Solutions to Odd-numbered Exercises

of *A First Look at Rigorous Probability Theory* (Rosenthal, 2006)

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Chapter 2: Probability Triples

Exercise (2.2.3)

Prove that the collection $\mathcal{J} = \{$all intervals contained in $[0,1]\}$ is a semialgebra *of subsets of Ω, meaning that it contains $\emptyset$ and Ω, it is closed under finite intersection, and the complement of any element of $\mathcal{J}$ is equal to a finite disjoint union of elements of $\mathcal{J}$.*

Solution

Show it obeys the requirements

- by definition $\mathcal{J}$ contains $\emptyset$ and Ω
- the intersection of any finite number of intervals is an interval
- complement of any interval can be made by taking the union of disjoint intervals (only at most two are needed: before and after).

Exercise (2.2.5)

(a) *Prove that $\mathcal{B}_0 = \{$all finite unions of elements of $\mathcal{J}\}$ is an* algebra *(or* field*) of subsets of Ω, meaning that it contains Ω and $\emptyset$, and is closed under formation of complements and of* finite *unions and intersections.*

(b) *Prove thet $\mathcal{B}_0$ is <u>not</u> a σ-algebra.*

Solution

(a)
- by definition $\mathcal{B}_0$ it contains $\emptyset$ and Ω
- it is closed under formation of complements because the complement of any element of $\mathcal{J}$ is a finite disjoint union of elements of $\mathcal{J}$, which is in $\mathcal{B}_0$ by definition
- likewise it is closed under finite unions, since any finite union of elements of $\mathcal{B}_0$ is a finite union of elements of $\mathcal{J}$.
- for any finite set $\{A_i\}_i$ of elements of $\mathcal{B}_0$, the intersection $\bigcap A_i = (\bigcup A_i^c)^c \in \mathcal{B}_0$ by closure under complements and finite unions.

(b) Assume $\mathcal{B}$ is a σ-algebra. Then it is closed under formation of countable unions. Take the set $N = \bigcup_{n\in\mathbb{N}}(\frac{1}{2^{n+1}}, \frac{1}{2^n})$, which is a countable union of intervals in $[0,1]$, so it is in $\mathcal{B}$. However, the set N consists of an infinite number of disjoint intervals, so it cannot be constructed from finite union of intervals, thus is not in $\mathcal{B}_0$. Contradiction.

References

Rosenthal, Jeffrey S (Nov. 2006). *A First Look at Rigorous Probability Theory*. 2nd ed. World Scientific.