(E(r))$. Combining this with the previous inequality we obtain (2.2):

$$\sum_r p_r \log 1/p_r \leq H_1 < \sum_r (1 + \log 1/p_r) p_r = 1 + \sum_r p_r \log 1/p_r.$$

Problem and Lacuna: Shannon observes, "Messages have *meaning* [. . . however . . .] the semantic aspects of communication are irrelevant to the engineering problem." In other words, can we answer a question like "what is the information in this book" by viewing it as an element of a set of possible books with a probability distribution on it? Or that the individual sections in this book form a random sequence with stochastic relations that damp out rapidly over a distance of several pages? And how to measure the quantity of hereditary information in biological organisms, as encoded in DNA? Again there is the possibility of seeing a particular form of animal as one of a set of possible forms with a probability distribution on it. This seems to be contradicted by the fact that the calculation of all possible lifeforms in existence at any one time on earth would give a ridiculously low figure like 2^{100}.

Shannon's classical information theory assigns a quantity of information to an ensemble of possible messages. All messages in the ensemble being equally probable, this quantity is the number of bits needed to count all possibilities. This expresses the fact that each message in the ensemble can be communicated using this number of bits. However, it does not say anything about the number of bits needed to convey any individual message in the ensemble. To illustrate this, consider the ensemble consisting of all binary strings of length 9999999999999999. ten quadrillion -1

By Shannon's measure, we require 9999999999999999 bits on the average to encode a string in such an ensemble. However, the string consisting of 9999999999999999 1's can be encoded in about 55 bits by expressing 9999999999999999 in binary and adding the repeated pattern "1." A requirement for this to work is that we have agreed on an algorithm that decodes the encoded string. We can compress the string still further when we note that 9999999999999999 equals $3^2 \times 1111111111111111$, and that 1111111111111111 consists of 2^4 1's.

Thus, we have discovered an interesting phenomenon: the description of some strings can be compressed considerably, provided they exhibit enough regularity. However, if regularity is lacking, it becomes more cumbersome to express large numbers. For instance, it seems easier to compress the number "one billion," than the number "one billion seven hundred thirty-five million two hundred sixty-eight thousand and three hundred ninety-four," even though they are of the same order of magnitude.

We are interested in a measure of information that, unlike Shannon's, does not rely on (often untenable) probabilistic assumptions, and that takes into account the phenomenon that 'regular' strings are compressible. Thus, we aim for a measure of information content of an *individual finite object*, and in the information conveyed about an individual finite object by another individual finite object. Here, we want the information content of an object x to be an attribute of x *alone*, and not to depend on, for instance, the means chosen to describe this information content. Surprisingly, this turns out to be possible, at least to a large extent. The resulting theory of information is based on Kolmogorov complexity, a notion independently proposed by Solomonoff (1964), Kolmogorov (1965) and Chaitin (1969); Li and Vitányi (1997) describe the history of the subject.

2.2 Kolmogorov Complexity

Suppose we want to describe a given object by a finite binary string. We do not care whether the object has many descriptions; however, each description should describe but one object. From among all descriptions of an object we can take the length of the shortest description as a measure of the object's complexity. It is natural to call an object "simple" if it has at least one short description, and to call it "complex" if all of its descriptions are long.

As in Section 1.3, consider a description method D, to be used to transmit messages from a sender to a receiver. If D is known to both a sender and receiver, then a message x can be transmitted from sender to receiver by transmitting the description y with $D(y) = x$. The cost of this transmission is measured by $l(y)$, the length of y. The least cost of transmission of x is determined by the length function $L(x)$: recall that $L(x)$ is the length of the shortest y such that $D(y) = x$. We choose this length function as the descriptional complexity of x under specification method D.

Obviously, this descriptional complexity of x depends crucially on D. The general principle involved is that the syntactic framework of the description language determines the succinctness of description.

In order to objectively compare descriptional complexities of objects, to be able to say "x is more complex than z," the descriptional complexity of x should depend on x alone. This complexity can be viewed as related to a universal description method that is a priori assumed by all senders and receivers. This complexity is optimal if no other description method assigns a lower complexity to any object.

We are not really interested in optimality with respect to all description methods. For specifications to be useful at all it is necessary that the mapping from y to $D(y)$ can be executed in an effective manner. That is, it can at least in principle be performed by humans or machines. This notion has been formalized as that of "partial recursive functions", also known simply as "computable functions", which are formally defined later. According to generally accepted mathematical viewpoints it coincides with the intuitive notion of effective computation.

has minimum description length

The set of partial recursive functions contains an optimal function that minimizes description length of every other such function. We denote this function by D_0. Namely, for any other recursive function D, for all objects x, there is a description y of x under D_0 that is shorter than any description z of x under D. (That is, shorter up to an additive constant that is independent of x.) Complexity with respect to D_0 minorizes the complexities with respect to all partial recursive functions.

be a lower bound

We identify the length of the description of x with respect to a fixed specification function D_0 with the "algorithmic (descriptional) complexity" of x. The optimality of D_0 in the sense above means that the complexity of an object x is invariant (up to an additive constant independent of x) under transition from one optimal specification function to another. Its complexity is an objective attribute of the described object alone: it is an intrinsic property of that object, and it does not depend on the description formalism. This complexity can be viewed as "absolute information content": the amount of information that needs to be transmitted between all senders and receivers when they communicate the message in absence of any other a priori knowledge that restricts the domain of the message. Thus, we have outlined the program for a general theory of algorithmic complexity. The three major innovations are as follows:

1. In restricting ourselves to formally effective descriptions, our definition covers every form of description that is intuitively acceptable as being effective according to general viewpoints in mathematics and logic.

2. The restriction to effective descriptions entails that there is a universal description method that minorizes the description length or complexity with respect to any other effective description method. Significantly, this implies Item 3.

3. The description length or complexity of an object is an intrinsic attribute of the object independent of the particular description method or formalizations thereof.

2.2.1 Formal Details

The Kolmogorov complexity $K(x)$ of a finite object x will be defined as the length of the shortest effective binary description of x. Broadly speaking, $K(x)$ may be thought of as the length of the shortest computer program that prints x and then halts. This computer program may be written in C, Java, LISP or any other universal language: we shall see that, for any two universal languages, the resulting program lengths differ at most by a constant not depending on x.

To make this precise, let $T_1, T_2, \ldots$ be a standard enumeration [18] of all Turing machines, and let $\phi_1, \phi_2, \ldots$ be the enumeration of corresponding functions which are computed by the respective Turing machines. That is, T_i computes ϕ_i. These functions are the *partial recursive* functions or *computable* functions, Section 1.2. For technical reasons we are interested in the so-called prefix complexity, which is associated with Turing machines for which the set of programs (inputs) resulting in a halting computation is prefix free[1]. We can realize this by equipping the Turing machine with a one-way input tape, a separate work tape, and a one-way output tape. Such Turing machines are called prefix machines since the halting programs for any one of them form a prefix free set.

We first define $K_{T_i}(x)$, the prefix Kolmogorov complexity of x relative to a given prefix machine T_i, where T_i is the i-th prefix machine in a standard enumeration of them. $K_{T_i}(x)$ is defined as the length of the shortest input sequence y such that $T_i(y) = \phi_i(y) = x$. If no such input sequence exists, $K_{T_i}(x)$ remains undefined. Of course, this preliminary definition is still highly sensitive to the particular prefix machine T_i that we use. But now the 'universal prefix machine' comes to our rescue. Just as there exists universal ordinary Turing machines, there also exist universal prefix machines. These have the remarkable property that they can simulate every other prefix machine. More specifically, there exists a prefix machine U such that, with as input the pair $\langle i, y \rangle$, it outputs $\phi_i(y)$ and then halts. We now fix, once and for all, a prefix machine U with this property and call U the *reference machine*. The Kolmogorov complexity $K(x)$ of x is defined as $K_U(x)$.

Let us formalize this definition. Let $\langle \cdot \rangle$ be a standard invertible effective one-one encoding from $\mathcal{N} \times \mathcal{N}$ to a prefix-free subset of $\mathcal{N}$. $\langle \cdot \rangle$ may be thought of as the encoding function of a prefix code. For example, we can set $\langle x, y \rangle = x'y'$. Comparing to the definition of in Section 1.2, we note that from now on, we require $\langle \cdot \rangle$ to map to a prefix-free set. We insist on prefix-freeness and effectiveness because we want a universal Turing machine to be able to read an image under $\langle \cdot \rangle$ from left to right and determine where it ends.

Definition 2.6 Let U be our reference prefix machine satisfying for all $i \in \mathcal{N}, y \in \{0,1\}^*$, $U(\langle i, y \rangle) = \phi_i(y)$. The *prefix Kolmogorov complexity* of x is

$$\begin{aligned} K(x) &= \min_z \{ l(z) : U(z) = x, z \in \{0,1\}^* \} = \\ &= \min_{i,y} \{ l(\langle i, y \rangle) : \phi_i(y) = x, y \in \{0,1\}^*, i \in \mathcal{N} \}. \end{aligned} \tag{2.3}$$

We can alternatively think of z as a program that prints x and then halts, or as $z = \langle i, y \rangle$ where y is a program such that, when T_i is input program y, it prints x and then halts.

Thus, by definition $K(x) = l(x^*)$, where x^* is the lexicographically first shortest self-delimiting (prefix) program for x with respect to the reference prefix machine. Consider the mapping E^* defined by $E^*(x) = x^*$.

[1] There exists a version of Kolmogorov complexity corresponding to programs that are not necessarily prefix-free, but we will not go into it here.

This may be viewed as the encoding function of a prefix-code (decoding function) D^* with $D^*(x^*) = x$. By its definition, D^* is a very parsimonious code. The reason for working with prefix rather than standard Turing machines is that, for many of the subsequent developments, we need D^* to be prefix.

Though defined in terms of a particular machine model, the Kolmogorov complexity is machine-independent up to an additive constant and acquires an asymptotically universal and absolute character through Church's thesis, from the ability of universal machines to simulate one another and execute any effective process. The Kolmogorov complexity of an object can be viewed as an absolute and objective quantification of the amount of information in it.

2.2.2 Intuition

To develop some intuitions, it is useful to think of $K(x)$ as the shortest program for x in some standard programming language such as LISP or Java. Consider the lexicographical enumeration of all syntactically correct LISP programs $\lambda_1, \lambda_2, \ldots$, and the lexicographical enumeration of all syntactically correct Java programs $\pi_1, \pi_2, \ldots$. We assume that both these programs are encoded in some standard prefix-free manner. With proper definitions we can view the programs in both enumerations as computing partial recursive functions from their inputs to their outputs. Choosing reference machines in both enumerations we can define complexities $K_{\text{LISP}}(x)$ and $K_{\text{Java}}(x)$ completely analogous to $K(x)$. All of these measures of the descriptional complexities of x coincide up to a fixed additive constant. Let us show this directly for $K_{\text{LISP}}(x)$ and $K_{\text{Java}}(x)$. Since LISP is universal, there exists a LISP program λ_P implementing a Java-to-LISP compiler. λ_P translates each Java program to an equivalent LISP program. Consequently, for all x, $K_{\text{LISP}}(x) \leq K_{\text{Java}}(x) + 2l(P)$. Similarly, there is a Java program π_L that is a LISP-to-Java compiler, so that for all x, $K_{\text{Java}}(x) \leq K_{\text{LISP}}(x) + 2l(L)$. It follows that $|K_{\text{Java}}(x) - K_{\text{LISP}}(x)| \leq 2l(P) + 2l(L)$ for all x!

The programming language view immediately tells us that $K(x)$ must be small for 'simple' or 'regular' objects x. For example, there exists a fixed-size program that, when input n, outputs the first n bits of π and then halts. Specification of n takes at most $L_{\mathcal{N}}(n) = \log n + 2 \log \log n + 1$ bits. Thus, if x consists of the first n binary digits of π, then $K(x) \overset{+}{<} \log n + 2 \log \log n$. Similarly, if 0^n denotes the string consisting of n 0's, then $K(0^n) \overset{+}{<} \log n + 2 \log \log n$.

On the other hand, for all x, there exists a program 'print x; halt'. This shows that for all $K(x) \overset{+}{<} l(x)$. As was previously noted, for any prefix code, there are no more than 2^m strings x which can be described by m or less bits. In particular, this holds for the prefix code E^* whose length function is $K(x)$. Thus, the fraction of strings x of length n with $K(x) \leq m$ is at most 2^{m-n}: the overwhelming majority of sequences cannot be compressed by more than a constant. Specifically, if x is determined by n independent tosses of a fair coin, then with overwhelming probability, $K(x) \approx l(x)$. Thus, while for very regular strings, the Kolmogorov complexity is small (sub-linear in the length of the string), *most* strings are 'random' and have Kolmogorov complexity about equal to their own length.

2.2.3 Kolmogorov complexity of sets, functions and probability distributions

Finite sets: The class of *finite sets* consists of the set of finite subsets $S \subseteq \{0,1\}^*$. The *complexity of the finite set* S is $K(S)$—the length (number of bits) of the shortest binary program p from which the reference universal prefix machine U computes a listing of the elements of S and then halts. That is, if $S = \{x_1, \ldots, x_n\}$, then $U(p) = \langle x_1, \langle x_2, \ldots, \langle x_{n-1}, x_n \rangle \ldots \rangle \rangle$. The *conditional complexity* $K(x \mid S)$ of x given S, is the length (number of bits) in the shortest binary program p from which the reference universal prefix machine U, given S literally as auxiliary information, computes x.

Integer-valued functions: The (prefix-) complexity $K(f)$ of a partial recursive function f is defined by $K(f) = \min_i \{K(i) : \text{Turing machine } T_i \text{ computes } f\}$. If f^* is a shortest program for computing the function f (if there is more than one of them then f^* is the first one in enumeration order), then $K(f) = l(f^*)$.

Remark 2.7 In the above definition of $K(f)$, the objects being described are functions instead of finite binary strings. To unify the approaches, we can consider a finite binary string x as corresponding to a function having value x for argument 0. Note that we can upper semi-compute (Section 1.2) x^* given x, but we cannot upper semi-compute f^* given f (as an oracle), since we should be able to verify agreement of a program for a function and an oracle for the target function, on all infinitely many arguments. ◇

Probability Distributions: In this text we identify probability distributions on finite and countable sets $\mathcal{X}$ with their corresponding mass functions (Section 1.2). Since any (sub-) probability mass function f is a total real-valued function, $K(f)$ is defined in the same way as above.

2.2.4 Kolmogorov Complexity and the Universal Distribution

Following the definitions above we now consider lower semi-computable and computable probability mass functions (Section 1.2). By the fundamental Kraft's inequality, Theorem 1.2, we know that if $l_1, l_2, \ldots$ are the code-word lengths of a prefix code, then $\sum_x 2^{-l_x} \leq 1$. Therefore, since $K(x)$ is the length of a prefix-free program for x, we can interpret $2^{-K(x)}$ as a sub-probability mass function, and we define $\mathbf{m}(x) = 2^{-K(x)}$. This is the so-called universal distribution—a rigorous form of Occam's razor. The following two theorems are to be considered as major achievements in the theory of Kolmogorov complexity, and will be used again and again in the sequel. For the proofs we refer to [18].

Theorem 2.8 *Let f represent a lower semi-computable (sub-) probability distribution on the natural numbers (equivalently, finite binary strings). (This implies $K(f) < \infty$.) Then, $2^{c_f}\mathbf{m}(x) > f(x)$ for all x, where $c_f = K(f) + O(1)$. We call $\mathbf{m}$ a* universal distribution.

The family of lower semi-computable sub-probability mass functions contains all distributions with computable parameters which have a name, or in which we could conceivably be interested, or which have ever been considered[2]. In particular, it contains the computable distributions. We call $\mathbf{m}$ "universal" since it assigns at least as much probability to each object as any other lower semi-computable distribution (up to a multiplicative factor), and is itself lower semi-computable.

Theorem 2.9

$$\log 1/\mathbf{m}(x) = K(x) \pm O(1). \tag{2.4}$$

That means that $\mathbf{m}$ assigns high probability to simple objects and low probability to complex or random objects. For example, for $x = 00\ldots0$ (n 0's) we have $K(x) = K(n) \pm O(1) \leq \log n + 2\log\log n + O(1)$ since the program

```
print n_times a ''0''
```

prints x. (The additional $2\log\log n$ term is the penalty term for a prefix encoding.) Then, $1/(n\log^2 n) = O(\mathbf{m}(x))$. But if we flip a coin to obtain a string y of n bits, then with overwhelming probability $K(y) \geq n \pm O(1)$ (because y does not contain effective regularities which allow compression), and hence $\mathbf{m}(y) = O(1/2^n)$.

Problem and Lacuna: Unfortunately $K(x)$ is not a recursive function: the Kolmogorov complexity is not computable in general. This means that there exists no computer program that, when input an arbitrary string, outputs the Kolmogorov complexity of that string and then halts. While Kolmogorov complexity is upper semi-computable (Section 1.2), it cannot be approximated in general in a practically useful sense; and even though there exist 'feasible', resource-bounded forms of Kolmogorov complexity (Li and Vitányi 1997), these lack some of the elegant properties of the original, uncomputable notion.

Now suppose we are interested in efficient storage and transmission of long sequences of data. According to Kolmogorov, we can compress such sequences in an essentially optimal way by storing or transmitting the shortest program that generates them. Unfortunately, as we have just seen, we cannot find such a program in general. According to Shannon, we can compress such sequences optimally in an average sense (and therefore, it turns out, also with high probability) if they are distributed according to some P and we know P. Unfortunately, in practice, P is often unknown, it may not be computable—bringing us in the same conundrum as with the Kolmogorov complexity approach—or worse, it may be nonexistent. In Appendix A, we consider *universal coding*, which can be considered a sort of middle ground between Shannon information and Kolmogorov complexity. In contrast to both these approaches, universal codes can be directly applied for practical data compression. Some basic knowledge of universal codes will be very helpful in providing intuition for the next section, in which we relate Kolmogorov complexity and Shannon entropy. Nevertheless,

[2] To be sure, in statistical applications, one often works with model classes containing distributions that are neither upper- nor lower semi-computable. An example is the Bernoulli model class, containing the distributions with $P(X = 1) = \theta$ for all $\theta \in [0, 1]$. However, every concrete *parameter estimate* or *predictive distribution* based on the Bernoulli model class that has ever been considered or in which we could be conceivably interested, is in fact computable; typically, θ is then rational-valued. See also Example A.2 in Appendix A.

universal codes are not directly needed in any of the statements and proofs of the next section or, in fact, anywhere else in the paper, which is why delegated their treatment to an appendix.

2.3 Expected Kolmogorov Complexity Equals Shannon Entropy

Suppose the source words x are distributed as a random variable X with probability $P(X = x) = f(x)$. While $K(x)$ is fixed for each x and gives the shortest code word length (but only up to a fixed constant) and is *independent* of the probability distribution P, we may wonder whether K is also universal in the following sense: If we weigh each individual code word length for x with its probability $f(x)$, does the resulting f-expected code word length $\sum_x f(x)K(x)$ achieve the minimal average code word length $H(X) = \sum_x f(x) \log 1/f(x)$? Here we sum over the entire support of f; restricting summation to a small set, for example the singleton set $\{x\}$, can give a different result. The reasoning above implies that, under some mild restrictions on the distributions f, the answer is yes. This is expressed in the following theorem, where, instead of the quotient we look at the difference of $\sum_x f(x)K(x)$ and $H(X)$. This allows us to express really small distinctions.

Theorem 2.10 *Let f be a computable probability mass function (Section 1.2) $f(x) = P(X = x)$ on sample space $\mathcal{X} = \{0,1\}^*$ associated with a random source X and entropy $H(X) = \sum_x f(x) \log 1/f(x)$. Then,*

$$0 \leq \left(\sum_x f(x)K(x) - H(X) \right) \leq K(f) + O(1).$$

Proof. Since $K(x)$ is the code word length of a prefix-code for x, the first inequality of the Noiseless Coding Theorem 2.5 states that

$$H(X) \leq \sum_x f(x)K(x).$$

Since $f(x) \leq 2^{K(f)+O(1)}\mathbf{m}(x)$ (Theorem 2.8) and $\log \mathbf{m}(x) = K(x) + O(1)$ (Theorem 2.9), we have $\log 1/f(x) \geq K(x) - K(f) - O(1)$. It follows that

$$\sum_x f(x)K(x) \leq H(X) + K(f) + O(1).$$

Set the constant c_f to

$$c_f := K(f) + O(1),$$

and the theorem is proved. As an aside, the constant implied in the $O(1)$ term depends on the lengths of the programs occurring in the proof of the cited Theorems 2.8, 2.9 (Theorems 4.3.1 and 4.3.2 in [18]). These depend only on the reference universal prefix machine. □

The theorem shows that for simple (low complexity) distributions the expected Kolmogorov complexity is close to the entropy, but these two quantities may be wide apart for distributions of high complexity. This explains the apparent problem arising in considering a distribution f that concentrates all probability on an element x of length n. Suppose we choose $K(x) > n$. Then $f(x) = 1$ and hence the entropy $H(f) = 0$. On the other hand the term $\sum_{x \in \{0,1\}^*} f(x)K(x) = K(x)$. Therefore, the discrepancy between the expected Kolmogorov complexity and the entropy exceeds the length n of x. One may think this contradicts the theorem, but that is not the case: The complexity of the distribution is at least that of x, since we can reconstruct x given f (just compute $f(y)$ for all y of length n in lexicographical order until we meet one that has probability 1). Thus, $c_f = K(f) + O(1) \geq K(x) + O(1) \geq n + O(1)$. Thus, if we pick a probability distribution with a complex support, or a trickily skewed probability distribution, than this is reflected in the complexity of that distribution, and as consequence in the closeness between the entropy and the expected Kolmogorov complexity.

For example, bringing the discussion in line with the universal coding counterpart of Appendix A by considering f's that can be interpreted as sequential information sources and denoting the conditional version of f restricted to strings of length n by $f^{(n)}$ as in Section 1.2, we find by the same proof as the theorem that for all n,

$$0 \leq \sum_{x \in \{0,1\}^n} f^{(n)}(x)K(x) - H(f^{(n)}) \leq c_{f^{(n)}},$$

where $c_{f^{(n)}} = K(f^{(n)}) + O(1) \leq K(f) + K(n) + O(1)$ is now a constant depending on both f and n. On the other hand, we can eliminate the complexity of the distribution, or its recursivity for that matter, and / or restrictions to a conditional version of f restricted to a finite support A (for example $A = \{0,1\}^n$), denoted by f^A, in the following conditional formulation (this involves a peek in the future since the precise meaning of the "$K(\cdot \mid \cdot)$" notation is only provided in Definition 3.2):

$$0 \leq \sum_{x \in A} f^A(x) K(x \mid f, A) - H(f^A) = O(1). \tag{2.5}$$

The Shannon-Fano code for a computable distribution is itself computable. Therefore, for every computable distribution f, the universal code D^* whose length function is the Kolmogorov complexity compresses on average at least as much as the Shannon-Fano code for f. This is the intuitive reason why, no matter what computable distribution f we take, its expected Kolmogorov complexity is close to its entropy.

3 Mutual Information

3.1 Probabilistic Mutual Information

How much information can a random variable X convey about a random variable Y? Taking a purely combinatorial approach, this notion is captured as follows: If X ranges over $\mathcal{X}$ and Y ranges over $\mathcal{Y}$, then we look at the set U of possible events $(X = x, Y = y)$ consisting of joint occurrences of event $X = x$ and event $Y = y$. If U does not equal the Cartesian product $\mathcal{X} \times \mathcal{Y}$, then this means there is some dependency between X and Y. Considering the set $U_x = \{(x,u) : (x,u) \in U\}$ for $x \in \mathcal{X}$, it is natural to define the *conditional entropy* of Y given $X = x$ as $H(Y|X = x) = \log d(U_x)$. This suggests immediately that the information given by $X = x$ about Y is

$$I(X = x : Y) = H(Y) - H(Y|X = x).$$

For example, if $U = \{(1,1),(1,2),(2,3)\}$, $U \subseteq \mathcal{X} \times \mathcal{Y}$ with $\mathcal{X} = \{1,2\}$ and $\mathcal{Y} = \{1,2,3,4\}$, then $I(X = 1 : Y) = 1$ and $I(X = 2 : Y) = 2$.

In this formulation it is obvious that $H(X|X = x) = 0$, and that $I(X = x : X) = H(X)$. This approach amounts to the assumption of a *uniform distribution* of the probabilities concerned.

We can generalize this approach, taking into account the frequencies or probabilities of the occurrences of the different values X and Y can assume. Let the *joint probability* $f(x,y)$ be the "probability of the joint occurrence of event $X = x$ and event $Y = y$." The *marginal probabilities* $f_1(x)$ and $f_2(y)$ are defined by $f_1(x) = \sum_y f(x,y)$ and $f_2(y) = \sum_x f(x,y)$ and are "the probability of the occurrence of the event $X = x$" and the "probability of the occurrence of the event $Y = y$", respectively. This leads to the self-evident formulas for joint variables X, Y:

$$\begin{aligned} H(X,Y) &= \sum_{x,y} f(x,y) \log 1/f(x,y), \\ H(X) &= \sum_x f(x) \log 1/f(x), \\ H(Y) &= \sum_y f(y) \log 1/f(y), \end{aligned}$$

where summation over x is taken over all outcomes of the random variable X and summation over y is taken over all outcomes of random variable Y. One can show that

$$H(X,Y) \leq H(X) + H(Y), \tag{3.1}$$

with equality only in the case that X and Y are independent. In all of these equations the entropy quantity on the left-hand side increases if we choose the probabilities on the right-hand side more equally.

Conditional entropy: We start the analysis of the information in X about Y by first considering the *conditional entropy* of Y given X as the average of the entropy for Y for each value of X weighted by the probability of getting that particular value:

$$\begin{aligned} H(Y|X) &= \sum_x f_1(x) H(Y|X=x) \\ &= \sum_x f_1(x) \sum_y f(y|x) \log 1/f(y|x) \\ &= \sum_{x,y} f(x,y) \log 1/f(y|x). \end{aligned}$$

Here $f(y|x)$ is the conditional probability mass function as defined in Section 1.2.

The quantity on the left-hand side tells us how uncertain we are on average about the outcome of Y when we know an outcome of X. With

$$\begin{aligned} H(X) &= \sum_x f_1(x) \log 1/f_1(x) \\ &= \sum_x \left(\sum_y f(x,y) \right) \log \sum_y 1/f(x,y) \\ &= \sum_{x,y} f(x,y) \log \sum_y 1/f(x,y), \end{aligned}$$

and substituting the formula for $f(y|x)$, we find $H(Y|X) = H(X,Y) - H(X)$. Rewrite this expression as the Entropy Equality

$$H(X,Y) = H(X) + H(Y|X). \tag{3.2}$$

This can be interpreted as, "the uncertainty of the joint event (X,Y) is the uncertainty of X plus the uncertainty of Y given X." Combining Equations 3.1, 3.2 gives $H(Y) \geq H(Y|X)$, which can be taken to imply that, on average, knowledge of X can never increase uncertainty of Y. In fact, uncertainty in Y will be decreased unless X and Y are independent.

Information: The *information* in the outcome $X = x$ about Y is defined as

$$I(X = x : Y) = H(Y) - H(Y|X = x). \tag{3.3}$$

Here the quantities $H(Y)$ and $H(Y|X = x)$ on the right-hand side of the equations are always equal to or less than the corresponding quantities under the uniform distribution we analyzed first. The values of the quantities $I(X = x : Y)$ under the assumption of uniform distribution of Y and $Y|X = x$ versus any other distribution are not related by inequality in a particular direction. The equalities $H(X|X = x) = 0$ and $I(X = x : X) = H(X)$ hold under any distribution of the variables. Since $I(X = x : Y)$ is a function of outcomes of X, while $I(Y = y : X)$ is a function of outcomes of Y, we do not compare them directly. However, forming the expectation defined as

$$\begin{aligned} \mathbf{E}(I(X = x : Y)) &= \sum_x f_1(x) I(X = x : Y), \\ \mathbf{E}(I(Y = y : X)) &= \sum_y f_2(y) I(Y = y : X), \end{aligned}$$

and combining Equations 3.2, 3.3, we see that the resulting quantities are equal. Denoting this quantity by $I(X;Y)$ and calling it the *mutual information* in X and Y, we see that this information is *symmetric*:

$$I(X;Y) = \mathbf{E}(I(X = x : Y)) = \mathbf{E}(I(Y = y : X)). \tag{3.4}$$

Writing this out we find that the *mutual information* $I(X;Y)$ is defined by:

$$I(X;Y) = \sum_x \sum_y f(x,y) \log \frac{f(x,y)}{f_1(x) f_2(y)}. \tag{3.5}$$

Another way to express this is as follows: a well-known criterion for the difference between a given distribution $f(x)$ and a distribution $g(x)$ it is compared with is the so-called *Kullback-Leibler divergence*

$$D(f \parallel g) = \sum_x f(x) \log f(x)/g(x). \tag{3.6}$$

It has the important property that

$$D(f \parallel g) \geq 0 \tag{3.7}$$

with equality only iff $f(x) = g(x)$ for all x. This is called the *information inequality* in [4], p. 26. Thus, the mutual information is the Kullback-Leibler divergence between the joint distribution and the product $f_1(x)f_2(y)$ of the two marginal distributions. If this quantity is 0 then $f(x,y) = f_1(x)f_2(y)$ for every pair x, y, which is the same as saying that X and Y are independent random variables.

Example 3.1 Suppose we want to exchange the information about the outcome $X = x$ and it is known already that outcome $Y = y$ is the case, that is, x has property y. Then we require (using the Shannon-Fano code) about $\log 1/P(X = x|Y = y)$ bits to communicate x. On average, over the joint distribution $P(X = x, Y = y)$ we use $H(X|Y)$ bits, which is optimal by Shannon's noiseless coding theorem. In fact, exploiting the mutual information paradigm, the expected information $I(Y;X)$ that outcome $Y = y$ gives about outcome $X = x$ is the same as the expected information that $X = x$ gives about $Y = y$, and is never negative. Yet there may certainly exist *individual* y such that $I(Y = y : X)$ is negative. For example, we may have $\mathcal{X} = \{0,1\}$, $\mathcal{Y} = \{0,1\}$, $P(X = 1|Y = 0) = 1$, $P(X = 1|Y = 1) = 1/2$, $P(Y = 1) = \epsilon$. Then $I(Y;X) = H(\epsilon, 1-\epsilon)$ whereas $I(Y = 1 : X) = H(\epsilon, 1-\epsilon) + \epsilon - 1$. For small ϵ, this quantity is smaller than 0. ♢

Problem and Lacuna: The quantity $I(Y;X)$ symmetrically characterizes to what extent random variables X and Y are correlated. An inherent problem with probabilistic definitions is that — as we have just seen — although $I(Y;X) = \mathbf{E}(I(Y = y : X))$ is always positive, for some probability distributions and some y, $I(Y = y : X)$ can turn out to be negative—which definitely contradicts our naive notion of information content. The *algorithmic* mutual information we introduce below can *never* be negative, and in this sense is closer to the intuitive notion of information content.

3.2 Algorithmic Mutual Information

For individual objects the information about one another is possibly even more fundamental than for random sources. Kolmogorov [10]:

> Actually, it is most fruitful to discuss the quantity of information "conveyed by an object" (x) "about an object" (y). It is not an accident that in the probabilistic approach this has led to a generalization to the case of continuous variables, for which the entropy is finite but, in a large number of cases,
>
> $$I_W(x,y) = \int\int P_{xy}(dx\ dy) \log_2 \frac{P_{xy}(dx\ dy)}{P_x(dx)P_y(dy)}$$
>
> is finite. The real objects that we study are very (infinitely) complex, but the relationships between two separate objects diminish as the schemes used to describe them become simpler. While a map yields a considerable amount of information about a region of the earth's surface, the microstructure of the paper and the ink on the paper have no relation to the microstructure of the area shown on the map."

In the discussions on Shannon mutual information, we first needed to introduce a conditional version of entropy. Analogously, to prepare for the definition of algorithmic mutual information, we need a notion of conditional Kolmogorov complexity.

Intuitively, the conditional prefix Kolmogorov complexity $K(x|y)$ of x given y can be interpreted as the shortest prefix program p such that, when y is given to the program p as input, the program prints x and then halts. The idea of providing p with an input y is realized by putting $\langle p, y \rangle$ rather than just p on the input tape of the universal prefix machine U.

Definition 3.2 The *conditional prefix Kolmogorov complexity* of x given y (for free) is

$$K(x|y) = \min_p \{l(p) : U(\langle p, y\rangle) = x, p \in \{0,1\}^*\}.$$

We define

$$K(x) = K(x|\epsilon). \tag{3.8}$$

Note that we just redefined $K(x)$ so that the unconditional Kolmogorov complexity is *exactly* equal to the conditional Kolmogorov complexity with empty input. This does not contradict our earlier definition: we can choose a reference prefix machine U such that $U(\langle p, \epsilon\rangle) = U(p)$. Then (3.8) holds automatically.

We now have the technical apparatus to express the relation between entropy inequalities and Kolmogorov complexity inequalities. Recall that the entropy expresses the expected information to transmit an outcome of a known random source, while the Kolmogorov complexity of every such outcome expresses the specific information contained in that outcome. This makes us wonder to what extend the entropy-(in)equalities hold for the corresponding Kolmogorov complexity situation. In the latter case the corresponding (in)equality is a far stronger statement, implying the same (in)equality in the entropy setting. It is remarkable, therefore, that similar inequalities hold for both cases, where the entropy ones hold exactly while the Kolmogorov complexity ones hold up to a logarithmic, and in some cases $O(1)$, additive precision.

Additivity: By definition, $K(x, y) = K(\langle x, y\rangle)$. Trivially, the symmetry property holds: $K(x, y) \overset{+}{=} K(y, x)$. Another interesting property is the "Additivity of Complexity" property that, as we explain further below, is equivalent to the "Symmetry of Algorithmic Mutual Information" property. Recall that x^* denotes the first (in a standard enumeration order) shortest prefix program that generates x and then halts.

Theorem 3.3 (Additivity of Complexity/Symmetry of Mutual Information)

$$K(x, y) \overset{+}{=} K(x) + K(y \mid x^*) \overset{+}{=} K(y) + K(x \mid y^*). \tag{3.9}$$

This is the Kolmogorov complexity equivalent of the entropy equality (3.2). That this latter equality holds is true by simply rewriting both sides of the equation according to the definitions of averages of joint and marginal probabilities. In fact, potential individual differences are averaged out. But in the Kolmogorov complexity case we do nothing like that: it is truly remarkable that additivity of algorithmic information holds for individual objects. It was first proven by Kolmogorov and Leonid A. Levin for the plain (non-prefix) version of Kolmogorov complexity, where it holds up to an additive logarithmic term, and reported in [29]. The prefix-version (3.9), holding up to an $O(1)$ additive term is due to [6], can be found as Theorem 3.9.1 in [18], and has a difficult proof.

Symmetry: To define the algorithmic mutual information between two individual objects x and y with no probabilities involved, it is instructive to first recall the probabilistic notion (3.5). Rewriting (3.5) as

$$\sum_x \sum_y f(x, y)[\log 1/f(x) + \log 1/f(y) - \log 1/f(x, y)],$$

and noting that $\log 1/f(s)$ is very close to the length of the prefix-free Shannon-Fano code for s, we are led to the following definition. The *information in y about x* is defined as

$$I(y : x) = K(x) - K(x \mid y^*) \overset{+}{=} K(x) + K(y) - K(x, y), \tag{3.10}$$

where the second equality is a consequence of (3.9) and states that this information is symmetrical, $I(x : y) \overset{+}{=} I(y : x)$, and therefore we can talk about *mutual information.*[3]

[3] The notation of the algorithmic (individual) notion $I(x : y)$ distinguishes it from the probabilistic (average) notion $I(X; Y)$. We deviate slightly from [18] where $I(y : x)$ is defined as $K(x) - K(x \mid y)$.

Precision $-$ $O(1)$ **vs.** $O(\log n)$**:** The version of (3.9) with just x and y in the conditionals doesn't hold with $\stackrel{+}{=}$, but holds up to additive logarithmic terms that cannot be eliminated. To gain some further insight in this matter, first consider the following lemma:

Lemma 3.4 *x^* has the same information as the pair $x, K(x)$, that is, $K(x^* \mid x, K(x)), K(x, K(x) \mid x^*) = O(1)$.*

Proof. Given $x, K(x)$ we can run all programs simultaneously in dovetailed fashion and select the first program of length $K(x)$ that halts with output x as x^*. (Dovetailed fashion means that in phase k of the process we run all programs i for j steps such that $i + j = k$, $k = 1, 2, \ldots$) □

Thus, x^* provides more information than x. Therefore, we have to be very careful when extending Theorem 3.3. For example, the conditional version of (3.9) is:

$$K(x, y \mid z) \stackrel{+}{=} K(x \mid z) + K(y \mid x, K(x \mid z), z). \tag{3.11}$$

Note that a naive version

$$K(x, y \mid z) \stackrel{+}{=} K(x \mid z) + K(y \mid x^*, z)$$

is incorrect: taking $z = x$, $y = K(x)$, the left-hand side equals $K(x^* \mid x)$ which can be as large as $\log n - \log\log n + O(1)$, and the right-hand side equals $K(x \mid x) + K(K(x) \mid x^*, x) \stackrel{+}{=} 0$.

But up to logarithmic precision we do not need to be that careful. In fact, it turns out that *every* linear entropy inequality holds for the corresponding Kolmogorov complexities within a logarithmic additive error, [9]:

Theorem 3.5 *All linear (in)equalities that are valid for Kolmogorov complexity are also valid for Shannon entropy and vice versa—provided we require the Kolmogorov complexity (in)equalities to hold up to additive logarithmic precision only.*

3.3 Expected Algorithmic Mutual Information Equals Probabilistic Mutual Information

Theorem 2.10 gave the relationship between entropy and ordinary Kolmogorov complexity; it showed that the entropy of distribution P is approximately equal to the expected (under P) Kolmogorov complexity. Theorem 3.6 gives the analogous result for the mutual information (to facilitate comparison to Theorem 2.10, note that x and y in (3.12) below may stand for strings of arbitrary length n).

Theorem 3.6 *Given a computable probability distribution $f(x, y)$ over (x, y) we have*

$$\begin{aligned} I(X;Y) - K(f) &\stackrel{+}{<} \sum_x \sum_y f(x,y) I(x:y) \\ &\stackrel{+}{<} I(X;Y) + 2K(f), \end{aligned} \tag{3.12}$$

Proof. Rewrite the expectation

$$\begin{aligned} \sum_x \sum_y f(x,y) I(x:y) \stackrel{+}{=} \sum_x \sum_y & f(x,y)[K(x) \\ & + K(y) - K(x,y)]. \end{aligned}$$

Define $\sum_y f(x,y) = f_1(x)$ and $\sum_x f(x,y) = f_2(y)$ to obtain

$$\begin{aligned} \sum_x \sum_y f(x,y) I(x:y) \stackrel{+}{=} & \sum_x f_1(x) K(x) + \sum_y f_2(y) K(y) \\ & - \sum_{x,y} f(x,y) K(x,y). \end{aligned}$$

Given the program that computes f, we can approximate $f_1(x)$ by $q_1(x, y_0) = \sum_{y \leq y_0} f(x, y)$, and similarly for f_2. That is, the distributions f_i $(i = 1, 2)$ are lower semicomputable. Because they sum to 1 it can be shown they must also be computable. By Theorem 2.10, we have $H(g) \overset{+}{<} \sum_x g(x)K(x) \overset{+}{<} H(g) + K(g)$ for every computable probability mass function g.

Hence, $H(f_i) \overset{+}{<} \sum_x f_i(x)K(x) \overset{+}{<} H(f_i) + K(f_i)$ $(i = 1, 2)$, and $H(f) \overset{+}{<} \sum_{x,y} f(x, y)K(x, y) \overset{+}{<} H(f) + K(f)$. On the other hand, the probabilistic mutual information (3.5) is expressed in the entropies by $I(X;Y) = H(f_1) + H(f_2) - H(f)$. By construction of the f_i's above, we have $K(f_1), K(f_2) \overset{+}{<} K(f)$. Since the complexities are positive, substitution establishes the lemma. □

Can we get rid of the $K(f)$ error term? The answer is affirmative; by putting $f(\cdot)$ in the conditional, and applying (2.5), we can even get rid of the computability requirement.

Lemma 3.7 *Given a joint probability distribution $f(x, y)$ over (x, y) (not necessarily computable) we have*

$$I(X;Y) \overset{+}{=} \sum_x \sum_y f(x, y)I(x : y \mid f),$$

where the auxiliary f means that we can directly access the values $f(x, y)$ on the auxiliary conditional information tape of the reference universal prefix machine.

Proof. The lemma follows from the definition of conditional algorithmic mutual information, if we show that $\sum_x f(x)K(x \mid f) \overset{+}{=} H(f)$, where the $O(1)$ term implicit in the $\overset{+}{=}$ sign is independent of f.

Equip the reference universal prefix machine, with an $O(1)$ length program to compute a Shannon-Fano code from the auxiliary table of probabilities. Then, given an input r, it can determine whether r is the Shannon-Fano code word for some x. Such a code word has length $\overset{+}{=} \log 1/f(x)$. If this is the case, then the machine outputs x, otherwise it halts without output. Therefore, $K(x \mid f) \overset{+}{<} \log 1/f(x)$. This shows the upper bound on the expected prefix complexity. The lower bound follows as usual from the Noiseless Coding Theorem. □

Thus, we see that the expectation of the algorithmic mutual information $I(x : y)$ is close to the probabilistic mutual information $I(X;Y)$ — which is important: if this were not the case then the algorithmic notion would not be a sharpening of the probabilistic notion to individual objects, but something else.

4 Mutual Information Non-Increase

4.1 Probabilistic Version

Is it possible to increase the mutual information between two random variables, by processing the outcomes in some deterministic manner? The answer is negative: For every function T we have

$$I(X;Y) \geq I(X;T(Y)), \tag{4.1}$$

that is, mutual information between two random variables cannot be increased by processing their outcomes in any deterministic way. The same holds in an appropriate sense for randomized processing of the outcomes of the random variables. This fact is called the *data processing inequality* [4], Theorem 2.8.1. The reason why it holds is that (3.5) is expressed in terms of probabilities $f(a, b), f_1(a), f_2(b)$, rather than in terms of the arguments. Processing the arguments a, b will not increase the value of the expression in the right-hand side. If the processing of the arguments just renames them in a one-to-one manner then the expression keeps the same value. If the processing eliminates or merges arguments then it is easy to check from the formula that the expression value doesn't increase.

4.2 Algorithmic Version

In the algorithmic version of mutual information, the notion is expressed in terms of the individual arguments instead of solely in terms of the probabilities as in the probabilistic version. Therefore, the reason for (4.1) to hold is not valid in the algorithmic case. Yet it turns out that the data processing inequality also holds between individual objects, by far more subtle arguments and not precisely but with a small tolerance. The first to observe this fact was Leonid A. Levin who proved his "information non-growth," and "information conservation inequalities" for both finite and infinite sequences under both deterministic and randomized data processing, [15, 16].

4.2.1 A Triangle Inequality

We first discuss some useful technical lemmas. The additivity of complexity (symmetry of information) (3.9) can be used to derive a "directed triangle inequality" from [8], that is needed later.

Theorem 4.1 *For all x, y, z,*

$$K(x \mid y^*) \overset{+}{<} K(x, z \mid y^*) \overset{+}{<} K(z \mid y^*) + K(x \mid z^*).$$

Proof. Using (3.9), an evident inequality introducing an auxiliary object z, and twice (3.9) again:

$$\begin{aligned} K(x, z \mid y^*) &\overset{+}{=} K(x, y, z) - K(y) \\ &\overset{+}{<} K(z) + K(x \mid z^*) + K(y \mid z^*) - K(y) \\ &\overset{+}{=} K(y, z) - K(y) + K(x \mid z^*) \\ &\overset{+}{=} K(x \mid z^*) + K(z \mid y^*). \end{aligned}$$

□

Remark 4.2 This theorem has bizarre consequences. These consequences are not simple unexpected artifacts of our definitions, but, to the contrary, they show the power and the genuine contribution to our understanding represented by the deep and important mathematical relation (3.9).

Denote $k = K(y)$ and substitute $k = z$ and $K(k) = x$ to find the following counterintuitive corollary: To determine the complexity of the complexity of an object y it suffices to give both y and the complexity of y. This is counterintuitive since in general we cannot compute the complexity of an object from the object itself; if we could this would also solve the so-called "halting problem", [18]. This noncomputability can be quantified in terms of $K(K(y) \mid y)$ which can rise to almost $K(K(y))$ for some y. But in the seemingly similar, but subtly different, setting below it is possible.

Corollary 4.3 *As above, let k denote $K(y)$. Then, $K(K(k) \mid y, k) \overset{+}{=} K(K(k) \mid y^*) \overset{+}{<} K(K(k) \mid k^*) + K(k \mid y, k) \overset{+}{=} 0$.*

◇

Now back to whether mutual information in one object about another one cannot be increased. In the probabilistic setting this was shown to hold for random variables. But does it also hold for individual outcomes? In [15, 16] it was shown that the information in one individual string about another cannot be increased by any deterministic algorithmic method by more than a constant. With added randomization this holds with overwhelming probability. Here, we follow the proof method of [8] and use the triangle inequality of Theorem 4.1 to recall, and to give proofs of this information non-increase.

4.2.2 Deterministic Data Processing:

Recall the definition 3.10 and Theorem 3.12. We prove a strong version of the information non-increase law under deterministic processing (later we need the attached corollary):

Theorem 4.4 *Given x and z, let q be a program computing z from x^*. Then*

$$I(z : y) \overset{+}{<} I(x : y) + K(q). \tag{4.2}$$

Proof. By the triangle inequality,

$$\begin{aligned} K(y \mid x^*) &\overset{+}{<} K(y \mid z^*) + K(z \mid x^*) \\ &\overset{+}{=} K(y \mid z^*) + K(q). \end{aligned}$$

Thus,

$$\begin{aligned} I(x:y) &= K(y) - K(y \mid x^*) \\ &\overset{+}{>} K(y) - K(y \mid z^*) - K(q) \\ &= I(z:y) - K(q). \end{aligned}$$

□

This also implies the slightly weaker but intuitively more appealing statement that the mutual information between strings x and y cannot be increased by processing x and y separately by deterministic computations.

Corollary 4.5 *Let f, g be recursive functions. Then*

$$I(f(x):g(y)) \overset{+}{<} I(x:y) + K(f) + K(g). \tag{4.3}$$

Proof. It suffices to prove the case $g(y) = y$ and apply it twice. The proof is by replacing the program q that computes a particular string z from a particular x^* in (4.2). There, q possibly depends on x^* and z. Replace it by a program q_f that first computes x from x^*, followed by computing a recursive function f, that is, q_f is independent of x. Since we only require an $O(1)$-length program to compute x from x^* we can choose $l(q_f) \overset{+}{=} K(f)$.

By the triangle inequality,

$$\begin{aligned} K(y \mid x^*) &\overset{+}{<} K(y \mid f(x)^*) + K(f(x) \mid x^*) \\ &\overset{+}{=} K(y \mid f(x)^*) + K(f). \end{aligned}$$

Thus,

$$\begin{aligned} I(x:y) &= K(y) - K(y \mid x^*) \\ &\overset{+}{>} K(y) - K(y \mid f(x)^*) - K(f) \\ &= I(f(x):y) - K(f). \end{aligned}$$

□

4.2.3 Randomized Data Processing:

It turns out that furthermore, randomized computation can increase information only with negligible probability. Recall from Section 2.2.4 that the *universal probability* $\mathbf{m}(x) = 2^{-K(x)}$ is maximal within a multiplicative constant among lower semicomputable semimeasures. So, in particular, for each computable measure $f(x)$ we have $f(x) \leq c_1 \mathbf{m}(x)$, where the constant factor c_1 depends on f. This property also holds when we have an extra parameter, like y^*, in the condition.

Suppose that z is obtained from x by some randomized computation. We assume that the probability $f(z \mid x)$ of obtaining z from x is a semicomputable distribution over the z's. Therefore it is upperbounded by $\mathbf{m}(z \mid x) \leq c_2 \mathbf{m}(z \mid x^*) = 2^{-K(z \mid x^*)}$. The information increase $I(z:y) - I(x:y)$ satisfies the theorem below.

Theorem 4.6 *There is a constant c_3 such that for all x, y, z we have*

$$\mathbf{m}(z \mid x^*) 2^{I(z:y) - I(x:y)} \leq c_3 \mathbf{m}(z \mid x^*, y, K(y \mid x^*)).$$

Remark 4.7 For example, the probability of an increase of mutual information by the amount d is $O(2^{-d})$. The theorem implies $\sum_z \mathbf{m}(z \mid x^*) 2^{I(z:y) - I(x:y)} = O(1)$, the $\mathbf{m}(\cdot \mid x^*)$-expectation of the exponential of the increase is bounded by a constant. ♢

Proof. We have

$$\begin{aligned} I(z:y) - I(x:y) &= K(y) - K(y \mid z^*) - (K(y) - K(y \mid x^*)) \\ &= K(y \mid x^*) - K(y \mid z^*). \end{aligned}$$

The negative logarithm of the left-hand side in the theorem is therefore

$$K(z \mid x^*) + K(y \mid z^*) - K(y \mid x^*).$$

Using Theorem 4.1, and the conditional additivity (3.11), this is

$$\overset{+}{>} K(y, z \mid x^*) - K(y \mid x^*) \overset{+}{=} K(z \mid x^*, y, K(y \mid x^*)).$$

□

Remark 4.8 An example of the use of algorithmic mutual information is as follows [17]. A celebrated result of K. Gödel states that Peano Arithmetic is incomplete in the sense that it cannot be consistently extended to a complete theory using recursively enumerable axiom sets. (Here 'complete' means that every sentence of Peano Arithmetic is decidable within the theory; for further details on the terminology used in this example, we refer to [18]). The essence is the non-existence of total recursive extensions of a universal partial recursive predicate. This is usually taken to mean that mathematics is undecidable. Non-existence of an algorithmic solution need not be a problem when the requirements do not imply unique solutions. A perfect example is the generation of strings of high Kolmogorov complexity, say of half the length of the strings. There is no deterministic effective process that can produce such a string; but repeatedly flipping a fair coin we generate a desired string with overwhelming probability. Therefore, the question arises whether randomized means allow us to bypass Gödel's result. The notion of mutual information between two finite strings can be refined and extended to infinite sequences, so that, again, it cannot be increased by either deterministic or randomized processing. In [17] the existence of an infinite sequence is shown that has infinite mutual information with all total extensions of a universal partial recursive predicate. As Levin states "it plays the role of password: no substantial information about it can be guessed, no matter what methods are allowed." This "forbidden information" is used to extend the Gödel's incompleteness result to also hold for consistent extensions to a complete theory by randomized means with non-vanishing probability. ◊

Problem and Lacuna: Entropy, Kolmogorov complexity and mutual (algorithmic) information are concepts that do not distinguish between different *kinds* of information (such as 'meaningful' and 'meaningless' information). In the remainder of this paper, we show how these more intricate notions can be arrived at, typically by *constraining* the description methods with which strings are allowed to be encoded (Section 5.2) and by considering *lossy* rather than lossless compression (Section 6). Nevertheless, the basic notions entropy, Kolmogorov complexity and mutual information continue to play a fundamental rôle.

5 Sufficient Statistic

In introducing the notion of sufficiency in classical statistics, Fisher [5] stated:

> "The statistic chosen should summarize the whole of the relevant information supplied by the sample. This may be called the Criterion of Sufficiency ... In the case of the normal curve of distribution it is evident that the second moment is a sufficient statistic for estimating the standard deviation."

A "sufficient" statistic of the data contains all information in the data about the model class. Below we first discuss the standard notion of (probabilistic) sufficient statistic as employed in the statistical literature. We show that this notion has a natural interpretation in terms of Shannon mutual information, so that we may just as well think of a probabilistic sufficient statistic as a concept in Shannon information theory. Just as in the other sections of this paper, there is a corresponding notion in the Kolmogorov complexity literature: the algorithmic sufficient statistic which we introduce in Section 5.2. Finally, in Section 5.3 we connect the statistical/Shannon and the algorithmic notions of sufficiency.

5.1 Probabilistic Sufficient Statistic

Let $\{P_\theta\}$ be a family of distributions, also called a *model class*, of a random variable X that takes values in a finite or countable *set of data* $\mathcal{X}$. Let $\mathbf{\Theta}$ be the set of parameters θ parameterizing the family $\{P_\theta\}$. Any function $S : \mathcal{X} \to \mathcal{S}$ taking values in some set $\mathcal{S}$ is said to be a *statistic* of the data in $\mathcal{X}$. A *statistic* S is said to be *sufficient* for the family $\{P_\theta\}$ if, for every $s \in \mathcal{S}$, the conditional distribution

$$P_\theta(X = \cdot \mid S(x) = s) \tag{5.1}$$

is invariant under changes of θ. This is the standard definition in the statistical literature, see for example [2]. Intuitively, (5.1) means that all information about θ in the observation x is present in the (coarser) observation $S(x)$, in line with Fisher's quote above.

The notion of 'sufficient statistic' can be equivalently expressed in terms of probability mass functions. Let $f_\theta(x) = P_\theta(X = x)$ denote the probability mass of x according to P_θ. We identify distributions P_θ with their mass functions f_θ and denote the model class $\{P_\theta\}$ by $\{f_\theta\}$. Let $f_\theta(x|s)$ denote the probability mass function of the conditional distribution (5.1), defined as in Section 1.2. That is,

$$f_\theta(x|s) = \begin{cases} f_\theta(x) / \sum_{x \in \mathcal{X}: S(x)=s} f_\theta(x) & \text{if } S(x) = s \\ 0 & \text{if } S(x) \neq s. \end{cases}$$

The requirement of S to be sufficient is equivalent to the existence of a function $g : \mathcal{X} \times \mathcal{S} \to \mathcal{R}$ such that

$$g(x \mid s) = f_\theta(x \mid s), \tag{5.2}$$

for every $\theta \in \mathbf{\Theta}$, $s \in \mathcal{S}$, $x \in \mathcal{X}$. (Here we change the common notation '$g(x, s)$' to '$g(x \mid s)$' which is more expressive for our purpose.)

Example 5.1 Let $\mathcal{X} = \{0, 1\}^n$, let $X = (X_1, \ldots, X_n)$. Let $\{P_\theta : \theta \in (0, 1)\}$ be the set of n-fold Bernoulli distributions on $\mathcal{X}$ with parameter θ. That is,

$$f_\theta(x) = f_\theta(x_1 \ldots x_n) = \theta^{S(x)}(1 - \theta)^{n - S(x)}$$

where $S(x)$ is the number of 1's in x. Then $S(x)$ is a sufficient statistic for $\{P_\theta\}$. Namely, fix an arbitrary P_θ with $\theta \in (0, 1)$ and an arbitrary s with $0 < s < n$. Then all x's with s ones and $n - s$ zeroes are equally probable. The number of such x's is $\binom{n}{s}$. Therefore, the probability $P_\theta(X = x \mid S(x) = s)$ is equal to $1/\binom{n}{s}$, and this does not depend on the parameter θ. Equivalently, for all $\theta \in (0, 1)$,

$$f_\theta(x \mid s) = \begin{cases} 1/\binom{n}{s} & \text{if } S(x) = s \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Since (5.3) satisfies (5.2) (with $g(x|s)$ the uniform distribution on all x with exactly s ones), $S(x)$ is a sufficient statistic relative to the model class $\{P_\theta\}$. In the Bernoulli case, $g(x|s)$ can be obtained by starting from the *uniform* distribution on $\mathcal{X}$ ($\theta = \frac{1}{2}$), and conditioning on $S(x) = s$. But g is not necessarily uniform. For example, for the Poisson model class, where $\{f_\theta\}$ represents the set of Poisson distributions on n observations, the observed mean is a sufficient statistic and the corresponding g is far from uniform. All information about the parameter θ in the observation x is already contained in $S(x)$. In the Bernoulli case, once we know the number $S(x)$ of 1's in x, all further details of x (such as the order of 0s and 1s) are irrelevant for determination of the Bernoulli parameter θ.

To give an example of a statistics that is not sufficient for the Bernoulli model class, consider the statistic $T(x)$ which counts the number of 1s in x that are followed by a 1. On the other hand, for every statistic U, the combined statistic $V(x) := (S(x), U(x))$ with $S(x)$ as before, is sufficient, since it contains all information in $S(x)$. But in contrast to $S(x)$, a statistic such as $V(x)$ is typically not *minimal*, as explained further below. ♢

It will be useful to rewrite (5.2) as

$$\log 1/f_\theta(x \mid s) = \log 1/g(x|s). \tag{5.4}$$

Definition 5.2 A function $S : \mathcal{X} \to \mathcal{S}$ is a *probabilistic sufficient statistic* for $\{f_\theta\}$ if there exists a function $g : \mathcal{X} \times \mathcal{S} \to \mathcal{R}$ such that (5.4) holds for every $\theta \in \mathbf{\Theta}$, every $x \in \mathcal{X}$, every $s \in \mathcal{S}$ (Here we use the convention $\log 1/0 = \infty$).

Expectation-version of definition: The standard definition of probabilistic sufficient statistics is ostensibly of the 'individual-sequence'-type: for S to be sufficient, (5.4) has to hold for *every* x, rather than merely in expectation or with high probability. However, the definition turns out to be equivalent to an expectation-oriented form, as shown in Proposition 5.3. We first introduce an a priori distribution over Θ, the parameter set for our model class $\{f_\theta\}$. We denote the probability density of this distribution by p_1. This way we can define a joint distribution $p(\theta, x) = p_1(\theta) f_\theta(x)$.

Proposition 5.3 The following two statements are equivalent to Definition 5.2: (1) For every $\theta \in \mathbf{\Theta}$,

$$\sum_x f_\theta(x) \log 1/f_\theta(x \mid S(x)) = \sum_x f_\theta(x) \log 1/g(x \mid S(x)) \ . \tag{5.5}$$

(2) For *every* prior $p_1(\theta)$ on $\mathbf{\Theta}$,

$$\sum_{\theta,x} p(\theta, x) \log 1/f_\theta(x \mid S(x)) = \sum_{\theta,x} p(\theta, x) \log 1/g(x \mid S(x)) \ . \tag{5.6}$$

Proof. *Definition 5.2* $\Rightarrow$ (5.5)*:* Suppose (5.4) holds for every $\theta \in \mathbf{\Theta}$, every $x \in \mathcal{X}$, every $s \in \mathcal{S}$. Then it also holds in expectation for every $\theta \in \mathbf{\Theta}$:

$$\sum_x f_\theta(x) \log 1/f_\theta(x|S(x)) = \sum_x f_\theta(x) \log 1/g(x|S(x))]. \tag{5.7}$$

(5.5)$\Rightarrow$ *Definition 5.2:* Suppose that for every $\theta \in \mathbf{\Theta}$, (5.7) holds. Denote

$$f_\theta(s) = \sum_{y \in \mathcal{X}: S(y)=s} f_\theta(y). \tag{5.8}$$

By adding $\sum_x f_\theta(x) \log 1/f_\theta(S(x))$ to both sides of the equation, (5.7) can be rewritten as

$$\sum_x f_\theta(x) \log 1/f_\theta(x) = \sum_x f_\theta(x) \log 1/g_\theta(x), \tag{5.9}$$

with $g_\theta(x) = f_\theta(S(x)) \cdot g(x|S(x))]$. By the information inequality (3.7), the equality (5.9) can only hold if $g_\theta(x) = f_\theta(x)$ for every $x \in \mathcal{X}$. Hence, we have established (5.4).

(5.5) $\Leftrightarrow$ (5.6)*:* follows by linearity of expectation. □

Mutual information-version of definition: After some rearranging of terms, the characterization (5.6) gives rise to the intuitively appealing definition of probabilistic sufficient statistic in terms of mutual information (3.5). The resulting formulation of sufficiency is as follows [4]: S is sufficient for $\{f_\theta\}$ iff for all priors p_1 on $\mathbf{\Theta}$:

$$I(\Theta; X) = I(\Theta; S(X)) \tag{5.10}$$

for all distributions of θ.

Thus, a statistic $S(x)$ is sufficient if the probabilistic mutual information is invariant under taking the statistic (5.10).

Minimal Probabilistic Sufficient Statistic: A sufficient statistic may contain information that is not relevant: for a normal distribution the sample mean is a sufficient statistic, but the pair of functions which give the mean of the even-numbered samples and the odd-numbered samples respectively, is also a sufficient statistic. A statistic $S(x)$ is a *minimal* sufficient statistic with respect to an indexed model class $\{f_\theta\}$, if it is a function of all other sufficient statistics: it contains no irrelevant information and maximally compresses the information in the data about the model class. For the family of normal distributions the sample mean is a minimal sufficient statistic, but the sufficient statistic consisting of the mean of the even samples in combination with the mean of the odd samples is not minimal. Note that one cannot improve on sufficiency: The data processing inequality (4.1) states that $I(\Theta; X) \geq I(\Theta; S(X))$, for every function S, and that for randomized functions S an appropriate related expression holds. That is, mutual information between data random variable and model random variable cannot be increased by processing the data sample in any way.

Problem and Lacuna: We can think of the probabilistic sufficient statistic as extracting those patterns in the data that are relevant in determining the parameters of a statistical model class. But what if we do not want to commit ourselves to a simple finite-dimensional parametric model class? In the most general context, we may consider the model class of all computable distributions, or all computable sets of which the observed data is an element. Does there exist an analogue of the sufficient statistic that automatically summarizes *all* information in the sample x that is relevant for determining the "best" (appropriately defined) model for x within this enormous class of models? Of course, we may consider the literal data x as a statistic of x, but that would not be satisfactory: we would still like our generalized statistic, at least in many cases, to be considerably coarser, and much more concise, than the data x itself. It turns out that, to some extent, this is achieved by the *algorithmic* sufficient statistic of the data: it summarizes *all* conceivably relevant information in the data x; at the same time, many types of data x admit an algorithmic sufficient statistic that is concise in the sense that it has very small Kolmogorov complexity.

5.2 Algorithmic Sufficient Statistic

5.2.1 Meaningful Information

The information contained in an individual finite object (like a finite binary string) is measured by its Kolmogorov complexity—the length of the shortest binary program that computes the object. Such a shortest program contains no redundancy: every bit is information; but is it meaningful information? If we flip a fair coin to obtain a finite binary string, then with overwhelming probability that string constitutes its own shortest program. However, also with overwhelming probability all the bits in the string are meaningless information, random noise. On the other hand, let an object x be a sequence of observations of heavenly bodies. Then x can be described by the binary string pd, where p is the description of the laws of gravity and the observational parameter setting, while d accounts for the measurement errors: we can divide the information in x into meaningful information p and accidental information d. The main task for statistical inference and learning theory is to distill the meaningful information present in the data. The question arises whether it is possible to separate meaningful information from accidental information, and if so, how. The essence of the solution to this problem is revealed when we write Definition 2.6 as follows:

$$K(x) = \min_{p,i}\{K(i) + l(p) : T_i(p) = x\} + O(1), \tag{5.11}$$

where the minimum is taken over $p \in \{0,1\}^*$ and $i \in \{1,2,\ldots\}$. The justification is that for the fixed reference universal prefix Turing machine $U(\langle i,p\rangle) = T_i(p)$ for all i and p. Since i^* denotes the shortest self-delimiting program for i, we have $|i^*| = K(i)$. The expression (5.11) emphasizes the two-part code nature of Kolmogorov complexity. In a randomly truncated initial segment of a time series

$$x = 10101010101010101010101010,$$

we can encode x by a small Turing machine printing a specified number of copies of the pattern "01." This way, $K(x)$ is viewed as the shortest length of a two-part code for x, one part describing a Turing machine T, or *model*, for the *regular* aspects of x, and the second part describing the *irregular* aspects of x in the form of a program p to be interpreted by T. The regular, or "valuable," information in x is constituted by the bits in the "model" while the random or "useless" information of x constitutes the remainder. This leaves open the crucial question: How to choose T and p that together describe x? In general, many combinations of T and p are possible, but we want to find a T that describes the meaningful aspects of x.

5.2.2 Data and Model

We consider only finite binary data strings x. Our model class consists of Turing machines T that enumerate a finite set, say S, such that on input $p \leq |S|$ we have $T(p) = x$ with x the pth element of T's enumeration of S, and $T(p)$ is a special *undefined* value if $p > |S|$. The "best fitting" model for x is a Turing machine T that reaches the minimum description length in (5.11). There may be many such T, but, as we will see, if chosen properly, such a machine T embodies the amount of useful information contained in x. Thus, we have divided a shortest program x^* for x into parts $x^* = T^*(p)$ such that T^* is a shortest self-delimiting program for T. Now suppose we consider only low complexity finite-set models, and under these constraints the shortest two-part description happens to be longer than the shortest one-part description. For example, this can happen if the data is generated by a model that is too complex to be in the contemplated model class. Does the model minimizing the two-part description still capture all (or as much as possible) meaningful information? Such considerations require study of the relation between the complexity limit on the contemplated model classes, the shortest two-part code length, and the amount of meaningful information captured.

In the following we will distinguish between "models" that are finite sets, and the "shortest programs" to compute those models that are finite strings. The latter will be called 'algorithmic statistics'. In a way the distinction between "model" and "statistic" is artificial, but for now we prefer clarity and unambiguousness in the discussion. Moreover, the terminology is customary in the literature on algorithmic statistics. Note that strictly speaking, neither an algorithmic statistic nor the set it defines is a statistic in the probabilistic sense: the latter was defined as a *function* on the set of possible data samples of given length. Both notions are unified in Section 5.3.

5.2.3 Typical Elements

Consider a string x of length n and prefix complexity $K(x) = k$. For every finite set $S \subseteq \{0,1\}^*$ containing x we have $K(x|S) \leq \log |S| + O(1)$. Indeed, consider the prefix code of x consisting of its $\lceil \log |S| \rceil$ bit long index of x in the lexicographical ordering of S. This code is called *data-to-model code*. We identify the *structure* or *regularity* in x that are to be summarized with a set S of which x is a *random* or *typical* member: given S containing x, the element x cannot be described significantly shorter than by its maximal length index in S, that is, $K(x \mid S) \geq \log |S| + O(1)$.

Definition 5.4 Let $\beta \geq 0$ be an agreed upon, fixed, constant. A finite binary string x is a *typical* or *random* element of a set S of finite binary strings, if $x \in S$ and

$$K(x \mid S) \geq \log |S| - \beta. \tag{5.12}$$

We will not indicate the dependence on β explicitly, but the constants in all our inequalities ($O(1)$) will be allowed to be functions of this β.

This definition requires a finite S. In fact, since $K(x \mid S) \leq K(x) + O(1)$, it limits the size of S to $O(2^k)$. Note that the notion of typicality is not absolute but depends on fixing the constant implicit in the O-notation.

Example 5.5 Consider the set S of binary strings of length n whose every odd position is 0. Let x be an element of this set in which the subsequence of bits in even positions is an incompressible string. Then x is a typical element of S (or by with some abuse of language we can say S is typical for x). But x is also a typical element of the set $\{x\}$. $\diamondsuit$

5.2.4 Optimal Sets

Let x be a binary data string of length n. For every finite set $S \ni x$, we have $K(x) \leq K(S) + \log |S| + O(1)$, since we can describe x by giving S and the index of x in a standard enumeration of S. Clearly this can be implemented by a Turing machine computing the finite set S and a program p giving the index of x in S. The size of a set containing x measures intuitively the number of properties of x that are represented: The largest set is $\{0,1\}^n$ and represents only one property of x, namely, being of length n. It clearly "underfits" as explanation or model for x. The smallest set containing x is the singleton set $\{x\}$ and represents all conceivable properties of x. It clearly "overfits" as explanation or model for x.

There are two natural measures of suitability of such a set as a model for x. We might prefer either the simplest set, or the smallest set, as corresponding to the most likely structure 'explaining' x. Both the largest set $\{0,1\}^n$ (having low complexity of about $K(n)$) and the singleton set $\{x\}$ (having high complexity of about $K(x)$), while certainly statistics for x, would indeed be considered poor explanations. We would like to balance simplicity of model versus size of model. Both measures relate to the optimality of a two-stage description of x using a finite set S that contains it. Elaborating on the two-part code:

$$\begin{aligned} K(x) \leq K(x,S) &\leq K(S) + K(x \mid S) + O(1) \\ &\leq K(S) + \log |S| + O(1), \end{aligned} \tag{5.13}$$

where only the final substitution of $K(x \mid S)$ by $\log |S| + O(1)$ uses the fact that x is an element of S. The closer the right-hand side of (5.13) gets to the left-hand side, the better the description of x is in terms of the set S. This implies a trade-off between meaningful model information, $K(S)$, and meaningless "noise" $\log |S|$. A set S (containing x) for which (5.13) holds with equality

$$K(x) = K(S) + \log |S| + O(1), \tag{5.14}$$

is called *optimal.* A data string x can be typical for a set S without that set S being optimal for x. This is the case precisely when x is typical for S (that is $K(x|S) = \log S + O(1)$) while $K(x, S) > K(x)$.

5.2.5 Sufficient Statistic

Intuitively, a model expresses the essence of the data if the two-part code describing the data consisting of the model and the data-to-model code is as concise as the best one-part description.

Mindful of our distinction between a finite set S and a program that describes S in a required representation format, we call a shortest program for an optimal set with respect to x an *algorithmic sufficient statistic* for x. Furthermore, among optimal sets, there is a direct trade-off between complexity and log-size, which together sum to $K(x) + O(1)$.

Example 5.6 It can be shown that the set S of Example 5.5 is also optimal, and so is $\{x\}$. Sets for which x is typical form a much wider class than optimal sets for x: the set $\{x, y\}$ is still typical for x but with most y, it will be too complex to be optimal for x.

For a perhaps less artificial example, consider complexities conditional on the length n of strings. Let y be a random string of length n, let S_y be the set of strings of length n which have 0's exactly where y has, and let x be a random element of S_y. Then x has about 25% 1's, so its complexity is much less than n. The set S_y has x as a typical element, but is too complex to be optimal, since its complexity (even conditional on n) is still n. ♢

An algorithmic sufficient statistic is a sharper individual notion than a probabilistic sufficient statistic. An optimal set S associated with x (the shortest program computing S is the corresponding sufficient statistic associated with x) is chosen such that x is maximally random with respect to it. That is, the information in x is divided in a relevant structure expressed by the set S, and the remaining randomness with respect to that structure, expressed by x's index in S of $\log |S|$ bits. The shortest program for S is itself alone an algorithmic definition of structure, without a probabilistic interpretation.

Those optimal sets that admit the shortest possible program are called *algorithmic minimal sufficient statistics* of x. They will play a major role in the next section on the Kolmogorov structure function. Summarizing:

Definition 5.7 (Algorithmic sufficient statistic, algorithmic minimal sufficient statistic) *An* algorithmic sufficient statistic *of x is a shortest program for a set S containing x that is optimal, i.e. it satisfies (5.14). An algorithmic sufficient statistic with optimal set S is* minimal *if there exists no optimal set S' with $K(S') < K(S)$.*

Example 5.8 Let k be a number in the range $0, 1, \ldots, n$ of complexity $\log n + O(1)$ given n and let x be a string of length n having k ones of complexity $K(x \mid n, k) \geq \log \binom{n}{k}$ given n, k. This x can be viewed as a typical result of tossing a coin with a bias about $p = k/n$. A two-part description of x is given by the number k of 1's in x first, followed by the index $j \leq \log |S|$ of x in the set S of strings of length n with k 1's. This set is optimal, since $K(x \mid n) = K(x, k \mid n) = K(k \mid n) + K(x \mid k, n) = K(S) + \log |S|$.

Note that S encodes the number of 1s in x. The shortest program for S is an algorithmic minimal sufficient statistic for *most* x of length n with k 1's, since only a fraction of at most 2^{-m} x's of length n with k 1s can have $K(x) < \log |S| - m$ (Section 2.2). But of course there exist x's with k ones which have much more regularity. An example is the string starting with k 1's followed by $n - k$ 0's. For such strings, S is still optimal and the shortest program for S is still an algorithmic sufficient statistic, but not a minimal one. ♢

5.3 Relating Probabilistic and Algorithmic Sufficiency

We want to relate 'algorithmic sufficient statistics' (defined independently of any model class $\{f_\theta\}$) to probabilistic sufficient statistics (defined relative to some model class $\{f_\theta\}$ as in Section 5.1). We will show that, essentially, algorithmic sufficient statistics are probabilistic nearly-sufficient statistics with respect to *all* model families $\{f_\theta\}$. Since the notion of algorithmic sufficiency is only defined to within additive constants, we cannot expect algorithmic sufficient statistics to satisfy the requirements (5.4) or (5.5) for probabilistic sufficiency *exactly*, but only 'nearly[4]'.

Nearly Sufficient Statistics: Intuitively, we may consider a probabilistic statistic S to be nearly sufficient if (5.4) or (5.5) holds to within some constant. For long sequences x, this constant will then be negligible compared to the two terms in (5.4) or (5.5) which, for most practically interesting statistical model classes, typically grow linearly in the sequence length. But now we encounter a difficulty:

> whereas (5.4) and (5.5) are equivalent if they are required to hold exactly, they express something substantially different if they are only required to hold within a constant.

Because of our observation above, when relating probabilistic and algorithmic statistics we have to be very careful about what happens if n is allowed to change. Thus, we need to extend probabilistic and algorithmic statistics to strings of arbitrary length. This leads to the following generalized definition of a statistic:

Definition 5.9 A *sequential statistic* is a function $S : \{0,1\}^* \to 2^{\{0,1\}^*}$, such that for all n, all $x \in \{0,1\}^n$, (1) $S(x) \subseteq \{0,1\}^n$, and (2) $x \in S(x)$, and (3) for all n, the set

$$\{s \mid \text{There exists } x \in \{0,1\}^n \text{ with } S(x) = s\ \}$$

is a partition of $\{0,1\}^n$.

Algorithmic statistics are defined relative to individual x of some length n. Probabilistic statistics are defined as functions, hence for all x of given length, but still relative to given length n. Such algorithmic and probabilistic statistics can be extended to each n and each $x \in \{0,1\}^n$ in a variety of ways; the three conditions in Definition 5.9 ensure that the extension is done in a reasonable way. Now let $\{f_\theta\}$ be a model class of sequential information sources (Section 1.2), i.e. a statistical model class defined for sequences of arbitrary length rather than just fixed n. As before, $f_\theta^{(n)}$ denotes the marginal distribution of f_θ on $\{0,1\}^n$.

Definition 5.10 We call sequential statistic S *nearly-sufficient for* $\{f_\theta\}$ *in the probabilistic-individual sense* if there exist functions $g^{(1)}, g^{(2)}, \ldots$ and a constant c such that for all θ, all n, every $x \in \{0,1\}^n$,

$$\left| \log 1/f_\theta^{(n)}(x \mid S(x)) - \log 1/g^{(n)}(x|S(x)) \right| \leq c. \tag{5.15}$$

We say S is *nearly-sufficient for* $\{f_\theta\}$ *in the probabilistic-expectation sense* if there exists functions $g^{(1)}, g^{(2)}, \ldots$ and a constant c' such that for all θ, all n,

$$\left| \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \left[\log 1/f_\theta^{(n)}(x \mid S(x)) - \log 1/g^{(n)}(x|S(x)) \right] \right| \leq c'. \tag{5.16}$$

Inequality (5.15) may be read as '(5.4) holds within a constant', whereas (5.16) may be read as '(5.5) holds within a constant'.

Remark 5.11 Whereas the individual-sequence definition (5.4) and the expectation-definition (5.5) are equivalent if we require exact equality, they become quite different if we allow equality to within a constant as in Definition 5.10. To see this, let S be some sequential statistic such that for all large n, for some θ_1, θ_2, for some $x \in \{0,1\}^n$,

$$f_{\theta_1}^{(n)}(x \mid S(x)) \gg f_{\theta_2}^{(n)}(x \mid S(x)),$$

while for all $x' \neq x$ of length n, $f_{\theta_1}^{(n)}(x|S(x)) \approx f_{\theta_2}^{(n)}(x|S(x))$. If x has very small but nonzero probability according to some $\theta \in \Theta$, then with very small f_θ-probability, the difference between the left-hand and right-hand side of (5.4) is very large, and with large f_θ-probability, the difference between the left-hand and right-hand side of (5.4) is about 0. Then S will be nearly sufficient in expectation, but not in the individual sense. ◊

[4] We use 'nearly' rather than 'almost' since 'almost' suggests things like 'almost everywhere/almost surely/with probability 1'. Instead, 'nearly' means, roughly speaking, 'to within $O(1)$'.

In the theorem below we focus on probabilistic statistics that are 'nearly sufficient in an expected sense'. We connect these to algorithmic sequential statistics, defined as follows:

Definition 5.12 A sequential statistic S is *sufficient in the algorithmic sense* if there is a constant c such that for all n, all $x \in \{0,1\}^n$, the program generating $S(x)$ is an algorithmic sufficient statistic for x (relative to constant c), i.e.

$$K(S(x)) + \log |S(x)| \leq K(x) + c. \tag{5.17}$$

In Theorem 5.13 we relate algorithmic to probabilistic sufficiency. In the theorem, S represents a sequential statistic, $\{f_\theta\}$ is a model class of sequential information sources and $g^{(n)}$ is the conditional probability mass function arising from the uniform distribution:

$$g^{(n)}(x|s) = \begin{cases} 1/|\{x \in \{0,1\}^n : S(x) = s\}| & \text{if } S(x) = s \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.13 (algorithmic sufficient statistic is probabilistic sufficient statistic) Let S be a sequential statistic that is sufficient in the algorithmic sense. Then for every θ with $K(f_\theta) < \infty$, there exists a constant c, such that for all n, inequality (5.16) holds with $g^{(n)}$ the uniform distribution. Thus, if $\sup_{\theta \in \Theta} K(f_\theta) < \infty$, then S is a nearly-sufficient statistic for $\{f_\theta\}$ in the probabilistic-expectation sense, with g equal to the uniform distribution.

Proof. The definition of algorithmic sufficiency, (5.17) directly implies that there exists a constant c such that for all θ, all n,

$$\sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \big[K(S(x)) + \log |S(x)| \big] \leq \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) K(x) + c. \tag{5.18}$$

Now fix any θ with $K(f_\theta) < \infty$. It follows (by the same reasoning as in Theorem 2.10) that for some $c_\theta \approx K(f_\theta)$, for all n,

$$0 \leq \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) K(x) - \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta(x) \leq c_\theta. \tag{5.19}$$

Essentially, the left inequality follows by the information inequality (3.7): no code can be more efficient in expectation under f_θ than the Shannon-Fano code with lengths $\log 1/f_\theta(x)$; the right inequality follows because, since $K(f_\theta) < \infty$, the Shannon-Fano code can be implemented by a computer program with a fixed-size independent of n. By (5.19), (5.18) becomes: for all n,

$$\sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta(x) \leq \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \big[K(S(x)) + \log |S(x)| \big] \leq \sum_x f_\theta^{(n)}(x) \log 1/f_\theta(x) + c_\theta. \tag{5.20}$$

For $s \subseteq \{0,1\}^n$, we use the notation $f_\theta^{(n)}(s)$ according to (5.8). Note that, by requirement (3) in the definition of sequential statistic,

$$\sum_{s: \exists x \in \{0,1\}^n : S(x) = s} f_\theta^{(n)}(s) = 1,$$

whence $f_\theta^{(n)}(s)$ is a probability mass function on $\mathcal{S}$, the set of values the statistic S can take on sequences of length n. Thus, we get, once again by the information inequality (3.7),

$$\sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) K(S(x)) \geq \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta^{(n)}(S(x)). \tag{5.21}$$

Now note that for all n,

$$\sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \big[\log 1/f_\theta^{(n)}(S(x)) + \log 1/f_\theta^{(n)}(x \mid S(x)) \big] = \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta(x). \tag{5.22}$$

Consider the two-part code which encodes x by first encoding $S(x)$ using $\log 1/f_\theta^{(n)}(S(x))$ bits, and then encoding x using $\log |S(x)|$ bits. By the information inequality, (3.7), this code must be less efficient than the Shannon-Fano code with lengths $\log 1/f_\theta(x)$, so that if follows from (5.22) that, for all n,

$$\sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log |S(x)| \geq \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta^{(n)}(x \mid S(x)). \tag{5.23}$$

Now defining

$$\begin{aligned} u &= \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) K(S(x)) \\ v &= \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log |S(x)| \\ u' &= \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta^{(n)}(S(x)) \\ v' &= \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta^{(n)}(x \mid S(x)) \\ w &= \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta(x), \end{aligned}$$

we find that (5.20), (5.22), (5.21) and (5.23) express, respectively, that $u + v \stackrel{+}{=} w$, $u' + v' = w$, $u \geq u'$, $v \geq v'$. It follows that $v \stackrel{+}{=} v'$, so that (5.23) must actually hold with equality up to a constant. That is, there exist a c' such that for all n,

$$\Big| \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log |S(x)| - \sum_{x \in \{0,1\}^n} f_\theta^{(n)}(x) \log 1/f_\theta^{(n)}(x \mid S(x)) \Big| \leq c'. \tag{5.24}$$

The result now follows upon noting that (5.24) is just (5.16) with $g^{(n)}$ the uniform distribution. □

6 Rate Distortion and Structure Function

We continue the discussion about meaningful information of Section 5.2.1. This time we a priori restrict the number of bits allowed for conveying the essence of the information. In the probabilistic situation this takes the form of allowing only a "rate" of R bits to communicate as well as possible, on average, the outcome of a random variable X, while the set $\mathcal{X}$ of outcomes has cardinality possibly exceeding 2^R. Clearly, not all outcomes can be communicated without information loss, the average of which is expressed by the "distortion". This leads to the so-called "rate–distortion" theory. In the algorithmic setting the corresponding idea is to consider a set of models from which to choose a single model that expresses the "meaning" of the given individual data x as well as possible. If we allow only R bits to express the model, while possibly the Kolmogorov complexity $K(x) > R$, we suffer information loss—a situation that arises for example with "lossy" compression. In the latter situation, the data cannot be perfectly reconstructed from the model, and the question arises in how far the model can capture the meaning present in the specific data x. This leads to the so-called "structure function" theory.

The limit of R bits to express a model to capture the most meaningful information in the data is an individual version of the average notion of "rate". The remaining less meaningful information in the data is the individual version of the average-case notion of "distortion". If the R bits are sufficient to express all meaning in the data then the resulting model is called a "sufficient statistic", in the sense introduced above. The remaining information in the data is then purely accidental, random, noise. For example, a sequence of outcomes of n tosses of a coin with computable bias p, typically has a sufficient statistic of $K(p)$ bits, while the remaining random information is typically at least about $pn - K(p)$ bits (up to an $O(\sqrt{n})$ additive term).

6.1 Rate Distortion

Initially, Shannon [22] introduced rate-distortion as follows: "Practically, we are not interested in exact transmission when we have a continuous source, but only in transmission to within a given tolerance. The question is, can we assign a definite rate to a continuous source when we require only a certain fidelity of recovery, measured in a suitable way." Later, in [23] he applied this idea to lossy data compression of discrete memoryless sources—our topic below. As before, we consider a situation in which sender A wants to communicate the outcome of random variable X to receiver B. Let X take values in some set $\mathcal{X}$, and the distribution P of X be known to both A and B. The change is that now A is only allowed to use a finite number, say R bits, to communicate, so that A can only send 2^R different messages. Let us denote by Y the encoding function used by A. This Y maps $\mathcal{X}$ onto some set $\mathcal{Y}$. We require that $|\mathcal{Y}| \leq 2^R$. If $|\mathcal{X}| > 2^R$ or if $\mathcal{X}$ is continuous-valued, then necessarily some information is lost during the communication. There is no decoding function $D : \mathcal{Y} \to \mathcal{X}$ such that $D(Y(x)) = x$ for all x. Thus, A and B cannot ensure that x can always be reconstructed. As the next best thing, they may agree on a code such that for all x, the value $Y(x)$ contains as much useful information about x as is possible—what exactly 'useful' means depends on the situation at hand; examples are provided below. An easy example would be that $Y(x)$ is a finite list of elements, one of which is x. We assume that the 'goodness' of $Y(x)$ is gaged by a *distortion function* $d : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$. This distortion function may be any nonnegative function that is appropriate to the situation at hand. In the example above it could be the logarithm of the number of elements in the list $Y(x)$. Examples of some common distortion functions are the Hamming distance and the squared Euclidean distance. We can view Y as a a random variable on the space $\mathcal{Y}$, a coarse version of the random variable X, defined as taking value $Y = y$ if $X = x$ with $Y(x) = y$. Write $f(x) = P(X = x)$ and $g(y) = \sum_{x:Y(x)=y} P(X = x)$. Once the distortion function d is fixed, we define the *expected* distortion by

$$\begin{aligned}\mathbf{E}[d(X,Y)] &= \sum_{x \in \mathcal{X}} f(x) d(x, Y(x)) \qquad (6.1)\\ &= \sum_{y \in \mathcal{Y}} g(y) \sum_{x:Y(x)=y} f(x)/g(y) d(x,y).\end{aligned}$$

If X is a continuous random variable, the sum should be replaced by an integral.

Example 6.1 In most standard applications of rate distortion theory, the goal is to compress x in a 'lossy' way, such that x can be reconstructed 'as well as possible' from $Y(x)$. In that case, $\mathcal{Y} \subseteq \mathcal{X}$ and writing $\hat{x} = Y(x)$, the value $d(x, \hat{x})$ measures the similarity between x and $\hat{x}$. For example, with $\mathcal{X}$ is the set of real numbers and $\mathcal{Y}$ is the set of integers, the squared difference $d(x, \hat{x}) = (x - \hat{x})^2$ is a viable distortion function. We may interpret $\hat{x}$ as an estimate of x, and $\mathcal{Y}$ as the set of values it can take. The reason we use the notation Y rather than $\hat{X}$ (as in, for example, [4]) is that further below, we mostly concentrate on slightly non-standard applications where $\mathcal{Y}$ should *not* be interpreted as a subset of $\mathcal{X}$. ♢

We want to determine the optimal code Y for communication between A and B under the constraint that there are no more than 2^R messages. That is, we look for the encoding function Y that minimizes the expected distortion, under the constraint that $|\mathcal{Y}| \leq 2^R$. Usually, the minimum achievable expected distortion is nonincreasing as a function of increasing R.

Example 6.2 Suppose X is a real-valued, normally (Gaussian) distributed random variable with mean $\mathbf{E}[X] = 0$ and variance $\mathbf{E}[X - \mathbf{E}[X]]^2 = \sigma^2$. Let us use the squared Euclidean distance $d(x,y) = (x-y)^2$ as a distortion measure. If A is allowed to use R bits, then $\mathcal{Y}$ can have no more than 2^R elements, in contrast to $\mathcal{X}$ that is uncountably infinite. We should choose $\mathcal{Y}$ and the function Y such that (6.1) is minimized. Suppose first $R = 1$. Then the optimal Y turns out to be

$$Y(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\sigma^2 & \text{if } x \geq 0 \\ -\sqrt{\frac{2}{\pi}}\sigma^2 & \text{if } x < 0. \end{cases}$$

Thus, the domain $\mathcal{X}$ is partitioned into two regions, one corresponding to $x \geq 0$, and one to $x < 0$. By the symmetry of the Gaussian distribution around 0, it should be clear that this is the best one can do. Within each of the two region, one picks a 'representative point' so as to minimize (6.1). This mapping allows B to estimate x as well as possible.

Similarly, if $R = 2$, then $\mathcal{X}$ should be partitioned into 4 regions, each of which are to be represented by a single point such that (6.1) is minimized. An extreme case is $R = 0$: how can B estimate X if it is always given the same information? This means that $Y(x)$ must take the same value for all x. The expected distortion (6.1) is then minimized if $Y(x) \equiv 0$, the mean of X, giving distortion equal to σ^2. ♢

In general, there is no need for the space of estimates $\mathcal{Y}$ to be a subset of $\mathcal{X}$. We may, for example, also lossily encode or 'estimate' the actual value of x by specifying a set in which x must lie (Section 6.2) or a probability distribution (see below) on $\mathcal{X}$.

Example 6.3 Suppose receiver B wants to estimate the actual x by a probability distribution P on $\mathcal{X}$. Thus, if R bits are allowed to be used, one of 2^R different distributions on $\mathcal{X}$ can be sent to receiver. The most accurate that can be done is to partition $\mathcal{X}$ into 2^R subsets $\mathcal{A}_1, \ldots, \mathcal{A}_{2^R}$. Relative to any such partition, we introduce a new random variable Y and abbreviate the event $x \in \mathcal{A}_y$ to $Y = y$. Sender observes that $Y = y$ for some $y \in \mathcal{Y} = \{1, \ldots, 2^R\}$ and passes this information on to receiver. The information y actually means that X is now distributed according to the conditional distribution $P(X = x \mid x \in \mathcal{A}_y) = P(X = x \mid Y = y)$.

It is now natural to measure the quality of the transmitted distribution $P(X = x \mid Y = y)$ by its conditional entropy, i.e. the expected additional number of bits that sender has to transmit before receiver knows the value of x with certainty. This can be achieved by taking

$$d(x, y) = \log 1/P(X = x \mid Y = y), \tag{6.2}$$

which we abbreviate to $d(x, y) = \log 1/f(x|y)$. In words, the distortion function is the Shannon-Fano code length for the communicated distribution. The expected distortion then becomes equal to the conditional entropy $H(X \mid Y)$ as defined in Section 3.1 (rewrite according to (6.1), $f(x|y) = f(x)/g(y)$ for $P(X = x|Y(x) = y)$ and $g(y)$ defined earlier, and the definition of conditional probability):

$$\begin{aligned} \mathbf{E}[d(X,Y)] &= \sum_{y \in \mathcal{Y}} g(y) \sum_{x:Y(x)=y} (f(x)/g(y)) d(x,y) \\ &= \sum_{y \in \mathcal{Y}} g(y) \sum_{x:Y(x)=y} f(x|y) \log 1/f(x|y) \\ &= H(X|Y). \end{aligned} \tag{6.3}$$

How is this related to lossless compression? Suppose for example that $R = 1$. Then the optimal distortion is achieved by partitioning $\mathcal{X}$ into two sets $\mathcal{A}_1, \mathcal{A}_2$ in the most 'informative' possible way, so that the conditional entropy

$$H(X|Y) = \sum_{y=1,2} P(Y = y) H(X|Y = y)$$

is minimized. If Y itself is encoded with the Shannon-Fano code, then $H(Y)$ bits are needed to communicate Y. Rewriting $H(X|Y) = \sum_{y \in \mathcal{Y}} P(Y = y) H(X|Y = y)$ and $H(X|Y = y) = \sum_{x:Y(x)=y} f(x|y) \log 1/f(x|y)$ with $f(x|y) = P(X = x)/P(Y = y)$ and rearranging, shows that for all such partitions of $\mathcal{X}$ into $|\mathcal{Y}|$ subsets defined by $Y : \mathcal{X} \to \mathcal{Y}$ we have

$$H(X|Y) + H(Y) = H(X). \tag{6.4}$$

The minimum rate distortion is obtained by choosing the function Y that minimizes $H(X|Y)$. By (6.4) this is also the Y maximizing $H(Y)$. Thus, the average total number of bits we need to send our message in this way is still equal to $H(X)$—the more we save in the second part, the more we pay in the first part. $\diamondsuit$

Rate Distortion and Mutual Information: Already in his 1948 paper, Shannon established a deep relation between mutual information and minimum achievable distortion for (essentially) *arbitrary* distortion functions. The relation is summarized in Theorem 6.8 below. To prepare for the theorem, we need to slightly extend our setting by considering *independent repetitions of the same scenario*. This can be motivated in various ways such as (a) it often corresponds to the situation we are trying to model; (b) it allows us to consider non-integer rates R, and (c) it greatly simplifies the mathematical analysis.

Definition 6.4 Let $\mathcal{X}, \mathcal{Y}$ be two sample spaces. The distortion of $y \in \mathcal{Y}$ with respect to $x \in \mathcal{X}$ is defined by a nonnegative real-valued function $d(x, y)$ as above. We extend the definition to sequences: the distortion of $(y_1, \ldots, y_n)$ with respect to $(x_1, \ldots, x_n)$ is

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i). \tag{6.5}$$

Let $X_1, \ldots, X_n$ be n independent identically distributed random variables on outcome space $\mathcal{X}$. Let $\mathcal{Y}$ be a set of code words. We want to find a sequence of functions $Y_1, \ldots, Y_n : \mathcal{X} \to \mathcal{Y}$ so that the message $(Y_1(x_1), \ldots, Y_n(x_n)) \in \mathcal{Y}^n$ gives as much expected information about the sequence of outcomes $(X_1 = x_1, \ldots, X_n = x_n)$ as is possible, under the constraint that the message takes at most $R \cdot n$ bits (so that R bits are allowed on average per outcome of X_i). Instead of $Y_1, \ldots, Y_n$ above write $Z_n : \mathcal{X}^n \to \mathcal{Y}^n$. The *expected distortion* $\mathbf{E}[d(X^n, Z_n)]$ for Z_n is

$$\mathbf{E}[d(X^n, Z_n)] = \sum_{(x_1,\ldots,x_n)\in\mathcal{X}^n} P(X^n = (x_1, \ldots, x_n)) \cdot \frac{1}{n} \sum_{i=1}^{n} d(x_i, Y_i(x_i)). \tag{6.6}$$

Consider functions Z_n with range $\mathcal{Z}_n \subseteq \mathcal{Y}^n$ satisfying $|\mathcal{Z}_n| \leq 2^{nR}$. Let for $n \geq 1$ random variables a choice $Y_1, \ldots, Y_n$ minimize the expected distortion under these constraints, and let the corresponding value $D^*_n(R)$ of the expected distortion be defined by

$$D^*_n(R) = \min_{Z_n : |\mathcal{Z}_n| \leq 2^{nR}} \mathbf{E}(d(X^n, Z_n)). \tag{6.7}$$

Lemma 6.5 *For every distortion measure, and all $R, n, m \geq 1$, $(n+m)D^*_{n+m}(R) \leq nD^*_n(R) + mD^*_m(R)$.*

Proof. Let $Y_1, \ldots, Y_n$ achieve $D^*_n(R)$ and $Y'_1, \ldots, Y'_m$ achieve $D^*_m(R)$. Then, $Y_1, \ldots, Y_n, Y'_1, \ldots, Y'_m$ achieves $(nD^*_n(R) + mD^*_m(R))/(n+m)$. This is an upper bound on the minimal possible value $D^*_{n+m}(R)$ for $n+m$ random variables. □

It follows that for all $R, n \geq 1$ we have $D^*_{2n}(R) \leq D^*_n(R)$. The inequality is typically strict; [4] gives an intuitive explanation of this phenomenon. For fixed R the value of $D^*_1(R)$ is fixed and it is finite. Since also $D^*_n(R)$ is necessarily positive for all n, we have established the existence of the limit

$$D^*(R) = \lim \inf_{n\to\infty} D^*_n(R). \tag{6.8}$$

The value of $D^*(R)$ is the minimum achievable distortion at rate (number of bits/outcome) R. Therefore, $D^*(\cdot)$ It is called the *distortion-rate function.* In our Gaussian Example 6.2, $D^*(R)$ quickly converges to 0 with increasing R. It turns out that for general d, when we view $D^*(R)$ as a function of $R \in [0, \infty)$, it is *convex and nonincreasing.*

Example 6.6 Let $\mathcal{X} = \{0, 1\}$, and let $P(X = 1) = p$. Let $\mathcal{Y} = \{0, 1\}$ and take the Shannon-Fano distortion function $d(x, y) = \log 1/f(x \mid y)$ with notation as in Example 6.3. Let Y be a function that achieves the minimum expected Shannon-Fano distortion $D^*_1(R)$. As usual we write Y for the random variable $Y(x)$ induced by X. Then, $D^*_1(R) = \mathbf{E}[d(X, Y)] = \mathbf{E}[\log 1/f(X|Y)] = H(X|Y)$. At rate $R = 1$, we can set $Y = X$ and the minimum achievable distortion is given by $D^*_1(1) = H(X|X) = 0$. Now consider some rate R with $0 < R < 1$, say $R = \frac{1}{2}$. Since we are now forced to use less than $2^R < 2$ messages in communicating, only a fixed message can be sent, no matter what outcome of the random variable X is realized. This means that no communication is possible at all and the minimum achievable distortion is $D^*_1(\frac{1}{2}) = H(X) = H(p, 1-p)$. But clearly, if we consider n repetitions of the same scenario and are allowed to send a message out of $\lfloor 2^{nR} \rfloor$ candidates, then some useful information can be communicated after all, even if $R < 1$. In Example 6.9 we will show that if $R > H(p, 1-p)$, then $D^*(R) = 0$; if $R \leq H(p, 1-p)$, then $D^*(R) = H(p, 1-p) - R$. ♢

Up to now we studied the minimum achievable distortion D as a function of the rate R. For technical reasons, it is often more convenient to consider the minimum achievable rate R as a function of the distortion D. This is the more celebrated version, the *rate-distortion function* $R^*(D)$. Because $D^*(R)$ is convex and nonincreasing, $R^*(D) : [0, \infty) \to [0, \infty]$ is just the *inverse* of the function $D^*(R)$.

It turns out to be possible to relate distortion to the Shannon mutual information. This remarkable fact, which Shannon proved already in [22, 23], illustrates the fundamental nature of Shannon's concepts. Up till now, we only considered *deterministic* encodings $Y : \mathcal{X} \to \mathcal{Y}$. But it is hard to analyze the rate-distortion, and distortion-rate, functions in this setting. It turns out to be advantageous to follow an indirect route by bringing information-theoretic techniques into play. To this end, we generalize the setting to *randomized* encodings. That is, upon observing $X = x$ with probability $f(x)$, the sender may use a randomizing device (e.g. a coin) to decide which code word in $y \in \mathcal{Y}$ he is going to send to the receiver. A randomized encoding Y thus maps each $x \in \mathcal{X}$ to $y \in \mathcal{Y}$ with probability $g_x(y)$, denoted in conditional probability format as $g(y|x)$. Altogether we deal with a joint distribution $g(x, y) = f(x)g(y|x)$ on the joint sample space $\mathcal{X} \times \mathcal{Y}$. (In the deterministic case we have $g(Y(x) \mid x) = 1$ for the given function $Y : \mathcal{X} \to \mathcal{Y}$.)

Definition 6.7 Let X and Y be joint random variables as above, and let $d(x,y)$ be a distortion measure. The *expected distortion* $D(X,Y)$ of Y with respect to X is defined by

$$D(X,Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} g(x,y) d(x,y). \tag{6.9}$$

Note that for a given problem the source probability $f(x)$ of outcome $X = x$ is fixed, but the randomized encoding Y, that is the conditional probability $g(y|x)$ of encoding source word x by code word y, can be chosen to advantage. We define the auxiliary notion of *information rate distortion function* $R^{(I)}(D)$ by

$$R^{(I)}(D) = \inf_{Y: D(X,Y) \leq D} I(X;Y). \tag{6.10}$$

That is, for random variable X, among *all* joint random variables Y with expected distortion to X less than or equal to D, the information rate $R^{(I)}(D)$ equals the minimal mutual information with X.

Theorem 6.8 (Shannon) *For every random source X and distortion measure d:*

$$R^*(D) = R^{(I)}(D) \tag{6.11}$$

This remarkable theorem states that the best deterministic code achieves a rate-distortion that equals the minimal information rate possible for a randomized code, that is, the minimal mutual information between the random source and a randomized code. Note that this does not mean that $R^*(D)$ is independent of the distortion measure. In fact, the source random variable X, together with the distortion measure d, determines a random code Y for which the joint random variables X and Y reach the infimum in (6.10). The proof of this theorem is given in [4]. It is illuminating to see how it goes: It is shown first that, for a random source X and distortion measure d, every deterministic code Y with distortion $\leq D$ has rate $R \geq R^{(I)}(D)$. Subsequently, it is shown that there exists a deterministic code that, with distortion $\leq D$, achieves rate $R^*(D) = R^{(I)}(D)$. To analyze deterministic $R^*(D)$ therefore, we can determine the best randomized code Y for random source X under distortion constraint D, and then we know that simply $R^*(D) = I(X;Y)$.

Example 6.9 *(Example 6.6, continued)* Suppose we want to compute $R^*(D)$ for some D between 0 and 1. If we only allow encodings Y that are deterministic functions of X, then either $Y(x) \equiv x$ or $Y(x) \equiv |1-x|$. In both cases $\mathbf{E}[d(X,Y)] = H(X|Y) = 0$, so Y satisfies the constraint in (6.10). In both cases, $I(X,Y) = H(Y) = H(X)$. With (6.11) this shows that $R^*(D) \leq H(X)$. However, $R^*(D)$ is actually smaller: by allowing randomized codes, we can define Y_α as $Y_\alpha(x) = x$ with probability α and $Y_\alpha(x) = |1-x|$ with probability $1-\alpha$. For $0 \leq \alpha \leq \frac{1}{2}$, $\mathbf{E}[d(X,Y_\alpha)] = H(X|Y_\alpha)$ increases with α, while $I(X;Y_\alpha)$ decreases with α. Thus, by choosing the α^* for which the constraint $\mathbf{E}[d(X,Y_\alpha)] \leq D$ holds with equality, we find $R^*(D) = I(X;Y_{\alpha^*})$. Let us now calculate $R^*(D)$ and $D^*(R)$ explicitly.

Since $I(X,Y) = H(X) - H(X|Y)$, we can rewrite $R^*(D)$ as

$$R^*(D) = H(X) - \sup_{Y: D(X,Y) \leq D} H(X|Y).$$

In the special case where D is itself the Shannon-Fano distortion, this can in turn be rewritten as

$$R^*(D) = H(X) - \sup_{Y: H(X|Y) \leq D} H(X \mid Y) = H(X) - D.$$

Since $D^*(R)$ is the inverse of $R^*(D)$, we find $D^*(R) = H(X) - R$, as announced in Example 6.6. $\diamondsuit$

Problem and Lacuna: In the Rate-Distortion setting we allow (on average) a rate of R bits to express the data as well as possible in some way, and measure the average of loss by some distortion function. But in many cases, like lossy compression of images, one is interested in the individual cases. The average over all possible images may be irrelevant for the individual cases one meets. Moreover, one is not particularly interested in bit-loss, but rather in preserving the essence of the image as well as possible. As another example, suppose the distortion function is simply to supply the remaining bits of the data. But this can be unsatisfactory: we are given an outcome of a measurement as a real number of n significant bits. Then the R most significant bits carry most of the meaning of the data, while the remaining $n-R$ bits may be irrelevant. Thus, we are lead to the elusive notion of a distortion function that captures the amount of

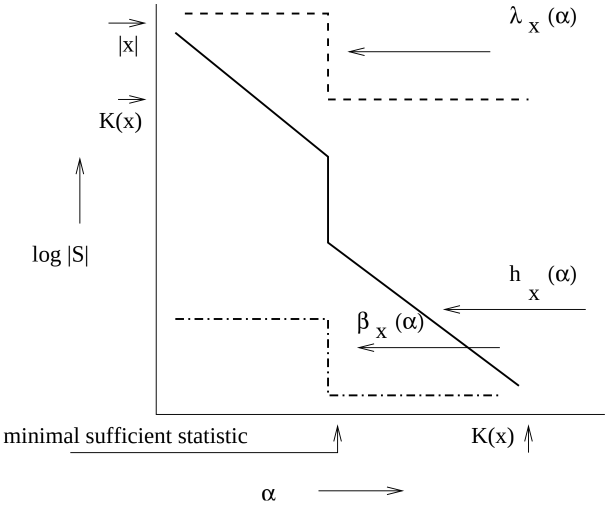


Figure 1: Structure functions $h_x(i), \beta_x(\alpha), \lambda_x(\alpha)$, and minimal sufficient statistic.

"meaning" that is not included in the R rate bits. These issues are taken up by Kolmogorov's proposal of the structure function. This cluster of ideas puts the notion of Rate–Distortion in an individual algorithmic (Kolmogorov complexity) setting, and focuses on the meaningful information in the data. In the end we can recycle the new insights and connect them to Rate-Distortion notions to provide new foundations for statistical inference notions as maximum likelihood (ML) [5], minimum message length (MML) [28], and minimum description length (MDL) [20].

6.2 Structure Function

There is a close relation between functions describing three, a priori seemingly unrelated, aspects of modeling individual data, depicted in Figure 1. One of these was introduced by Kolmogorov at a conference in Tallinn 1974 (no written version) and in a talk at the Moscow Mathematical Society in the same year of which the abstract [11] is as follows (this is the only writing by Kolmogorov about this circle of ideas):

> "To each constructive object corresponds a function $\Phi_x(k)$ of a natural number k—the log of minimal cardinality of x-containing sets that allow definitions of complexity at most k. If the element x itself allows a simple definition, then the function Φ drops to 1 even for small k. Lacking such definition, the element is "random" in a negative sense. But it is positively "probabilistically random" only when function Φ having taken the value Φ_0 at a relatively small $k = k_0$, then changes approximately as $\Phi(k) = \Phi_0 - (k - k_0)$."

Kolmogorov's Φ_x is commonly called the "structure function" and is here denoted as h_x and defined in (6.14). The structure function notion entails a proposal for a non-probabilistic approach to statistics, an individual combinatorial relation between the data and its model, expressed in terms of Kolmogorov complexity. It turns out that the structure function determines all stochastic properties of the data in the sense of determining the best-fitting model at every model-complexity level, the equivalent notion to "rate" in the Shannon theory. A consequence is this: minimizing the data-to-model code length (finding the ML estimator or MDL estimator), in a class of contemplated models of prescribed maximal (Kolmogorov) complexity, *always* results in a model of best fit, irrespective of whether the source producing the data is in the model class considered. In this setting, code length minimization *always* separates optimal model information from the remaining accidental information, and not only with high probability. The function that maps the maximal allowed model complexity to the goodness-of-fit (expressed as minimal "randomness deficiency") of the best model cannot itself be monotonically approximated. However, the shortest one-part or two-part code above can—implicitly optimizing this elusive goodness-of-fit.

In probabilistic statistics the goodness of the selection process is measured in terms of expectations over probabilistic ensembles. For current applications, average relations are often irrelevant, since the part of the support of the probability mass function that will ever be observed has about zero measure. This may be the case in, for example, complex video and sound analysis. There arises the problem that for individual cases the selection performance may be bad although the performance is good on average, or vice versa. There is also the problem of what probability means, whether it is subjective, objective, or exists at all. Kolmogorov's proposal strives for the firmer and less contentious ground of finite combinatorics and effective computation.

Model Selection: It is technically convenient to initially consider the simple model class of finite sets to obtain our results, just as in Section 5.2. It then turns out that it is relatively easy to generalize everything to the model class of computable probability distributions (Section 6.2.1). That class is very large indeed: perhaps it contains every distribution that has ever been considered in statistics and probability theory, as long as the parameters are computable numbers—for example rational numbers. Thus the results are of great generality; indeed, they are so general that further development of the theory must be aimed at restrictions on this model class.

Below we will consider various model selection procedures. These are approaches for finding a model S (containing x) for arbitrary data x. The goal is to find a model that captures all meaningful information in the data x . All approaches we consider are at some level based on coding x by giving its index in the set S, taking $\log |S|$ bits. This codelength may be thought of as a particular distortion function, and here lies the first connection to Shannon's rate-distortion:

Example 6.10 A model selection procedure is a function Z_n mapping binary data of length n to finite sets of strings of length n, containing the mapped data, $Z_n(x) = S$ $(x \in S)$. The range of Z_n satisfies $\mathcal{Z}_n \subseteq 2^{\{0,1\}^n}$, The distortion function d is defined to be $d(x, Y(x)) = \frac{1}{n} \log |S|$. To define the rate–distortion function we need that x is the outcome of a random variable X. Here we treat the simple case that X represents n flips of a fair coin; this is substantially generalized in Section 6.3. Since each outcome of a fair coin can be described by one bit, we set the rate R at $0 < R < 1$. Then, $D_n^*(R) = \min_{Z_n:|\mathcal{Z}_n| \leq 2^{nR}} \sum_{|x|=n} 2^{-n} \frac{1}{n} \log |Z_n(x)|$ For the minimum of the right-hand side we can assume that if $y \in Z_n(x)$ then $Z_n(y) = Z_n(x)$ (the distinct $Z_n(x)$'s are disjoint). Denote the distinct $Z_n(x)$'s by $Z_{n,i}$ with $i = 1, \ldots, k$ for some $k \leq 2^{nR}$. Then, $D_n^*(R) = \min_{Z_n:|\mathcal{Z}_n| \leq 2^{nR}} \sum_{1=1}^{k} |Z_{n,i}| 2^{-n} \frac{1}{n} \log |Z_{n,i}|$. The right-hand side reaches its minimum for all $Z_{n,i}$'s having the same cardinality and $k = 2^{nR}$. Then, $D_n^*(R) = 2^{nR} 2^{(1-R)n} 2^{-n} \frac{1}{n} \log 2^{(1-R)n} = 1 - R$. Therefore, $D^*(R) = 1 - R$ and therefore $R^*(D) = 1 - D$.

Alternatively, and more in line with the structure-function approach below, one may consider repetitions of a random variable X with outcomes in $\{0,1\}^n$. Then, a model selection procedure is a function Y mapping binary data of length n to finite sets of strings of length n, containing the mapped data, $Y(x) = S$ $(x \in S)$. The range of Y satisfies $\mathcal{Y} \subseteq 2^{\{0,1\}^n}$, The distortion function d is defined by $d(x, Y(x)) = \log |S|$. To define the rate–distortion function we need that x is the outcome of a random variable X, say a toss of a fair 2^n-sided coin. Since each outcome of a fair coin can be described by n bits, we set the rate R at $0 < R < n$. Then, for outcomes $\overline{x} = x_1 \ldots x_m$ $(|x_i| = n)$, resulting from m i.i.d. random variables $X_1, \ldots, X_m$, we have $d(\overline{x}, Z_m(\overline{x})) = \frac{1}{m} \sum_{i=1}^{m} \log |Y_i(x_i)| = \frac{1}{m} \log |Y_1(x_1) \times \cdots \times Y_m(x_m)|$. Then, $D_m^*(R) = \min_{Z_m:|\mathcal{Z}_m| \leq 2^{mR}} \sum_{\overline{x}} 2^{-mn} d(\overline{x}, Z_m(\overline{x}))$. Assume that $\overline{y} \in Z_m(\overline{x})$ if $Z_m(\overline{y}) = Z_m(\overline{x})$: the distinct $Z_m(\overline{x})$'s are disjoint and partition $\{0,1\}^{mn}$ into disjoint subsets $Z_{m,i}$, with $i = 1, \ldots, k$ for some $k \leq 2^{mR}$. Then, $D_m^*(R) = \min_{Z_m:|\mathcal{Z}_m| \leq 2^{mR}} \sum_{i=1,\ldots,k} |Z_{m,i}| 2^{-mn} \frac{1}{m} \log |Z_{m,i}|$. The right-hand side reaches its minimum for all $Z_{m,i}$'s having the same cardinality and $k = 2^{mR}$, so that $D_m^*(R) = 2^{(n-R)m} 2^{mR} 2^{-mn} \frac{1}{m} \log 2^{(n-R)m} = n - R$. Therefore, $D^*(R) = n - R$ and $R^*(D) = n - D$. In Example 6.13 we relate these numbers to the structure function approach described below. $\diamondsuit$

Model Fitness: A distinguishing feature of the structure function approach is that we want to formalize what it means for an element to be "typical" for a set that contains it. For example, if we flip a fair coin n times, then the sequence of n outcomes, denoted by x, will be an element of the set $\{0,1\}^n$. In fact, most likely it will be a "typical" element in the sense that it has all properties that hold on average for an element of that set. For example, x will have $\frac{n}{2} \pm O(\sqrt{n})$ frequency of 1's, it will have a run of about $\log n$ consecutive 0's, and so on for many properties. Note that the sequence $x = 0 \ldots 01 \ldots 1$, consisting of one half 0's followed by one half ones, is very untypical, even though it satisfies the two properties described explicitly. The question arises how to formally define "typicality". We do this as follows: The lack of typicality of x with respect to a finite set S (the model) containing it, is the amount by which $K(x|S)$ falls short of the length $\log |S|$ of the data-to-model code (Section 5.2). Thus, the *randomness deficiency* of x in S is defined by

$$\delta(x|S) = \log |S| - K(x|S), \tag{6.12}$$

for $x \in S$, and ∞ otherwise. Clearly, x can be typical for vastly different sets. For example, every x is typical for the singleton set $\{x\}$, since $\log |\{x\}| = 0$ and $K(x \mid \{x\}) = O(1)$. Yet the many x's that have $K(x) \geq n$ are also typical for $\{0,1\}^n$, but in another way. In the first example, the set is about as complex as x itself. In the second example, the set is vastly less complex than x: the set has complexity about $K(n) \leq \log n + 2 \log \log n$ while $K(x) \geq n$. Thus, very high complexity data may have simple sets for which they are typical. As we shall see, this is certainly not the case for all high complexity data. The question arises how typical data x of length n can be in the best case for a finite set of complexity R when R ranges from 0 to n. The function describing this dependency, expressed in terms of randomness deficiency to measure the optimal typicality, as a function of the complexity "rate" R $(0 \leq R \leq n)$ of the number of bits we can maximally spend to describe a finite set containing x, is defined as follows:

The *minimal randomness deficiency* function is

$$\beta_x(R) = \min_S \{\delta(x|S) : S \ni x, \ K(S) \leq R\}, \tag{6.13}$$

where we set $\min \emptyset = \infty$. If $\delta(x|S)$ is small, then x may be considered as a *typical* member of S. This means that S is a "best" model for x—a most likely explanation. There are no simple special properties that single it out from the majority of elements in S. We therefore like to call $\beta_x(R)$ the *best-fit estimator*. This is not just terminology: If $\delta(x|S)$ is small, then x satisfies *all* properties of low Kolmogorov complexity that hold with high probability (under the uniform distribution) for the elements of S. To be precise [26]: Consider strings of length n and let S be a subset of such strings. We view a *property* of elements in S as a function $f_P : S \to \{0,1\}$. If $f_P(x) = 1$ then x has the property represented by f_P and if $f_P(x) = 0$ then x does not have the property. Then: (i) If f_P is a property satisfied by all x with $\delta(x|S) \leq \delta(n)$, then f_P holds with probability at least $1 - 1/2^{\delta(n)}$ for the elements of S.

(ii) Let f_P be any property that holds with probability at least $1 - 1/2^{\delta(n)}$ for the elements of S. Then, every such f_P holds simultaneously for every $x \in S$ with $\delta(x|S) \leq \delta(n) - K(f_P|S) - O(1)$.

Example 6.11 Lossy Compression: The function $\beta_x(R)$ is relevant to lossy compression (used, for instance, to compress images) – see also Remark 6.19. Assume we need to compress x to R bits where $R \ll K(x)$. Of course this implies some loss of information present in x. One way to select redundant information to discard is as follows: Find a set $S \ni x$ with $K(S) \leq R$ and with small $\delta(x|S)$, and consider a compressed version S' of S. To reconstruct an x', a decompresser uncompresses S' to S and selects at random an element x' of S. Since with high probability the randomness deficiency of x' in S is small, x' serves the purpose of the message x as well as does x itself. Let us look at an example. To transmit a picture of "rain" through a channel with limited capacity R, one can transmit the indication that this is a picture of the rain and the particular drops may be chosen by the receiver at random. In this interpretation, $\beta_x(R)$ indicates how "random" or "typical" x is with respect to the best model at complexity level R—and hence how "indistinguishable" from the original x the randomly reconstructed x' can be expected to be. ♢

Remark 6.12 This randomness deficiency function quantifies the goodness of fit of the best model at complexity R for given data x. As far as we know no direct counterpart of this notion exists in Rate–Distortion theory, or, indeed, can be expressed in classical theories like Information Theory. But the situation is different for the next function we define, which, in almost contradiction to the previous statement, can be tied to the minimum randomness deficiency function, yet, as will be seen in Example 6.13 and Section 6.3, does have a counterpart in Rate–Distortion theory after all. ♢

Maximum Likelihood estimator: The *Kolmogorov structure* function h_x of given data x is defined by

$$h_x(R) = \min_S \{\log |S| : S \ni x, \ K(S) \leq R\}, \tag{6.14}$$

where $S \ni x$ is a contemplated model for x, and R is a nonnegative integer value bounding the complexity of the contemplated S's. The structure function uses models that are finite sets and the value of the structure function is the log-cardinality of the smallest such set containing the data. Equivalently, we can use uniform probability mass functions over finite supports (the former finite set models). The smallest set containing the data then becomes the uniform probability mass assigning the highest probability to the data—with the value of the structure function the corresponding negative log-probability. This motivates us to call h_x the *maximum likelihood estimator*. The treatment can be extended from uniform probability mass functions with finite supports, to probability models that are arbitrary computable probability mass functions, keeping all relevant notions and results essentially unchanged, Section 6.2.1, justifying the maximum likelihood identification even more.

Clearly, the Kolmogorov structure function is non-increasing and reaches $\log |\{x\}| = 0$ for the "rate" $R = K(x) + c_1$ where c_1 is the number of bits required to change x into $\{x\}$. It is also easy to see that for argument $K(|x|) + c_2$, where c_2 is the number of bits required to compute the set of all strings of length $|x|$ of x from $|x|$, the value of the structure function is at most $|x|$; see Figure 1

Example 6.13 Clearly the structure function measures for individual outcome x a distortion that is related to the one measured by $D_1^*(R)$ in Example 6.10 for the uniform average of outcomes x. Note that all strings x of length n satisfy $h_x(K(n) + O(1)) \leq n$ (since $x \in S_n = \{0,1\}^n$ and $K(S_n) = K(n) + O(1)$). For every R $(0 \leq R \leq n)$, we can describe every $x = x_1 x_2 \dots x_n$ as an element of the set $A_R = \{x_1 \dots x_R y_{R+1} \dots y_n : y_i \in \{0,1\}, R < i \leq n\}$. Then, $|A_R| = 2^{n-R}$ and $K(A_R) \leq R + K(n,R) + O(1) \leq R + O(\log n)$. This shows that $h_x(R) \leq n - R + O(\log n)$ for every x and every R with $0 \leq R \leq n$; see Figure 1.

For all x's and R's we can describe x in a two-part code by the set S witnessing $h_x(R)$ and x's index in that set. The first part describing S in $K(S) = R$ allows us to generate S, and given S we know $\log |S|$. Then, we can parse the second part of $\log |S| = h_x(R)$ bits that gives x's index in S. We also need a fixed $O(1)$ bit program to produce x from these descriptions. Since $K(x)$ is the lower bound on the length of effective descriptions of x, we have $h_x(R) + R \geq K(x) - O(1)$. There are $2^n - 2^{n-K(n)+O(1)}$ strings x of complexity $K(x) \geq n$, [18]. For all these strings $h_x(R) + R \geq n - O(1)$. Hence, the expected value $h_x(R)$ equals $2^{-n}\{(2^n - 2^{n-K(n)+O(1)})[n - R + O(\log n)] + 2^{n-K(n)+O(1)}O(n - R + O(\log n))\} = n - R + O(n - R/2^{-K(n)}) = n - R + o(n - R)$ (since $K(n) \to \infty$ for $n \to \infty$). That is, the expectation of $h_x(R)$ equals $(1 + o(1))D_1^*(R) = (1 + o(1))D^*(R)$, the Distortion-Rate function, where the $o(1)$ term goes to 0 with the length n of x. In Section 6.3 we extend this idea to non-uniform distributions on X. $\diamondsuit$

For every $S \ni x$ we have

$$K(x) \leq K(S) + \log |S| + O(1). \tag{6.15}$$

Indeed, consider the following *two-part code* for x: the first part is a shortest self-delimiting program p of S and the second part is $\lceil \log |S| \rceil$ bit long index of x in the lexicographical ordering of S. Since S determines $\log |S|$ this code is self-delimiting and we obtain (6.15) where the constant $O(1)$ is the length of the program to reconstruct x from its two-part code. We thus conclude that $K(x) \leq R + h_x(R) + O(1)$, that is, the function $h_x(R)$ never decreases more than a fixed independent constant below the diagonal *sufficiency line* L defined by $L(R) + R = K(x)$, which is a lower bound on $h_x(R)$ and is approached to within a constant distance by the graph of h_x for certain R's (for instance, for $R = K(x) + c_1$). For these R's we thus have $R + h_x(R) = K(x) + O(1)$. In the terminology we have introduced in Section 5.2.5 and Definition 5.7, a model corresponding to such an R (witness for $h_x(R)$) is an optimal set for x and a shortest program to compute this model is a sufficient statistic. It is *minimal* for the least such R for which the above equality holds.

MDL Estimator: The length of the minimal two-part code for x consisting of the model cost $K(S)$ and the length of the index of x in S, the complexity of S upper bounded by R, is given by the *MDL (minimum description length) function*:

$$\lambda_x(R) = \min_S \{\Lambda(S) : S \ni x,\ K(S) \leq R\}, \tag{6.16}$$

where $\Lambda(S) = \log |S| + K(S) \geq K(x) - O(1)$ is the total length of two-part code of x with help of model S. Clearly, $\lambda_x(R) \leq h_x(R) + R + O(1)$, but a priori it is still possible that $h_x(R') + R' < h_x(R) + R$ for $R' < R$. In that case $\lambda_x(R) \leq h_x(R') + R' < h_x(R) + R$. However, in [26] it is shown that $\lambda_x(R) = h_x(R) + R + O(\log n)$ for all x of length n. Even so, this doesn't mean that a set S that witnesses $\lambda_x(R)$ in the sense that $x \in S$, $K(S) \leq R$, and $K(S) + \log |S| = \lambda_x(R)$, also witnesses $h_x(R)$. It can in fact be the case that $K(S) \leq R - r$, and $\log |S| = h_x(R) + r$ for arbitrarily large $r \leq n$.

Apart from being convenient for the technical analysis in this work, $\lambda_x(R)$ is the celebrated two-part Minimum Description Length code length [20] with the model-code length restricted to at most R. When

R is large enough so that $\lambda_x(R) = K(x)$, then there is a set S that is a sufficient statistic, and the smallest such R has an associated witness set S that is a minimal sufficient statistic.

The most fundamental result in [26] is the equality

$$\beta_x(R) = h_x(R) + R - K(x) = \lambda_x(R) - K(x) \tag{6.17}$$

which holds within logarithmic additive terms in argument and value. Additionally, every set S that witnesses the value $h_x(R)$ (or $\lambda_x(R)$), also witnesses the value $\beta_x(R)$ (but not vice versa). It is easy to see that $h_x(R)$ and $\lambda_x(R)$ are upper semi-computable (Definition 1.1); but we have shown [26] that $\beta_x(R)$ is neither upper nor lower semi-computable (not even within a great tolerance). A priori there is no reason to suppose that a set that witnesses $h_x(R)$ (or $\lambda_x(R)$) also witnesses $\beta_x(R)$, for *every* R. But the fact that they do, vindicates Kolmogorov's original proposal and establishes h_x's pre-eminence over β_x – the pre-eminence of h_x over λ_x is discussed below.

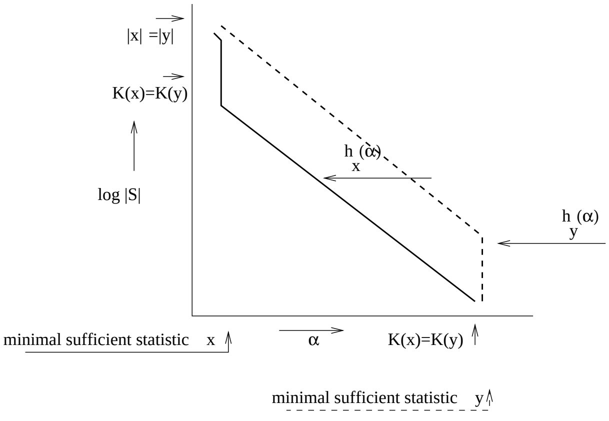
Remark 6.14 What we call 'maximum likelihood' in the form of h_x is really 'maximum likelihood' under a complexity constraint R on the models' as in $h_x(R)$. In statistics, it is a well-known fact that maximum likelihood often fails (dramatically overfits) when the models under consideration are of unrestricted complexity (for example, with polynomial regression with Gaussian noise, or with Markov chain model learning, maximum likelihood will always select a model with n parameters, where n is the size of the sample—and thus typically, maximum likelihood will dramatically overfit, whereas for example MDL typically performs well). The equivalent, in our setting, is that allowing models of unconstrained complexity for data x, say complexity $K(x)$, will result in the ML-estimator $h_x(K(x)+O(1)) = 0$—the witness model being the trivial, maximally overfitting, set $\{x\}$. In the MDL case, on the other hand, there may be a long constant interval with the MDL estimator $\lambda_x(R) = K(x)$ ($R \in [R_1, K(x)]$) where the length of the two-part code doesn't decrease anymore. Selecting the least complexity model witnessing this function value we obtain the, very significant, algorithmic *minimal* sufficient statistic, Definition 5.7. In this sense, MDL augmented with a bias for the least complex explanation, which we may call the 'Occam's Razor MDL', is superior to maximum likelihood and resilient to overfitting. If we don't apply bias in the direction of simple explanations, then – at least in our setting – MDL may be just as prone to overfitting as is ML. For example, if x is a typical random element of $\{0,1\}^n$, then $\lambda_x(R) = K(x) + O(1)$ for the entire interval $K(n) + O(1) \leq R \leq K(x) + O(1) \approx n$. Choosing the model on the left side, of simplest complexity, of complexity $K(n)$ gives us the best fit with the correct model $\{0,1\}^n$. But choosing a model on the right side, of high complexity, gives us a model $\{x\}$ of complexity $K(x) + O(1)$ that completely overfits the data by modeling all random noise in x (which in fact in this example almost completely consists of random noise).

Thus, it should be emphasized that 'ML = MDL' really only holds if complexities are constrained to a value R (that remains fixed as the sample size grows—note that in the Markov chain example above, the complexity grows linearly with the sample size); it certainly does not hold in an unrestricted sense (not even in the algorithmic setting). $\diamondsuit$

Remark 6.15 In a sense, h_x is more strict than λ_x: A set that witnesses $h_x(R)$ also witnesses $\lambda_x(R)$ but not necessarily vice versa. However, at those complexities R where $\lambda_x(R)$ drops (a little bit of added complexity in the model allows a shorter description), the witness set of λ_x is also a witness set of h_x. But if λ_x stays constant in an interval $[R_1, R_2]$, then we can trade-off complexity of a witness set versus its cardinality, keeping the description length constant. This is of course not possible with h_x where the cardinality of the witness set at complexity R is fixed at $h_x(R)$. $\diamondsuit$

The main result can be taken as a foundation and justification of common statistical principles in model selection such as maximum likelihood or MDL. The structure functions λ_x, h_x and β_x can assume all possible shapes over their full domain of definition (up to additive logarithmic precision in both argument and value), see [26]. (This establishes the significance of (6.17), since it shows that $\lambda_x(R) \gg K(x)$ is common for (x, R) pairs—in which case the more or less easy fact that $\beta_x(R) = 0$ for $\lambda_x(R) = K(x)$ is not applicable, and it is a priori unlikely that (6.17) holds: Why should minimizing a set containing x also minimize its randomness deficiency? Surprisingly, it does!) We have exhibited a—to our knowledge first—natural example, β_x, of a function that is not semi-computable but computable with an oracle for the halting problem.

Example 6.16 "Positive" and "Negative" Individual Randomness: In [8] we showed the existence of strings for which essentially the singleton set consisting of the string itself is a minimal sufficient statistic.



|x| =|y|
K(x)=K(y)
log |S|
h (α) x
h (α) y
minimal sufficient statistic x
α
K(x)=K(y)
minimal sufficient statistic y

Figure 2: Data string x is “positive random” or “stochastic” and data string y is just “negative random” or “non-stochastic”.

While a sufficient statistic of an object yields a two-part code that is as short as the shortest one part code, restricting the complexity of the allowed statistic may yield two-part codes that are considerably longer than the best one-part code (so the statistic is insufficient). In fact, for every object there is a complexity bound below which this happens—but if that bound is small (logarithmic) we call the object “stochastic” since it has a simple satisfactory explanation (sufficient statistic). Thus, Kolmogorov in [11] makes the important distinction of an object being random in the “negative” sense by having this bound high (it has high complexity and is not a typical element of a low-complexity model), and an object being random in the “positive, probabilistic” sense by both having this bound small and itself having complexity considerably exceeding this bound (like a string x of length n with $K(x) \geq n$, being typical for the set $\{0,1\}^n$, or the uniform probability distribution over that set, while this set or probability distribution has complexity $K(n) + O(1) = O(\log n)$). We depict the distinction in Figure 2. In simple terms: High Kolmogorov complexity of a data string just means that it is random in a *negative sense*; but a data string of high Kolmogorov complexity is *positively random* if the simplest satisfactory explanation (sufficient statistic) has low complexity, and it therefore is the typical outcome of a simple random process.

In [26] it is shown that for every length n and every complexity $k \leq n + K(n) + O(1)$ (the maximal complexity of x of length n) and every $R \in [0,k]$, there are x’s of length n and complexity k such that the minimal randomness deficiency $\beta_x(i) \geq n - k \pm O(\log n)$ for every $i \leq R \pm O(\log n)$ and $\beta_x(i) \pm O(\log n)$ for every $i > R \pm O(\log n)$. Therefore, the set of n-length strings of every complexity k can be partitioned in subsets of strings that have a Kolmogorov minimal sufficient statistic of complexity $\Theta(i \log n)$ for $i = 1, \ldots, k/\Theta(\log n)$. For instance, there are n-length non-stochastic strings of almost maximal complexity $n - \sqrt{n}$ having significant $\sqrt{n} \pm O(\log n)$ randomness deficiency with respect to $\{0,1\}^n$ or, in fact, every other finite set of complexity less than $n - O(\log n)$! $\diamondsuit$

6.2.1 Probability Models

The structure function (and of course the sufficient statistic) use properties of data strings modeled by finite sets, which amounts to modeling data by uniform distributions. As already observed by Kolmogorov himself, it turns out that this is no real restriction. Everything holds also for computable probability mass functions (probability models), up to additive logarithmic precision. Another version of h_x uses probability models f rather than finite set models. It is defined as $h'_x(R) = \min_f \{\log 1/f(x) : f(x) > 0, K(f) \leq R\}$. Since $h'_x(R)$ and $h_x(R)$ are close by Proposition 6.17 below, Theorem 6.27 and Corollary 6.29 also apply to h'_x and the distortion-rate function $D^*(R)$ based on a variation of the Shannon-Fano distortion measure defined by using encodings $Y(x) = f$ with f a computable probability distribution. In this context, the Shannon-Fano distortion measure is defined by

$$d'(x,f) = \log 1/f(x). \tag{6.18}$$

It remains to show that probability models are essentially the same as finite set models. We restrict ourselves to the model class of *computable probability distributions*. Within the present section, we assume these are

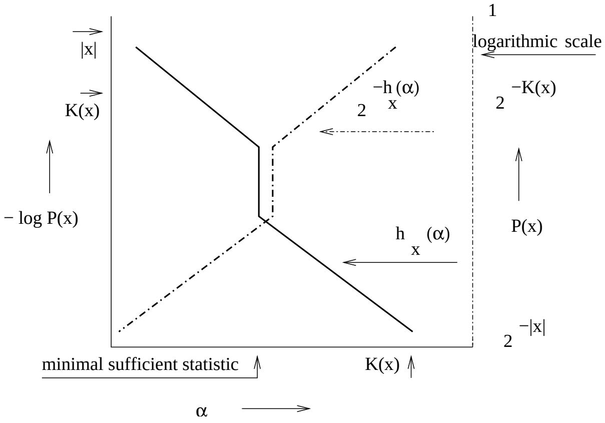


Figure 3: Structure function $h_x(i) = \min_f \{\log 1/f(x) : f(x) > 0,\ K(f) \leq i\}$ with f a computable probability mass function, with values according to the left vertical coordinate, and the maximum likelihood estimator $2^{-h_x(i)} = \max\{f(x) : p(x) > 0,\ K(f) \leq i\}$, with values according to the right-hand side vertical coordinate.

defined on strings of arbitrary length; so they are represented by mass functions $f : \{0,1\}^* \rightarrow [0,1]$ with $\sum f(x) = 1$ being computable according to Definition 1.1. A string x is typical for a distribution f if the randomness deficiency $\delta(x \mid f) = \log 1/f(x) - K(x \mid f)$ is small. The conditional complexity $K(x \mid f)$ is defined as follows. Say that a function A approximates f if $|A(y,\epsilon) - f(y)| < \epsilon$ for every y and every positive rational ϵ. Then $K(x \mid f)$ is the minimum length of a program that given every function A approximating f as an oracle prints x. Similarly, f is c-optimal for x if $K(f) + \log 1/f(x) \leq K(x) + c$. Thus, instead of the data-to-model code length $\log |S|$ for finite set models, we consider the data-to-model code length $\log 1/f(x)$ (the Shannon-Fano code). The value $\log 1/f(x)$ measures also how likely x is under the hypothesis f. The mapping $x \mapsto f_{\min}$ where $f_{\min}$ minimizes $\log 1/f(x)$ over f with $K(f) \leq R$ is a *maximum likelihood estimator*, see figure 3. Our results thus imply that that maximum likelihood estimator always returns a hypothesis with minimum randomness deficiency.

It is easy to show that for every data string x and a contemplated finite set model for it, there is an almost equivalent computable probability model. The converse is slightly harder: for every data string x and a contemplated computable probability model for it, there is a finite set model for x that has no worse complexity, randomness deficiency, and worst-case data-to-model code for x, up to additive logarithmic precision:

Proposition 6.17 *(a) For every x and every finite set $S \ni x$ there is a computable probability mass function f with $\log 1/f(x) = \log |S|$, $\delta(x \mid f) = \delta(x \mid S) + O(1)$ and $K(f) = K(S) + O(1)$.*

(b) There are constants c, C, such that for every string x, the following holds: For every computable probability mass function f there is a finite set $S \ni x$ such that $\log |S| < \log 1/f(x) + 1$, $\delta(x \mid S) \leq \delta(x \mid f) + 2\log K(f) + K(\lfloor \log 1/f(x) \rfloor) + 2\log K(\lfloor \log 1/f(x) \rfloor) + C$ and $K(S) \leq K(f) + K(\lfloor \log 1/f(x) \rfloor) + C$.

Proof. (a) Define $f(y) = 1/|S|$ for $y \in S$ and 0 otherwise.

(b) Let $m = \lfloor \log 1/f(x) \rfloor$, that is, $2^{-m-1} < f(x) \leq 2^{-m}$. Define $S = \{y : f(y) > 2^{-m-1}\}$. Then, $|S| < 2^{m+1} \leq 2/f(x)$, which implies the claimed value for $\log |S|$. To list S it suffices to compute all consecutive values of $f(y)$ to sufficient precision until the combined probabilities exceed $1 - 2^{-m-1}$. That is, $K(S) \leq K(f) + K(m) + O(1)$. Finally, $\delta(x \mid S) = \log |S| - K(x|S^*) < \log 1/f(x) - K(x \mid S^*) + 1 = \delta(x \mid f) + K(x \mid f) - K(x \mid S^*) + 1 \leq \delta(x \mid f) + K(S^* \mid f) + O(1)$. The term $K(S^* \mid f)$ can be upper bounded as $K(K(S)) + K(m) + O(1) \leq 2\log K(S) + K(m) + O(1) \leq 2\log(K(f) + K(m)) + K(m) + O(1) \leq 2\log K(f) + 2\log K(m) + K(m) + O(1)$, which implies the claimed bound for $\delta(x \mid S)$.

□

How large are the nonconstant additive complexity terms in Proposition 6.17 for strings x of length n? In item (b), we are commonly only interested in f such that $K(f) \leq n + O(\log n)$ and $\log 1/f(x) \leq n + O(1)$. Indeed, for every f there is f' such that $K(f') \leq \min\{K(f), n\} + O(\log n)$, $\delta(x \mid f') \leq \delta(x \mid f) + O(\log n)$, $\log 1/f'(x) \leq \min\{\log 1/f(x), n\} + 1$. Such f' is defined as follows: If $K(f) > n$ then $f'(x) = 1$ and $f'(y) = 0$

for every $y \neq x$; otherwise $f' = (f + U_n)/2$ where U_n stands for the uniform distribution on $\{0,1\}^n$. Then the additive terms in item (b) are $O(\log n)$.

6.3 Expected Structure Function Equals Distortion–Rate Function

In this section we treat the general relation between the expected value of $h_x(R)$, the expectation taken on a distribution $f(x) = P(X = x)$ of the random variable X having outcome x, and $D^*(R)$. This involves the development of a rate-distortion theory for individual sequences and arbitrary computable distortion measures. Following [27], we outline such a theory in Sections 6.3.1- 6.3.3. Based on this theory, we present in Section 6.3.4 a general theorem (Theorem 6.27) relating Shannon's $D^*(R)$ to the expected value of $h_x(R)$, for arbitrary random sources and computable distortion measures. This generalizes Example 6.13 above, where we analyzed the case of the distortion function

$$d(x, Y(x)) = \log |Y(x)|, \tag{6.19}$$

where $Y(x)$ is an x-containing finite set, for the uniform distribution. Below we first extend this example to arbitrary generating distributions, keeping the distortion function still fixed to (6.19. This will prepare us for the general development in Sections 6.3.1–6.3.3

Example 6.18 *In Example 6.13 it transpired that the distortion-rate function is the expected structure function, the expectation taken over the distribution on the x's. If, instead of using the uniform distribution on* $\{0,1\}^n$ *we use an arbitrary distribution* $f(x)$*, it is not difficult to compute the rate-distortion function* $R^*(D) = H(X) - \sup_{Y:d(X,Y)\leq D} H(X|Y)$ *where* Y *is a random vaiable with outcomes that are finite sets. Since* d *is a special type of Shannon-Fano distortion, with* $d(x,y) = P(X = x|Y = y) = \log |y|$ *if* $x \in y$*, and 0 otherwise, we have already met* $D^*(R)$ *for the distortion measure* (6.19) *in another guise. By the conclusion of Example 6.9, generalized to the random variable* X *having outcomes in* $\{0,1\}^n$*, and* R *being a rate in between 0 and* n*, we know that*

$$D^*(R) = H(X) - R. \tag{6.20}$$

♢

In the particular case analyzed above, the code word for a source word is a finite set containing the source word, and the distortion is the log-cardinality of the finite set. Considering the set of source words of length n, the distortion-rate function is the diagonal line from n to n. The structure functions of the individual data x of length n, on the other hand, always start at n, decrease at a slope of at least -1 until they hit the diagonal from $K(x)$ to $K(x)$, which they must do, and follow the diagonal henceforth. Above we proved that the average of the structure function is simply the straight line, the diagonal, between n and n. This is the case, since the strings x with $K(x) \geq n$ are the overwhelming majority. All of them have a minimal sufficient statistic (the point where the structure function hits the diagonal from $K(x)$ to $K(x)$. This point has complexity at most $K(n)$. The structure function for all these x's follows the diagonal from about n to n, giving overall an expectation of the structure function close to this diagonal, that is, the probabilistic distortion-rate function for this code and distortion measure.

6.3.1 Distortion Spheres

Modeling the data can be viewed as encoding the data by a model: the data are source words to be coded, and models are code words for the data. As before, the set of possible data is $\mathcal{X} = \{0,1\}^n$. Let $\mathcal{R}^+$ denote the set of non-negative real numbers. For every model class $\mathcal{Y}$ (particular set of code words) we choose an appropriate recursive function $d : \mathcal{X} \times \mathcal{Y} \to \mathcal{R}^+$ defining the *distortion* $d(x,y)$ between data $x \in \mathcal{X}$ and model $y \in \mathcal{Y}$.

Remark 6.19 (Lossy Compression) The choice of distortion function is a selection of which aspects of the data are relevant, or meaningful, and which aspects are irrelevant (noise). We can think of the distortion-rate function as measuring how far the model at each bit-rate falls short in representing the data. Distortion-rate theory underpins the practice of lossy compression. For example, lossy compression of a sound file gives as "model" the compressed file where, among others, the very high and very low inaudible frequencies have been suppressed. Thus, the rate-distortion function will penalize the deletion of the inaudible frequencies but lightly because they are not relevant for the auditory experience.

But in the traditional distortion-rate approach, we average twice: once because we consider a sequence of outcomes of m instantiations of the same random variable, and once because we take the expectation over the sequences. Essentially, the results deal with typical "random" data of certain simple distributions. This assumes that the data to a certain extent satisfy the behavior of repeated outcomes of a random source. Kolmogorov [10]:

> The probabilistic approach is natural in the theory of information transmission over communication channels carrying "bulk" information consisting of a large number of unrelated or weakly related messages obeying definite probabilistic laws. In this type of problem there is a harmless and (in applied work) deep-rooted tendency to mix up probabilities and frequencies within sufficiently long time sequence (which is rigorously satisfied if it is assumed that "mixing" is sufficiently rapid). In practice, for example, it can be assumed that finding the "entropy" of a flow of congratulatory telegrams and the channel "capacity" required for timely and undistorted transmission is validly represented by a probabilistic treatment even with the usual substitution of empirical frequencies for probabilities. If something goes wrong here, the problem lies with the vagueness of our ideas of the relationship between mathematical probabilities and real random events in general.
>
> But what real meaning is there, for example, in asking how much information is contained in "War and Peace"? Is it reasonable to include the novel in the set of "possible novels", or even to postulate some probability distribution for this set? Or, on the other hand, must we assume that the individual scenes in this book form a random sequence with "stocahstic relations" that damp out quite rapidly over a distance of several pages?

Currently, individual data arising in practice are submitted to analysis, for example sound or video files, where the assumption that they either consist of a large number of weakly related messages, or being an element of a set of possible messages that is susceptible to analysis, is clearly wrong. It is precisely the global related aspects of the data which we want to preserve under lossy compression. The rich versatility of the structure functions, that is, many different distortion-rate functions for different individual data, is all but obliterated in the averaging that goes on in the traditional distortion-rate function. In the structure function approach one focuses entirely on the stochastic properties of one data item. ♢

Below we follow [27], where we developed a rate-distortion theory for individual data for general computable distortion measures, with as specific examples the 'Kolmogorov' distortion below, but also Hamming distortion and Euclidean distortion. This individual rate-distortion theory is summarized in Sections 6.3.2 and 6.3.3. In Section 6.3.4, Theorem 6.27. we connect this indivual rate-distortion theory to Shannon's. We emphasize that the typical data items of i.i.d. distributed simple random variables, or simple ergodic stationary sources, which are the subject of Theorem 6.27, are generally unrelated to the higly globally structured data we want to analyze using our new rate-distortion theory for individual data. From the prespective of lossy compression, the typical data have the characteristics of random noise, and there is no significant "meaning" to be preserved under the lossy compression. Rather, Theorem 6.27 serves as a 'sanity check' showing that in the special, simple case of repetitive probabilistic data, the new theory behaves essentially like Shannon's probabilistic rate-distortion theory.

Example 6.20 Let us look at various model classes and distortion measures:

(i) The set of models are the finite sets of finite binary strings. Let $S \subseteq \{0,1\}^*$ and $|S| < \infty$. We define $d(x,S) = \log |S|$ if $x \in S$, and ∞ otherwise.

(ii) The set of models are the computable probability density functions f mapping $\{0,1\}^*$ to $[0,1]$. We define $d(x,S) = \log 1/f(x)$ if $f(x) > 0$, and ∞ otherwise.

(iii) The set of models are the total recursive functions f mapping $\{0,1\}^*$ to $\mathcal{N}$. We define $d(x,f) = \min\{l(d) : f(d) = x\}$, and ∞ if no such d exists.

All of these model classes and accompanying distortions [26], together with the "communication exchange" models in [1], are loosely called *Kolmogorov* models and distortion, since the graphs of their structure functions (individual distortion-rate functions) are all within a strip—of width logarithmic in the binary length of the data—of one another. ♢

If $\mathcal{Y}$ is a model class, then we consider *distortion spheres* of given radius r centered on $y \in \mathcal{Y}$:

$$B_y(r) = \{x : d(x,y) = r\}.$$

This way, every model class and distortion measure can be treated similarly to the canonical finite set case, which, however, is especially simple in that the radius not variable. That is, there is only one distortion sphere centered on a given finite set, namely the one with radius equal to the log-cardinality of that finite set. In fact, that distortion sphere equals the finite set on which it is centered.

6.3.2 Randomness Deficiency—Revisited

Let $\mathcal{Y}$ be a model class and d a distortion measure. Since in our definition the distortion is recursive, given a model $y \in \mathcal{Y}$ and diameter r, the elements in the distortion sphere of diameter r can be recursively enumerated from the distortion function. Giving the index of any element x in that enumeration we can find the element. Hence, $K(x|y,r) \overset{+}{<} \log |B_y(r)|$. On the other hand, the vast majority of elements x in the distortion sphere have complexity $K(x|y,r) \overset{+}{>} \log |B_y(r)|$ since, for every constant c, there are only $2^{\log |B_y(r)|-c} - 1$ binary programs of length $< \log |B_y(r)| - c$ available, and there are $|B_y(r)|$ elements to be described. We can now reason as in the similar case of finite set models. With data x and $r = d(x,y)$, if $K(x|y,d(x,y)) \overset{+}{>} \log |B_y(d(x,y))|$, then x belongs to every large majority of elements (has the property represented by that majority) of the distortion sphere $B_y(d(x,y))$, provided that property is simple in the sense of having a description of low Kolmogorov complexity.

Definition 6.21 The *randomness deficiency* of x with respect to model y under distortion d is defined as

$$\delta(x \mid y) = \log |B_y(d(x,y))| - K(x|y,d(x,y)).$$

Data x is *typical* for model $y \in \mathcal{Y}$ (and that model "typical" or "best fitting" for x) if

$$\delta(x \mid y) \overset{+}{=} 0. \tag{6.21}$$

If x is typical for a model y, then the shortest way to effectively describe x, given y, takes about as many bits as the descriptions of the great majority of elements in a recursive enumeration of the distortion sphere. So there are no special simple properties that distinguish x from the great majority of elements in the distortion sphere: they are all typical or random elements in the distortion sphere (that is, with respect to the contemplated model).

Example 6.22 Continuing Example 6.20 by applying (6.21) to different model classes:

(i) *Finite sets:* For finite set models S, clearly $K(x|S) \overset{+}{<} \log |S|$. Together with (6.21) we have that x is typical for S, and S best fits x, if the randomness deficiency according to (6.12) satisfies $\delta(x|S) \overset{+}{=} 0$.

(ii) *Computable probability density functions:* Instead of the data-to-model code length $\log |S|$ for finite set models, we consider the data-to-model code length $\log 1/f(x)$ (the Shannon-Fano code). The value $\log 1/f(x)$ measures how likely x is under the hypothesis f. For probability models f, define the conditional complexity $K(x \mid f, \lceil \log 1/f(x) \rceil)$ as follows. Say that a function A approximates f if $|A(x,\epsilon) - f(x)| < \epsilon$ for every x and every positive rational ϵ. Then $K(x \mid f, \lceil \log 1/f(x) \rceil)$ is defined as the minimum length of a program that, given $\lceil \log 1/f(x) \rceil$ and any function A approximating f as an oracle, prints x.

Clearly $K(x|f, \lceil \log 1/f(x) \rceil) \overset{+}{<} \log 1/f(x)$. Together with (6.21), we have that x is typical for f, and f best fits x, if $K(x|f, \lceil \log 1/f(x) \rceil) \overset{+}{>} \log |\{z : \log 1/f(z) \leq \log 1/f(x)\}|$. The right-hand side set condition is the same as $f(z) \geq f(x)$, and there can be only $\leq 1/f(x)$ such z, since otherwise the total probability exceeds 1. Therefore, the requirement, and hence typicality, is implied by $K(x|f, \lceil \log 1/f(x) \rceil) \overset{+}{>} \log 1/f(x)$. Define the randomness deficiency by $\delta(x \mid f) = \log 1/f(x) - K(x \mid f, \lceil \log 1/f(x) \rceil)$. Altogether, a string x is *typical for a distribution* f, or f is the *best fitting model* for x, if $\delta(x \mid f) \overset{+}{=} 0$. if $\delta(x \mid f) \overset{+}{=} 0$.

(iii) *Total Recursive Functions:* In place of $\log |S|$ for finite set models we consider the data-to-model code length (actually, the distortion $d(x,f)$ above)

$$l_x(f) = \min\{l(d) : f(d) = x\}.$$

Define the conditional complexity $K(x \mid f, l_x(f))$ as the minimum length of a program that, given $l_x(f)$ and an oracle for f, prints x.

Clearly, $K(x|f, l_x(f)) \overset{+}{<} l_x(f)$. Together with (6.21), we have that x is typical for f, and f best fits x, if $K(x|f, l_x(f)) \overset{+}{>} \log\{z : l_z(f) \leq l_x(f)\}$. There are at most $(2^{l_x(f)+1} - 1)$- many z satisfying the set condition since $l_z(f) \in \{0,1\}^*$. Therefore, the requirement, and hence typicality, is implied by $K(x|f, l_x(f)) \overset{+}{>} l_x(f)$. Define the randomness deficiency by $\delta(x \mid f) = l_x(f) - K(x \mid f, l_x(f))$. Altogether, a string x is *typical for a total recursive function* f, and f is the *best fitting recursive function model* for x if $\delta(x \mid f) \overset{\pm}{=} 0$, or written differently,

$$K(x|f, l_x(f)) \overset{\pm}{=} l_x(f). \tag{6.22}$$

Note that since $l_x(f)$ is given as conditional information, with $l_x(f) = l(d)$ and $f(d) = x$, the quantity $K(x|f, l_x(f))$ represents the number of bits in a shortest *self-delimiting* description of d. ♢

Remark 6.23 We required $l_x(f)$ in the conditional in (6.22). This is the information about the radius of the distortion sphere centered on the model concerned. Note that in the canonical finite set model case, as treated in [11, 8, 26], every model has a fixed radius which is explicitly provided by the model itself. But in the more general model classes of computable probability density functions, or total recursive functions, models can have a variable radius. There are subclasses of the more general models that have fixed radiuses (like the finite set models).

(i) In the computable probability density functions one can think of the probabilities with a finite support, for example $f_n(x) = 1/2^n$ for $l(x) = n$, and $f(x) = 0$ otherwise.

(ii) In the total recursive function case one can similarly think of functions with finite support, for example $f_n(x) = \sum_{i=1}^n x_i$ for $x = x_1 \ldots x_n$, and $f_n(x) = 0$ for $l(x) \neq n$.

The incorporation of the radius in the model will increase the complexity of the model, and hence of the minimal sufficient statistic below. ♢

6.3.3 Sufficient Statistic—Revisited

As with the probabilistic sufficient statistic (Section 5.1), a statistic is a function mapping the data to an element (model) in the contemplated model class. With some sloppiness of terminology we often call the function value (the model) also a statistic of the data. A statistic is called sufficient if the two-part description of the data by way of the model and the data-to-model code is as concise as the shortest one-part description of x. Consider a model class $\mathcal{Y}$.

Definition 6.24 *A model $y \in \mathcal{Y}$ is a* sufficient statistic *for x if*

$$K(y, d(x,y)) + \log |B_y(d(x,y))| \overset{\pm}{=} K(x). \tag{6.23}$$

Lemma 6.25 *If y is a sufficient statistic for x, then $K(x \mid y, d(x,y) \overset{\pm}{=} \log |B_y(d(x,y))|$, that is, x is typical for y.*

Proof. We can rewrite $K(x) \overset{+}{<} K(x, y, d(x,y)) \overset{+}{<} K(y, d(x,y)) + K(x|y, d(x,y)) \overset{+}{<} K(y, d(x,y)) + \log |B_y(d(x,y))| \overset{\pm}{=} K(x)$. The first three inequalities are straightforward and the last equality is by the assumption of sufficiency. Altogether, the first sum equals the second sum, which implies the lemma. □

Thus, if y is a sufficient statistic for x, then x is a typical element for y, and y is the best fitting model for x. Note that the converse implication, "typicality" implies "sufficiency," is not valid. Sufficiency is a special type of typicality, where the model does not add significant information to the data, since the preceding proof shows $K(x) \overset{\pm}{=} K(x, y, d(x,y))$. Using the symmetry of information (3.9) this shows that

$$K(y, d(x,y) \mid x) \overset{\pm}{=} K(y \mid x) \overset{\pm}{=} 0. \tag{6.24}$$

This means that:

(i) A sufficient statistic y is determined by the data in the sense that we need only an $O(1)$-bit program, possibly depending on the data itself, to compute the model from the data.

(ii) For each model class and distortion there is a universal constant c such that for every data item x there are at most c sufficient statistics.

Example 6.26 *Finite sets:* For the model class of finite sets, a set S is a sufficient statistic for data x if

$$K(S) + \log |S| \overset{\pm}{=} K(x).$$

Computable probability density functions: For the model class of computable probability density functions, a function f is a sufficient statistic for data x if

$$K(f) + \log 1/f(x) \stackrel{+}{=} K(x).$$

For the model class of *total recursive functions*, a function f is a *sufficient statistic* for data x if

$$K(x) \stackrel{+}{=} K(f) + l_x(f). \tag{6.25}$$

Following the above discussion, the meaningful information in x is represented by f (the model) in $K(f)$ bits, and the meaningless information in x is represented by d (the noise in the data) with $f(d) = x$ in $l(d) = l_x(f)$ bits. Note that $l(d) \stackrel{+}{=} K(d) \stackrel{+}{=} K(d|f^*)$, since the two-part code (f^*, d) for x cannot be shorter than the shortest one-part code of $K(x)$ bits, and therefore the d-part must already be maximally compressed. By Lemma 6.25, $l_x(f) \stackrel{+}{=} K(x \mid f^*, l_x(f))$, x is typical for f, and hence $K(x) \stackrel{+}{=} K(f) + K(x \mid f^*, l_x(f))$. $\diamondsuit$

6.3.4 Expected Structure Function

We treat the relation between the expected value of $h_x(R)$, the expectation taken on a distribution $f(x) = P(X = x)$ of the random variable X having outcome x, and $D^*(R)$, for arbitrary random sources provided the probability mass function $f(x)$ is recursive.

Theorem 6.27 *Let d be a recursive distortion measure. Given m repetitions of a random variable X with outcomes $x \in \mathcal{X}$ (typically, $\mathcal{X} = \{0,1\}^n$) with probability $f(x)$, where f is a total recursive function, we have*

$$\mathbf{E}\frac{1}{m} h_{\overline{x}}(mR + K(f,d,m,R) + O(\log n)) \leq D^*_m(R) \leq \mathbf{E}\frac{1}{m} h_{\overline{x}}(mR),$$

the expectations are taken over $\overline{x} = x_1 \ldots x_m$ where x_i is the outcome of the ith repetition of X.

Proof. As before, let $X_1, \ldots, X_m$ be m independent identically distributed random variables on outcome space $\mathcal{X}$. Let $\mathcal{Y}$ be a set of code words. We want to find a sequence of functions $Y_1, \ldots, Y_m : \mathcal{X} \to \mathcal{Y}$ so that the message $(Y_1(x_1), \ldots, Y_m(x_m)) \in \mathcal{Y}^m$ gives as much expected information about the sequence of outcomes $(X_1 = x_1, \ldots, X_m = x_m)$ as is possible, under the constraint that the message takes at most $R \cdot m$ bits (so that R bits are allowed on average per outcome of X_i). Instead of $Y_1, \ldots, Y_m$ above write $\overline{Y} : \mathcal{X}^m \to \mathcal{Y}^m$. Denote the cardinality of the range of $\overline{Y}$ by $\rho(\overline{Y}) = |\{\overline{Y}(\overline{x}) : \overline{x} \in \mathcal{X}^m\}|$. Consider distortion spheres

$$B_{\overline{y}}(d) = \{\overline{x} : d(\overline{x}, \overline{y}) = d\}, \tag{6.26}$$

with $\overline{x} = x_1 \ldots x_m \in \mathcal{X}^m$ and $\overline{y} \in \mathcal{Y}^m$.

Left Inequality: Keeping the earlier notation, for m i.i.d. random variables $X_1, \ldots, X_m$, and extending f to the m-fold Cartesian product of $\{0,1\}^n$, we obtain $D^*_m(R) = \frac{1}{m} \min_{\overline{Y}:\rho(\overline{Y}) \leq 2^{mR}} \sum_{\overline{x}} f(\overline{x}) d(\overline{x}, \overline{Y}(\overline{x}))$. By definition of $D^*_m(R)$ it equals the following expression in terms of a minimal canonical covering of $\{0,1\}^{nm}$ by disjoint nonempty spheres $B'_{\overline{y}_i}(d_i)$ $(1 \leq i \leq k)$ obtained from the possibly overlapping distortion spheres $B_{\overline{y}_i}(d_i)$ as follows. Every element $\overline{x}$ in the overlap between two or more spheres is assigned to the sphere with the smallest radius and removed from the other spheres. If there is more than one sphere of smallest radius, then we take the sphere of least index in the canonical covering. Empty B'-spheres are removed from the B'-covering. If $S \subseteq \{0,1\}^{nm}$, then $f(S)$ denotes $\sum_{x \in S} f(x)$. Now, we can rewrite

$$D^*_m(R) = \min_{\overline{y}_1, \ldots, \overline{y}_k; d_1, \ldots, d_k; k \leq 2^{mR}} \frac{1}{m} \sum_{i=1}^{k} f(B'_{\overline{y}_i}(d_i)) d_i. \tag{6.27}$$

In the structure function setting we consider some individual data $\overline{x}$ residing in one of the covering spheres. Given m, n, R and a program to compute f and d, we can compute the covering spheres centers $\overline{y}_1, \ldots, \overline{y}_k$, and radiuses $d_1, \ldots, d_k$, and hence the B'-sphere canonical covering. In this covering we can identify every pair $(\overline{y}_i, d_i)$ by its index $i \leq 2^{mR}$. Therefore, $K(\overline{y}_i, d_i) \leq mR + K(f,d,m,R) + O(\log n)$ $(1 \leq i \leq k)$. For $\overline{x} \in B'_{\overline{y}_i}(d_i)$ we have $h_{\overline{x}}(mR + K(f,d,m,R) + O(\log n)) \leq d_i$. Therefore, $\mathbf{E}\frac{1}{m} h_{\overline{x}}(mR + K(f,d,m,R) + O(\log n)) \leq D^*_m(R)$, the expectation taken over $f(\overline{x})$ for $\overline{x} \in \{0,1\}^{mn}$.

Right Inequality: Consider a covering of $\{0,1\}^{nm}$ by the (possibly overlapping) distortion spheres $B_{\overline{y}_i}(d_i)$ satisfying $K(B_{\overline{y}_i}(d_i)|mR) < mR - c$, with c an appropriate constant choosen so that the remainder of the argument goes through. If there are more than one spheres with different (center, radius)-pairs representing the same subset of $\{0,1\}^{nm}$, then we eliminate all of them except the one with the smallest radius. If there are more than one such spheres, then we only keep the one with the lexicographically least center. From this covering we obtain a canonical covering by nonempty disjoint spheres $B'_{\overline{y}_i}(d_i)$ similar to that in the previous paragraph, $(1 \leq i \leq k)$.

For every $\overline{x} \in \{0,1\}^{nm}$ there is a unique sphere $B'_{\overline{y}_i}(d_i) \ni \overline{x}$ $(1 \leq i \leq k)$. Choose the constant c above so that $K(B'_{\overline{y}_i}(d_i)|mR) < mR$. Then, $k \leq 2^{mR}$. Moreover, by construction, if $B'_{\overline{y}_i}(d_i)$ is the sphere containing $\overline{x}$, then $h_{\overline{x}}(mR) = d_i$. Define functions $\gamma : \{0,1\}^{nm} \rightarrow \mathcal{Y}^m$, $\delta : \{0,1\}^{nm} \rightarrow \mathcal{R}^+$ defined by $\gamma(\overline{x}) = \overline{y}_i$ and $\delta(\overline{x}) = d_i$ for $\overline{x}$ in the sphere $B'_{\overline{y}_i}(d_i)$. Then,

$$\mathbf{E}\frac{1}{m}h_{\overline{x}}(mR) = \frac{1}{m}\sum_{\overline{x} \in \{0,1\}^{mn}} f(\overline{x})d(\overline{x},\gamma(\overline{x})) = \frac{1}{m}\sum_{\overline{y}_1,\ldots,\overline{y}_k;d_1,\ldots,d_k} f(B'_{\overline{y}_i}(d_i))d_i. \tag{6.28}$$

The distortion $D^*_m(R)$ achieves the minimum of the expression in right-hand side of (6.27). Since $K(B'_{\gamma(\overline{x})}(\delta(\overline{x}))|mR) < mR$, the cover in the right-hand side of (6.28) is a possible partition satisfying the expression being minimized in the right-hand side of (6.27), and hence majorizes the minumum $D^*_m(R)$. Therefore, $\mathbf{E}\frac{1}{m}h_{\overline{x}}(mR) \geq D^*_m(R)$. □

Remark 6.28 A sphere is a subset of $\{0,1\}^{nm}$. The same subset may correspond to more than one spheres with different centers and radiuses: $B_{\overline{y_0}}(d_0) = B_{\overline{y_1}}(d_1)$ with $(y_0,d_0) \neq (y_1,d_1)$. Hence, $K(B_{\overline{y}}(d)) \leq K(\overline{y},d)) + O(1)$, but possibly $K(\overline{y},d)) > K(B_{\overline{y}}(d)) + O(1)$. However, in the proof we constructed the ordered sequence of B' spheres such that every sphere uniquely corresponds to a (center, radius)-pair. Therefore, $K(B'_{\overline{y}_i}(d_i)|mR) \stackrel{+}{=} K(\overline{y}_i, d_i|mR)$. ♢

Corollary 6.29 *It follows from the above theorem that, for a recursive distortion function d: (i) $\mathbf{E}h_x(R + K(f,d,R) + O(\log n)) \leq D^*_1(R) \leq \mathbf{E}h_x(R)$, for outcomes of a single repetition of random variable $X = x$ with $x \in \{0,1\}^n$, the expectation taken over $f(x) = P(X = x)$; and*

(ii) $\lim_{m\rightarrow\infty} \mathbf{E}\frac{1}{m}h_{\overline{x}}(mR) = D^*(R)$ for outcomes $\overline{x} = x_1 \ldots x_m$ of i.i.d. random variables $X_i = x_i$ with $x_i \in \{0,1\}^n$ for $1 \leq i \leq m$, the expectation taken over $f(\overline{x}) = P(X_i = x_i, i = 1,\ldots,m)$ (the extension of f to m repetitions of X).

This is the sense in which the expected value of the structure function is asymptotically equal to the value of the distortion-rate function, for arbitrary computable distortion measures. In the structure function approach we dealt with only two model classes, finite sets and computable probability density functions, and the associated quantities to be minimized, the log-cardinality and the negative log-probability, respectively. Translated into the distortion-rate setting, the models are code words and the minimalizable quantities are distortion measures. In [26] we also investigate the model class of total recursive functions, and in [1] the model class of communication protocols. The associated quantities to be minimized are then function arguments and communicated bits, respectively. All these models are equivalent up to logarithmic precision in argument and value of the corresponding structure functions, and hence their expectations are asymptotic to the distortion-rate functions of the related code-word set and distortion measure.

7 Conclusion

We have compared Shannon's and Kolmogorov's theories of information, highlighting the various similarities and differences. Some of this material can also be found in [4], the standard reference for Shannon information theory, as well as [18], the standard reference for Kolmogorov complexity theory. These books predate much of the recent material on the Kolmogorov theory discussed in the present paper, such as [9] (Section 3.2), [17] (Section 4.2), [8] (Section 5.2), [26, 27] (Section 6.2). The material in Sections 5.3 and 6.3 has not been published before. The present paper summarizes these recent contributions and systematically compares them to the corresponding notions in Shannon's theory.

Related Developments: There are two major practical theories which have their roots in both Shannon's and Kolmogorov's notions of information: first, *universal coding*, briefly introduced in Appendix A below, is a remarkably successful theory for practical lossless data compression. Second, Rissanen's *Minimum Description Length (MDL) Principle* [20, 7] is a theory of inductive inference that is both practical and successful. Note that direct practical application of Shannon's theory is hampered by the typically untenable assumption of a true and known distribution generating the data. Direct application of Kolmogorov's theory is hampered by the noncomputability of Kolmogorov complexity and the strictly asymptotic nature of the results. Both universal coding (of the individual sequence type, Appendix A) and MDL seek to overcome both problems by restricting the description methods used to those corresponding to a set of probabilistic predictors (thus making encodings and their lengths computable and nonasymptotic); yet when applying these predictors, the assumption that any one of them generates the data is never actually made. Interestingly, while in its current form MDL bases inference on universal codes, in recent work Rissanen and co-workers have sought to found the principle on a restricted form of the algorithmic sufficient statistic and Kolmogorov's structure function as discussed in Section 6.2 [21].

By looking at general types of prediction errors, of which codelengths are merely a special case, one achieves a generalization of the Kolmogorov theory that goes by the name of *predictive complexity*, pioneered by Vovk, Vyugin, Kalnishkan and others[5] [25]. Finally, the notions of 'randomness deficiency' and 'typical set' that are central to the algorithmic sufficient statistic (Section 5.2) are intimately related to the celebrated Martin-Löf-Kolmogorov theory of *randomness in individual sequences*, an overview of which is given in [18].

A Appendix: Universal Codes

Shannon's and Kolmogorov's idea are not directly applicable to most actual data compression problems. Shannon's theory is hampered by the typically untenable assumption of a true and known distribution generating the data. Kolmogorov's theory is hampered by the noncomputability of Kolmogorov complexity and the strictly asymptotic nature of the results. Yet there is a middle ground that is feasible: *universal codes* that may be viewed as both an generalized version of Shannon's, and a feasible approximation to Kolmogorov's theory. In introducing the notion of universal coding Kolmogorov says [10]:

> "A universal coding method that permits the transmission of any sufficiently long message [of length n] in an alphabet of s letters with no more nh [h is the empirical entropy] binary digits is not necessarily excessively complex; in particular, it is not essential to begin by determining the frequencies p_r for the entire message."

Below we repeatedly use the coding concepts introduced in Section 1.3. Suppose we are given a recursive enumeration of prefix codes $D_1, D_2, \ldots$. Let $L_1, L_2, \ldots$ be the length functions associated with these codes. That is, $L_i(x) = \min_y\{l(y) : D_i(y) = x\}$; if there exists no y with $D_i(y) = x$, then $L_i(y) = \infty$. We may encode x by first encoding a natural number k using the standard prefix code for the natural numbers. We then encode x itself using the code D_k. This leads to a so-called *two-part code* $\tilde{D}$ with lengths $\tilde{L}$. By construction, this code is prefix and its lengths satisfy

$$\tilde{L}(x) := \min_{k \in \mathcal{N}} L_{\mathcal{N}}(k) + L_k(x), \tag{A.1}$$

Let $\mathbf{x}$ be an infinite binary sequence and let $x_{[1:n]} \in \{0,1\}^n$ be the initial n-bit segment of this sequence. Since $L_{\mathcal{N}}(k) = O(\log k)$, we have for all k, all n:

$$\tilde{L}(x_{[1:n]}) \leq L_k(x_{[1:n]}) + O(\log k).$$

Recall that for each fixed L_k, the fraction of sequences of length n that can be compressed by more than m bits is less than 2^{-m}. Thus, typically, the codes L_k and the strings $x_{[1:n]}$ will be such that $L_k(x_{[1:n]})$ grows *linearly* with n. This implies that for every $\mathbf{x}$, the newly constructed $\tilde{L}$ is 'almost as good' as whatever code D_k in the list is best for that particular $\mathbf{x}$: the difference in code lengths is bounded by a constant depending on k but not on n. In particular, for each infinite sequence $\mathbf{x}$, for each fixed k,

$$\lim_{n \to \infty} \frac{\tilde{L}(x_{[1:n]})}{L_k(x_{[1:n]})} \leq 1. \tag{A.2}$$

[5] See `www.vovk.net` for an overview.

A code satisfying (A.2) is called a *universal code* relative to the *comparison class* of codes $\{D_1, D_2, \ldots\}$. It is 'universal' in the sense that it compresses every sequence essentially as well as the D_k that compresses that particular sequence the most. In general, there exist many types of codes that are universal: the 2-part universal code defined above is just one means of achieving (A.2).

Universal codes and Kolmogorov: In most practically interesting cases we may assume that for all k, the decoding function D_k is computable, i.e. there exists a prefix Turing machine which for all $y \in \{0,1\}^*$, when input y' (the prefix-free version of y), outputs $D_k(y)$ and then halts. Since such a program has finite length, we must have for all k,

$$l(E^*(x_{[1:n]})) = K(x_{[1:n]}) \leq^+ L_k(x_{[1:n]})$$

where E^* is the encoding function defined in Section 2.2, with $l(E^*(x)) = K(x)$. Comparing with (A.2) shows that the code D^* with encoding function E^* is a universal code relative to $D_1, D_2, \ldots$. Thus, we see that the Kolmogorov complexity K is just the length function of the universal code D^*. Note that D^* is an example of a universal code that is not (explicitly) two-part.

Example A.1 Let us create a universal two-part code that allows us to significantly compress all binary strings with frequency of 0's deviating significantly from $\frac{1}{2}$. For $n_0 < n_1$, let $D_{\langle n,n_0\rangle}$ be the code that assigns code words of equal (minimum) length to all strings of length n with n_0 zeroes, and no code words to any other strings. Then $D_{\langle n,n_0\rangle}$ is a prefix-code and $L_{\langle n,n_0\rangle}(x) = \lceil \log \binom{n}{n_0} \rceil$. The universal two part code $\tilde{D}$ relative to the set of codes $\{D_{\langle i,j\rangle} \ : \ i,j \in \mathcal{N}\}$ then achieves the following lengths (to within 1 bit): for all n, all $n_0 \in \{0, \ldots, n\}$, all $x_{[1:n]}$ with n_0 zeroes,

$$\tilde{L}(x_{[1:n]}) = \log n + \log n_0 + 2\log\log n + 2\log\log n_0 + \log \binom{n}{n_0} = \log \binom{n}{n_0} + O(\log n)$$

Using Stirling's approximation of the factorial, $n! \sim n^n e^{-n}\sqrt{2\pi n}$, we find that

$$\log \binom{n}{n_0} = \log n! - \log n_0! + \log(n - n_0)! =$$
$$n \log n - n_0 \log n_0 - (n - n_0)\log(n - n_0) + O(\log n) = nH(n_0/n) + O(\log n) \quad \text{(A.3)}$$

Note that $H(n_0/n) \leq 1$, with equality iff $n_0 = n$. Therefore, if the frequency deviates significantly from $\frac{1}{2}$, $\tilde{D}$ compresses $x_{[1:n]}$ by a factor linear in n. In all such cases, D^* compresses the data by at least the same linear factor. Note that (a) each individual code $D_{\langle n,n_0\rangle}$ is capable of exploiting a particular type of regularity in a sequence to compress that sequence, (b) the universal code $\tilde{D}$ may exploit *many* different types of regularities to compress a sequence, and (c) the code D^* with lengths given by the Kolmogorov complexity asymptotically exploits *all* computable regularities so as to maximally compress a sequence. $\diamondsuit$

Universal codes and Shannon: If a random variable X is distributed according to some known probability mass function $f(x) = P(X = x)$, then the optimal (in the average sense) code to use is the Shannon-Fano code. But now suppose it is only known that $f \in \{f\}$, where $\{f\}$ is some given (possibly very large, or even uncountable) set of candidate distributions. Now it is not clear what code is optimal. We may try the Shannon-Fano code for a particular $f \in \{f\}$, but such a code will typically lead to very large expected code lengths if X turns out to be distributed according to some $g \in \{f\}, g \neq f$. We may ask whether there exists another code that is 'almost' as good as the Shannon-Fano code for f, no matter what $f \in \{f\}$ actually generates the sequence? We now show that, provided $\{f\}$ is finite or countable, then (perhaps surprisingly), the answer is yes. To see this, we need the notion of an *sequential information source*, Section 1.2.

Suppose then that $\{f\}$ represents a finite or countable set of sequential information sources. Thus, $\{f\} = \{f_1, f_2, \ldots\}$ and $f_k \equiv (f_k^{(1)}, f_k^{(2)}, \ldots)$ represents a sequential information source, abbreviated to f_k. To each marginal distribution $f_k^{(n)}$, there corresponds a unique Shannon-Fano code defined on the set $\{0,1\}^n$ with lengths $L_{\langle n,k\rangle}(x) := \lceil \log 1/f_k^{(n)}(x) \rceil$ and decoding function $D_{\langle n,k\rangle}$.

For given $f \in \{f\}$, we define $H(f^{(n)}) := \sum_{x \in \{0,1\}^n} f^{(n)}(x)[\log 1/f^{(n)}(x)]$ as the entropy of the distribution of the first n outcomes.

Let E be a prefix-code assigning codeword $E(x)$ to source word $x \in \{0,1\}^n$. The Noiseless Coding Theorem 2.5 asserts that the minimal average codeword length $\bar{L}(f^{(n)}) = \sum_{x \in \{0,1\}^n} f^{(n)}(x) l(E(x))$ among all such prefix-codes E satisfies

$$H(f^{(n)}) \leq L(f^{(n)}) \leq H(f^{(n)}) + 1.$$

The entropy $H(f^{(n)})$ can therefore be interpreted as the expected code length of encoding the first n bits generated by the source f, when the optimal (Shannon-Fano) code is used.

We look for a prefix code $\tilde{D}$ with length function $\tilde{L}$ that satisfies, for all fixed $f \in \{f\}$:

$$\lim_{n \to \infty} \frac{\mathbf{E}_f \tilde{L}(X_{[1:n]})}{H(f^{(n)})} \leq 1. \tag{A.4}$$

where $\mathbf{E}_f \tilde{L}(X_{[1:n]}) = \sum_{x \in \{0,1\}^n} f^{(n)}(x) L(x)$. Define $\tilde{D}$ as the following two-part code: first, n is encoded using the standard prefix code for natural numbers. Then, among all codes $D_{\langle n,k \rangle}$, the k that minimizes $L_{\langle n,k \rangle}(x)$ is encoded (again using the standard prefix code); finally, x is encoded in $L_{\langle n,k \rangle}(x)$ bits. Then for all n, for all k, for *every* sequence $x_{[1:n]}$,

$$\tilde{L}(x_{[1:n]}) \leq L_{\langle n,k \rangle}(x_{[1:n]}) + L_{\mathcal{N}}(k) + L_{\mathcal{N}}(n) \tag{A.5}$$

Since (A.5) holds for all strings of length n, it must also hold in expectation for all possible distributions on strings of length n. In particular, this gives, for all $k \in \mathcal{N}$,

$$\mathbf{E}_{f_k} \tilde{L}(X_{[1:n]}) \leq \mathbf{E}_{f_k} L_{\langle n,k \rangle}(X_{[1:n]}) + O(\log n) = H(f_k^{(n)}) + O(\log n),$$

from which (A.4) follows.

Historically, codes satisfying (A.4) have been called *universal codes* relative to $\{f\}$; codes satisfying (A.2) have been considered in the literature only much more recently and are usually called 'universal codes for individual sequences' [19]. The two-part code $\tilde{D}$ that we just defined is universal both in an individual sequence and in an average sense: $\tilde{D}$ achieves code lengths within a constant of that achieved by $D_{\langle n,k \rangle}$ for *every individual sequence*, for *every* $k \in \mathcal{N}$; but $\tilde{D}$ also achieves expected code lengths within a constant of the Shannon-Fano code for f, for *every* $f \in \{f\}$. Note once again that the D^* based on Kolmogorov complexity does at least as well as $\tilde{D}$.

Example A.2 Suppose our sequence is generated by independent tosses of a coin with bias p of tossing "head" where $p \in (0,1)$. Identifying 'heads' with 1, the probability of $n - n_0$ outcomes "1" in an initial segment $x_{[1:n]}$ is then $(1-p)^{n_0} p^{n-n_0}$. Let $\{f\}$ be the set of corresponding information sources, containing one element for each $p \in (0,1)$. $\{f\}$ is an uncountable set; nevertheless, a universal code for $\{f\}$ exists. In fact, it can be shown that the code $\tilde{D}$ with lengths (A.3) in Example A.1 is universal for $\{f\}$, i.e. it satisfies (A.4). The reason for this is (roughly) as follows: if data are generated by a coin with bias p, then with probability 1, the frequency n_0/n converges to p, so that, by (A.3), $n^{-1}\tilde{L}(x_{[1:n]})$ tends to $n^{-1}H(f^{(n)}) = H(p, 1-p)$.

If we are interested in practical data-compression, then the assumption that the data are generated by a biased-coin source is very restricted. But there are much richer classes of distributions $\{f\}$ for which we can formulate universal codes. For example, we can take $\{f\}$ to be the class of all Markov sources of each order; here the probability that $X_i = 1$ may depend on arbitrarily many earlier outcomes. Such ideas form the basis of most data compression schemes used in practice. Codes which are universal for the class of all Markov sources of each order and which encode and decode in real-time can easily be implemented. Thus, while we cannot find the shortest program that generates a particular sequence, it is often possible to effectively find the shortest encoding within a quite sophisticated class of codes. ♢

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