Solution to Dummy Homework

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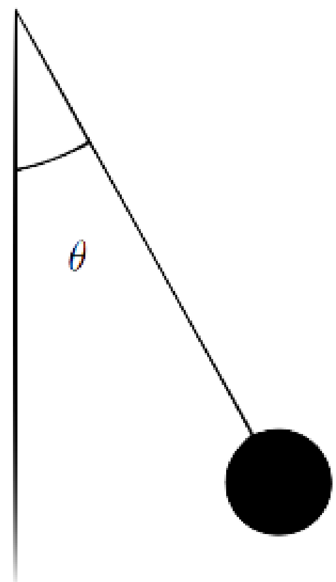
Description: Solutions to Exercise 1.1 of Nonlinear Control EL2620. This is a dummy homework, just to illustrate the expected reporting style

Useful material for creating live scripts in matlab

Read info at [Create Live Scripts]

Description of the problem



The nonlinear dynamic equation for a pendulum is given by

$$ml\ddot{\theta} = -mg\sin(\theta) - kl\dot{\theta}$$

where l is the length of the pendulum, m is the mass of the bob, and θ is the angle subtended by the rod and the vertical axis through the pivot point.

No Units?

Constants

```
% mass of the bob
m = 1;
% length of pendulum
l = 1;
% acceleration due to gravity
g = 10;
% friction coefficient (we will play with different values)
```

```
% k = 1 or k = 2
```

Exercise 1.1 a)

Choose appropriate state variables and write down the state equations.

We can choose θ (standing for the angular position) and ω (standing for the angular velocity) as state variables, with the state equations given by

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = f(\theta, \omega) := \begin{pmatrix} \omega \\ -\frac{g}{l}\sin(\theta) - kl\omega \end{pmatrix}.$$

We can give different names: x_1 (standing for the angular position) and x_2 (standing for the angular velocity) as state variables, with the state equations given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = f(x_1, x_2) := \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - klx_2 \end{pmatrix}.$$

Another option:

We can choose ξ_1 (standing for the angular position + angular velocity) and ξ_2 (standing for the angular position - angular velocity) as state variables, with the state equations given by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = g(\xi_1, \xi_2) := \begin{pmatrix} \frac{1}{2}(\xi_1 - \xi_2) - \frac{g}{l}\sin\left(\frac{1}{2}(\xi_1 + \xi_2)\right) - kl\frac{1}{2}(\xi_1 - \xi_2) \\ \frac{1}{2}(\xi_1 - \xi_2) + \frac{g}{l}\sin\left(\frac{1}{2}(\xi_1 + \xi_2)\right) + kl\frac{1}{2}(\xi_1 - \xi_2) \end{pmatrix}.$$

Maybe you should provide more details in this derivation

A good choice of state yields "simpler" state equations.

In defining the vector field in matlab, we introduce the dummy names:

. z_1 stands for the angular position (or θ)

. z_2 stands for the angular velocity (or $\dot{\theta}$, or ω)

```
f = @(z1,z2,k) [z2; - g/l*sin(z1) - k*l*z2];
```

Just for the purpose of display of the vector field (aka, state equations) we introduce symbolic variables

```
syms x1 x2 k
```

```
f(x1,x2,k)
```

ans =

$$\begin{pmatrix} x_2 \\ -10\sin(x_1) - k\,x_2 \end{pmatrix}$$

Exercise 1.1 b)

Find all equilibria of the system

Definition: Given state equations f, x^* is equilibrium if $f(x^*) = 0$.

Since $f(x_1, x_2) = \left(x_2, -\frac{g}{l}\sin(x_1) - klx_2\right)$, then $f(x_1^*, x_2^*) = 0$ implies that

1) $x_2^* = 0$ (from first component of vector field)

2) $-\frac{g}{l}\sin(x_1^*) - klx_2^* = 0 \Rightarrow \sin(x_1^*) = 0 \Leftrightarrow x_1^* = p\pi \forall p \in Z$ (recall 1))

We thus have infinite many equilibria.

Exercise 1.1 c)

Description: Linearize the system around the equilibrium points, and determine whether the system equilibria are stable or not (consider $k = 1$ in the case numerical calculations are required)

It suffices to check for equilibria $(x_1^*, x_2^*) \in \{(0,0), (\pi, 0)\}$ (explain why!).

You would need to explain why

Exercise 1.1 c) Alternatice 1 (preferred solution)

Do everything analytically

$$A(x) = \frac{df(x)}{dx} = \begin{pmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -kl \end{pmatrix}$$

You must detail your derivation steps

Thus, for $x^* = (x_1^*, x_2^*) = (0,0)$

$$A(x^*) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -kl \end{pmatrix} \Rightarrow \cdots \Rightarrow eigenavalues(A(x^*)) \in \left\{-\frac{kl}{2} + \sqrt{\left(\frac{kl}{2}\right)^2 - \frac{g}{l}}, -\frac{kl}{2} - \sqrt{\left(\frac{kl}{2}\right)^2 - \frac{g}{l}}\right\}$$

which means that $x^* = (x_1^*, x_2^*) = (0, 0)$ is stable, since the eigenvalues are on LHP (you would need to argue a bit more on why the eigenvalues are on the LHP).

On the other hand, for $x^* = (x_1^*, x_2^*) = (\pi, 0)$

$$A(x^*) = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -kl \end{pmatrix} \Rightarrow \cdots \Rightarrow eigenavalues(A(x^*)) \in \left\{ -\frac{kl}{2} + \sqrt{\left(\frac{kl}{2}\right)^2 + \frac{g}{l}}, -\frac{kl}{2} - \sqrt{\left(\frac{kl}{2}\right)^2 + \frac{g}{l}} \right\}$$

which means that $x^* = (x_1^*, x_2^*) = (\pi, 0)$ is ubstable, since one of the eigenvalues is on the RHP (you would need to argue a bit more on why one of the eigenvalues is on the RHP).

Exercise 1.1 c) Alternatice 2

Do symbolic calculations with the help of matlab

```
% create symbolic variables x1, x2 and k (you could let g and l be symbolic variables as well)
syms x1 x2 k
A = jacobian((f(x1,x2,k)), [x1, x2])
```

A =

$$\begin{pmatrix} 0 & 1 \\ -10\cos(x_1) & -k \end{pmatrix}$$

$(x_1^*, x_2^*) = (0, 0)$ is stable because eigenvalues are on LHP

```
% one may check the eigenvales for arbitrary k
A1 = subs(A, {x1, x2}, {0, 0});
eig(A1)
```

ans =

$$\begin{pmatrix} -\frac{k}{2} - \frac{\sqrt{k^2 - 40}}{2} \\ \frac{\sqrt{k^2 - 40}}{2} - \frac{k}{2} \end{pmatrix}$$

```
% or one may check the eigenvales for k = 1, for example
A1 = subs(A, {x1, x2, k}, {0, 0 1});
eig(A1)
```

ans =

$$\begin{pmatrix} -\frac{1}{2}-\frac{\sqrt{39}\,\mathrm{i}}{2} \\ -\frac{1}{2}+\frac{\sqrt{39}\,\mathrm{i}}{2} \end{pmatrix}$$

```
% print eigenvalues in ''easier'' to read format
double(eig(A1))
```

```
ans =
  -0.5000 - 3.1225i
  -0.5000 + 3.1225i
```

```

```

$(x_1, x_2) = (\pi, 0)$ is unstable because eigenvalues are not on LHP:

```
% one may check the eigenvales for arbitrary k
A2 = subs(A, {x1, x2}, {pi, 0});
eig(A2)
```

ans =

$$\begin{pmatrix} -\frac{k}{2}-\frac{\sqrt{k^2+40}}{2} \\ \frac{\sqrt{k^2+40}}{2}-\frac{k}{2} \end{pmatrix}$$

```
% or one may check the eigenvales for k = 1, for example
A2 = subs(A, {x1, x2, k}, {pi, 0, 1});
eig(A2)
```

ans =

$$\begin{pmatrix} -\frac{\sqrt{41}}{2}-\frac{1}{2} \\ \frac{\sqrt{41}}{2}-\frac{1}{2} \end{pmatrix}$$

```
% print eigenvalues in ''easier'' to read format
double(eig(A2))
```

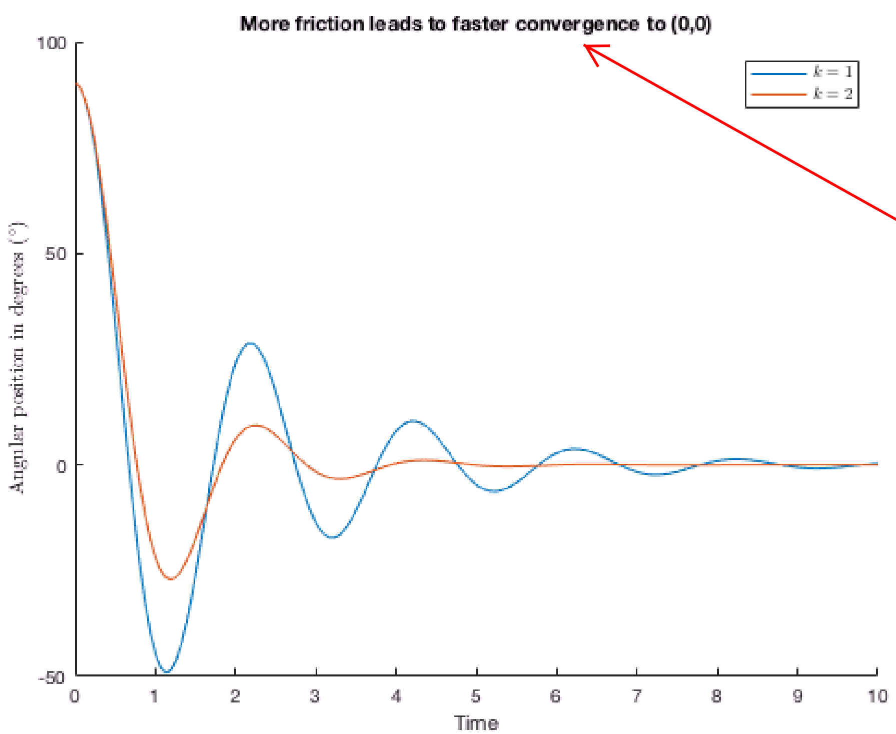
```
ans =

   -3.7016
    2.7016
```

Exercise 1.1 d) (this is not in the Exercise: it is just to further illustrate how to report)

Simulate system with $k = 1$ and $k = 2$ and with initial condition $x(0) = (x_1(0), x_2(0)) = (0.5\pi, 0.1)$. Discuss the results.

Exercise 1.1 d) ALTERNATIVE 1

```
% initial time instant
t0 = 0;
% final time instant
tF = 10;
% initial condition
x0 = [0.5*pi;0.1];

hold on
for friction = [1,2]
    % check documentation for ode45
    % remark: in general a vector field is a function of time and state, aka (t,x)
    [time_vector,state_vector] = ode45(@(t,x) f(x(1),x(2), friction),[t0 tF],x0);
    % angular position from radians to degrees
    plot(time_vector,state_vector(:,1)*180/pi)
end

% label axis
xlabel('Time')
ylabel('Angular position in degrees ($^\circ$)','interpreter','latex')
% legend for the plot
h = legend('$k=1$','$k=2$');
set(h,'interpreter','latex')
% title with the DISCUSSION
title('More friction leads to faster convergence to (0,0)')
```



This discussion of the results is not enough

No units

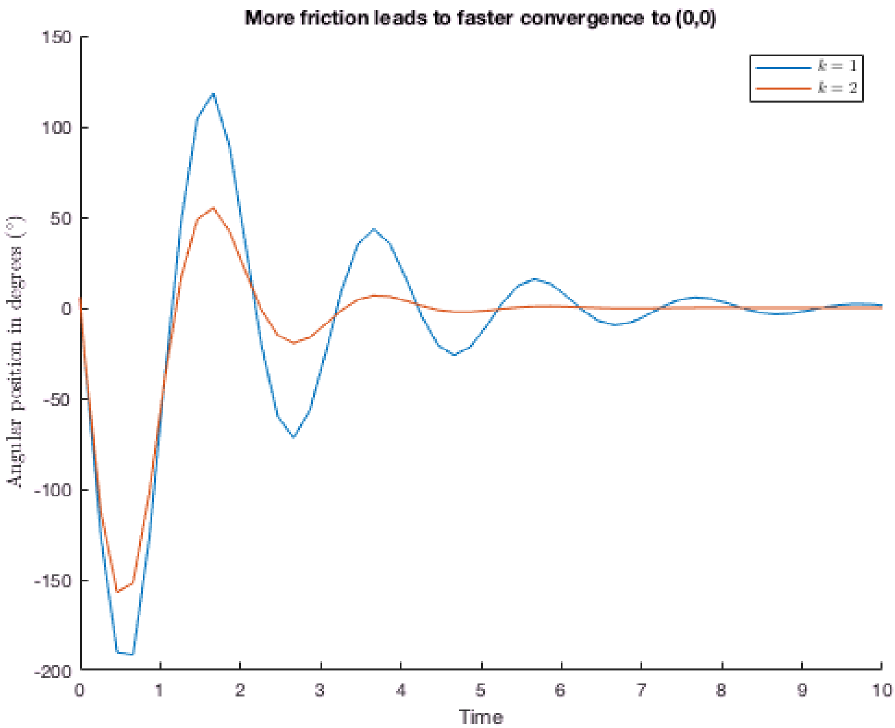
Exercise 1.1 d) ALTERNATIVE 2

Using simulink

```
% close previous figure
close all

for friction = [1,2]

    k = friction;
    % simulate model described in PendulumSimulation.slx
    % in this model, k is a parameter, which as been set to the same value
    % as friction
    %
    % if you wish, change step size, so that plot is not so cluncky
    sim('PendulumSimulation.slx');

    % vector with time instants (sec)
    time_vector = State.Time;
    % vector with state
    state_vector = State.Data;

    hold on
    plot(time_vector,state_vector(:,1)*180/pi)
end
```

```
Warning: Model 'PendulumSimulation' is using a default value of 0.2 for maximum step size.
You can disable this diagnostic by setting Automatic solver parameter selection to 'none'
Warning: Model 'PendulumSimulation' is using a default value of 0.2 for maximum step size.
You can disable this diagnostic by setting Automatic solver parameter selection to 'none'
```

```

% label axis
xlabel('Time')
ylabel('Angular position in degrees ($^\circ$)','interpreter','latex')
% legend for the plot
h = legend('$k=1$','$k=2$');
set(h,'interpreter','latex')
% title with the DISCUSSION
title('More friction leads to faster convergence to (0,0)')
```



WARNINGS: Always check documentation of functions

Example: select ode45 with cursor and press F1

WARNINGS: Latex

For quick typesetting, write "$", then write in latex, and when you close with "$", the latex expression is rendered.

WARNINGS: Other problems

Google is your friend