0.1 Homework 1

Suppose that there is a grassy island supporting populations of two species x and y. If the populations are large then it is reasonable to let the normalized populations be continuous functions of time (so that $x = 1$ might represent a population of e.g., 100 000 species). We propose the following simple model of the change in population:

$$\begin{aligned}\dot{x}(t) &= x(t)(a + bx(t) + cy(t)) \\ \dot{y}(t) &= y(t)(d + ex(t) + fy(t)),\end{aligned} \tag{1}$$

where x and y are non-negative functions and $a, b, \ldots, f$ are constants. Let us start with discussing the model heuristically to gain some insight. Consider the first equation when $y = 0$: $\dot{x} = x(a + bx)$. If the species x preys on y then the coefficient a must be non-positive ($a \leq 0$), since if $y = 0$ there is no available food so the population should die of starvation. If x instead eats grass then $a > 0$, so the population grows if the initial population is small. The coefficient b enables a non-zero equilibrium for the population of species x in the absence of the second species. We have $b < 0$ since the island is finite so large populations suffer from overcrowding. The coefficient c describes the effect of y on x. If $c > 0$ then the population growth increases (for example, if x feeds upon y), while if $c < 0$ then the population growth decreases (for example, if x and y compete for the same resource). Similar interpretations hold for d, e, and f.

1. [1p] Depending on the sign of c and e there are four different population models. Continue the discussion above and label these four models as

- predator-prey (x predator, y prey),
- prey-predator (x prey, y predator),
- competitive (x and y inhibit each other),
- symbiotic (x and y benefit each other).

2. [1p] Consider the population model (1) with $a = 3$, $b = f = -1$, and $d = 2$. Draw the phase portrait[1] for the following four cases

- $(c, e) = (-2, -1)$,
- $(c, e) = (-2, 1)$,
- $(c, e) = (2, -1)$,
- $(c, e) = (2, 1)$.

Determine (analytically) the type of each equilibrium in each case. Interpret your results and comment on their implications when the model is applied in a real investigation of the dynamics between two species.

3. [1p] Repeat the previous exercise for $a = e = 1$, $b = f = 0$, and $c = d = -1$.

4. [1p] Show that the x- and y-axes are invariant for all values of the parameters in (1).[2] Can you indentify another invariant set (apart from the x and y axis)? Why is this a necessary feature of a population model? Assume $a = e = 1$, $b = f = 0$, and $c = d = -1$ in (1) and show that a periodic orbit exists. (Hint: find a closed and bounded trajectory that is an invariant set and contains no equilibrium point. For that purpose, consider $\frac{dy}{dx}$ and attempt to solve the differential equation you find, which should yield a trajectory defined in an implicit form. [3])

5. [1p] Generalize the population model (1) to $N > 2$ species and explain the meaning of whatever coefficients you introduce.

[1] It might be a good idea to first make a sketch by hand and then use pplane8 in Matlab.
[2] A set $W \subset \mathbb{R}^n$ is invariant if $z(0) \in W$ implies $z(t) \in W$ for all $t \geq 0$.
[3] The Poincaré-Bendixson criterion then ensures that the trajectory is a periodic orbit.