Homework 0

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Description: Solutions to Exercise 1.1 of Nonlinear Control EL2620

Markdown cheat sheet

How to write in markdown: Markdown cheat sheet (https://github.com/adam-p/markdown-here/wiki/Markdown-Cheatsheet)

Description

The nonlinear dynamic equation for a pendulum is given by

$$ml\ddot{\theta} = -mg\sin(\theta) - kl\dot{\theta}$$

where l is the length of the pendulum, m is the mass of the bob, and θ is the angle subtended by the rod and the vertical axis through the pivot point

Pendulum Constants

In [1]:

```
# mass
m = 1
# length of pendulum
l = 1
# acceleration due to gravity
g = 10
# friction coefficient (we will play with different values)
# k = 1 or k = 2
```

Out[1]:

```
10
```

Exercise 1.1 a)

Choose appropriate state variables and write down the state equations.

z_1 stands for the angular position (or θ), and z_2 stands for the angular velocity (or $\dot{\theta}$, or ω)

In [2]:

```
function f(x1,x2,k)
    # pay attention to the scope of variables
    # note that g and l are defined, but k is not
    (x2,-g/l*sin(x1) - k*l*x2)
end
```

Out[2]:

```
f (generic function with 1 method)
```

Exercise 1.1 b)

Find all equilibria of the system

Definition: $x^\star$ is equilibrium of vector field f (aka, state equations) if $f(x^\star) = 0$.

Let $x^\star = (x_1^\star, x_2^\star)$:

$$f(x^\star) = (0,0) \Leftrightarrow \quad \Leftrightarrow (x_2^\star, -\frac{g}{l}\sin(x_1^\star) - klx_2^\star) = (0,0) \ \Leftrightarrow (x_2^\star, \sin(x_1^\star)) = (0,0) \ \Leftrightarrow (x_2^\star, x_1^\star) = (0, p\pi)$$

Indeed, notice that (next, I am taking $k = 2$, just for illustration purposes)

In [3]:

```
f(0*pi,0,2)
```

Out[3]:

```
(0,-0.0)
```

In [4]:

```
f(1*pi,0,2)
```

Out[4]:

```
(0,-1.2246467991473533e-15)
```

Exercise 1.1 c)

You must explain in detail your reasonings.

Linearize the system around the equilibrium points, and determine whether the system equilibria are stable or not (you may consider, if you wish, $k = 1$).

It suffices to check for equilibria $(x_1^\star, x_2^\star) = (0, 0)$ and $(x_1^\star, x_2^\star) = (\pi, 0)$ (explain why!)

$$
\begin{aligned}
A(x) &= \frac{df(y)}{dy}\Big|_{y=x} \\
&= \begin{bmatrix} \frac{\partial f_1(y_1,y_2)}{\partial y_1} & \frac{\partial f_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial f_2(y_1,y_2)}{\partial y_1} & \frac{\partial f_2(y_1,y_2)}{\partial y_2} \end{bmatrix}\Big|_{(y_1,y_2)=(x_1,x_2)} && f := (f_1, f_2), y := (y_1, y_2), x := (x_1, x_2) \\
&= \begin{bmatrix} \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_2} \\ \frac{\partial\left(-\frac{g}{l}\sin(y_1)-kly_2\right)}{\partial y_1} & \frac{\partial\left(-\frac{g}{l}\sin(y_1)-kly_2\right)}{\partial y_2} \end{bmatrix}\Big|_{(y_1,y_2)=(x_1,x_2)} && f(y_1, y_2) := \left(y_2, -\frac{g}{l}\sin(y_1) - kly_2\right) \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(y_1) & -kl \end{bmatrix}\Big|_{(y_1,y_2)=(x_1,x_2)} && \text{simple calculations} \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -kl \end{bmatrix} && \text{replacement}
\end{aligned}
$$

For the equilibria $x^\star = (x_1^\star, x_2^\star) = (0, 0)$, it follows that

$$
\begin{aligned}
A(x^\star) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -kl \end{bmatrix} &\Rightarrow \det(\lambda I - A(x^\star)) = 0 \\
&\Leftrightarrow \det \begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + kl \end{bmatrix} = 0 \\
&\Leftrightarrow \lambda^2 + kl\lambda + \frac{g}{l} = 0 \\
&\Leftrightarrow \lambda = -\frac{kl}{2} \pm \sqrt{\left(\frac{kl}{2}\right)^2 - \frac{g}{l}}
\end{aligned}
$$

1. If $\frac{g}{l} > \left(\frac{kl}{2}\right)^2$, the eigenvalues are complex conjugate and the real part is $-\frac{kl}{2}$: since they are on the LHP, equilibrium is stable.
2. If $0 < \frac{g}{l} < \left(\frac{kl}{2}\right)^2$, the eigenvalues are real and the largest real part is $-\frac{kl}{2} + \sqrt{\left(\frac{kl}{2}\right)^2 - \frac{g}{l}} < 0$: since they are on the LHP, equilibrium is stable.

For the equilibria $x^\star = (x_1^\star, x_2^\star) = (\pi, 0)$, it follows that

$$
\begin{aligned}
A(x^\star) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -kl \end{bmatrix} &\Rightarrow \det(\lambda I - A(x^\star)) = 0 \\
&\Leftrightarrow \det \begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + kl \end{bmatrix} = 0 \\
&\Leftrightarrow \lambda^2 + kl\lambda - \frac{g}{l} = 0 \\
&\Leftrightarrow \lambda = -\frac{kl}{2} \pm \sqrt{\left(\frac{kl}{2}\right)^2 + \frac{g}{l}}
\end{aligned}
$$

Since $\frac{g}{l} > 0$ it follows that $-\frac{kl}{2} + \sqrt{\left(\frac{kl}{2}\right)^2 + \frac{g}{l}} > 0$: since at least one eigenvalue is on the RHP, the equilibrium is unstable.

Exercise 1.1 d)

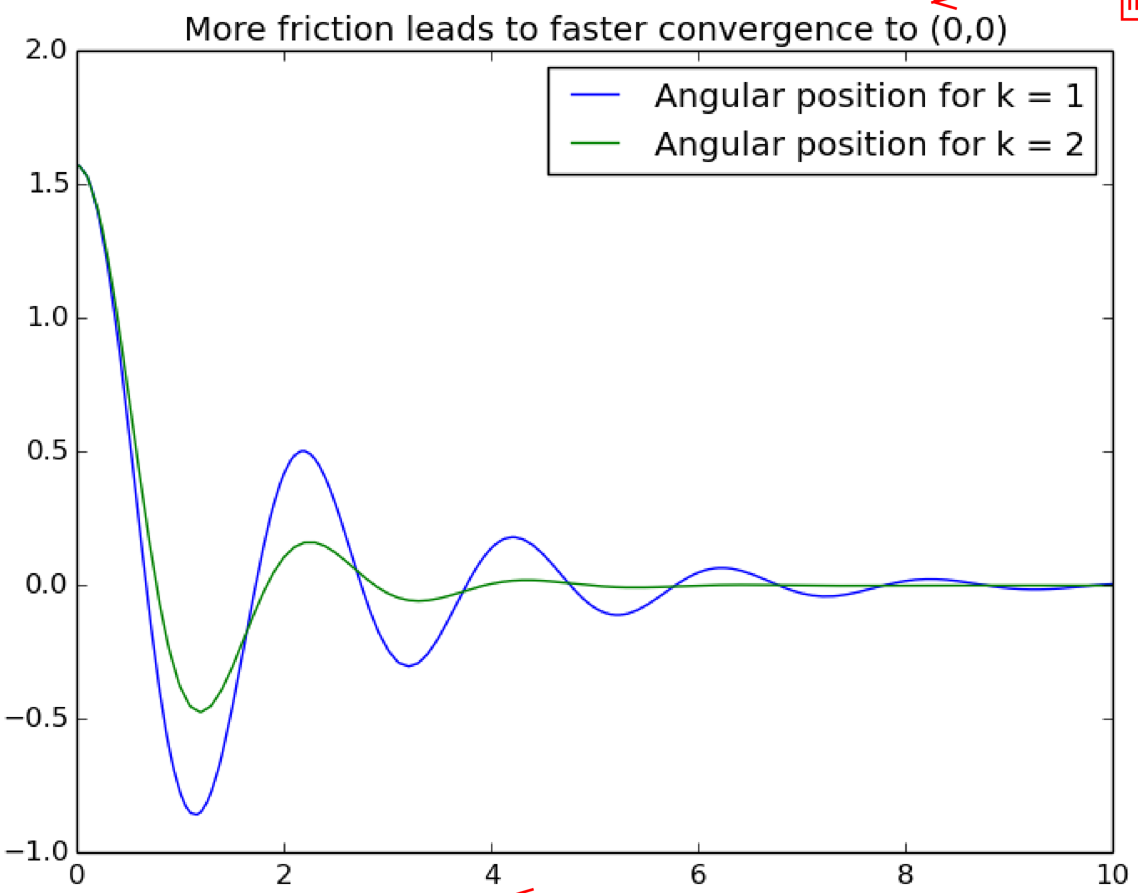
(this is not in the Exercise: it is just to further illustrate how to report)

Simulate system with $k = 1$ and $k = 2$ and with initial condition $x(0) = (x_1(0), x_2(0)) = (0.5\pi, 0.1)$. Discuss the results

In [5]:

```
using ODE;
using PyPlot

for friction = [1;2]

    function vector_field(t, y)
        # Return the derivatives as a vector
        derivative = f(y[1],y[2],friction)

        # Return the derivatives as a vector
        return [derivative[1]; derivative[2]]
    end;

    # Initial conditions
    x0 = [0.5*pi; 0.1];

    # Time vector going from 0 to 10 in 0.1 increments
    time = 0:0.1:10.0;

    t, x = ODE.ode45(vector_field, x0, time)

    x_1 = map(x -> x[1], x)
    x_2 = map(x -> x[2], x);

    PyPlot.plot(t, x_1,label= "Angular position for k = $(friction)")

end

legend();
PyPlot.title("More friction leads to faster convergence to (0,0)");
```

This discussion is insufficient.



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