Aerial slung load position tracking under unknown wind forces

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Abstract

We propose a dynamic controller for position tracking of a point-mass load, attached to an aerial vehicle by means of a cable. Both the load and the aerial vehicle are subject to unknown winds, which disturb their linear accelerations, and the effect of which is removed by the controller by means of estimators. After modeling the slung-load system, the latter is put into a canonical form, i.e., a form which is independent of the system's physical parameters (bodies' weights and the cable length), and a dynamic control law is developed for this canonical system, which, by means of an appropriate input transformation, can be used to construct the control law to be implemented on the real slung-load system. The wind force on the load is not input-additive, which means its estimation and the removal of its effect cannot be accomplished with standard disturbance removal techniques. Loosely speaking, the difference between the wind forces (on the load and the aerial vehicle) is an input-additive disturbance, and its removal, when compared to the latter, is more straightforward. The controller is designed by following a backstepping procedure, and four estimators are designed, since each of the two wind disturbances has two separate effects (an effect on the linear acceleration, and another on the angular acceleration, of the canonical form). Finally, we impose conditions on the desired position trajectory, and on the wind on the load, which guarantee that a well-defined equilibrium trajectory exists, and the designed controller guarantees that the cable, physically coupling the load and the aerial vehicle, remains taut provided the system is initialized in a (conservative) set, which we identify in the paper.

I. Introduction

Vertical take off and landing (VTOL) rotorcrafts, with hover capabilities, hereafter UAVs, are underactuated vehicles whose popularity stems from their ability to be used in small spaces, their reduced mechanical complexity, and inexpensive components. Slung-load transportation consists of a UAV physically coupled to a point-mass load, whose position we wish to track a given desired position trajectory. Slung-load transportation has been considered in the literature, but not while formally taking into account the presence of unknown wind forces acting on the load and on the aerial vehicle, which is an inevitable reality if such a transportation is to be accomplished outdoors.

Different solutions to slung-load transportation are found in the literature [1]–[15]. Regarding modeling, different approaches have been pursued, such as following an Euler-Lagrange formulation, an Hamiltonian formulation or Kane's method [1]–[5], and, the slung-load system, as a mechanical system, is known to be underactuated. Most works rely on local parametrizations of the configuration space, while others provide a coordinate-free modeling as well as a coordinate-free control law [6]–[9]. Some works have focused on the simpler problem of position stabilization [3], [4], [7], [8], while others have focused on a simplified two-dimensional setting [4], [16]. Vision has been used to estimate the load position with respect to the UAV [1], [10], [15], and a force sensor on the rope has been used to compensate and/or estimate the tension on the cable [1], [14]. Dynamic controllers, considering model uncertainties and/or input disturbances, are found in [2], [8], [9], with some relying on a discrete time model, and a flexible cable has also been considered [7], [8]. Trajectory planning has been considered by exploring (hybrid) differential flatness [7], [16], or by minimizing the allowed swing motion [2], [3]. Finally, slung-load transportation with multiple UAVs has also been considered [1], [11]–[13], which is however not the focus of this paper. In the coordinate-free cited works, the desired attitude for the cable is computed, which involves the normalization of a designed three dimensional vector, which is only valid if the latter vector is non-zero: in this paper, we guarantee this normalization is valid for any state in the state-space, which includes the internal states of the dynamic controller (the solution involves using a bounded linear acceleration control law, and a bounded disturbance estimator). Also, all the latter works assume, without guarantees, that the cable remains taut – this constraint must be satisfied, otherwise the load behaves as free-falling unactuated point-mass: in this paper, the designed control law guarantees that that physical constraint is satisfied, provided that the state is initialized in a set we identify in the paper.

Position tracking for the slung-load system shares similarities with position tracking for a standard UAV, and a review on different control strategies for the latter is found in [17]. In this paper, we use differential flatness [18] to compute the desired state and input trajectory and to describe feasible position trajectories. In designing a controller, we then follow a backstepping procedure, similar to that found in [19]–[23], but we do feedback linearize the system by dynamic augmentation of the thrust (in our case, tension), as done in [21]–[23]. In position tracking, it is known that an *a priori* bounded linear acceleration control law is necessary [9], [20], [21], a problem we also tackle in this paper; we also improve on these works by providing a smooth

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projector operator (instead of a sufficiently smooth operator). Moreover, when controlling a UAV, one must guarantee that the thrust remains positive (either because the UAV rotors can only spin in one direction; or because dynamic augmentation of the thrust so requires), and [23] provides a region of initial states for which such a constraint is satisfied. In a similar fashion, in a slung load system, one must guarantee that the cable remains taut, which is the case if the tension on the cable remains positive, and, in this paper, we provide a set of initial states for which this physical constraint is satisfied for the resulting closed-loop trajectory.

II. Contributions

Let us summarize here our problem solving strategy and the main contributions of this paper. In Section IV, we present the model of an aerial slung-load system, in the presence of winds acting on the UAV and on the load. In Section V, we show that the system is differentially flat with respect to the load's position, and, given some desired load's position trajectory, we compute the (two) desired system's trajectories, and the (two) desired input trajectories (one solution is physically feasible – cable is under tension – while the other is unfeasible – cable is under compression). In the same section, we introduce the notion of feasible trajectories, as trajectories where the load is not buoyant in the air at any time instant (Brocket's necessary condition is not satisfied when the load is buoyant in the air). In Section VI, we provide a coordinate transformation which transforms the system's vector field into a canonical form: this canonical form is agnostic to the system's physical parameters, and it highlights the cascaded structure of the system, which is explored during the controller design. We find out that there are two types of disturbances, where one is input additive (loosely speaking, this disturbance is associated with the difference between the winds on the UAV and on the load), while the other is not (this disturbance is associated with the wind on the load): the latter is the one that motivated us to present a smooth projector that guarantees that an estimator remains in a pre-specified domain, and whose derivatives, of any order, can be computed. Given some feasible trajectory, we perform a backstepping procedure, with six steps in Subsections VIII-A–VIII-F, by exploring the cascaded structure of the transformed problem. In some steps, the disturbances are assumed known, and these steps are immediately followed by steps where estimators for those disturbances are designed. In Subsection VIII-A, we provide a bounded analytic control law for a double integrator, with a companion analytic Lyapunov function whose "derivative" is negative definite; In Subsections VIII-C and VIII-D, we construct Lyapunov functions whose gradient is bounded, and which allow us to design estimators whose "derivative" does not grow unbounded in an unbounded domain (if the load starts *far away* from its desired position, the estimators do not immediately saturate). Also, at each step, we present the equilibria sets (which always come in pairs) and we study their stability and attractivity properties rigorously – Theorems 40, 50, 58, 67 and 73. It the end, the proposed controller is a dynamic one, and, ultimately, given some feasible trajectory, we verify that both the system's trajectory and the system's input converge to their desired trajectories, even if the estimators do not converge to the corresponding unknown disturbances. However, we provide an excitation criterion, which, if satisfied, guarantees that one of the estimators (the one associated to the non-input-additive disturbance) converges to its corresponding unknown disturbance. In our final theorem, Theorem 74, we show that the proposed controller also guarantees that the cable remains taut along a closed-loop trajectory, given that the system's initial condition lies in a set we identify therein. Finally, we note that the tracking problem we tackle cannot be put in an autonomous form (loosely speaking, the error differential equations depend explicitly on time, and not just on the errors themselves) and thus LaSalle's invariance principle cannot be invoked, in the aforementioned theorems, to conclude that tracking is accomplished asymptotically: we invoke Barbalat's lemma instead.

III. Notation

The map $\mathcal{S}:\mathbb{R}^3\ni x\mapsto\mathcal{S}(x)\in\mathbb{R}^{3\times 3}$ yields a skew-symmetric matrix and it satisfies $\mathcal{S}(a)\,b=a\times b$, for any $a,b\in\mathbb{R}^3$. $\mathbb{S}^2:=\{x\in\mathbb{R}^3:\|x\|=1\}$ denotes the set of unit vectors in $\mathbb{R}^3$. The map $\Pi:\mathbb{S}^2\to\mathbb{R}^{3\times3}$, defined as $\Pi(x)y:=y-\langle y,x\rangle x$ for any $y\in\mathbb{R}^3$ and $x\in\mathbb{S}^2$, yields a matrix that represents the projection of y onto the subspace orthogonal to x. For $n\in\mathbb{N}$, we denote by $e_1^n,\cdots,e_n^n\in\mathbb{R}^n$ the canonical basis vectors in $\mathbb{R}^n$; when clear from the context, we denote $e_i\equiv e_i^n$, i.e., we omit the super-script indicating the dimension of the Euclidean space, which the basis vector belongs to. Let n be a positive integer, r be a positive number, and denote

$$\mathbb{B}_r^n:=\{x\in\mathbb{R}^n:\|x\|<r\},\ \text{and} \tag{1a}$$
$$\mathbb{C}_r^n:=\{x\in\mathbb{R}^n:\|x\|>r\}=\mathbb{R}^n\backslash\bar{\mathbb{B}}_r^n, \tag{1b}$$

as the open-ball in $\mathbb{R}^n$ of radius r, and as the complement of a closed-ball in $\mathbb{R}^n$ of radius r, respectively. Given some $n,m\in\mathbb{N}$, and a function $f:\mathbb{R}^n\ni x\mapsto f(x)\in\mathbb{R}^m$, $df:\mathbb{R}^n\ni x\mapsto df(x)\in\mathbb{R}^{m\times n}$ denotes the derivative of f; when $m=1$, $\nabla f:\mathbb{R}^n\ni x\mapsto\nabla f(x)\in\mathbb{R}^n$ denotes the gradient of f, i.e., $\langle\nabla f(x),\dot{x}\rangle=df(x)\dot{x}$ for any $\dot{x}\in\mathbb{R}^n$. $d_if(x_1,\cdots,x_i,\cdots,x_n)$ denotes the derivative of f with respect to its ith entry. Let M and N be manifolds, and consider the differentiable map $f:M\to N$; for any point $m\in M$, $df(m):T_mM\to T_{f(m)}N$ denotes the derivative (which is a linear map) of the map f at the point m. Given some manifold M, some continuous map $f:M\to\mathbb{R}$, and some $r\in\mathbb{R}$,

$$f_{\leq r}:=\{m\in M:f(m)\leq r\}\,,\ \text{and} \tag{2a}$$
$$f_{<r}:=\{m\in M:f(m)<r\} \tag{2b}$$

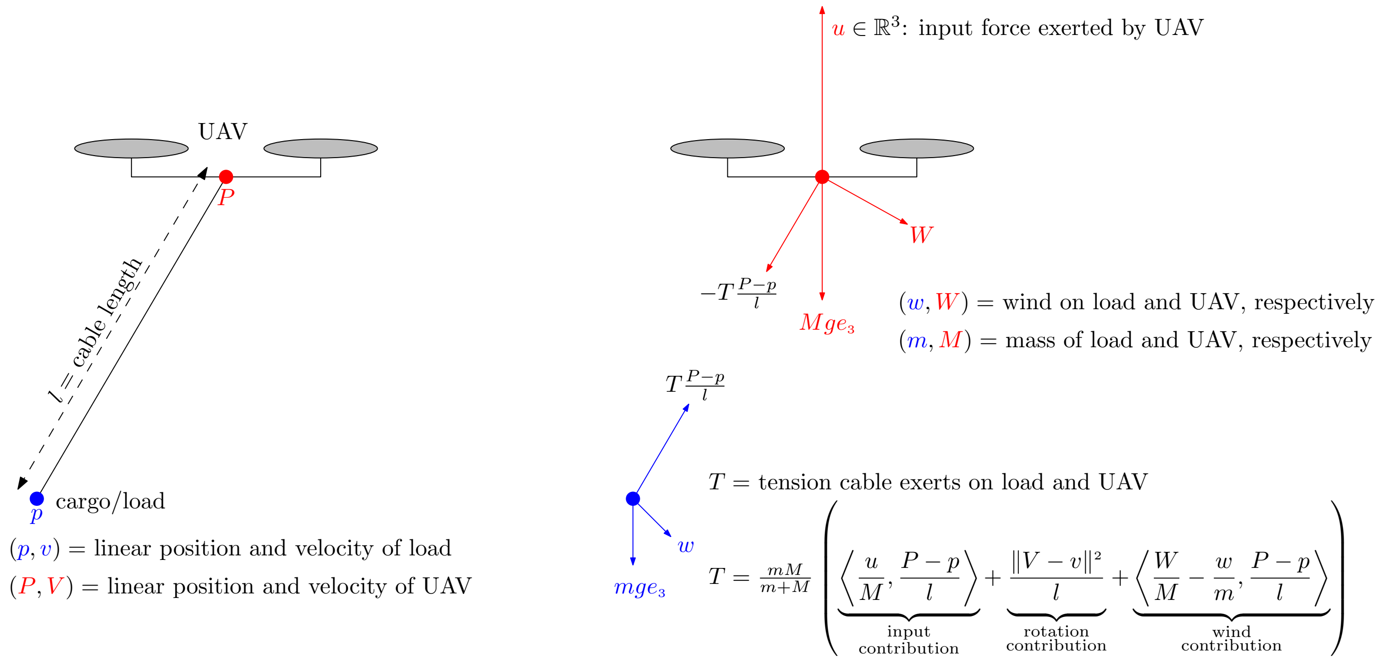


Fig. 1: Modeling of the aerial slung-load system subject to wind forces. Left: system of two point-masses physically coupled by inter-medium of cable. Right: distribution of forces on each point-mass (system composed of two point-masses).

denote the sublevel set of value r of the map f, and the same sublevel without its boundary, respectively.

Lemma 1: (Barbalat's Lemma [24, Lemma 4.2]) Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. If the limit $\lim_{t\to+\infty} f(t)$ exists, $\lim_{t\to\infty} f(t) \in \mathbb{R}$, and $\dot{f} : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $\lim_{t\to+\infty} \dot{f}(t) = 0 \in \mathbb{R}$.

Corollary 2: (Barbalat's Lemma [24, Lemma 4.2]) Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function. If the limit $\lim_{t\to+\infty} f(t)$ exists, $\lim_{t\to\infty} f(t) \in \mathbb{R}$, and $\sup_{t\geq 0} |\ddot{f}(t)| < \infty$, then $\lim_{t\to+\infty} \dot{f}(t) = 0 \in \mathbb{R}$.

Corollary 2 follows from Lemma 1, since $\sup_{t\geq 0} |\ddot{f}(t)| < \infty$ implies that $\dot{f} : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $(0, \infty)$. Later, we invoke Barbalat's Lemma to infer attractivity properties of sets (stability properties are also inferred, but they follow after (positive) invariance of sub-level sets, and not after invoking Barbalat's Lemma).

IV. Problem Description

We consider an aerial vehicle and a point-mass load physically coupled to each other by a massless cable, which behaves as a rigid link, and as illustrated in Fig. 1.

In this section, we let the aerial vehicle be fully-actuated (i.e., the attitude of the vehicle can be controlled instantaneously, and thus the vehicle can provide an arbitrary three dimensional force); in contrast, in Section X, we let the aerial vehicle be under-actuated (i.e., the vehicle can only provide a thrust force along a body direction, which can in turn be rotated by applying a torque) and we describe the necessary steps to cope with the aerial vehicle under-actuation.

Let then the system be described by a position

$$\mathcal{P} \in \mathbb{R}^6 :\Leftrightarrow \begin{bmatrix} p \\ P \end{bmatrix} = \begin{bmatrix} \text{linear position of the load} \in \mathbb{R}^3 \\ \text{linear position of the UAV} \in \mathbb{R}^3 \end{bmatrix}, \tag{3a}$$

and by a velocity

$$\mathcal{V} \in \mathbb{R}^6 :\Leftrightarrow \begin{bmatrix} v \\ V \end{bmatrix} = \begin{bmatrix} \text{linear velocity of the load} \in \mathbb{R}^3 \\ \text{linear velocity of the UAV} \in \mathbb{R}^3 \end{bmatrix}; \tag{3b}$$

and finally, for convenience, denote

$$z \in \mathbb{R}^{12} :\Leftrightarrow (\mathcal{P}, \mathcal{V}) \in \mathbb{R}^6 \times \mathbb{R}^6 :\Leftrightarrow (p, P, v, V) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{3c}$$

Because the system is composed of only (two) point-masses, the system kinematics are of the form

$$\dot{\mathcal{P}} = \mathcal{V}. \tag{4}$$

The cable connecting the load and the aerial vehicle imposes two constraints, one geometric constraint and a corresponding holonomic constraint. To be specific, given a $z \in \mathbb{R}^{12}$, those two constraints are described by $f(z) = 0_2$ where

$$\begin{aligned} f &: \mathbb{R}^{12} \ni z \mapsto f(z) \in \mathbb{R}^2 \\ f(z) &:= \begin{bmatrix} f_1(\mathcal{P}) \\ f_2(\mathcal{P}, \mathcal{V}) \end{bmatrix} := \begin{bmatrix} \frac{\|P-p\|^2}{l^2} - 1 \\ df_1(\mathcal{P})\mathcal{V} \end{bmatrix}, \end{aligned} \tag{5a}$$

with f_1 describing the geometric constraint (a constraint on the positions of the system), and which requires the positions of the load and of the aerial vehicle to be apart by the length of the cable; and with f_2 describing the holonomic constraint (a constraint on the velocities of the system), and which follows from differentiation of the geometric constraint composed with the kinematics of the system in (4). The constraints map f in (5a) allows us to define the state space of our system, namely

$$\mathbb{Z} := \{z \in \mathbb{R}^{12} : f(z) = 0_2\}, \tag{5b}$$

and, consequently, it also allows us to define the tangent set to the state space at each point of the state space, namely

$$T_z\mathbb{Z} := \{\dot{z} \in \mathbb{R}^{12} : df(z)\dot{z} = 0_2\}, \text{ for } z \in \mathbb{Z}. \tag{5c}$$

The linear accelerations, of the load and of the aerial vehicle, may be found by considering the net force applied on each point-mass. The distribution of forces is shown in Fig. 1. In Fig. 1,

$$u \in \mathbb{R}^3 \tag{6}$$

denotes the input force that one can apply on the aerial vehicle (which is fully-actuated); and

$$(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3 \tag{7}$$

denotes the pair of wind forces applied on each point-mass, where w is the wind force applied on the load, and W is the wind force applied on the aerial vehicle – the wind forces maybe be time-varying, but for the purposes of control design they are assumed to be constant (i.e., $\dot{w} = 0_3$ and $\dot{W} = 0_3$); and finally where $T = T(z, u)$ is the tension on the cable (which depends on the state of the system, on the input applied to the system, and on the wind forces). We then have that the linear accelerations are given by

$$\dot{\mathcal{V}} = \mathcal{A}(z, u) :\Leftrightarrow \begin{bmatrix} \dot{v} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} \frac{1}{m}T(z, u)\frac{P-p}{l} - ge_3 + \frac{w}{m} \\ \frac{u}{M} - \frac{1}{M}T(z, u)\frac{P-p}{l} - ge_3 + \frac{W}{M} \end{bmatrix}, \tag{8a}$$

which we can rewrite as

$$\begin{bmatrix} \dot{v} \\ \dot{V} \end{bmatrix} = \underbrace{\begin{bmatrix} -ge_3 + \frac{w}{m} \\ \frac{u}{M} - ge_3 + \frac{W}{M} \end{bmatrix}}_{=:\mathcal{G}(z,u) \in \mathbb{R}^6} + \underbrace{\begin{bmatrix} \frac{1}{m}\frac{P-p}{l} \\ -\frac{1}{M}\frac{P-p}{l} \end{bmatrix}}_{=:\mathcal{N}(\mathcal{P}) \in \mathbb{R}^6} T(z, u). \tag{8b}$$

The map $(z, u) \mapsto T(z, u)$ is shown in (11) (and also in Fig. 1), and how this map is inferred is explained next.

Combining the system kinematics in (4) and the system dynamics in (8a), we can then construct the system's vector field, which is given by[1]

$$\begin{aligned} Z_{w,W} &: \mathbb{Z} \times \mathbb{R}^3 \ni (z, u) \mapsto Z_{w,W}(z, u) \in T_z\mathbb{Z} \subset \mathbb{R}^{12} \\ \dot{z} = Z_{w,W}(z, u) &:\Leftrightarrow \begin{bmatrix} \dot{\mathcal{P}} \\ \dot{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} \mathcal{V} \\ \mathcal{A}(z, u) \end{bmatrix} = \begin{bmatrix} \text{kinematics} \\ \text{dynamics} \end{bmatrix}. \end{aligned} \tag{9}$$

Note that $Z_{w,W}(z, u) \in T_z\mathbb{Z}$, for any $(z, u) \in \mathbb{Z} \times \mathbb{R}^3$, because we require $\mathbb{Z}$ to be invariant (that is, a solution to the differential equation $\dot{z} = Z_{w,W}(z, u)$ starting in $\mathbb{Z}$ remains in $\mathbb{Z}$). Recalling the tangent set definition in (5c), it then follows that

$$\begin{aligned} Z_{w,W}(z, u) \in T_z\mathbb{Z} &\Leftrightarrow df(z)Z_{w,W}(z, u) = 0_2 \\ &\Leftrightarrow \begin{bmatrix} df_1(\mathcal{P})\mathcal{V} \\ d_1f_2(\mathcal{P}, \mathcal{V})\mathcal{V} + d_2f_2(\mathcal{P}, \mathcal{V})\mathcal{A}(z, u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{10}$$

The satisfaction of the first equation in (10) is immediate, since it corresponds to the holonomic equation f_2 in (5a). On the other hand, the second equation in (10), combined with the dynamics as expressed in (8b) and the fact that f_2 is linear in its second entry (i.e., linear with respect to the velocities), may be rewritten as

$$\begin{aligned} & d_1f_2(\mathcal{P}, \mathcal{V})\mathcal{V} + d_2f_2(\mathcal{P}, \mathcal{V})\mathcal{A}(z, u) = 0 \Leftrightarrow \\ \Leftrightarrow & d_1f_2(\mathcal{P}, \mathcal{V})\mathcal{V} + df_1(\mathcal{P})\mathcal{A}(z, u) = 0 && \because d_2f_2 = df_1 \\ \Leftrightarrow & d_1f_2(\mathcal{P}, \mathcal{V})\mathcal{V} + df_1(\mathcal{P})(\mathcal{G}(z, u) + \mathcal{N}(\mathcal{P})T(z, u)) = 0 && \because \text{(8b)} \\ \Leftrightarrow & T(z, u) = -\left(df_1(\mathcal{P})\mathcal{N}(\mathcal{P})\right)^{-1}\left(df_1(\mathcal{P})\mathcal{G}(z, u) + d_1f_2(\mathcal{P}, \mathcal{V})\mathcal{V}\right) \\ \Leftrightarrow & T(z, u) = \frac{m}{m+M}\left(\left\langle u, \frac{P-p}{l}\right\rangle + \frac{M}{l}\|V - v\|^2 + M\left\langle \frac{W}{M} - \frac{w}{m}, \frac{P-p}{l}\right\rangle\right), \end{aligned} \tag{11}$$

[1]We index the vector field with $\mathcal{W}$ to emphasize the fact that it depends on $\mathcal{W}$, while the control law that we design later cannot and does not.

where (11) expresses how the tension (exerted by the cable on the load and aerial vehicle) depends on the state and the input (and also on the wind forces)[2].

At this point, we can state the problem that we wish to solve, which will be further refined later in this paper.

Problem 1: Let $\mathbb{R} \ni t \mapsto p_\star(t) \in \mathbb{R}^3$ be some given desired position trajectory, and consider the open-loop vector field $Z_{w,W}$ in (9), for some *unknown (by the controller)* wind forces $(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3$. Design a control law $(t, z) \mapsto u^{cl}(t, z)$ such that $\lim_{t\to\infty}(p(t) - p_\star(t)) = 0_3$ along a trajectory of $\dot{z}(t) = Z_{w,W}(z(t), u^{cl}(t, z(t)))$ with $z(t_0) \in \mathbb{Z}$ and for any $t_0 \in \mathbb{R}$.

Remark 3: We will assume hereafter that the wind force applied on the load does not match the weight of the load; i.e., $(w, W) \in (\mathbb{R}^3\backslash\{mge_3\}) \times \mathbb{R}^3$. The reasoning behind this assumption lies on the fact that, if $w = mge_3$, then the load would be buoyant in the air, and this makes it impossible to stabilize (with a continuous control law) the load around any point in space[3].

Indeed, let $n \in \mathbb{S}^2$ and $\bar{p} \in \mathbb{R}^3$. Note then that, if $w = mge_3$, it follows that

$$Z_{w,W}(z^\star, u^\star)|_{w=mge_3} = 0_{12} \text{ with } z^\star = (\bar{p}, \bar{p} + ln, 0_3, 0_3) \in \mathbb{Z} \text{ and } u^\star = Mge_3 - W \in \mathbb{R}^3,$$

that is, $z^\star$ is an equilibrium point of the open-loop vector field $z \mapsto Z(z, u^\star)$. In order for a continuous control law $z \mapsto u(z)$ to exist, which renders $z^\star$ as an asymptotically stable equilibrium point of $z \mapsto Z(z, u^\star + u(z))$, it must be the case that (Brockett's necessary condition – see the third necessary condition in [25, Theorem 1])

$$\{(z, u) \in (\text{Neighborhood of } z^\star \text{ in } \mathbb{Z}) \times \mathbb{R}^3 : Z_{w,W}(z, u^\star + u)\} = \text{Neighborhood of } 0_{12} \text{ in } T_{z^\star}\mathbb{Z}.$$

However, this condition is not satisfied: indeed, note that, given some arbitrarily small $\lambda \in \mathbb{R}$, it is the case that

$$\dot{z}^\star = (0_3, 0_3, \lambda\tilde{n}, \lambda\tilde{n}) \in \text{ Neighborhood of } 0_{12} \text{ in } T_{z^\star}\mathbb{Z}$$

for any $\tilde{n}$ not in a neighborhood of $\pm n$; nonetheless, there exists no $(z, u) \in$ (Neighborhood of $z^\star$ in $\mathbb{Z}$) $\times \mathbb{R}^3$ such that $Z_{w,W}(z, u^\star + u)|_{w=mge_3} = \dot{z}^\star$: in fact

$$\begin{aligned}
&Z_{w,W}(z, u^\star + u)|_{w=mge_3} = \dot{z}^\star \Leftrightarrow \\
&\Leftrightarrow \begin{bmatrix} v \\ V \\ \left\langle \frac{u}{m+M}, \frac{P-p}{l} \right\rangle \frac{P-p}{l} \\ \frac{u}{M} - \frac{m}{M}\left\langle \frac{u}{m+M}, \frac{P-p}{l} \right\rangle \frac{P-p}{l} \end{bmatrix} = \begin{bmatrix} 0_3 \\ 0_3 \\ \lambda\tilde{n} \\ \lambda\tilde{n} \end{bmatrix} \Rightarrow \begin{bmatrix} p \\ P \\ v \\ V \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{p} \\ \bar{p} \pm l\tilde{n} \\ 0_3 \\ 0_3 \end{bmatrix}}_{\substack{\text{not in neighborhood of} \\ z^\star \text{ in } \mathbb{Z} \text{ regardless of } \lambda}} \quad \text{and } u = \pm\lambda(m + M)\tilde{n}.
\end{aligned}$$

This proves that it is impossible to stabilize the load, with a continuous control law, around the position $\bar{p}$.

Remark 4: We say that the pair of wind forces $(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3$ is *non-aggressive* if the vertical component of the wind applied on the load does not exceed the load's weight, i.e., $\langle w, e_3\rangle < mg$; it is *aggressive* if $\langle w, e_3\rangle \geq mg$.

The requirements set in Remarks 3 and 4 will become clear the next section.

Remark 5: The tension $T(z, u)$ in (8a) is an internal force to the system (composed of two point-masses)[4]. Indeed, if we define the total linear momentum

$$\begin{aligned}
&L : \{\mathcal{V} \in \mathbb{R}^6 : df_1(\mathcal{P})\mathcal{V} = 0_1\} \to \mathbb{R}^3 \\
&L(\mathcal{V}) := mv + MV,
\end{aligned}$$

it follows immediately that

$$\text{“}\frac{d}{dt}L(\mathcal{V})\text{”} = dL(\mathcal{V})\mathcal{A}(z, u) \overset{\text{(8b)}}{=} dL(\mathcal{V})\mathcal{G}(z, u) + \underbrace{dL(\mathcal{V})\mathcal{N}(\mathcal{P})}_{=0_3} T(z, u).$$

That is, the tension does not contribute to any change in the total linear momentum, and this is true regardless of the map $(z, u) \mapsto T(z, u)$.

[2]The inverse in (11) is well-defined: indeed, for any $z \in \mathbb{Z}$, $df_1(\mathcal{P})\mathcal{N}(\mathcal{P}) = -2\frac{m+M}{lmM} \neq 0$.

[3]Problem 1 is concern with position tracking, which is a generalization of position stabilization. Indeed, stabilization of the load around a point $\bar{p} \in \mathbb{R}^3$ is described in Problem 1 if one considers the desired position trajectory $\mathbb{R} \ni t \mapsto p_\star(t) := \bar{p} \in \mathbb{R}^3$.

[4]When we refer to tension, we are implicitly referring to the pair of forces applied by the cable, one applied on the load, and the other applied on the aerial vehicle.

V. Differential Flatness

Recall Problem 1, where we have introduced $\mathbb{R} \ni t \mapsto p_\star(t) \in \mathbb{R}^3$ as the desired position trajectory that we wish the load to track. Let us impose the following constraints on the desired trajectory,

$$p_\star \in \mathcal{C}^4 \text{ and} \tag{12a}$$

$$\inf_{t\in\mathbb{R}} \|mp_\star^{(2)}(t) + mge_3 - w\| > 0, \text{ and} \tag{12b}$$

$$\sup_{t\in\mathbb{R}} \|p_\star^{(i)}(t)\| < \infty \text{ for } i \in \{2,3,4,5\}, \tag{12c}$$

whose necessity will be made clear soon. We say that a desired trajectory is *feasible* if (12b) is satisfied, and it is *infeasible* if (12b) is not satisfied. In particular, if the wind applied on the load does not match the load's weight (see Remark 1), then trajectories with *small* accelerations are feasible: i.e., (12b) is satisfied if $\sup_{t\in\mathbb{R}} \|p_\star^{(2)}(t)\| \leq \left\|ge_3 - \frac{w}{m}\right\| \neq 0$. As a special case, we thus have that constant speed (whose acceleration is zero) trajectories are non-aggressive trajectories. The condition in (12a) is required, because the first five derivatives (position, velocity, acceleration, jerk and snap) are required next (and they must be continuous).

Let us then start by proving that the system we wish to control is differentially flat with respect to the position of the load. That is, if in Problem 1 we require that $p(t) = p_\star(t)$ for all time instants $t \in \mathbb{R}$, then we can determine uniquely[5] the whole system trajectory $\mathbb{R} \ni t \mapsto z(t) \in \mathbb{Z}$.

Loosely speaking, it follows from the system dynamics in (8a) that

$$\dot{v} \text{ in (8a)} \Rightarrow T(z,u)\frac{P-p}{l} = m\dot{v} + mge_3 - w \Rightarrow \begin{cases} P = p \pm l\frac{m\dot{v}+mge_3-w}{\|m\dot{v}+mge_3-w\|} \\ T(z,u) = \pm\|m\dot{v} + mge_3 - w\| \end{cases}, \tag{13a}$$

$$M\dot{V} + m\dot{v} \text{ in (8a)} \Rightarrow u = M\dot{V} + m\dot{v} + (M+m)ge_3 - (W+w). \tag{13b}$$

With the latter in mind, and given a feasible trajectory $p_\star$, define the pairs $(z_{\star,+}, u_{\star,+})$ and $(z_{\star,-}, u_{\star,-})$ given by

$$t \mapsto z_{\star,\pm}(t) \overset{(3c)}{:\Leftrightarrow} t \mapsto \begin{bmatrix} p_\star(t) \\ P_{\star,\pm}(t) \\ v_\star(t) \\ V_{\star,\pm}(t) \end{bmatrix} := \begin{bmatrix} p_\star^{(0)}(t) \\ p_\star^{(0)}(t) \pm l\frac{mp_\star^{(2)}(t)+mge_3-w}{\|mp_\star^{(2)}(t)+mge_3-w\|} \\ p_\star^{(1)}(t) \\ p_\star^{(1)}(t) \pm l\Pi\left(\frac{mp_\star^{(2)}(t)+mge_3-w}{\|mp_\star^{(2)}(t)+mge_3-w\|}\right)\frac{mp_\star^{(3)}(t)}{\|mp_\star^{(2)}(t)+mge_3-w\|} \end{bmatrix}, \tag{14a}$$

$$t \mapsto u_{\star,\pm}(t) := MV_{\star,\pm}^{(1)}(t) + mv_\star^{(1)}(t) + (M+m)ge_3 - (W+w). \tag{14b}$$

It is now clear from the definition of $(z_{\star,\pm}, u_{\star,\pm})$ why we required the conditions on (12) to be satisfied. Condition (12b) guarantees that the unit vector $\frac{a}{\|a\|}$ with $a = mp_\star^{(2)}(t) + mge_3 - w$ is always well-defined; while condition (12a) guarantees that $(z_{\star,\pm}, u_{\star,\pm})$ is well-defined and continuous on $\mathbb{R}$. It is simple to verify that $t \mapsto z_{\star,\pm}(t)$ satisfies $\dot{z}_{\star,\pm}(t) = Z_{w,W}(z_{\star,\pm}(t), u_{\star,\pm}(t))$ for all $t \in \mathbb{R}$, which implies that the system is differentially flat with respect to the load's linear position. Moreover, $(z_{\star,+}, u_{\star,+})$ and $(z_{\star,-}, u_{\star,-})$ can be thought as the two equilibria options (equilibrium state trajectory, and equilibrium open-loop input trajectory) that guarantee that the load tracks the desired position trajectory. The visualization of these two options is illustrated in Fig. 2, where the equilibrium configuration depends on whether the wind is aggressive or not. In particular, note that given the definitions in (14a), it follows that

$$\langle e_3, P_{\star,\pm}(t) - p_\star(t)\rangle = \pm\frac{\langle e_3, mp_\star^{(2)}(t)\rangle + mg - \langle e_3, w\rangle}{\|mp_\star^{(2)}(t) + mge_3 - w\|}; \tag{15a}$$

and, moreover, given the tension function, as defined in (11), it follows that (the proof of the equality below is found at the end of this section)

$$t \mapsto T(z_{\star,\pm}(t), u_{\star,\pm}(t)) = \pm\|mp_\star^{(2)}(t) + mge_3 - w\|. \tag{15b}$$

Firstly, it follows from (15b) that we may discard the pair $(z_{\star,-}, u_{\star,-})$ as this yields a negative tension in the cable (this solution would only be feasible if the cable were replaced by a massless rigid link, which can undergo compressive forces). For that reason, and hereafter, we simplify the notation and denote $(z_\star, u_\star) := (z_{\star,+}, u_{\star,+})$. Secondly, it follows from (15a), that if the wind is non-aggressive and if the desired trajectory has a lower bounded vertical acceleration (i.e., $\inf_{t\in\mathbb{R}}\langle e_3, p_\star^{(2)}(t)\rangle \geq -g + \left\langle e_3, \frac{w}{m}\right\rangle > 0$) then the UAV will always be above the load (i.e., $\langle e_3, P_{\star,+}(t) - p_\star(t)\rangle > 0$). We note now that requiring the wind to be non-aggressive is not necessary (in order for definitions to be well-defined): the point we wish to emphasize is that if the wind is aggressive, and if the load is required, for example, to track a constant speed trajectory (say, to stay at rest at some point), then the UAV will need to be under the load in order for the cable to be under tension – as illustrated in Fig 2.

Proof of (15b)*:* For brevity, in what follows, we drop all time dependencies and we also drop the index $\star$ in $(z_{\star,\pm}, u_{\star,\pm})$ and $p_\star$. Denote

$$n = \frac{m\ddot{p} + mge_3 - w}{\|m\ddot{p} + mge_3 - w\|} \in \mathbb{S}^2. \tag{16a}$$

[5] We obtain two solutions: however, only one is feasible, which is the one that satisfies the requirement that the cable remains taut.

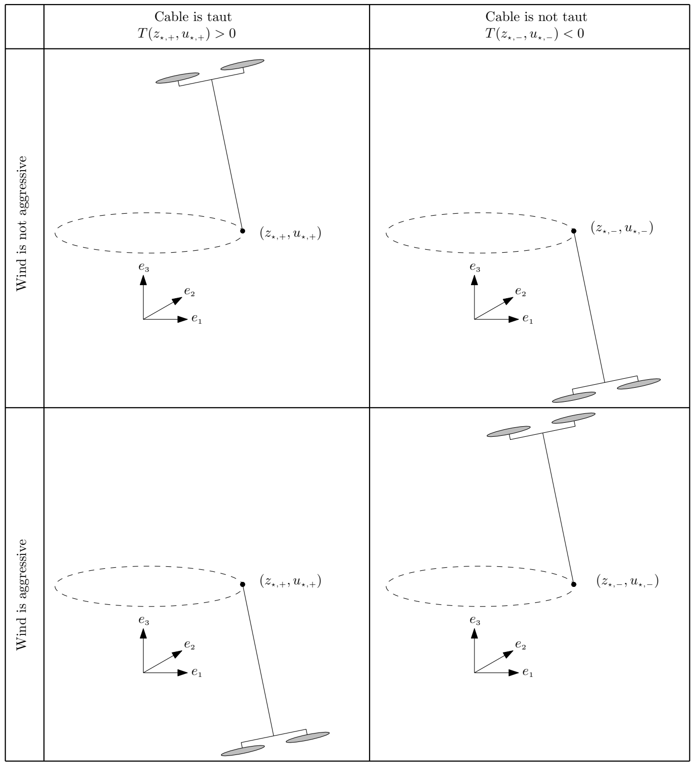


Fig. 2: Graphical representation of the equilibrium trajectories $(z_{\star,+}, u_{\star,+})$ and $(z_{\star,-}, u_{\star,-})$ when the load is required to track a desired trajectory (in the figure it is a circular path marked with a dashed line) and depending on whether the wind is aggressive or not. If the desired acceleration is appropriately lower bounded (i.e., $\inf_{t\in\mathbb{R}}\langle e_3, p_\star^{(2)}(t)\rangle \geq -g + \left\langle e_3, \frac{w}{m}\right\rangle > 0$) and if the wind is non-aggressive, then, for the equilibrium $(z_{\star,+}, u_{\star,+})$, the UAV is always above the load, i.e., $\langle e_3, P_{\star,+}(t) - p_\star(t)\rangle > 0$ for all $t \in \mathbb{R}$.

Because n is a unit vector, it satisfies the equality $\langle n, n\rangle = 1$ and, therefore, by differentiation, it also satisfies the two equalities

$$\langle n, \dot{n}\rangle = 0, \tag{16b}$$

$$\langle n, \ddot{n}\rangle + \langle \dot{n}, \dot{n}\rangle = 0, \tag{16c}$$

We can then rewrite $(z_{\star,\pm}, u_{\star,\pm})$, defined in (14), in a more compact fashion as

$$\begin{aligned} z_\pm &= (p, P_\pm, v, V_\pm) = (p, p \pm ln, \dot{p}, \dot{p} \pm l\dot{n}), && (16d)\\ u_\pm &= M\dot{V}_\pm + m\dot{v} + (M+m)ge_3 - (W + w) \\ &= (M+m)(\ddot{p} + ge_3) - (W+w) \pm l\ddot{n} && \because (16d)\\ &= (M+m)\left(\ddot{p} + ge_3 - \frac{w}{m}\right) - M\left(\frac{W}{M} - \frac{w}{m}\right) \pm l\ddot{n}. && (16e) \end{aligned}$$

Composing the tension function, as defined in (11), with the pair defined above, we then have that

$$\begin{aligned} T(z_\pm, u_\pm) &= \frac{m}{m+M}\left(\langle u_\pm, \pm n\rangle + Ml\langle \pm\dot{n}, \pm\dot{n}\rangle + M\left\langle \frac{W}{M} - \frac{w}{m}, \pm n\right\rangle\right) \\ &= m\left\langle \ddot{p} + ge_3 - \frac{w}{m}, \pm n\right\rangle + Ml\left(\langle \pm n, \pm\ddot{n}\rangle + \langle \pm\dot{n}, \pm\dot{n}\rangle\right) && \because (16e)\\ &= \langle m\ddot{p} + mge_3 - w, \pm n\rangle && \because (16c)\\ &= \pm\|m\ddot{p} + mge_3 - w\|. && \because (16a) \end{aligned}$$

■

VI. Change of coordinates

Let us now perform a change of coordinates which will simplify the exposition of the controller design, and which is illustrated in Fig. 3. Consider then the following state space and state decomposition

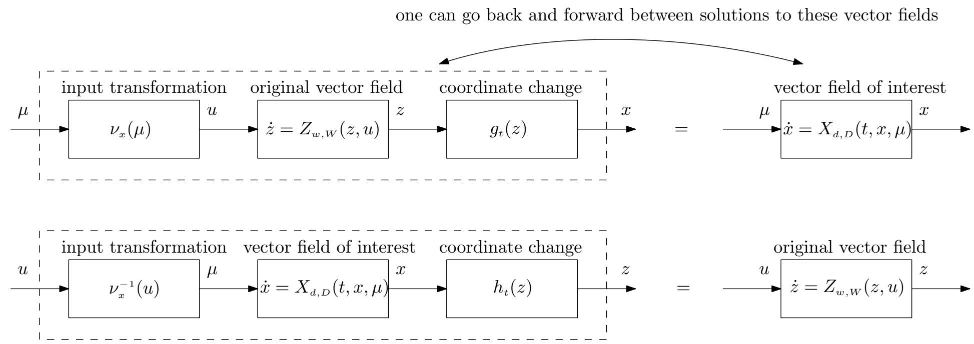


Fig. 3: Change of coordinates that allows us to obtain the vector field $X_{\mathcal{W}}$: the exposition of the controller design is simpler in the new vector field.

$$
\begin{aligned}
&\mathbb{X} := \{(p,v,n,\omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : \langle n,n\rangle = 1, \langle n,\omega\rangle = 0\} = \mathbb{R}^6 \times T\mathbb{S}^2, && \text{(17a)}\\
&x \in \mathbb{X} :\Leftrightarrow (p,v,n,\omega) \in \mathbb{X}. && \text{(17b)}
\end{aligned}
$$

Given a time instant $t \in \mathbb{R}$, consider then the change of coordinates

$$
\begin{aligned}
g_t &: \mathbb{Z} \ni z \mapsto g_t(z) \in \mathbb{X}, && \text{(18a)}\\
h_t &: \mathbb{X} \ni x \mapsto h_t(x) \in \mathbb{Z}, && \text{(18b)}
\end{aligned}
$$

defined as

$$
g_t(z) := \begin{bmatrix} p - p_\star^{(0)}(t) \\ v - p_\star^{(1)}(t) \\ \frac{P-p}{l} \\ \mathcal{S}\left(\frac{P-p}{l}\right)\frac{V-v}{l} \end{bmatrix}, h_t(x) := \begin{bmatrix} p + p_\star^{(0)}(t) \\ p + ln \\ v + p_\star^{(1)}(t) \\ p + l\mathcal{S}(\omega)n \end{bmatrix}. \tag{18c}
$$

The maps g_t and h_t are smooth, and it is easy to verify that $g_t \circ h_t = \mathrm{id}_{\mathbb{X}}$ and that $h_t \circ g_t = \mathrm{id}_{\mathbb{Z}}$. As such g_t and h_t are diffeomorphisms, and each one is the inverse of the other.

We can then write the vector field $Z_{w,W}$ in (9) in the new coordinates, i.e., in the state space $\mathbb{X}$, namely

$$
\begin{aligned}
&\tilde X_{w,W} : \mathbb{R} \times \mathbb{X} \times \mathbb{R}^3 \ni (t,x,u) \mapsto \tilde X_{w,W}(t,x,u) \in T_x\mathbb{X}\\
&\tilde X_{w,W}(t,x,u) := \left(\frac{d}{dt}g_t(z) + dg_t(z)Z_{\mathcal{W}}(z,u)\right)|_{z=h_t(x)}
\end{aligned}, \tag{19a}
$$

where we have included the dependency on the unknown wind forces as an subindex (as a reminder that the vector field $\tilde X_{w,W}$ is partially unknown to the controller). Because it will prove useful, we show the explicit vector field $\tilde X_{w,W}$, which is obtained from (19a),

$$
\begin{aligned}
\dot x = \tilde X_{w,W}(t,x,u) :\Leftrightarrow \begin{bmatrix} \dot p \\ \dot v \\ \dot n \\ \dot\omega \end{bmatrix} &= \begin{bmatrix} v \\ T(h_t(x),u)n - g(t) + \frac{w}{m} \\ \mathcal{S}(\omega)n \\ \mathcal{S}(n)\left(\frac{u}{Ml} + \frac{1}{l}\left(\frac{W}{M} - \frac{w}{m}\right)\right) \end{bmatrix}\\
\Leftrightarrow \begin{bmatrix} \dot p \\ \dot v \\ \dot n \\ \dot\omega \end{bmatrix} &\overset{(11)}{=} \begin{bmatrix} v \\ \left(\frac{1}{M+m}\langle n,u\rangle + \frac{M}{M+m}\left(l\langle\omega,\omega\rangle + \langle n, \frac{W}{M} - \frac{w}{m}\rangle\right)\right)n - g(t) + \frac{w}{m} \\ \mathcal{S}(\omega)n \\ \mathcal{S}(n)\left(\frac{u}{Ml} + \frac{1}{l}\left(\frac{W}{M} - \frac{w}{m}\right)\right) \end{bmatrix},
\end{aligned} \tag{19b}
$$

where $g : \mathbb{R} \to \mathbb{R}^3$, hereafter called *time-varying gravity acceleration*, is defined as

$$
\begin{aligned}
&g : \mathbb{R} \to \mathbb{R}^3\\
&g(t) := p_\star^{(2)}(t) + ge_3
\end{aligned}. \tag{19c}
$$

Because it is convenient, hereafter, given some $x \in \mathbb{X}$, denote

$$
\mu \in \mathbb{V}_x :\Leftrightarrow (T,\tau) \in \mathbb{R} \times T_n\mathbb{S}^2. \tag{20a}
$$

The vector field in (19b) motivates the introduction of the following input transformation

$$
\begin{aligned}
&\nu_x : \mathbb{V}_x \ni \mu \mapsto \nu(\mu) \in \mathbb{R}^3\\
&\nu_x(\mu) := ((m+M)T - Ml\langle\omega,\omega\rangle)n - Ml\mathcal{S}(n)\tau
\end{aligned}, \tag{20b}
$$

and its inverse ($\nu_x \circ \nu_x^{-1} = \mathrm{id}_{\mathbb{R}^3}$ and $\nu_x^{-1} \circ \nu_x = \mathrm{id}_{\mathbb{V}_x}$)

$$\begin{aligned} &\nu_x^{-1} : \mathbb{R}^3 \ni u \mapsto \nu_x^{-1}(u) \in \mathbb{V}_x \\ &\nu_x^{-1}(u) := \left(\frac{1}{M+m}\langle n, u\rangle + \frac{M}{M+m} l\langle \omega, \omega\rangle, \mathcal{S}\left(n\right)\frac{u}{Ml} \right). \end{aligned} \tag{20c}$$

The same vector field motivates the shorter notation

$$\begin{aligned} &\begin{cases} d = \frac{w}{m} \in \mathbb{R}^3 & \text{(physical dimensions of a linear acceleration)} \\ D = \frac{M}{M+m}\left(\frac{W}{M} - \frac{w}{m}\right) \in \mathbb{R}^3 & \text{(physical dimensions of a linear acceleration)} \end{cases}, \\ &\begin{cases} \phi(n) := n \in \mathbb{R}^3 \\ \Phi(n) := \frac{1}{l}\frac{M+m}{M}\mathcal{S}\left(n\right) \in \mathbb{R}^{3\times 3} \end{cases}. \end{aligned} \tag{21}$$

With (20b) and (21) in mind, it follows that

$$\begin{aligned} \tilde{X}_{w,W}(t, x, \nu_x(\mu)) &= X_{d,D}(t, x, \mu), \\ \tilde{X}_{w,W}(t, x, u) &= X_{d,D}(t, x, \nu^{-1}(x, u)), \end{aligned}$$

where $X_{d,D}$ is the final vector field of interest, and it is given by

$$\begin{aligned} &X_{d,D} : \{(t, x, \mu) : \mathbb{R} \times \mathbb{X} \times \mathbb{R}^4 : \mu \in \mathbb{V}_x\} \ni (t, x, \mu) \mapsto X_{d,D}(t, x, \mu) \in T_x\mathbb{X} \\ &\dot{x} = X_{d,D}(t, x, \mu) :\Leftrightarrow \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{n} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ (T + \langle \phi(n), D\rangle)\, n - g(t) + d \\ \mathcal{S}\left(\omega\right) n \\ \Pi\left(n\right)\left(\tau + \Phi(n) D\right) \end{bmatrix}, \end{aligned} \tag{22}$$

where $(d, D) \in \mathbb{R}^3 \times \mathbb{R}^3$ are unknown (by the controller) disturbances. The vector field $X_{d,D}$ has a cascaded structure, which can be visualized in Fig. 4. This is the canonical vector field for the slung-load system, for which we will be designing the controller in the next sections, and it does not depend on the system's physical parameters – see next Remark (Remark 6). Obviously, the physical parameters need to be known when controlling the vector field of the real physical system (because they are used in the coordinate change g_t in (17), and in the input change ν_x in (20b)), but they do not need to be known when controlling the canonical vector field.

Remark 6: The vector field $X_{d,D}$ in (22) is not a true canonical form, because the map Φ in (21) actually depends on the system's physical parameters. Indeed, note that $\Phi(n) := \gamma\mathcal{S}\left(n\right)$, with $\gamma = \frac{1}{l}\frac{M+m}{M}$: thus, formally speaking, two slung-load systems have the same canonical form only if the constant γ is the same for both.

In a true canonical form, we would define the disturbances

$$\begin{cases} d = \frac{w}{m} \in \mathbb{R}^3 & \text{(physical dimensions of a linear acceleration)} \\ D_T = \frac{M}{M+m}\left(\frac{W}{M} - \frac{w}{m}\right) \in \mathbb{R}^3 & \text{(physical dimensions of a linear acceleration)} \\ D_\tau = \frac{1}{l}\left(\frac{W}{M} - \frac{w}{m}\right) \in \mathbb{R}^3 & \text{(physical dimensions of an angular acceleration)} \end{cases},$$

and we would define the vector field

$$\begin{aligned} &X_{d,D_T,D_\tau} : \{(t, x, \mu) : \mathbb{R} \times \mathbb{X} \times \mathbb{R}^4 : \mu \in \mathbb{V}_x\} \ni (t, x, \mu) \mapsto X_{d,D_T,D_\tau}(t, x, \mu) \in T_x\mathbb{X} \\ &\dot{x} = X_{d,D_T,D_\tau}(t, x, \mu) :\Leftrightarrow \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{n} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ (T + \langle n, D_T\rangle)\, n - g(t) + d \\ \mathcal{S}\left(\omega\right) n \\ \Pi\left(n\right)\left(\tau + \mathcal{S}\left(n\right) D_\tau\right) \end{bmatrix}, \end{aligned} \tag{23}$$

where X_{d,D_T,D_τ} is truly independent of all slung load physical constants. That is, in order to consider a true canonical form, we need to define three unknown disturbances, instead of just two. However, in order to simplify the notation and the exposition that follows, rather than working with the vector field X_{d,D_T,D_τ} in (23), we will work with the vector field $X_{d,D}$ in (22).

Finally recall the results from the previous section, concerning the differential flatness properties of our system, whose conclusions naturally extend to the system in the new coordinates. For that reason, given the definition of the equilibrium

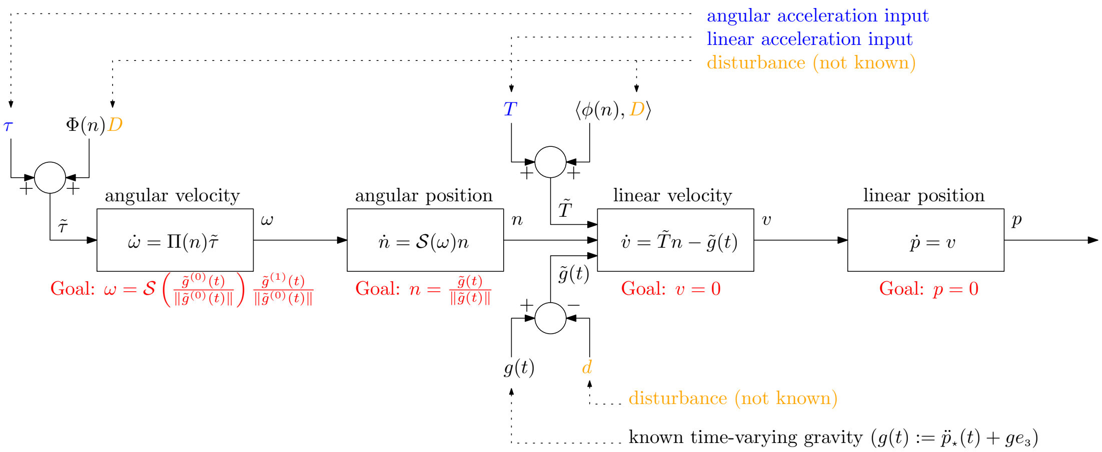


Fig. 4: Cascaded structure of the vector field $X_{d,D}$. Note that if set $p = 0$ (which is our goal) then the disturbance d propagates backwards: i.e., it propagates to ω (part of the state), and it propagates to τ (part of the input), even though τ is not immediately affected by d.

trajectories and inputs $t \mapsto (z_{\star,\pm}(t), u_{\star,\pm}(t))$ in (14), we can also define the equilibrium trajectories and inputs in the new coordinates, namely

$$t \mapsto x_{\star,\pm}(t) := g_t(z_{\star,\pm}(t)) \overset{(17b)}{:\Leftrightarrow}$$
$$t \mapsto \begin{bmatrix} p_\star(t) \\ v_\star(t) \\ n_{\star,\pm}(t) \\ \omega_\star(t) \end{bmatrix} := \underbrace{\begin{bmatrix} 0_3 \\ 0_3 \\ \pm\frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|} \\ \mathcal{S}\left(\frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|}\right)\frac{\dot{\tilde{g}}_d(t)}{\|\tilde{g}_d(t)\|} \end{bmatrix}}_{\tilde{g}_d(t):=g(t)-d}, \tag{24a}$$

and

$$t \mapsto \mu_{\star,\pm}(t) := \nu^{-1}_{x(t)}(u_{\star,\pm}(t)) \overset{(20a)}{:\Leftrightarrow}$$
$$t \mapsto \begin{bmatrix} T_{\star,\pm}(t) \\ \tau_{\star,\pm}(t) \end{bmatrix} := \underbrace{\begin{bmatrix} \pm\|\tilde{g}_d(t)\| - \left\langle \phi\left(\pm\frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|}\right), D\right\rangle \\ \mathcal{S}\left(\frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|}\right)\left(\frac{\ddot{\tilde{g}}_d(t)}{\|\tilde{g}_d(t)\|} - \left\langle \frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|}, \frac{\dot{\tilde{g}}_d(t)}{\|\tilde{g}_d(t)\|}\right\rangle \frac{\dot{\tilde{g}}_d(t)}{\|\tilde{g}_d(t)\|}\right) - \Phi\left(\pm\frac{\tilde{g}_d(t)}{\|\tilde{g}_d(t)\|}\right)D \end{bmatrix}}_{\tilde{g}_d(t):=g(t)-d}, \tag{24b}$$

which can be easily deduced by inspection of Fig. 4.

We can then restate Problem 1 in the new coordinates.

Problem 2: Let $(d, D) \in \mathbb{R}^3 \times \mathbb{R}^3$ be some unknown disturbance, and $\mathbb{R} \ni t \mapsto g(t) \in \mathbb{R}^3$ be some time-varying gravity acceleration satisfying

$$g \in \mathcal{C}^2, \text{ and} \tag{25a}$$
$$\inf_{t\in\mathbb{R}} \|g(t) - d\| > 0, \text{ and} \tag{25b}$$
$$\sup_{t\in\mathbb{R}} \|g^{(i)}(t)\| < 0 \text{ for } i \in \{0, 1, 2\}. \tag{25c}$$

Consider then the vector field $X_{d,D}$ in (22), and the desired trajectory $\mathbb{R} \ni t \mapsto x_{\star,+}(t) \in \mathbb{X}$ as defined in (24). Design a control law $\mu^{cl} : \mathbb{R} \times \mathbb{X} \ni (t, x) \mapsto \mu^{cl}(t, x) \in \mathbb{V}_x$ such that, along a solution $t \mapsto x(t)$ of $\dot{x}(t) = X_{d,D}(t, x(t), \mu^{cl}(t, x(t)))$ with $x(0) \in \mathbb{X}_0 \subset \mathbb{X}$, it follows that $\lim_{t\to\infty} \|x(t) - x_{\star,+}(t)\|_{\mathbb{R}^{12}} = 0$ for some non-empty set of initial conditions $\mathbb{X}_0$ (and where $\mathbb{X}_0$ is dense in $\mathbb{X}$).

Remark 7: The conditions in (25a)–(25c) are a restatement (i.e., they are equivalent) to the conditions in (12a)–(12c), which guarantees well-posedness of the equilibrium trajectories.

Note that we do not expect to have $\mathbb{X}_0 = \mathbb{X}$, because there are two equilibrium trajectories. As such, we should expect that, at best, $\mathbb{X}_0 \subset \mathbb{X}\backslash\{x_{\star,-}(0)\}$, with $x_{\star,-}$ being the undesired equilibrium trajectory. Nonetheless, as we shall prove later, the trajectory $t \mapsto x_{\star,-}(t)$ is unstable. We also note that devising a control law such that $\mathbb{X}_0 = \mathbb{X}$ cannot be accomplished with a continuous control law due to the fact that $\mathbb{X}$ is a non-contractible set (which imposes a topological constraint). Finally note that (25b), from a controller design perspective, is not useful, since it depends on the unknown disturbance d. This leads us to further refine our problem statement.

Problem 3: Let the *unknown* disturbance $d \in \mathbb{R}^3$ be upper bounded in norm by some *known* $\bar{d} \geq 0$, i.e., $\|d\|_{\mathbb{R}^3} \leq \bar{d}$. We wish to solve the same problem as stated in Problem 2 with the exception that (25b) is replaced by

$$\inf_{t \in \mathbb{R}} \|g(t)\| > \bar{d}, \tag{25d}$$

where the satisfaction of (25d) implies the satisfaction of (25b), but not vice-versa.
Condition (25d), when compared with (25b), is a more restrictive condition imposed on the set of trajectories that the load can track (recall, from (19c), that $g(t) := ge_3 + p_\star^{(2)}(t)$). Such a conservative approach is unavoidable, since the disturbance d is unknown.

Remark 8: Note that the time-dependency of the vector field $X_{d,D}$ in (22) comes exclusively from the time-varying gravity acceleration $t \mapsto g(t)$. Also note that the equilibrium state trajectory $t \mapsto x_{\star,\pm}(t)$ and input trajectory $t \mapsto u_{\star,\pm}(t)$, in (24), depend on $t \mapsto g^{(0)}(t), g^{(1)}(t), g^{(2)}(t)$. In order to simplify the proof in our final result, rather than dealing with explicit time-dependency, we include the time-varying gravity as part of the state: to be specific, we include $t \mapsto g^{(0)}(t), g^{(1)}(t), g^{(2)}(t)$ in the state definition, which we will denote, respectively, by g^0, g^1, g^2.

Remark 9: Note in (24) that only the disturbance $d \in \mathbb{R}^3$ impacts the equilibrium trajectory, while the disturbance $D \in \mathbb{R}^3$ does not. That indicates that the disturbance d has a different nature than that of the disturbance D, which will become clearer in the next section. In order to gain some intuition, suppose that d and D are not constant, but rather time-varying. Then the equilibrium state and input trajectories defined in (24) are easily updated to (see Fig. 4)

$$\begin{aligned}
&t \mapsto x_{\star,\pm}(t) := g_t(z_{\star,\pm}(t)) \overset{(17b)}{:\Leftrightarrow} \\
&t \mapsto \begin{bmatrix} p_\star(t) \\ v_\star(t) \\ n_{\star,\pm}(t) \\ \omega_\star(t) \end{bmatrix} := \underbrace{\begin{bmatrix} 0_3 \\ 0_3 \\ \pm\frac{\tilde{g}(t)}{\|\tilde{g}(t)\|} \\ \mathcal{S}\left(\frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}\right) \frac{\dot{g}(t) - \dot{d}(t)}{\|\tilde{g}(t)\|} \end{bmatrix}}_{\tilde{g}(t) := g(t) - d(t)},
\end{aligned} \tag{26a}$$

and

$$\begin{aligned}
&t \mapsto \mu_{\star,\pm}(t) := \nu_{x(t)}^{-1}(u_{\star,\pm}(t)) \overset{(20a)}{:\Leftrightarrow} \\
&t \mapsto \begin{bmatrix} T_{\star,\pm}(t) \\ \tau_{\star,\pm}(t) \end{bmatrix} := \underbrace{\begin{bmatrix} \pm\|\tilde{g}(t)\| - \left\langle \phi\left(\pm\frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}\right), D(t) \right\rangle \\ \mathcal{S}\left(\frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}\right)\left(\frac{\ddot{g}(t) - \ddot{d}(t)}{\|\tilde{g}(t)\|} - \left\langle \frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}, \frac{\dot{g}(t) - \dot{d}(t)}{\|\tilde{g}(t)\|} \right\rangle \frac{\dot{g}(t) - \dot{d}(t)}{\|\tilde{g}(t)\|}\right) - \Phi\left(\pm\frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}\right) D(t) \end{bmatrix}}_{\tilde{g}(t) := g(t) - d(t)}.
\end{aligned} \tag{26b}$$

This equation highlights the different nature of the disturbances: indeed, note that the equilibrium state trajectory $x_{\star,\pm}$ depends on both d and $\dot{d}$, while the equilibrium input depends on d, $\dot{d}$ and $\ddot{d}$. On the other hand, the equilibrium state trajectory $x_{\star,\pm}$ does not depend on D, while the equilibrium input only depends on D (and none of its derivatives). That is, loosely speaking, the disturbance d propagates backwards through the system, while the disturbance D only really affects the input. This can also be visualized in Fig. 4. Moreover, note that if replace d with an estimate $\hat{d}$ that we update according to some appropriate update-law, then such an update law must satisfy two important criteria

- the estimate $\hat{d}$ must remain in some ball of appropriate radius, so that $\inf_{t \in \mathbb{R}} \|g(t) - \hat{d}\| > 0$ (which will guarantee that the unit vector $\frac{g(t) - \hat{d}}{\|g(t) - \hat{d}\|}$ is well-defined);
- the estimate update law (that is, $\dot{\hat{d}}$) must be at least $\mathcal{C}^1$ in its domain, so that both the state and input are continuous (that is, we need $\hat{d}^{(0)}$, $\hat{d}^{(1)}$ and $\hat{d}^{(2)}$ to be continuous in time, where $\hat{d}^{(1)}$ is what we design).

VII. Backstepping: Preliminaries

A. Input-additive and non-input-additive disturbances

In order to illustrate some properties of the vector field $X_{d,D}$ in (22), let us rewrite it as

$$X_{d,D}(t, x, \mu) = \underbrace{\begin{bmatrix} v \\ -g(t) \\ \mathcal{S}(\omega)\, n \\ 0_3 \end{bmatrix}}_{\text{drift vector field (v.f.)}} + \underbrace{\begin{bmatrix} 0_3 & 0_{3\times 3} \\ n & 0_{3\times 3} \\ 0_3 & 0_{3\times 3} \\ 0_3 & \Pi(n) \end{bmatrix}}_{\substack{\text{control v.f. of} \\ \text{the input } \mu}} \left(\underbrace{\begin{bmatrix} T \\ \tau \end{bmatrix}}_{\mu} + \underbrace{\begin{bmatrix} \langle \phi(n), D \rangle \\ \Phi(n) D \end{bmatrix}}_{=:\Phi(x)D} \right) + \underbrace{\begin{bmatrix} 0_{3\times 3} \\ I_3 \\ 0_{3\times 3} \\ 0_{3\times 3} \end{bmatrix}}_{\substack{\text{control v.f. of} \\ \text{the disturbance } d}} d, \tag{27}$$

where one identifies that:

- there exists a non-zero drift vector field;
- the vector field is input affine;

- the vector field is affine in the disturbances;
- the disturbance D is an input-additive disturbance (its control vector field is in the image space of the input control vector field);
- the disturbance d is not an input-additive disturbance (its control vector field is not in the image space of the input control vector field).

The main challenge in accomplishing Problem 3 lies on how to compensate for the unknown disturbance $(d, D) \in \mathbb{R}^3 \times \mathbb{R}^3$. One point of remark is that the disturbance d is fundamentally different from the disturbance D. Indeed note that the disturbance D enters the system the same way the input does (input-additive disturbance), i.e.,

$$X_{d,D}(t, x, \mu) = X_{d,0_3}(t, x, \mu + \Phi(x)D), \tag{28}$$

and, as such, if the disturbance D were known, it could be immediately canceled by the input μ (by choosing $\mu = \mu' - \Phi(x)D$). Input-additive disturbances are easier to remove when compared to disturbances that enter the system in a non-input-additive manner:

- for an input-additive disturbance, we can design a controller ignoring its existence, and, afterwards, we can an estimator for that disturbance (the control design is decoupled from the estimation design);
- for an non-input-additive disturbance, on the other hand, the control design cannot be accomplished by ignoring its existence.

It turns out that for the disturbance d (non-input-additive disturbance) we will need to design two estimators, while for the disturbance D (input-additive disturbance) one estimator would suffice (but, for simplicity, we design two estimators for D).

B. Disturbance estimator update law design

Consider a system

$$\dot{x} = f(t, x), \tag{29a}$$

with $x \in \mathbb{R}^n$ (for some $n \in \mathbb{N}$), and suppose a Lyapunov function has been constructed

$$V : \mathbb{R} \times \mathbb{R}^n \ni (t, x) \mapsto V(t, x) \in [0, \infty), \tag{29b}$$

for which it holds that

$$\text{“}\dot{V}(t, x)\text{”} \equiv W(t, x) := d_1 V(t, x) + d_2 V(t, x) \underbrace{f(t, x)}_{\dot{x} \text{ in (29a)}} \leq 0. \tag{29c}$$

Consider now the system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{a}} \end{bmatrix} = \begin{bmatrix} f(t, x) \\ 0_m \end{bmatrix} + \underbrace{\begin{bmatrix} \tilde{E}_\alpha(t, x, \hat{a})(a - \hat{a}) \\ E_\alpha(t, x, \hat{a}) \end{bmatrix}}_{\text{top is “linear” in } a - \hat{a}}, \tag{29d}$$

where

- $a \in \mathbb{R}^m$ (for some $m \in \mathbb{N}$) is an unknown disturbance;
- $\hat{a} \in \mathbb{R}^m$ is an estimate of the disturbance $a \in \mathbb{R}^m$ whose dynamics we wish to design;
- $\tilde{E}_\alpha(t, x, \hat{a}) \in \mathbb{R}^{n \times m}$ determines how the estimation error (i.e., $a - \hat{a}$) affects the dynamics $\dot{x}$, and which can depend on the time instant t, the state x and the estimate $\hat{a}$ (but, it cannot depend on the unknown disturbance a);
- $E_\alpha(t, x, \hat{a}) \in \mathbb{R}^m$ is the estimator dynamics we wish to design, which may depend on the time instant t, the state x and the estimate $\hat{a}$ (but, it cannot depend on the unknown disturbance a).

Given the Lyapunov function V in (29b), let k be some positive (integral) gain, and consider then the estimator dynamics

$$E_\alpha(t, x, \hat{a}) := k \tilde{E}_\alpha(t, x, \hat{a})^T \nabla_2 V(t, x). \tag{29e}$$

It then follows that for the extended Lyapunov function

$$\begin{aligned} &\bar{V} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty), \\ &\bar{V}(t, x, \hat{a}) := V(t, x) + \frac{1}{k} \frac{\|a - \hat{a}\|^2}{2}, \end{aligned} \tag{29f}$$

it holds that

$$\begin{aligned} \text{“}\dot{\bar{V}}(t, x, \hat{a})\text{”} &= d_1 \bar{V}(t, x, \hat{a}) + d_2 \bar{V}(t, x, \hat{a}) \dot{x} + d_3 \bar{V}(t, x, \hat{a}) \dot{\hat{a}} \\ &= W(t, x) + d_2 V(t, x) \tilde{E}_\alpha(t, x, \hat{a})(a - \hat{a}) + \frac{1}{k} \langle a - \hat{a}, -\tilde{E}_\alpha(t, x, \hat{a}) \rangle \quad \because \text{(29d) and (29c)} \\ &= W(t, x) - \frac{1}{k} \left\langle a - \hat{a}, E_\alpha(t, x) - k \tilde{E}_\alpha(t, x, \hat{a})^T \nabla_2 V(t, x) \right\rangle \\ &= W(t, x) \quad \because \text{(29e)} \\ &\leq 0. \quad \because \text{(29c)} \end{aligned}$$

This procedure describes the steps we will be following four times in the later sections (where we design four estimators).

C. Necessity of a p-differentiable update law (for some $p \in \mathbb{N}$)

As emphasized in Remark 9, in the what follows, a 2-differentiable update law will be required, which also guarantees that the estimate obtained from integration of the update law remains in some ball of pre-specified radius. With the latter in mind, let

- $e \in \mathbb{R}^n$ be an unknown disturbance, and $\mathfrak{e} \in \mathbb{R}$ be a known upper-bound on its norm, i.e., $\|e\| \leq \mathfrak{e}$;
- $\hat{e} \in \mathbb{R}^n$ be an estimate of the unknown disturbance, and $\hat{\mathfrak{e}} \in \mathbb{R}$ be a pre-specified upper-bound on its norm, i.e., $\|\hat{e}\| < \hat{\mathfrak{e}}$, where the only constraint to be satisfied is that $\mathfrak{e} < \hat{\mathfrak{e}}$;

We then assume that an update law exists, of the form

$$\mathbb{R}^n \times \mathbb{R}^n \ni (\dot{\hat{e}}, \hat{e}) \mapsto \text{Proj}_{\hat{\mathfrak{e}},p}\left(\dot{\hat{e}}, \hat{e}\right) \in \mathbb{R}^n, \tag{30a}$$

which satisfies the following properties (recall that $\mathbb{B}_r^n := \{x \in \mathbb{R}^n : \|x\| < r\}$)

1) for any $\mu \in \mathbb{R}^n$ (it may be time-varying) and for some $r \in (\mathfrak{e}, \hat{\mathfrak{e}})$,
$$\bar{\mathbb{B}}_r^n \text{ is positively invariant for } \dot{\hat{e}} = \text{Proj}_{\mathfrak{e},p}(\mu, \hat{e}). \tag{30b}$$
2) for all $e \in \bar{\mathbb{B}}_{\mathfrak{e}}^n$ and for all $(\mu, \hat{e}) \in \mathbb{R}^n \times \mathbb{B}_{\hat{\mathfrak{e}}}^n$,
$$-\left\langle e - \hat{e}, \text{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) - \dot{\hat{e}} \right\rangle \leq 0 \Leftrightarrow \left\langle e - \hat{e}, \text{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) \right\rangle \geq \langle e - \hat{e}, \mu \rangle. \tag{30c}$$
3) $\text{Proj}_{\hat{\mathfrak{e}},p}$ is $\mathcal{C}^p$ in $\mathbb{R}^n \times \mathbb{B}_{\hat{\mathfrak{e}}}^n$.
4) If $\mu : \mathbb{R} \to \mathbb{R}^n$ is continuous and $\lim_{t\to\infty} \mu(t) = 0_n$, then, along a solution of $\dot{\hat{e}}(t) = \text{Proj}_{\hat{\mathfrak{e}},p}(\mu(t), \hat{e}(t))$ with $\hat{e}(0) \in \bar{\mathbb{B}}_{\mathfrak{e}}^n$, it follows that $\lim_{t\to\infty} \dot{\hat{e}}(t) = 0_n$ (which does not imply that $\lim_{t\to\infty} \hat{e}(t)$ exists).

Loosely speaking, $\text{Proj}_{\hat{\mathfrak{e}},p}(\dot{\hat{e}}, \hat{e})$ accepts $\dot{\hat{e}}$ from a standard update law, and modifies it if the estimate $\hat{e}$ exits the ball $\bar{\mathbb{B}}_{\mathfrak{e}}^n$ (within which the unknown disturbance e is known to belong to) and such that the estimate $\hat{e}$ remains in the pre-specified ball $\mathbb{B}_{\hat{\mathfrak{e}}}^n$.

Remark 10: Nothing that the unknown disturbance e is upper-bound by $\mathfrak{e}$, i.e. that $\|e\| \leq \mathfrak{e} \Leftrightarrow e \in \bar{\mathbb{B}}_{\mathfrak{e}}^n$, is only important when guaranteeing that (30c) is satisfied: this is the property that is latter used to guarantee that a given Lyapunov function is non-increasing along solutions of the system.

1) Example 1 of $\text{Proj}_{\hat{\mathfrak{e}},p}$: All the projectors used later in the simulations have this form, but we emphasize that the specific form they take is not important, provided that the requirements set above are satisfied. A projector, which is smooth (infinitely differentiable), is given by

$$\begin{aligned}
&\text{Proj}_{\hat{\mathfrak{e}},1} \equiv \text{Proj}_{\hat{\mathfrak{e}},2} \equiv \cdots \equiv \text{Proj}_{\hat{\mathfrak{e}},\infty} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \\
&\text{Proj}_{\hat{\mathfrak{e}},1}(\mu, \hat{e}) \equiv \text{Proj}_{\hat{\mathfrak{e}},2}(\mu, \hat{e}) \equiv \cdots \equiv \text{Proj}_{\hat{\mathfrak{e}},\infty}(\mu, \hat{e}) := \mu - f\left(\frac{\|\hat{e}\|^2 - \mathfrak{e}^2}{\hat{\mathfrak{e}}^2 - \mathfrak{e}^2}\right) \frac{1}{2} \left(\left\langle \frac{\hat{e}}{\mathfrak{e}}, \mu \right\rangle + \sqrt{\left\langle \frac{\hat{e}}{\mathfrak{e}}, \mu \right\rangle^2 + \bar{\mu}^2} \right) \frac{\hat{e}}{\mathfrak{e}},
\end{aligned} \tag{31a}$$

where $\mathfrak{e}, \hat{\mathfrak{e}}$ are positive constants (whose meaning has been provided before), $\bar{\mu} > 0$ is also a positive constant, and where f is a smooth function (not analytic though) given by

$$\begin{aligned}
&f : \mathbb{R} \to \mathbb{R} \\
&f(x) := \begin{cases} 0 & \text{if } x \leq 0\,, \\ \tilde{f}(x) := e^{\kappa \frac{x-1}{x}} & \text{if } x > 0 \end{cases}
\end{aligned} \tag{31b}$$

for some positive κ. All the four properties listed above are indeed satisfied the projector function in (31a) – see Remark 11. The phase plot of this projector is shown in Fig. 5, for $\kappa = 1$, and the impact of $\bar{\mu}$ in the update law can be visualized in the same figure – the bigger the $\bar{\mu}$, the stronger the "repulsion" is which keeps the estimate $\hat{e}$ away from the boundary where $\hat{e} = \hat{\mathfrak{e}}$.

2) Example 2 of $\text{Proj}_{\hat{\mathfrak{e}},p}$: Let us provide another example of a possible projector operator, satisfying the properties required above, which is taken from [26]. Consider then the projector

$$\begin{aligned}
&\text{Proj}_{\hat{\mathfrak{e}},p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \\
&\text{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) := \mu - f\left(\frac{\|\hat{e}\|^2 - \mathfrak{e}^2}{\hat{\mathfrak{e}}^2 - \mathfrak{e}^2}\right) \frac{1}{2} \left(\left\langle \frac{\hat{e}}{\mathfrak{e}}, \mu \right\rangle + \sqrt{\left\langle \frac{\hat{e}}{\mathfrak{e}}, \mu \right\rangle^2 + \bar{\mu}^2} \right) \frac{\hat{e}}{\mathfrak{e}},
\end{aligned} \tag{32a}$$

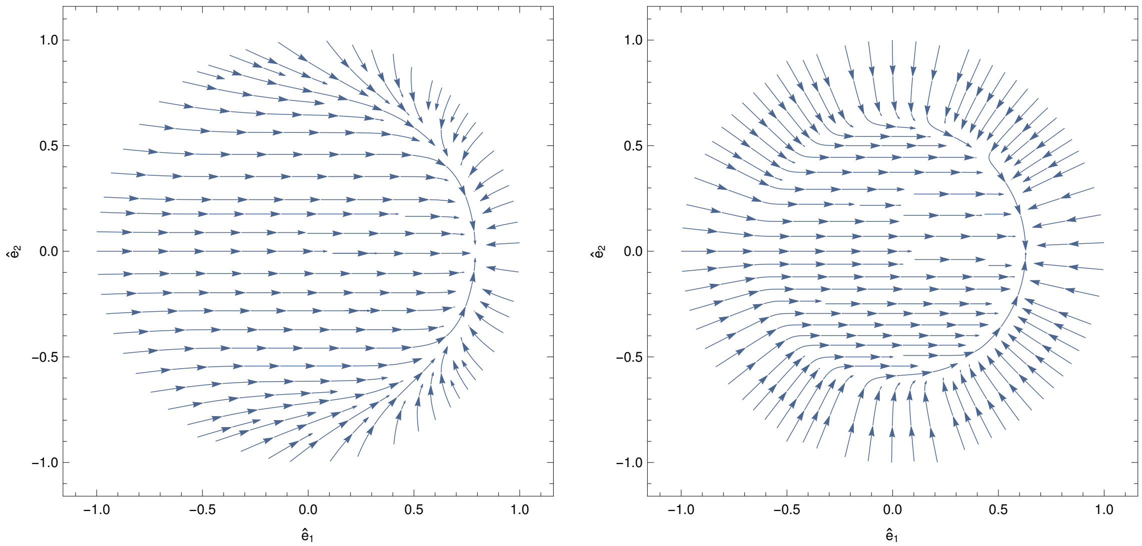


Fig. 5: Phase-plot for two dimensional adaption law, namely, $\dot{\hat{e}} = \mathrm{Proj}_{1,\infty}(\mu, \hat{e})$ with $\mu = (0.1, 0) \in \mathbb{R}^2$ (and $\mathfrak{e} = 0.5$, $\hat{\mathfrak{e}} = 1$, $\kappa = 1$): left – $\bar{\mu} = 0.1$; right – $\bar{\mu} = 10$.

where $\mathfrak{e}, \hat{\mathfrak{e}}$ are positive constants (whose meaning has been provided before), $\bar{\mu} > 0$ is also a positive constant, and where $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^p$ function given by

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ \tilde{f}(x) := x^{p+1} & \text{if } x > 0 \end{cases} \tag{32b}$$

All the four properties listed above are indeed satisfied the projector function in (32a) – see Remark 11.

Remark 11: Let us prove that both projectors function in (31a) and in (32a) satisfy the four properties listed at the beginning of this subsection. The proofs that follow are inspired on the proofs found in [26], but we note, however, that the fourth property listed at the beginning of this section is not studied in [26]. In order to prove the condition in (30b), consider the function

$$V : \mathbb{R}^n \ni \hat{e} \mapsto V(\hat{e}) := \frac{1}{2}\|\hat{e}\|^2 \in [0, \infty),$$

whose sub-level sets are closed balls around the origin. We are interested in showing positive invariance of a certain closed ball around the origin, i.e., positive invariance of $\bar{\mathbb{B}}^n_r$ for some $r < \hat{\mathfrak{e}}$; therefore it proves useful to consider the evolution of the function $t \mapsto V(\hat{e}(t))$ along a solution of $\frac{d}{dt}\hat{e}(t) = \mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu(t), \hat{e}(t))$ with initial condition $\hat{e}(0) \in \bar{\mathbb{B}}^n_r$ and for an arbitrary exogenous input $t \mapsto \mu(t)$. For that purpose, we compute the time derivative of $t \mapsto V(\hat{e}(t))$, which is given by

$$\dot{V}(\hat{e}) = dV(\hat{e})\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) = \left\langle \hat{e}, \mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) \right\rangle. \tag{33a}$$

Let us then consider the supremum (it turns out to be a maximum) of that derivative along the boundary of the sub-level set $\bar{\mathbb{B}}^n_{\hat{\mathfrak{e}}}$, i.e.

$$\begin{aligned}
\sup_{\|\hat{e}\| = \hat{\mathfrak{e}}, \mu \in \mathbb{R}^n} dV(\hat{e})\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) &= \sup_{\|\hat{e}\| = \hat{\mathfrak{e}}, \mu \in \mathbb{R}^n} \left\langle \hat{e}, \mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu, \hat{e}) \right\rangle && \because (33a) \\
&= \sup_{\|\hat{e}\| = \hat{\mathfrak{e}}, \mu \in \mathbb{R}^n} \langle \hat{e}, \mu \rangle - f\left(\frac{\|\hat{e}\|^2 - \mathfrak{e}^2}{\hat{\mathfrak{e}}^2 - \mathfrak{e}^2}\right) \frac{1}{2}\left(\langle \hat{e}, \mu \rangle + \sqrt{\langle \hat{e}, \mu \rangle^2 + (\mathfrak{e}\bar{\mu})^2}\right) \frac{\|\hat{e}\|^2}{\mathfrak{e}^2} && \because (31a) \text{ or } (32a) \\
&= \sup_{\|\hat{e}\| = \hat{\mathfrak{e}}, \mu \in \mathbb{R}^n} \langle \hat{e}, \mu \rangle - \frac{1}{2}\left(\langle \hat{e}, \mu \rangle + \sqrt{\langle \hat{e}, \mu \rangle^2 + (\mathfrak{e}\bar{\mu})^2}\right) \frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2} && \because \|\hat{e}\| = \hat{\mathfrak{e}} \\
&= \sup_{x \in \mathbb{R}} \left(x - \frac{1}{2}\left(x + \sqrt{x + (\mathfrak{e}\bar{\mu})^2}\right) \frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}\right) && \because x = \langle \hat{e}, \mu \rangle \\
&= -\bar{\mu}\sqrt{\hat{\mathfrak{e}}^2 - \mathfrak{e}^2} < 0. && \because \hat{\mathfrak{e}} > \mathfrak{e}
\end{aligned} \tag{33b}$$

That is, the maximum time derivative of the Lyapunov function on the boundary of the sub-level set $\bar{\mathbb{B}}^n_{\hat{\mathfrak{e}}}$ is negative, and that suffices to conclude (based on the continuity properties of V and $\mathrm{Proj}_{\hat{\mathfrak{e}},p}$) that there exists an $r < \hat{\mathfrak{e}}$ such that $\bar{\mathbb{B}}^n_r$ is positively invariant. In particular, if $\hat{e}(0) = 0_n \in \bar{\mathbb{B}}^n_r$, then $\hat{e}(t) \in \bar{\mathbb{B}}^n_r$ for all $t \geq 0$.

In order to prove the condition in (30c), let us compute

$$\begin{aligned}\langle e-\hat{e}, -\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e})+\mu\rangle &= \left\langle e-\hat{e}, f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{1}{2}\left(\langle\hat{e},\mu\rangle+\sqrt{\langle\hat{e},\mu\rangle^2+(\mathfrak{e}\bar{\mu})^2}\right)\frac{\hat{e}}{\mathfrak{e}^2}\right\rangle \qquad \because \text{(31a) or (32a)}\\ &= \underbrace{f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)}_{(\star)}\underbrace{\frac{1}{2}\left(\langle\hat{e},\mu\rangle+\sqrt{\langle\hat{e},\mu\rangle^2+(\mathfrak{e}\bar{\mu})^2}\right)}_{\geq 0}\underbrace{\frac{\langle e,\hat{e}\rangle-\langle\hat{e},\hat{e}\rangle}{\mathfrak{e}^2}}_{(\dagger)}.\end{aligned}$$

Note then that, when $\|\hat{e}\|^2 \leq \mathfrak{e}^2$, then $(\star)=0$ (while $(\dagger)$ may be positive, negative or zero); on the other hand, when $\|\hat{e}\|^2 > \mathfrak{e}^2$, then $(\star)>0$ and $(\dagger)\leq 0$ because $\|e\|\leq\mathfrak{e}$. All together, this implies that $\langle e-\hat{e}, -\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e})\rangle \leq \langle e-\hat{e},-\mu\rangle$, as required by (30c).

In order to prove the third condition, it suffices to check that the function $\mathbb{R}\ni x\mapsto f(x)$ is $\mathcal{C}^p$ (the function f in (31b) is $\mathcal{C}^\infty$, and thus it is $\mathcal{C}^p$; however, the function f in (32b) is only $\mathcal{C}^p$).

In order to prove the fourth and final condition, first note that

$$\mathrm{Proj}_{\hat{\mathfrak{e}},p}(0_n,\hat{e}) = \begin{cases} 0_n & \text{if } \|\hat{e}\|\leq\mathfrak{e}\\ -\frac{\bar{\mu}}{2}\tilde{f}\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{\hat{e}}{\mathfrak{e}} & \text{if } \|\hat{e}\|>\mathfrak{e}\end{cases},$$

and that (application of the mean value theorem, below)

$$\begin{aligned}&\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e}) = \mathrm{Proj}_{\hat{\mathfrak{e}},p}(0_n,\hat{e}) + \int_0^1 d_1\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e})d\sigma\mu, \text{ where}\\ &d_1\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e}) = I_n - f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{1}{2}\left(1+\frac{\langle\hat{e},\mu\rangle}{\sqrt{\langle\hat{e},\mu\rangle^2+(\mathfrak{e}\bar{\mu})^2}}\right)\frac{\hat{e}\hat{e}^T}{\mathfrak{e}^2}\in\mathbb{R}^{n\times n}.\end{aligned}$$

In the latter in mind, it follows that, for any $\hat{e}\in\bar{\mathbb{B}}^n_{\hat{\mathfrak{e}}}$,

$$\begin{aligned}\dot{V}(\hat{e}) &:= dV(\hat{e})\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e})\\ &= \left\langle\hat{e},\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\mu,\hat{e})\right\rangle\\ &= \left\langle\hat{e},\mathrm{Proj}_{\hat{\mathfrak{e}},p}(0_n,\hat{e})+\int_0^1 d_1\mathrm{Proj}_{\hat{\mathfrak{e}},p}(\sigma\mu,\hat{e})d\sigma\mu\right\rangle\\ &= \left\langle\hat{e},\mathrm{Proj}_{\hat{\mathfrak{e}},p}(0_n,\hat{e})\right\rangle + \int_0^1 \underbrace{\left(1-\underbrace{f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)}_{0<\ldots\leq 1}\underbrace{\frac{1}{2}\left(1+\frac{\langle\hat{e},\sigma\mu\rangle}{\sqrt{\langle\hat{e},\sigma\mu\rangle^2+(\mathfrak{e}\bar{\mu})^2}}\right)}_{-1<\ldots\leq+1}\frac{\langle\hat{e},\hat{e}\rangle}{\mathfrak{e}^2}\right)}_{|\ldots|\leq 1+\frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}}d\sigma\langle\hat{e},\mu\rangle\\ &\leq -f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{\|\hat{e}\|^2}{\mathfrak{e}}\frac{\bar{\mu}}{2}+\left(1+\frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}\right)\|\hat{e}\|\|\mu\|\\ &= -\|\hat{e}\|\frac{\bar{\mu}}{2}\left(f\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{\|\hat{e}\|}{\mathfrak{e}}-\frac{1}{2}\left(1+\frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}\right)\frac{\|\mu\|}{\bar{\mu}}\right).\end{aligned}$$

Let us then define

$$(\mathfrak{e},\infty)\ni x\mapsto h(x):=\tilde{f}\left(\frac{x^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{x}{\mathfrak{e}}$$

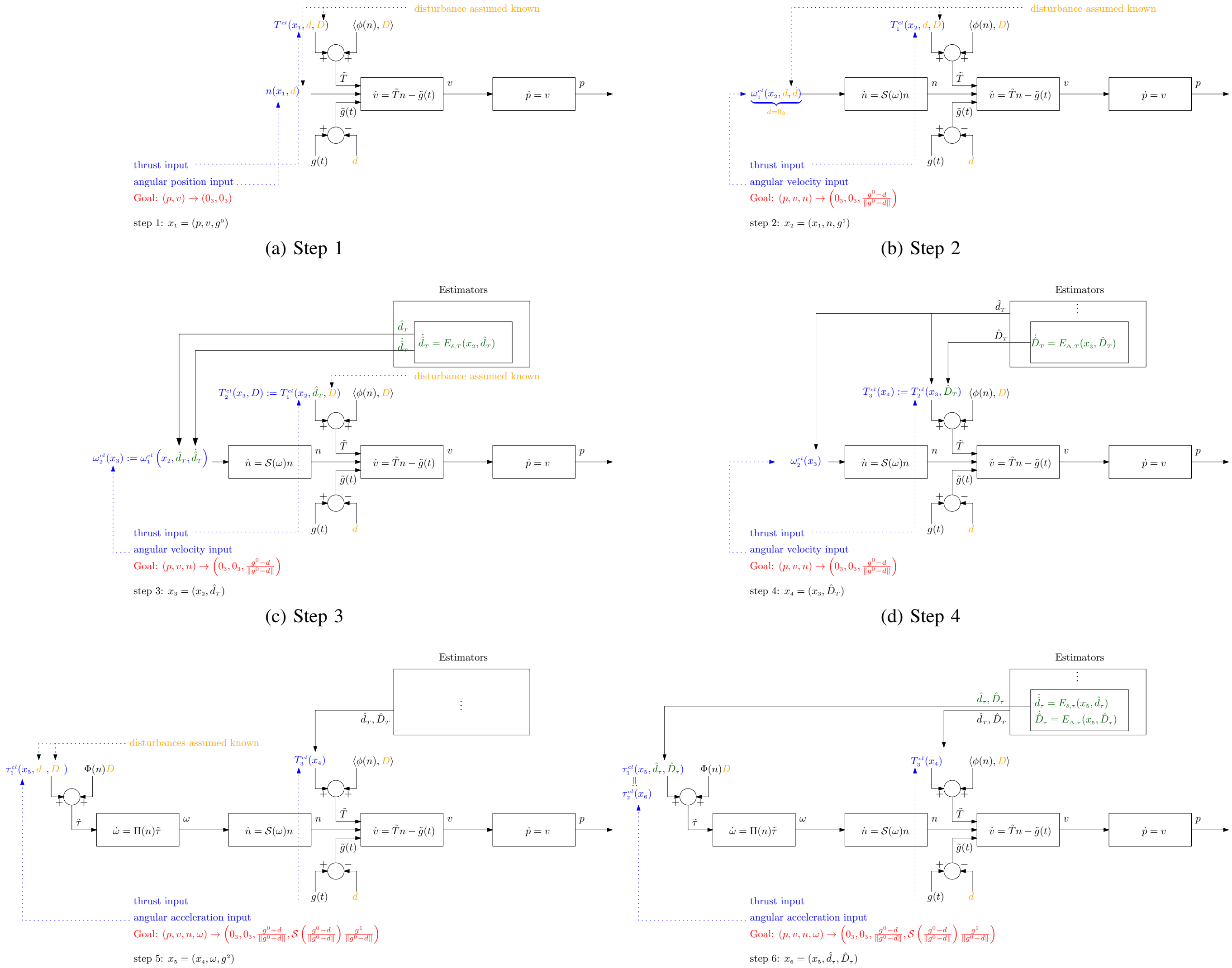
where we emphasize that and inverse exists (even if it cannot be provided explicitly) since it consists of an increasing function (the product of the two positive increasing functions yields an increasing function), and, moreover, the inverse is also an increasing function. With the latter in mind, and given a $\sigma\in(0,1)$, if we further restrict

$$\|\hat{e}\|>h^{-1}\left(\frac{1}{\sigma}\frac{1}{2}\left(1+\frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}\right)\frac{\|\mu\|}{\bar{\mu}}\right)=:\chi(\|\mu\|)\Leftrightarrow\frac{1}{2}\left(1+\frac{\hat{\mathfrak{e}}^2}{\mathfrak{e}^2}\right)\frac{\|\mu\|}{\bar{\mu}}<\sigma\tilde{f}\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{\|\hat{e}\|}{\mathfrak{e}}$$

where we emphasize that $\chi(0)=\mathfrak{e}$, it follows that

$$\begin{aligned}\dot{V}(\hat{e}) &\leq -(1-\sigma)\frac{\bar{\mu}}{2}\tilde{f}\left(\frac{\|\hat{e}\|^2-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{\|\hat{e}\|^2}{\mathfrak{e}}<0 \qquad \text{for } \|\hat{e}\|>\chi(\|\mu\|),\\ &= -(1-\sigma)\frac{\bar{\mu}}{2}\tilde{f}\left(\frac{2V(\hat{e})-\mathfrak{e}^2}{\hat{\mathfrak{e}}^2-\mathfrak{e}^2}\right)\frac{2V(\hat{e})}{\mathfrak{e}}<0 \quad \text{for } V(\hat{e})>\frac{1}{2}\chi(\|\mu\|)^2.\end{aligned}\tag{33c}$$

Thus given a exogenous $\mu:\mathbb{R}\to\mathbb{R}$, which is continuous and which vanishes asymptotically (i.e., $\lim_{t\to\infty}\mu(t)=0_n$), we can then conclude that $\limsup_{t\to\infty}V(\hat{e}(t))\leq\frac{1}{2}\chi(0)^2\Rightarrow\limsup_{t\to\infty}\|\hat{e}(t)\|\leq\mathfrak{e}$.



(a) Step 1

(b) Step 2

(c) Step 3

(d) Step 4

(e) Step 5

(f) Step 6

Fig. 6: Backstepping steps

We can then finally conclude that

$$
\begin{aligned}
\lim_{t\to\infty} \dot{\hat{e}}(t) &= \lim_{t\to\infty} \mathrm{Proj}_{\mathfrak{e},p}(\mu(t), \hat{e}(t)) \\
&= \lim_{t\to\infty} \mathrm{Proj}_{\mathfrak{e},p}(0_n, \hat{e}(t)) && \because \lim_{t\to\infty} \mu(t) = 0_n \\
&= 0_n && \because \limsup_{t\to\infty} \|\hat{e}(t)\| \leq \mathfrak{e} \text{ and (33c).}
\end{aligned}
$$

As a final remark, we emphasize that this is an ISS-like property (see [27]).

D. Backstepping steps

Remark 12: For convenience, we need to introduce some constants, that need to be specified and/or known for the purposes of the controller design.

- we pick $\underline{g} > 0$ such that $\underline{g} < \inf_{t\in\mathbb{R}} \|g(t)\|$ $(\inf_{t\in\mathbb{R}} \|g(t)\| > 0$ – see (25d));

- the disturbance $d \in \mathbb{R}^3$ needs be bounded in norm by some known upper bound $\bar{d} > 0$ (i.e., $\|d\|_{\mathbb{R}^3} \leq \bar{d}$)[6], and we pick $\bar{\hat{d}}$ such that $\bar{\hat{d}} > \bar{d}$ – later we design an estimator $\hat{d}$ for the disturbance d, and $\hat{d}$ will be guaranteed to remain bounded in norm by some constant smaller than $\bar{\hat{d}}$ (see Section VII-C);
- we pick a constant $\bar{u} > 0$ such that $\underline{g} - (\bar{u} + \bar{\hat{d}}) > 0$ – loosely speaking, $\bar{u}$ will represent how much linear acceleration the system will be able to provide ($\|\dot{v}\| \lessapprox \bar{u}$);
- the disturbance $D \in \mathbb{R}^3$ also needs be bounded in norm by some known upper bound $\bar{D} > 0$ (i.e., $\|D\|_{\mathbb{R}^3} \leq \bar{D}$)[7], and we pick $\bar{\hat{D}}$ such that $\bar{\hat{D}} > \bar{D}$;

The choice of constants $\underline{g}, \bar{\hat{d}}, \bar{u}$ is always feasible if $\inf_{t\in\mathbb{R}} \|g(t)\| - d > 0$ as required in Problem 3: however, the closer $\inf_{t\in\mathbb{R}} \|g(t)\| - d$ is to zero, the less input action $\bar{u} > 0$ is available, which means a slower convergence ($\|\dot{v}\| \lessapprox \bar{u}$); and the less disturbance overshoot $\bar{\hat{d}} - \bar{d} > 0$ is available, which means the estimator update law $\hat{d}$ can be very stiff, in order to guarantee that the estimate $\hat{d}$ norm does not overshoot over $\bar{\hat{d}}$ once the same norm crosses $\bar{d}$.

Let n be a positive integer, r be a positive number, and recall the definitions

$$\mathbb{B}^n_r := \{x \in \mathbb{R}^n : \|x\| < r\},$$
$$\mathbb{C}^n_r := \{x \in \mathbb{R}^n : \|x\| > r\} = \mathbb{R}^n \backslash \bar{\mathbb{B}}^n_r,$$

of the open-ball in $\mathbb{R}^n$ of radius r, and of the complement of a closed-ball in $\mathbb{R}^n$ of radius r, respectively.

For the controller design, we follow a backsteeping procedure with six steps, which we illustrate in Fig. 6. The necessity and importance of each step will be motivated throughout this section. Recall Remark 8, and consider then the following notation for the states of each step[8]

$$x_1 \in \mathbb{X}_1 :\Leftrightarrow (p, v, g^0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{C}^3_{\underline{g}} \tag{34a}$$
$$x_2 \in \mathbb{X}_2 :\Leftrightarrow (x_1, n, g^1) \in \mathbb{X}_1 \times \mathbb{S}^2 \times \mathbb{R}^3 \tag{34b}$$
$$x_3 \in \mathbb{X}_3 :\Leftrightarrow (x_2, \hat{d}_T) \in \mathbb{X}_2 \times \mathbb{B}^3_{\bar{\hat{d}}} \tag{34c}$$
$$x_4 \in \mathbb{X}_4 :\Leftrightarrow (x_3, \hat{D}_T) \in \mathbb{X}_3 \times \mathbb{R}^3 \tag{34d}$$
$$x_5 \in \mathbb{X}_5 :\Leftrightarrow (x_4, \omega, g^2) \in \mathbb{X}_4 \times T_n\mathbb{S}^2 \times \mathbb{R}^3 \tag{34e}$$
$$x_6 \in \mathbb{X}_6 :\Leftrightarrow (x_5, \hat{d}_\tau, \hat{D}_\tau) \in \mathbb{X}_5 \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{34f}$$

Consider then the corresponding vector fields. In the first step, we consider the system

$$\dot{x}_1 = X_{1,d,D}(x_1, (T, n, g^1)) :\Leftrightarrow \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{g}^0 \end{bmatrix} = \begin{bmatrix} v \\ (T + \langle \phi(n), D \rangle)\, n - g^0 + d \\ g^1 \end{bmatrix}, \tag{35a}$$

where we design the input $(T, n) = (T^{cl}(x_1, d, D), n^{cl}(x_1, d))$ – note the dependencies, assuming that d and D are known, and such that $(p, v) \to (0_3, 0_3)$, as illustrated in Fig. 6a. In the second step, we consider the system

$$\dot{x}_2 = X_{2,d,D}(x_2, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} = \begin{bmatrix} X_{1,d,D}(x_1, (T, n, g^1)) \\ \mathcal{S}(\omega)\, n \\ g^2 \end{bmatrix}, \tag{35b}$$

where we lift the assumption that we control the angular position n and modify the thrust control law to $T = T^{cl}_1(x_2, d, D)$, and design the angular velocity input $\omega = \omega^{cl}_1(x_2, d, \dot{d})|_{\dot{d}=0_3}$, assuming that both d and D are known, and such that $(p, v, n) \to \left(0_3, 0_3, \frac{g^0 - d}{\|g^0 - d\|}\right)$, as illustrated in Fig. 6b. In the third step, we consider the system

$$\dot{x}_3 = X_{3,d,D}(x_3, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{\hat{d}}_T \end{bmatrix} = \begin{bmatrix} X_{2,d,D}(x_2, (T, \omega, g^2)) \\ E_{\delta,T}(x_2, \hat{d}_T) \end{bmatrix}, \tag{35c}$$

where we lift the assumption that the disturbance d is known in the control laws ($\omega = \omega^{cl}_1(x_2, d, \dot{d})|_{\dot{d}=0_3}$ and $T = T^{cl}_1(x_2, d, D)$) and replaced it with its estimate $\hat{d}_T$ (i.e., $\omega = \omega^{cl}_2(x_3) := \omega^{cl}_1(x_2, \hat{d}_T, , \dot{\hat{d}}_T)$ and $T = T^{cl}_2(x_3, D) := T^{cl}_1(x_2, \hat{d}_T, D)$); and we

[6]From the physical system, we know that $d = \frac{w}{m}$; thus, one needs to know that the wind force on the load $w \in \mathbb{R}^3$ is bounded in norm by some known upper bound $\bar{w} > 0$ – i.e., $\|w\|_{\mathbb{R}^3} \leq \bar{w}$ – in which case, $\bar{d} = \frac{\bar{w}}{m}$

[7]From the physical system, we know that $D = \frac{W}{M} - \frac{w}{m}$; thus, one needs to know that the wind forces $w, W \in \mathbb{R}^3$ are bounded in norm by some known upper bounds $\bar{w}, \bar{W} > 0$ (i.e., $\|w\|_{\mathbb{R}^3} \leq \bar{w}$ and $\|W\|_{\mathbb{R}^3} \leq \bar{W}$), in which case we can pick $\bar{D} = \frac{\bar{W}}{M} + \frac{\bar{w}}{m}$ (i.e., $\|D\|_{\mathbb{R}^3} \leq \bar{D}$). We note, however, that this is a very conservative estimate as it assumes that the winds W and w blow from opposite orientations, which is unlikely.

[8]Technically speaking, the set $\mathbb{X}_5$ in (34e) cannot be expressed as a Cartesian product $\mathbb{X}_4 \times T_n\mathbb{S}^2 \times \mathbb{R}^3$. The correct formulation is $\{(p, v, g^0, n, g^1, \hat{d}_T, \hat{D}_T, \omega, g^2) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : g^0 \in \mathbb{C}^3_{\underline{g}} \text{ and } n \in \mathbb{S}^2 \Leftrightarrow \langle n, n \rangle = 1 \text{ and } \omega \in T_n\mathbb{S}^2 \Leftrightarrow \langle \omega, n \rangle = 0\}$

design the estimator $\hat{d}_T$ with dynamics $E_{\delta,T}$ (see Remark 13), and such that $(p, v, n) \to \left(0_3, 0_3, \frac{g^0 - d}{\|g^0 - d\|}\right)$, as illustrated in Fig. 6c. In the fourth step, we consider the system

$$\dot{x}_4 = X_{4,d,D}(x_4, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_3 \\ \dot{\hat{D}}_T \end{bmatrix} = \begin{bmatrix} X_{3,d,D}(x_3, (T, \omega, g^2)) \\ E_{\Delta,T}(x_3, \hat{D}_T) \end{bmatrix}, \tag{35d}$$

where we lift the assumption that the disturbance D is known in the thrust control law ($T = T_2^{cl}(x_3, D)$) and replaced with with its estimate $\hat{D}_T$ (i.e., $T = T_3^{cl}(x_4) := T_2^{cl}(x_3, \hat{D}_T)$); and design an estimator $\hat{D}_T$ with dynamics $E_{\Delta,T}$, and such that $(p, v, n) \to \left(0_3, 0_3, \frac{g^0 - d}{\|g^0 - d\|}\right)$, as illustrated in Fig. 6d. In the fifth step, we consider the system

$$\dot{x}_5 = X_{5,d,D}(x_5, (T, \tau, g^3)) :\Leftrightarrow \begin{bmatrix} \dot{x}_4 \\ \dot{\omega} \\ \dot{g}^2 \end{bmatrix} = \begin{bmatrix} X_{4,d,D}(x_4, (T, \omega, g^2)) \\ \Pi(n)(\tau + \Phi(n) D) \\ g^3 \end{bmatrix}, \tag{35e}$$

where we lift the assumption that we control the angular velocity ω and design the angular acceleration input $\tau = \tau_1^{cl}(x_5, d, D)$, assuming that both d and D are known, and such that $(p, v, n, \omega) \to \left(0_3, 0_3, \frac{g^0 - d}{\|g^0 - d\|}, \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right) \frac{g^1}{\|g^0 - d\|}\right)$, as illustrated in Fig. 6e. And finally, in the sixth step, we consider the system

$$\dot{x}_6 = X_{6,d,D}(x_6, (T, \tau, g^3)) :\Leftrightarrow \begin{bmatrix} \dot{x}_5 \\ \dot{\hat{d}}_\tau \\ \dot{\hat{D}}_\tau \end{bmatrix} = \begin{bmatrix} X_{5,d,D}(x_5, (T, \tau, g^3)) \\ E_{\delta,\tau}(x_5, \hat{d}_T) \\ E_{\Delta,\tau}(x_5, \hat{D}_T) \end{bmatrix}, \tag{35f}$$

where we lift the assumption that the disturbances d and D are known in the angular acceleration control law ($\tau = \tau_1^{cl}(x_5, d, D)$) and replaced them with with theirs estimates $\hat{d}_\tau$ and $\hat{D}_\tau$ (i.e., $\tau = \tau_2^{cl}(x_6) := \tau_1^{cl}(x_5, \hat{d}_\tau, \hat{D}_\tau)$); and design the estimators $\hat{d}_\tau$ and $\hat{D}_\tau$ with dynamics $E_{\delta,\tau}$ and $E_{\Delta,\tau}$ respectively, and such that $(p, v, n, \omega) \to \left(0_3, 0_3, \frac{g^0 - d}{\|g^0 - d\|}, \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right) \frac{g^1}{\|g^0 - d\|}\right)$, as illustrated in Fig. 6f.

Remark 13: The estimators $\hat{d}_T$, $\hat{D}_T$, $\hat{d}_\tau$, $\hat{D}_\tau$ have dynamics named $E_{\delta,T}$, $E_{\Delta,T}$, $E_{\delta,\tau}$, $E_{\Delta,\tau}$. We emphasize that those dynamics do not depend on d or D, which are unknown. The reason behind their naming is the following: δ is associated to d, and Δ is associated to D; T is associated to the estimators designed at the thrust input level, and τ is associated to the estimators designed at the angular acceleration input level.

Remark 14: Notice the estimator dynamics $\dot{\hat{D}}_T$ depends on the estimate $\hat{d}_T$, while the estimator dynamics $\dot{\hat{d}}_T$ is independent of the estimate $\hat{D}_T$. This is a reflection of the different nature of the disturbances d and D, as discussed in Remark 9.

Remark 15: In (35a)–(35f), we introduced the vector fields for each step, and naturally they depend on the unknown disturbances d and D. In order to illustrate this dependence, we denoted $X_{i,d,D}$ as the vector field at step $i \in \{1, \cdots, 6\}$, with d and D as sub-indexes. One point to emphasize is that these vector fields are affine in both disturbances, i.e.,

$$\underbrace{X_{i,d,D}}_{\text{unknown}} = \underbrace{X_{i,\hat{d},\hat{D}}}_{\text{known}} + \underbrace{E_{\delta,i}}_{\text{known}}(d - \hat{d}) + \underbrace{E_{\Delta,i}}_{\text{known}}(D - \hat{D}),$$

for some known $E_{\delta,i}$ and $E_{\Delta,i}$, and for any $\hat{d}$ and $\hat{D}$. In brief, we cannot use the vector field $X_{i,d,D}$ for the purposes of control, but we can use instead $X_{i,\hat{d},\hat{D}}$ where we replace d and D by some estimates $\hat{d}$ and $\hat{D}$. The error when making this approximation ($X_{i,d,D} - X_{i,\hat{d},\hat{D}}$) is linear with respect to the estimation errors $d - \hat{d}$ and $D - \hat{D}$, and we use the matrices $E_{\delta,i}$ and $E_{\Delta,i}$ to design update laws for the estimates $\hat{d}$ and $\hat{D}$, respectively.

Throughout all the steps in the backstepping procedure, we define the open-loop vectors field in the form

$$\begin{aligned} X_{i,d,D} : (\text{state}, \text{input}) &\mapsto X_{i,d,D}(\text{state}, \text{input}) \\ (x_i, u_i) &\mapsto X_{i,d,D}(x_i, u_i), \end{aligned}$$

and once we define a control law $u_i^{cl} : x_i \mapsto u_i^{cl}(x_i)$, we denote the closed-loop vector field by

$$X_{i,d,D}^{cl} : x_i \mapsto X_{i,d,D}^{cl}(x_i) := X_{i,d,D}(x_i, u_i^{cl}(x_i)).$$

E. Preliminary Propositions

Let us provide two propositions that will be invoked multiple times throughout the next subsections.

Loosely speaking, In the first proposition states that given a continuous Lypuanov function whose sub-level sets are compact, it is then the case that the smaller those sub-levels are, the closer the state is to the set where the Lyapunov vanishes: in other words, compactness of all sub-levels set of a non-negative (Lyapunov) function, guarantees that such a (Lyapunov) function can be used to construct a neighborhood around the set where the function vanishes; for a Lyapunov function, with non-positive time-derivative, this can then be used to conclude stability of the set where the Lyapunov vanishes (the equilibrium set). The subtlety can be understood from the simple example: suppose that $\mathbb{R} \ni x \mapsto V(x) := \frac{x^2}{1+x^4} \in [0, \infty)$, where no sub-level set is compact, apart from the zero sub-level $V_{\leq 0} := \{x \in \mathbb{R} : V(x) \leq 0\} = \{0\}$; then if the Lypuanov function V approaches 0,

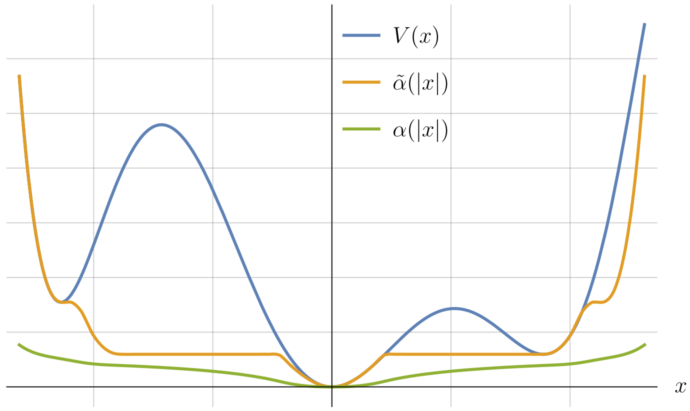


Fig. 7: Illustration of result in Proposition 16, where a Lyapunov function V with compact sub-level sets is necessarily lower bounded by an increasing function: $\alpha(|x|) \leq \tilde{\alpha}(|x|) \leq V(x)$ for any $x \in \mathbb{R}$.

one cannot conclude that the state x also approaches the set $V_{\leq 0} = \{0\}$ (the equilibrium set), since it may instead approach instead $\{\pm\infty\}$. In conclusion, having a Lypuanov function whose sub-level sets are compact precludes this behaviour.

In what follows, we say that a map $\alpha : [0, \infty) \to [0, \infty)$ belongs to $\mathcal{K}^{\infty}$, i.e., $\alpha \in \mathcal{K}^{\infty}$, if α is continuous, and $\alpha(0) = 0$, and $\alpha(r) > 0$ for any $r > 0$, and α is increasing in $(0, \infty)$. The results that follows, and their proofs, are based on [28, Proposition 2.2].

Proposition 16: Let

- M be a Riemannian manifold;
- $V : M \to [0, \infty)$ be a continuous map, and denote

$$\begin{aligned} M^{\star} &:= \{m \in M : V(m) = 0\}, \\ V_{\leq V_0} &:= \{m \in M : V(m) \leq V_0\}, \end{aligned}$$

  where $M^{\star}$ stands for the set where V vanishes, and $V_{\leq V_0}$ stands for the sub-level of V of value $V_0 \geq 0$ (as a remark, note that $M^{\star} = V_{\leq 0}$).
- for any $V_0 \geq 0$, the sub-level set $V_{\leq V_0}$ is a compact subset of M.

Then, there exists a continuous $\alpha \in \mathcal{K}_{\infty}$ such that

$$\alpha\left(\text{dist}_M(m, M^{\star})\right) \leq V(m) \text{ for all } m \in M.$$

Proof: We assume, without loss of generality, that $\sup_{m \in M} \text{dist}_M(m, M^{\star}) = \infty$. Consider then the function

$$\begin{aligned} &\tilde{\alpha} : [0, \infty) \to [0, \infty) \\ &\tilde{\alpha}(\epsilon) := \inf_{m \in M_{\epsilon}} V(m), \text{ where } M_{\epsilon} := \{m \in M : \text{dist}_M(m, M^{\star}) \geq \epsilon\} \end{aligned}.$$

Note then that

- given any $\epsilon \geq 0$, pick $V_0 > \tilde{\alpha}(\epsilon) \in [0, \infty)$; it is then the case that $V_{\leq V_0}$ is compact and so is $M_{\epsilon} \cap V_{\leq V_0}$; as such, $\inf_{m \in M_{\epsilon}} V(m) = \inf_{m \in M_{\epsilon} \cap V_{\leq V_0}} V(m) = \min_{m \in M_{\epsilon} \cap V_{\leq V_0}} V(m)$ and therefore, $\tilde{\alpha}$ is continuous, given that V is continuous on M and $M_{\epsilon} \cap V_{\leq V_0}$ is a compact subset of M, which changes in a continuous fashion with respect to ϵ;
- $\tilde{\alpha}(0) = 0$ because $m \in M^{\star} \Rightarrow V(m) = 0$;
- $\tilde{\alpha}(\epsilon) > 0$ for any $\epsilon > 0$ because $m \notin M^{\star} \Rightarrow V(m) > 0$;
- $\tilde{\alpha}$ is non-decreasing in $(0, \infty)$, since $M_{\epsilon_2} \subset M_{\epsilon_1}$ for any $0 < \epsilon_1 < \epsilon_2 < \infty$, and thus $\tilde{\alpha}(\epsilon_1) \leq \tilde{\alpha}(\epsilon_2)$.

Consider then the function

$$\begin{aligned} &\alpha : [0, \infty) \to [0, \infty) \\ &\alpha(\epsilon) := \frac{1}{\epsilon + 1} \int_0^{\epsilon} \tilde{\alpha}(s) ds. \end{aligned}$$

It is then the case that α is continuous (in fact, it is $\mathcal{C}^1$), that $\alpha(0) = 0$, and that

- $\alpha(\epsilon) < \tilde{\alpha}(\epsilon)$ for any $\epsilon > 0$ since $\frac{1}{\epsilon+1} \int_0^{\epsilon} \tilde{\alpha}(s) ds < \frac{\epsilon}{\epsilon+1} \tilde{\alpha}(\epsilon) < \tilde{\alpha}(\epsilon)$ (in the first inequality, we invoked the fact that $\tilde{\alpha}$ is positive and non-decreasing, i.e., $\sup_{s \in (0,\epsilon)} \tilde{\alpha}(s) = \tilde{\alpha}(\epsilon)$);
- α is increasing, since, for any $\epsilon > 0$, $d\alpha(\epsilon) = \frac{1}{\epsilon+1}(\tilde{\alpha}(\epsilon) - \alpha(\epsilon)) > 0$ (where the latter inequality follows from the conclusion in the item above).

An illustration of the functions $\tilde{\alpha}$ and α are found in Fig. 7. ■

Note that in a backstepping procedure, one constructs, at each step, a *new* Lyapunov function based on an *old* Lyapunov function (the one from the previous step)[9] Since we follow a backstepping procedure with six steps, it is convenient to provide a generic result that can be invoked at each step to guarantee the stability of the equilibria set.

Proposition 17: Let

- M, N be Riemannian manifolds;
- $f : M \to N$ be a continuous map;
- $V : M \to [0, \infty)$ be a map, satisfying the same properties as in Proposition 16

If we define

$$\begin{aligned}&\tilde{V} : M \times N \to [0, \infty)\\&\tilde{V}(m, n) := V(m) + \frac{1}{2}\left(\text{dist}_N(n, f(m))\right)^2,\end{aligned}$$

then

1) $\tilde{V}_{\leq V_0} := \{(m, n) \in M \times N : \tilde{V}(m, n) \leq V_0\}$ defines a compact subset of $M \times N$;
2) and there exists $\alpha \in \mathcal{K}^\infty$, such that

$$\alpha\left(\text{dist}_{M\times N}((m, n), (M \times N)^\star)\right) \leq \tilde{V}(m, n) \text{ for all } (m, n) \in M \times N,$$

where $(M \times N)^\star := \{(m, n) \in M \times N : \tilde{V}(m, n) = 0\} = \{(m, n) \in M \times N : m \in M^\star \text{ and } n = f(m)\}$.

Proof: For the first claim, pick $V_0 \in [0, \infty)$ and then pick $(m, n) \in \tilde{V}_{\leq V_0}$. Then note that $\tilde{V}(m, n) \leq V_0 \Rightarrow m \in V_{\leq V_0}$, where the latter sub-level set is compact by assumption. As such, $f(V_{\leq V_0}) := \{n \in N : n = f(m) \text{ with } m \in V_{\leq V_0}\}$ is also compact, given that f is continuous in M and $V_{\leq V_0}$ is a compact subset of M. It then follows that $\tilde{V}(m, n) \leq V_0 \Rightarrow n \in \{n \in N : \text{dist}_N(n, f(V_{\leq V_0})) \leq \sqrt{2V_0}\}$, where the latter set is also compact (closed since the set is described with $\leq$ and where all functions involved are continuous, and bounded since $f(V_{\leq V_0})$ is bounded).

For the second claim, it suffices to invoke the first claim in tandem with Proposition 16. ■

Remark 18: There is a problem we wish to avoid later on, and let us motivate its solution at this point. For that purpose, pick some positive $V_0 > 0$, and let us define

$$\begin{aligned}&\Gamma_{V_0} : [0, \infty) \to [0, \infty)\\&\Gamma_{V_0}(V) := V_0 \log\left(\frac{V + V_0}{V_0}\right),\end{aligned} \tag{36a}$$

where $d\Gamma_{V_0}(V) = (1 + \frac{V}{V_0})^{-1} > 0$ for all $V \in [0, \infty)$. The purpose of the map Γ_{V_0} is the following: let $V : \mathbb{R}^n \to [0, \infty)$ be some Lyapunov function satisfying $\lim_{\|x\|\to\infty} V(x) = \infty$ and $\lim_{\|x\|\to\infty} \frac{\nabla V(x)}{V(x)} = 0_n$ (loosely speaking, $V(x)$ does not grow exponentially with the norm of x); it is then the case that

$$\lim_{\|x\|\to\infty} \underbrace{d\Gamma_{V_0}(V(x))\nabla V(x)}_{=:\nabla_{\Gamma,V_0} V(x)} = \lim_{\|x\|\to\infty} \frac{\nabla V(x)}{1 + \frac{V(x)}{V_0}} = 0_n, \tag{36b}$$

where we introduce the notation ∇_{Γ,V_0}, in order to shorten the presentation later on.

Suppose now we are tasked with steering $x \in \mathbb{R}$ to 0 for a system of the form $\dot{x} = u + d$, where $u \in \mathbb{R}$ is the control input, and $d \in \mathbb{R}$ is an unknown disturbance. One could propose the dynamic control law $u = -x - \hat{d}$ with $\dot{\hat{d}} = x$, which accomplishes the task (consider the Lyapunov function $V_2(x, \hat{d}) := V_1(x) + \frac{1}{2}(d - \hat{d})^2$ with $V_1(x) := \frac{x^2}{2}$), but it has one drawback: the bigger $|x(0)|$ is, the bigger $|\dot{\hat{d}}(0)|$ also is, and thus the estimator will change drastically initially, even if the real disturbance d is small. As an alternative, one could propose the dynamic control law $u = -x - \hat{d}$ with $\dot{\hat{d}} = d\Gamma_{V_0}(V_1(x))\nabla V_1(x)$, which also accomplishes the previous task (consider the Lyapunov function $V_2(x, \hat{d}) := \Gamma_{V_0}(V_1(x)) + \frac{1}{2}(d - \hat{d})^2)$, but, because (36b) is satisfied, it does not suffer from the problem of the previous approach. Loosely speaking, the estimator only "starts working" once the state is *small*, where this notion of *small* is determined by the constant V_0 (in particular, note that if $V_0 = \infty$, then Γ_{V_0} in (36a) corresponds to the identity function, and we recover the previous approach). The introduction of the map Γ_{V_0} is also beneficial in a backstepping procedure, where the backstepping terms have a form like $d\Gamma_{V_0}(V(x))\nabla V(x)$.

When performing a backstepping procedure, or a disturbance-removal procedure, we can use the Γ_{V_0} map to construct a Lyapunov function – loosely speaking, $V_{i+1}(x_{i+1}) = \Gamma_{V_0}(V_i(x_i)) + \|\text{error}(x_{i+1})\|^2)$ – and, in doing so, the backstepping/disturbance-removal terms will be of form $d\Gamma_{V_0}(V_i(x_i))\nabla V_i(x_i)$. Because (36b) is satisfied, it then follows that the backstepping/disturbance-removal terms are "inactive" when the norm of x_i is "big", and only "start working" when the norm of x_i is "small"; and where the sensitivity parameter (what determines what "big" and "small" are) is the constant V_0 (in particular, note that if $V_0 = \infty$, then the function Γ_{V_0} in (36a) corresponds to the identity function, and we recover the previous strategy).

[9] Note that only the Lyapunov function associated to the final backstepping step constitutes a true Lyapunov function, in the sense that its derivative along a solution is non-positive.

VIII. BACKSTEPPING PROCEDURE

A. Step 1

Throughout this section, keep in mind the scheme illustrated in Fig. 6a. Consider then the vector field

$$\dot{x}_1 = X_{1,d,D}(x_1,(T,n,g^1)) :\Leftrightarrow \begin{bmatrix}\dot{p}\\ \dot{v}\\ \dot{g}^0\end{bmatrix} = \begin{bmatrix} v \\ (T+\langle\phi(n),D\rangle)\,n - g^0 + d \\ g^1\end{bmatrix}, \tag{37}$$

where

- $x_1 \in \mathbb{X}_1 :\Leftrightarrow (p,v,g^0) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{C}^3_{\underline{g}}$ is the state, composed of the linear position p, the linear velocity v and the (time-varying) gravity g^0 (recall that $\mathbb{C}^3_{\underline{g}} := \{x\in\mathbb{R}^3 : \|x\| > \underline{g}\}$);
- T and n (thrust and angular position) are the inputs to the vector field; and g^1 is the time derivative of the time-varying gravity;
- the disturbances d and D are assumed to be known;
- and denote $\mathbb{X}_1^\star := \{x_1\in\mathbb{X}_1 : p = 0_3 \text{ and } v = 0_3\}$ as the equilibrium set (recall (26a)).

Assumption 19: We assume we have available a bounded control law for a double integrator

$$u_{di} : \mathbb{R}^3\times\mathbb{R}^3 \ni (p,v) \mapsto u_{di}(p,v) \in \bar{\mathbb{B}}^3_{\bar{u}} := \{u\in\mathbb{R}^3 : \|u\| \leq \bar{u}\} \tag{38a}$$

for some chosen bound $\bar{u}$ (as we made clear before, $\bar{u}$ must be chosen such that $\underline{g} - (\bar{u}+\bar{\bar{d}}) > 0$ – for more context, see the discussion at the beginning of Subsection VII-D), which comes equipped with a Lyapunov function

$$V_{di} : \mathbb{R}^3\times\mathbb{R}^3 \ni (p,v) \mapsto V_{di}(p,v) \in [0,\infty), \tag{38b}$$

and such that

1) $u_{di}(0_3,0_3) = 0_3$ and $V_{di}(p,v) = 0 \Leftrightarrow (p,v) = (0_3,0_3)$;
2) the control law needs to be at least $\mathcal{C}^4$ in $\mathbb{R}^3\times\mathbb{R}^3$, and the Lyapunov function needs to be at least $\mathcal{C}^3$ in $\mathbb{R}^3\times\mathbb{R}^3$;
3) any sub-level set of V_{di} defines a compact set in $\mathbb{R}^3\times\mathbb{R}^3$ (see implication resulting from Proposition 16);
4) $\dot{V}_{di}(p,v) \equiv W_{di}(p,v) := d_1V_{di}(p,v)v + d_2V_{di}(p,v)u_{di}(p,v) < 0$ for all $(p,v)\in(\mathbb{R}^3\times\mathbb{R}^3)\backslash\{(0_3,0_3)\}$.

Let us define what one may label the desired three dimensional force, i.e.,

$$\begin{aligned} &T^{3d} : \mathbb{X}_1\times\mathbb{B}^3_{\bar{\bar{d}}} \to \mathbb{C}^3_\epsilon, \text{ where } \epsilon := \underline{g} - (\bar{u}+\bar{\bar{d}}) > 0, \\ &T^{3d}(x_1,d) := u_{di}(p,v) + g^0 - d \end{aligned}. \tag{39}$$

Proposition 20: If $(x_1,d)\in\mathbb{X}_1\times\mathbb{B}_{\bar{\bar{d}}}$ (recall that $\mathbb{B}_{\bar{d}} \subset \mathbb{B}_{\bar{\bar{d}}}$), then $T^{3d}(x_1,d) \in \mathbb{C}^3_\epsilon := \{T\in\mathbb{R}^3 : \|T\| > \epsilon\}$, where $\epsilon := \underline{g} - (\bar{u}+\bar{\bar{d}}) > 0$. Also, if $(x_1,d)\in\mathbb{X}_1^\star\times\mathbb{B}_{\bar{d}}$, then $T^{3d}(x_1,d) = g^0 - d$.

Proof: Let $(x_1,d)\in\mathbb{X}_1\times\mathbb{B}_{\bar{\bar{d}}}$; then, note that (i) $g^0\in\mathbb{C}^3_{\underline{g}}$; (ii) $\|u_{di}(p,v)\| \leq \bar{u}$ – see (38a); (iii) $d\in\mathbb{B}_{\bar{\bar{d}}}$; and (iv) $\underline{g} - (\bar{u}+\bar{\bar{d}}) > 0$. Combining (i)–(iv), it follows that $\|T^{3d}(x_1,d)\| > \underline{g} - (\bar{u}+\bar{\bar{d}}) =: \epsilon > 0$, which implies that $T^{3d}(x_1,d)$ belongs indeed to $\mathbb{C}^3_\epsilon := \{T\in\mathbb{R}^3 : \|T\| > \epsilon\}$.

The final claim follows from the fact that $u_{di}(0_3,0_3) = 0_3$. ■

Note that T^{3d} never vanishes in its domain, and thus we can define the unit vector $\frac{T^{3d}(x_1,d)}{\|T^{3d}(x_1,d)\|}$ for any $(x_1,d)\in\mathbb{X}_1\times\mathbb{B}^3_{\bar{\bar{d}}}$. With the latter in mind, if we pick the thrust control law

$$\begin{aligned} &T^{cl} : \mathbb{X}_1\times\mathbb{B}^3_{\bar{\bar{d}}}\times\mathbb{R}^3 \to \mathbb{R} \\ &T^{cl}(x_1,d,D) := \|T^{3d}(x_1,d)\| - \left\langle \phi\left(\frac{T^{3d}(x_1,d)}{\|T^{3d}(x_1,d)\|}\right), D\right\rangle \end{aligned}, \tag{40a}$$

and the angular position control law

$$\begin{aligned} &n^{cl} : \mathbb{X}_1\times\mathbb{B}^3_{\bar{\bar{d}}} \to \mathbb{S}^2 \\ &n^{cl}(x_1,d) := \frac{T^{3d}(x_1,d)}{\|T^{3d}(x_1,d)\|} \end{aligned}, \tag{40b}$$

it follows immediately that composing the vector field $X_{1,d,D}$ in (37) with the control laws T^{cl} and n^{cl} in (40) yields

$$\dot{x}_1 = \underbrace{X_{1,d,D}(x_1,(T^{cl}(x_1,d,D),n^{cl}(x_1,d),g^1))}_{\text{independent of } d \text{ and } D} =: X_1^{cl}(x_1,g^1) :\Leftrightarrow \begin{bmatrix}\dot{p}\\ \dot{v}\\ \dot{g}^0\end{bmatrix} = \begin{bmatrix} v\\ u_{di}(p,v)\\ g^1\end{bmatrix}, \tag{41}$$

and therefore the objective of steering the position to the origin is accomplished. Since we are in the first step (and in order to have a coherent notation among all the sections/steps that follow), we define

$$\mathbb{X}_1 \ni x_1 \mapsto V_1(x_1) := V_{di}(p, v), \tag{42a}$$

$$\mathbb{X}_1 \ni x_1 \mapsto W_1(x_1) := W_{di}(p, v) = \underbrace{dV_1(x_1)X_1^{cl}(x_1, g^1)}_{\text{independent of } g^1} \leq 0, \tag{42b}$$

as the Lyapunov function and its derivative at the end of step 1.

Remark 21: The control law in (40) is well defined in its domain.

Proposition 22: Consider the time-varying gravity acceleration g in (19c), and let us define the set

$$U_1 := \left\{ x_1 \in \mathbb{X}_1 : \inf_{t \in \mathbb{R}} \|g^{(0)}(t)\| \leq \|g^0\| \leq \sup_{t \in \mathbb{R}} \|g^{(0)}(t)\| \right\},$$

where by assumption $\underline{g} < \inf_{t \in \mathbb{R}} \|g^{(0)}(t)\|$ and $\sup_{t \in \mathbb{R}} \|g^{(0)}(t)\| < \infty$. Consider also a sub-level set of V_1, i.e., for some positive constant V_0, consider $(V_1)_{\leq V_0} := \{x_1 \in \mathbb{X}_1 : V_1(x_1) \leq V_0\}$. Then $(V_1)_{\leq V_0} \cap U_1$ defines a compact subset of the state space $\mathbb{X}_1$.

Proof: The sub-level set $(V_1)_{\leq V_0}$ does not define a compact subset of $\mathbb{X}_1$, since V_1 does not depend on g^0 – see (42). However, U_1 restricts g^0 to be contained in a compact set (within $\mathbb{C}^3_{\underline{g}}$), while $(V_1)_{\leq V_0}$ restricts (p, v) to be contained in a compact set (within $\mathbb{R}^3 \times \mathbb{R}^3$) – see Assumption 19. As such, $(V_1)_{\leq V_0} \cap U_1$ defines a compact subset of the state space $\mathbb{X}_1$.

As a remark, we recall that g^0 is (a time-varying signal) specified by the user, and thus we can require that g^0 be restricted to a compact set. ■

We emphasize that the description that follows in the next backstepping steps is *agnostic* to the specific form of the functions u_{di} and V_{di}: we only require that they satisfy the conditions in Assumption 19. However, for purposes of completeness, we present, in the next remarks, functions that satisfy the requirements in Assumption 19. Also, for the the specific functions that are implemented in the simulations, we refer to Remark 25.

Remark 23: Let k_p, k_v be positive constants (proportional and derivative gains), and let also σ_p, σ_v be positive constants (proportional and derivative saturations). Consider then the double integrator control law u_{di} described by (denote $\text{sat}_\sigma(x) := x\frac{\sigma}{\sqrt{\sigma^2 + \|x\|^2}}$, as a saturation function)

$$\begin{aligned} &u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to \bar{\mathbb{B}}^3_{\bar{u}} \text{ with } \bar{u} = k_p\sigma_p + k_v\sigma_v \\ &u_{di}(p, v) := -k_p \text{sat}_{\sigma_p}(p) - k_v \text{sat}_{\sigma_v}(v) \end{aligned} \quad , \tag{43a}$$

and the Lyapunov function V_{di} described by

$$\begin{aligned} &V_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty) \\ &V_{di}(p, v) := k_p\sigma_p \left(\sqrt{\langle p, p \rangle + \sigma_p^2} - \sigma_p \right) + \beta \langle \text{sat}_{\sigma_p}(p), \text{sat}_{\sigma_v}(v) \rangle + \frac{\langle v, v \rangle}{2} \end{aligned} \quad , \tag{43b}$$

for some positive constant $\beta < k_v \left(1 + k_v^2 (4k_p)^{-1}\right)^{-1}$ (gain that guarantees that *(1)* $\dot{V}_{di}(p, v) = W_{di}(p, v)$ is negative definite and that *(2)* the sub-levels sets of V_{di} are compact). The functions u_{di} and V_{di} satisfy the requirements set in Assumption 19, and a detailed proof on why that is indeed the case is found in Section VIII-A1.

Remark 24: Let $k_{p,h}, k_{v,h}, k_{p,z}, k_{v,z}$ be positive constants (proportional and derivative gains, for the **h**orizontal and **v**ertical motions), and let also $\sigma_{p,h}, \sigma_{v,h}, \sigma_{p,z}, \sigma_{v,z}$ be positive constants (proportional and derivative saturations, for the **h**orizontal and **v**ertical motions). Consider then the double integrator control law u_{di} described by (denote $\text{sat}_\sigma(x) := x\frac{\sigma}{\sqrt{\sigma^2 + \|x\|^2}}$)

$$\begin{aligned} &u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to \bar{\mathbb{B}}^3_{\bar{u}} \text{ with } \bar{u} = \sqrt{(k_{p,h}\sigma_{p,h} + k_{v,h}\sigma_{v,h})^2 + (k_{p,z}\sigma_{p,z} + k_{v,z}\sigma_{v,z})^2} \\ &u_{di}(p, v) := \begin{bmatrix} \tilde{u}_{di}(p, v)|_{k_p = k_{p,h}, k_v = k_{v,h}, \sigma_p = \sigma_{p,h}, \sigma_v = \sigma_{v,h}}|_{p = \left(\langle e_1^3, p \rangle, \langle e_2^3, p \rangle\right), v = \left(\langle e_1^3, v \rangle, \langle e_2^3, v \rangle\right)} \\ \tilde{u}_{di}(p, v)|_{k_p = k_{p,z}, k_v = k_{v,z}, \sigma_p = \sigma_{p,z}, \sigma_v = \sigma_{v,z}}|_{p = \langle e_3^3, p \rangle, v = \langle e_3^3, v \rangle} \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{bmatrix} \quad , \\ &\tilde{u}_{di}(p, v) := -k_p \text{sat}_{\sigma_p}(p) - k_v \text{sat}_{\sigma_v}(v) \end{aligned} \tag{44a}$$

and the chosen Lyapunov function V_{di} is described by

$$\begin{aligned} &V_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty) \\ &V_{di}(p, v) := \tilde{V}_{di}(p, v)|_{k_p = k_{p,h}, k_v = k_{v,h}, \sigma_p = \sigma_{p,h}, \sigma_v = \sigma_{v,h}, \beta = \beta_h}|_{p = \left(\langle e_1^3, p \rangle, \langle e_2^3, p \rangle\right), v = \left(\langle e_1^3, v \rangle, \langle e_2^3, v \rangle\right)} + \\ &\qquad \tilde{V}_{di}(p, v)|_{k_p = k_{p,z}, k_v = k_{v,z}, \sigma_p = \sigma_{p,z}, \sigma_v = \sigma_{v,z}, \beta = \beta_z}|_{p = \langle e_3^3, p \rangle, v = \langle e_3^3, v \rangle} \quad , \\ &\tilde{V}_{di}(p, v) := k_p\sigma_p \left(\sqrt{\|p\|^2 + \sigma_p^2} - \sigma_p \right) + \beta \langle \text{sat}_{\sigma_p}(p), \text{sat}_{\sigma_v}(v) \rangle + \frac{\|v\|^2}{2} \end{aligned} \tag{44b}$$

for some positive constants $k_{p,h}, k_{p,z}$ (proportional gains), $k_{v,h}, k_{v,z}$ (derivative gains), $\sigma_{p,h}, \sigma_{p,z}$ (position saturations), $\sigma_{v,h}, \sigma_{v,z}$ (velocity saturations), and for $\beta_h < k_{v,h} \left(1 + k_{v,h}^2 (4k_{p,h})^{-1}\right)^{-1}$, $\beta_z < k_{v,z} \left(1 + k_{v,z}^2 (4k_{p,z})^{-1}\right)^{-1}$ (gains that guarantee that *(1)*

$\dot{V}_{di}(p,v) = W_{di}(p,v)$ is negative definite and that *(2)* the sub-levels sets of V_{di} are compact). For a detailed explanation on why the conditions required on Assumption 19 are satisfied, we refer to Section VIII-A2. This double integrator control law, as opposed to that found in Remark 23, decouples the horizontal motion from the vertical motion, and it provides us with the opportunity to select different gains to cope with the horizontal and the vertical motions.

Remark 25: Let $k_{p,h}, k_{v,h}, k_{p,z}, k_{v,z}$ be positive constants (proportional and derivative gains, for the **h**orizontal and **v**ertical motions), and let also $\sigma_{p,h}, \sigma_{v,h}, \sigma_{p,z}, \sigma_{v,z}$ be positive constants (proportional and derivative saturations, for the **h**orizontal and **v**ertical motions). Consider then the double integrator control law u_{di} described by (denote $\text{sat}_\sigma(x) := x\frac{\sigma}{\sqrt{\sigma^2+\|x\|^2}}$)

$$\begin{aligned}
&u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to \bar{\mathbb{B}}^3_{\bar{u}} \text{ with } \bar{u} = \sqrt{(k_{p,h}\sigma_{p,h})^2 + (k_{v,h}\sigma_{v,h})^2 + (k_{p,z}\sigma_{p,z})^2 + (k_{v,z}\sigma_{v,z})^2} \\
&u_{di}(p,v) := \begin{bmatrix} \tilde{u}_{di}(p,v)|_{k_p=k_{p,h}, k_v=k_{v,h}, \sigma_p=\sigma_{p,h}, \sigma_v=\sigma_{v,h}}|_{p=(\langle e_1^3,p\rangle, \langle e_2^3,p\rangle), v=(\langle e_1^3,v\rangle, \langle e_2^3,v\rangle)} \\ \tilde{u}_{di}(p,v)|_{k_p=k_{p,z}, k_v=k_{v,z}, \sigma_p=\sigma_{p,z}, \sigma_v=\sigma_{v,z}}|_{p=\langle e_3^3,p\rangle, v=\langle e_3^3,v\rangle} \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{bmatrix} \quad, \\
&\tilde{u}_{di}(p,v) := -\frac{\sigma_v}{\sqrt{\sigma_v^2 + \|v\|^2}} k_p \text{sat}_{\sigma_p}(p) - k_v \text{sat}_{\sigma_v}(v)
\end{aligned} \tag{45a}$$

and the chosen Lyapunov function V_{di} is described by

$$\begin{aligned}
&V_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty) \\
&V_{di}(p,v) := \tilde{V}_{di}(p,v)|_{k_p=k_{p,h}, k_v=k_{v,h}, \sigma_p=\sigma_{p,h}, \sigma_v=\sigma_{v,h}, \beta=\beta_h}|_{p=(\langle e_1^3,p\rangle, \langle e_2^3,p\rangle), v=(\langle e_1^3,v\rangle, \langle e_2^3,v\rangle)} + \\
&\qquad\qquad \tilde{V}_{di}(p,v)|_{k_p=k_{p,z}, k_v=k_{v,z}, \sigma_p=\sigma_{p,z}, \sigma_v=\sigma_{v,z}, \beta=\beta_z}|_{p=\langle e_3^3,p\rangle, v=\langle e_3^3,v\rangle} \quad, \\
&\tilde{V}_{di}(p,v) := k_p\sigma_p\left(\sqrt{\|p\|^2+\sigma_p^2} - \sigma_p\right) + \beta\langle \text{sat}_{\sigma_p}(p), \text{sat}_{\sigma_v}(v)\rangle + \frac{1}{3}\sigma_v^2\left(\sqrt{1+\frac{\|v\|^2}{\sigma^2}}^3 - 1\right)
\end{aligned} \tag{45b}$$

for some positive constants $k_{p,h}, k_{p,z}$ (proportional gains), $k_{v,h}, k_{v,z}$ (derivative gains), $\sigma_{p,h}, \sigma_{p,z}$ (position saturations), $\sigma_{v,h}, \sigma_{v,z}$ (velocity saturations), and for $\beta_h < k_{v,h}\left(1 + k_{v,h}^2(4k_{p,h})^{-1}\right)^{-1}, \beta_z < k_{v,z}\left(1 + k_{v,z}^2(4k_{p,z})^{-1}\right)^{-1}$ (gains that guarantee that *(1)* $\dot{V}_{di}(p,v) = W_{di}(p,v)$ is negative definite and that *(2)* the sub-levels sets of V_{di} are compact). For a detailed explanation on why the conditions required on Assumption 19 are satisfied, we refer to Section VIII-A2. This double integrator control law

- as opposed to that found in Remark 23, decouples the horizontal motion from the vertical motion, and it provides us with the opportunity to select different gains to cope with the horizontal and vertical motions;
- as opposed to that found in Remark 24, it satisfies the property that $\lim_{\|v\|\to\pm\infty} \tilde{u}_{di}(p,v) = \lim_{\|v\|\to\pm\infty} -k_v\sigma_v\frac{v}{\|v\|}$ is independent of the position p (that is, when the velocity is *large*, all the control law is focused in steering the velocity to zero, without any focus on the position).

This is the double integrator control law implemented in the simulations presented later.

1) Bounded double integrator: Let the following positive constants be given:

- k_p as a position/proportional gain, and k_v as a velocity/derivative gain;
- σ_p as a position/proportional saturation, and σ_v as a velocity/derivative saturation;
- β as a positive constant that will guarantee positive definiteness of the Lyapunov function, and negative definiteness of its derivative; with β upper bounded as

$$\beta < \beta^\star := \frac{k_v}{1+\frac{k_v^2}{4k_p}} \leq \min\left(\sqrt{k_p}, \frac{k_v}{1+\frac{k_v^2}{4k_p}}\right) \tag{46a}$$

  where
  - $\beta < \sqrt{k_p}$ guarantees that the sub-level sets of the proposed Lyapunov function are compact;
  - $\beta < k_v\left(1 + k_v^2(4k_p)^{-1}\right)^{-1}$ guarantees that the time derivative of the proposed Lyapunov function is negative definite (with respect to the state of the double integrator).

Remark 26: Let $\omega > 0$ be a natural frequency and $\xi > 0$ be a damping. If we parametrize the proportional and derivative gains k_p, k_v as $k_p = \omega^2$ and $k_v = 2\xi\omega$ (and conversely, $\omega = \sqrt{k_p}$ and $\xi = k_v(2\sqrt{k_p})^{-1}$ – that is, there is a bijection between (k_p, k_v) and (ω, ξ), provided that k_p, ω are positive), then (46a) reads simply as

$$\beta < \beta^\star := \left(1 + \frac{(\xi-1)^2}{2\xi}\right)^{-1}\omega \leq \min\left(1, \left(1+\frac{(\xi-1)^2}{2\xi}\right)^{-1}\right)\omega$$

We emphasize that designing a bounded control law for a double integrator is *relatively* easy, if one is satisfied with a Lyapunov function whose derivative is semi-negative definite. The challenge, however, relies on finding a companion Lyapunov function whose derivative is negative definite: this is essential since we are concerned with designing disturbance/update-laws that rely on gradients of the Lyapunov function.

Let us then present the bounded control law for a double integrator, and a companion Lyapunov function satisfying the requirements set in Assumption 19.

For that purpose, define the saturation function

$$\begin{aligned} &\mathrm{sat}_\sigma : \mathbb{R}^n \to \mathbb{B}^n_\sigma := \{u \in \mathbb{R}^n : \|u\| < \sigma\} \\ &\mathrm{sat}_\sigma(x) := \frac{x}{\sqrt{1+\frac{\langle x,x\rangle}{\sigma^2}}} \end{aligned},$$

where σ is some positive constant that describes the bound on the norm of the saturation, i.e., $\|\mathrm{sat}_\sigma(x)\| < \sigma$ for any $x \in \mathbb{R}^n$: loosely speaking, if $\|x\| \ll \sigma$ then $\mathrm{sat}_\sigma(x) \approx x$, and if $\|x\| \gg \sigma$ then $\mathrm{sat}_\sigma(x) \approx \sigma\frac{x}{\|x\|}$.

The proposed bounded double integrator control law is then given by

$$\begin{aligned} &u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to \bar{\mathbb{B}}^3_{\bar{u}} \text{ with } \bar{u} = k_p\sigma_p + k_v\sigma_v \\ &u_{di}(p,v) := -k_p\mathrm{sat}_{\sigma_p}(p) - k_v\mathrm{sat}_{\sigma_v}(v) \end{aligned}. \tag{46b}$$

Proposition 27: The double integrator control law u_{di} in (46b)

- is analytic (and thus $\mathcal{C}^\infty$) in $\mathbb{R}^n \times \mathbb{R}^n$;
- is bounded with bound $\bar{u} = k_p\sigma_p + k_v\sigma_v$;
- and its power series expansion at $(p,v) = (0_3, 0_3)$ to order $\|(p,v)\|^2$ is given by

$$u_{di}(p,v) = -k_p p - k_v v + \mathcal{O}(\|(p,v)\|^2).$$

For brevity, denote

$$\begin{aligned} &n_\sigma : \mathbb{R}^n \to [1,\infty) \\ &n_\sigma(x) := \sqrt{1+\frac{\langle x,x\rangle}{\sigma^2}} \end{aligned}. \tag{46c}$$

Proposition 28: The function f_σ

$$\begin{aligned} &f_\sigma : \mathbb{R} \to (0,1] \\ &f_\sigma(x) := \begin{cases} \frac{n_\sigma(x)-1}{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} & x \neq 0 \\ 1 & x = 0 \end{cases} \end{aligned},$$

with n_σ as defined in (46c), is analytic.

Proof: To conclude that f_σ is analytic it suffices to look at the Taylor series of $x \mapsto n_\sigma(x) - 1$ (which is an analytic function) around 0: since $n_\sigma(x) - 1 = \frac{1}{2}\left(\frac{x}{\sigma}\right)^2 + \mathcal{O}(|x|^4)$, that conclusion follows immediately.

To conclude that the image space of f_σ is $(0,1]$ it suffices to note that: (i) $f_\sigma(x) > 0$ because $n_\sigma(x) > 1$ for $x \neq 0$; and (ii) given some $\mathfrak{f} > 0$, it follows that $f(x) = \mathfrak{f} \Leftrightarrow x = \pm 2\sigma\mathfrak{f}^{-1}\sqrt{1-\mathfrak{f}}$, which is only well-defined for $\mathfrak{f} \in (0,1]$. ∎

Consider now the Lyapunov function

$$\begin{aligned} &V_{di} : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty) \\ &V_{di}(p,v) := k_p\sigma_p^2\left(n_{\sigma_p}(p) - 1\right) + \beta\langle \mathrm{sat}_{\sigma_p}(p), \mathrm{sat}_{\sigma_v}(v)\rangle + \frac{\langle v,v\rangle}{2} \end{aligned}. \tag{46d}$$

Proposition 29: The Lyapunov function V_{di} is analytic (and thus $\mathcal{C}^\infty$) in $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, any sub-level set of the Lyapunov function V_{di} forms a compact set, i.e., $\{(p,v) \in \mathbb{R}^n \times \mathbb{R}^n : V_{di}(p,v) \leq V_0\}$ is compact for any $V_0 \geq 0$.

Proof: That V_{di} is analytic follows immediately from the fact that all its component functions are analytic (sums, products, and compositions of analytic functions are analytic).

In order to study a sub-level set of the Lyapunov function V_{di}, note that V_{di} in (46d) is equivalently expressed as

$$V_{di}(p,v) = \frac{1}{2}\begin{bmatrix} p \\ v \end{bmatrix}^T \left(\underbrace{\begin{bmatrix} k_p f_{\sigma_p}(\|p\|) & \beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1} \\ \beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1} & 1 \end{bmatrix}}_{=:A} \otimes I_n\right)\begin{bmatrix} p \\ v \end{bmatrix},$$

with f_{σ_p} as defined in Proposition 28. To infer that a sub-level set of the Lyapunov function V_{di} forms a compact set, it suffices to prove that the matrix $A \in \mathbb{R}^{2\times 2}$ above is positive definite for any $(p,v) \in \mathbb{R}^n \times \mathbb{R}^n$. Since both diagonal entries of A are positive, it suffices to check that its Schur complement is also positive:

$$\begin{aligned} &k_p f_{\sigma_p}(\|p\|) - \left(\beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1}\right)^2 > 0 \Leftrightarrow \\ \Leftrightarrow &\beta^2 < k_p \underbrace{f_{\sigma_p}(\|p\|)(n_{\sigma_p}(p))^2}_{\geq 1}\underbrace{(n_{\sigma_v}(v))^2}_{\geq 1}. \end{aligned}$$

The condition above is indeed satisfied since $\beta < \beta^\star$ (see (46a)). ∎

Given the Lyapunov function V_{di} in (46d), and the control law u_{di} in (46b), it follows that $\frac{d}{dt}V(p,v) = d_1V_{di}(p,v)v + d_2V_{di}(p,v)u_{di}(p,v) = W_{di}(p,v)$, where

$$W_{di}:\mathbb{R}^n\times\mathbb{R}^n\to(-\infty,0]$$

$$W_{di}(p,v) := -(n_{\sigma_v}(v))^{-1}\begin{bmatrix}\text{sat}_{\sigma_p}(p)\\ v\end{bmatrix}^T \underbrace{\begin{bmatrix}k_p\beta n_{\sigma_v}(v)d\text{sat}_{\sigma_v}(v) & \frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v)\\ \frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v) & k_vI_n-\beta d\text{sat}_{\sigma_p}(p)\end{bmatrix}}_{=:A\in\mathbb{R}^{(2n)\times(2n)}}\begin{bmatrix}\text{sat}_{\sigma_p}(p)\\ v\end{bmatrix}\quad . \tag{46e}$$

Proposition 30: The function W_{di} is negative definite, that is, $W_{di}(p,v)<0$ for any $(p,v)\in(\mathbb{R}^n\times\mathbb{R}^n)\backslash\{(0_n,0_n)\}$, and $W_{di}(0_n,0_n)=0$.

Proof: First note that

$$d\text{sat}_\sigma(x) = \frac{1}{n_\sigma(x)}\left(I_n - \frac{\text{sat}_\sigma(x)}{\sigma}\frac{\text{sat}_\sigma(x)}{\sigma}^T\right)\in\mathbb{R}^{n\times n},$$

and, therefore,

$$\xi^Td\text{sat}_\sigma(x)\xi = \frac{1}{n_\sigma(x)}\left(\|\xi\|^2 - \left\langle\xi,\frac{\text{sat}_\sigma(x)}{\sigma}\right\rangle^2\right)\Rightarrow 0 \overset{\because\|\text{sat}_\sigma(\cdot)\|<\sigma}{<} \xi^Td\text{sat}_\sigma(x)\xi \leq \frac{\|\xi\|^2}{n_\sigma(x)} \overset{\because\|n_\sigma(\cdot)\|\geq 1}{\leq} \|\xi\|^2, \tag{46f}$$

for any $\xi\in\mathbb{R}^n\backslash\{0_n\}$. Moreover, since $\beta<\beta^\star \overset{(46a)}{\Rightarrow} \beta<k_v$, it follows that the matrix $k_vI_n-\beta d\text{sat}_{\sigma_p}(p)$ is positive definite, since

$$\xi^T\left(k_vI_n-\beta d\text{sat}_{\sigma_p}(p)\right)\xi = k_v\|\xi\|^2-\beta\xi^Td\text{sat}_\sigma(x)\xi \geq (k_v-\beta)\|\xi\|^2>0, \tag{46g}$$

for any $\xi\in\mathbb{R}^n\backslash\{0_n\}$. Finally note that $d\text{sat}_\sigma(x)$ is positive definite (see (46f)), and therefore invertible.

Now, in order to prove that the function W_{di} is negative definite, it suffices to prove that the matrix $A\in\mathbb{R}^{(2n)\times(2n)}$ in (46e) is positive definite, i.e., positive for any $(p,v)\in(\mathbb{R}^n\times\mathbb{R}^n)\backslash\{(0_n,0_n)\}$. Since both diagonal entries of A are positive definite (proved above), it suffices to check that its Schur complement is also positive definite:

$$k_vI_n-\beta d\text{sat}_{\sigma_p}(p) - \left(\frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v)\right)\left(k_p\beta n_{\sigma_v}(v)d\text{sat}_{\sigma_v}(v)\right)^{-1}\left(\frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v)\right)>0\Leftrightarrow$$

$$\Leftrightarrow k_vI_n-\beta\left(d\text{sat}_{\sigma_p}(p)+\frac{k_v^2}{4k_p}\frac{1}{n_{\sigma_v}(v)}d\text{sat}_{\sigma_v}(v)\right)>0.$$

Following the exact same steps as in (46g), we conclude that the Schur complement is positive definite if $\beta<\frac{k_v}{1+\frac{k_v^2}{4k_p}\frac{1}{n_{\sigma_v}(v)}}$, which is indeed satisfied since $\beta<\beta^\star$ (note that $n_\sigma(\cdot)\geq 1$). ■

Remark 31: Let $[0,\infty)\ni t\mapsto(p(t),v(t))\in\mathbb{R}^n\times\mathbb{R}^n$ be a solution of $(\dot{p}(t),\dot{v}(t))=(v(t),u_{di}(p(t),v(t)))$ with $(p(0),v(0))=(p_0,v_0)\in\mathbb{R}^n\times\mathbb{R}^n$. That a complete solution exits (solution for all positive time instants) follows from the fact that u_{di} is continuous and the fact that the solution is trapped in a sub-level set of V_{di} ($\{(p,v)\in\mathbb{R}^n\times\mathbb{R}^n: V_{di}(p,v)\leq V_{di}(p_0,v_0)\}$), which is compact.

Note that it can be the case that $\lim_{\|(p,v)\|\to\infty}W_{di}(p,v)=0$, and therefore one cannot simply conclude that $\lim_{t\to\infty}W_{di}(p(t),v(t))=0$ implies that $\lim_{t\to\infty}(p(t),v(t))=(0_n,0_n)$. However, given that the solution is trapped in a sub-level set of V_{di}, which is compact, it follows that indeed $\lim_{t\to\infty}W_{di}(p(t),v(t))=0$ implies that $\lim_{t\to\infty}(p(t),v(t))=(0_n,0_n)$.

2) Bounded double integrator with crossed terms: Let the following positive constants be given:

- k_p as a position/proportional gain, and k_v as a velocity/derivative gain;
- σ_p as a position/proportional saturation, and σ_v as a velocity/derivative saturation;
- β as a positive constant that will guarantee positive definiteness of the Lyapunov function, and negative definiteness of its derivative; with β upper bounded as

$$\beta<\beta^\star := \frac{k_v}{1+\frac{k_v^2}{4k_p}} \leq \min\left(\sqrt{k_p},\frac{k_v}{1+\frac{k_v^2}{4k_p}}\right) \tag{47a}$$

  where
  - $\beta<\sqrt{k_p}$ guarantees that the sub-level sets of the proposed Lyapunov function are compact;
  - $\beta<k_v\left(1+k_v^2(4k_p)^{-1}\right)^{-1}$ guarantees that the time derivative of the proposed Lyapunov function is negative definite (with respect to the state of the double integrator).

Remark 32: Let $\omega>0$ be a natural frequency and $\xi>0$ be a damping. If we parametrize the proportional and derivative gains k_p,k_v as $k_p=\omega^2$ and $k_v=2\xi\omega$ (and conversely, $\omega=\sqrt{k_p}$ and $\xi=k_v(2\sqrt{k_p})^{-1}$ – that is, there is a bijection between (k_p,k_v) and (ω,ξ), provided that k_p,ω are positive), then (47a) reads simply as

$$\beta<\beta^\star := \left(1+\frac{(\xi-1)^2}{2\xi}\right)^{-1}\omega \leq \min\left(1,\left(1+\frac{(\xi-1)^2}{2\xi}\right)^{-1}\right)\omega$$

We emphasize that designing a bounded control law for a double integrator is *relatively* easy, if one is satisfied with a Lyapunov function whose derivative is semi-negative definite. The challenge, however, relies on finding a companion Lyapunov function whose derivative is negative definite: this is essential since we are concerned with designing disturbance/update-laws that rely on gradients of the Lyapunov function.

Let us then present the bounded control law for a double integrator, and a companion Lyapunov function satisfying the requirements set in Assumption 19.

For that purpose, define the saturation function

$$\begin{aligned} &\mathrm{sat}_\sigma : \mathbb{R}^n \to \mathbb{B}^n_\sigma := \{u \in \mathbb{R}^n : \|u\| < \sigma\} \\ &\mathrm{sat}_\sigma(x) := \frac{x}{\sqrt{1+\frac{\langle x, x\rangle}{\sigma^2}}} \end{aligned} ,$$

where σ is some positive constant that describes the bound on the norm of the saturation, i.e., $\|\mathrm{sat}_\sigma(x)\| < \sigma$ for any $x \in \mathbb{R}^n$: loosely speaking, if $\|x\| \ll \sigma$ then $\mathrm{sat}_\sigma(x) \approx x$, and if $\|x\| \gg \sigma$ then $\mathrm{sat}_\sigma(x) \approx \sigma \frac{x}{\|x\|}$. For brevity, denote

$$\begin{aligned} &n_\sigma : \mathbb{R}^n \to [1, \infty) \\ &n_\sigma(x) := \sqrt{1 + \frac{\langle x, x\rangle}{\sigma^2}} \end{aligned} . \tag{47b}$$

The proposed bounded double integrator control law is then given by

$$\begin{aligned} &u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \to \bar{\mathbb{B}}^3_{\bar{u}} \text{ with } \bar{u} = \sqrt{(k_p\sigma_p)^2 + (k_v\sigma_v)^2} \\ &u_{di}(p, v) := -\frac{k_p}{n_{\sigma_v}(v)}\mathrm{sat}_{\sigma_p}(p) - k_v \mathrm{sat}_{\sigma_v}(v) \end{aligned} . \tag{47c}$$

Proposition 33: The double integrator control law u_{di} in (47c)

- is analytic (and thus $\mathcal{C}^\infty$) in $\mathbb{R}^n \times \mathbb{R}^n$;
- is bounded with bound $\bar{u} = \sqrt{(k_p\sigma_p)^2 + (k_v\sigma_v)^2}$;
- satisfies

$$\lim_{\lambda \to \pm\infty} u_{di}(p, \lambda\mu) = -k_v\sigma_v\mu \text{ for any } p \in \mathbb{R}^3, \mu \in \mathbb{S}^2, \tag{47d}$$

  and (power series expansion of u_{di} at $(p, v) = (0_3, 0_3)$ to order $\|(p, v)\|^2$)

$$u_{di}(p, v) = -k_p p - k_v v + \mathcal{O}(\|(p, v)\|^2).$$

The control law u_{di} in (47c), when compared to that in (46b), has an advantage and a disadvantage: when the velocity is *big*, all the input is focused on steering the velocity to zero (see (47d)); however, this comes at the cost of introducing cross terms, that is $d_2d_1u_{di}(p, v) = d_1d_2u_{di}(p, v) \neq 0_{3\times3\times3} \in \mathbb{R}^{3\times3\times3}$ (while $d_2d_1u_{di}(p, v) = d_1d_2u_{di}(p, v) \neq 0_{3\times3\times3} \in \mathbb{R}^{3\times3\times3}$ with the u_{di} in (46b)), which increases the computational complexity (since these derivatives need to be known in the backstepping steps that follow).

Proposition 34: The functions f_σ and g_σ

$$\begin{aligned} &f_\sigma : \mathbb{R} \to (0, 1] & &g_\sigma : \mathbb{R} \to [1, \infty) \\ &f_\sigma(x) := \begin{cases} \frac{n_\sigma(x)-1}{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} & x \neq 0 \\ 1 & x = 0 \end{cases} , & &g_\sigma(x) := \begin{cases} \frac{\frac{1}{3}\left((n_\sigma(x))^3-1\right)}{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} & x \neq 0 \\ 1 & x = 0 \end{cases} \end{aligned} ,$$

with n_σ as defined in (47b), are analytic.

Proof: To conclude that f_σ is analytic it suffices to look at the Taylor series of $x \mapsto n_\sigma(x) - 1$ around 0: since $n_\sigma(x) - 1 = \frac{1}{2}\left(\frac{x}{\sigma}\right)^2 + \mathcal{O}(|x|^4)$, that conclusion follows immediately.

To conclude that the image space of f_σ is $(0, 1]$ it suffices to note that: (i) $f_\sigma(x) > 0$ because $n_\sigma(x) > 1$ for $x \neq 0$; and (ii) given some $\mathfrak{f} > 0$, it follows that $f(x) = \mathfrak{f} \Leftrightarrow x = \pm 2\sigma\mathfrak{f}^{-1}\sqrt{1 - \mathfrak{f}}$, which is only well-defined for $\mathfrak{f} \in (0, 1]$.

Same reasoning applies in regards to g_σ. ■

Consider now the Lyapunov function

$$\begin{aligned} &V_{di} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \\ &V_{di}(p, v) := \underbrace{k_p\sigma_p^2\left(n_{\sigma_p}(p) - 1\right)}_{=\int_{s=0}^{s=\|p\|} k_p \frac{s}{n_{\sigma_p}(s)} ds} + \beta\langle \mathrm{sat}_{\sigma_p}(p), \mathrm{sat}_{\sigma_v}(v)\rangle + \underbrace{\frac{1}{3}\sigma_v^2\left((n_{\sigma_v}(v))^3 - 1\right)}_{=\int_{s=0}^{s=\|v\|} n_{\sigma_v}(s) s ds} \end{aligned} . \tag{47e}$$

Proposition 35: The Lyapunov function V_{di} is analytic (and thus $\mathcal{C}^\infty$) in $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, any sub-level set of the Lyapunov function V_{di} forms a compact set, i.e., $\{(p,v) \in \mathbb{R}^n \times \mathbb{R}^n : V_{di}(p,v) \leq V_0\}$ is compact for any $V_0 \geq 0$.

Proof: That V_{di} is analytic follows immediately from the fact that all its component functions are analytic (sums, products, and compositions of analytic functions are analytic).

In order to study a sub-level set of the Lyapunov function V_{di}, note that V_{di} in (47e) is equivalently expressed as

$$V_{di}(p,v) = \frac{1}{2}\begin{bmatrix} p \\ v \end{bmatrix}^T \left(\underbrace{\begin{bmatrix} k_p f_{\sigma_p}(\|p\|) & \beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1} \\ \beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1} & g_{\sigma_v}(\|v\|) \end{bmatrix}}_{=:A} \otimes I_n \right) \begin{bmatrix} p \\ v \end{bmatrix},$$

with f_{σ_p} and g_{σ_v} as defined in Proposition 34. To infer that a sub-level set of the Lyapunov function V_{di} forms a compact set, it suffices to prove that the matrix $A \in \mathbb{R}^{2\times 2}$ above is positive definite for any $(p,v) \in \mathbb{R}^n \times \mathbb{R}^n$. Since both diagonal entries of A are positive, it suffices to check that its Schur complement is also positive:

$$\begin{aligned} & k_p f_{\sigma_p}(\|p\|) - \left(\beta(n_{\sigma_p}(p))^{-1}(n_{\sigma_v}(v))^{-1}\right)^2 (g_{\sigma_v}(\|v\|))^{-1} > 0 \Leftrightarrow \\ \Leftrightarrow & \beta^2 < k_p \underbrace{f_{\sigma_p}(\|p\|)(n_{\sigma_p}(p))^2}_{\geq 1} \underbrace{(n_{\sigma_v}(v))^2}_{\geq 1} \underbrace{g_{\sigma_v}(\|v\|)}_{\geq 1}. \end{aligned}$$

The condition above is indeed satisfied since $\beta < \beta^\star$ (see (47a)). ∎

Given the Lyapunov function V_{di} in (47e), and the control law u_{di} in (47c), it follows that $\frac{d}{dt}V(p,v) = d_1 V_{di}(p,v)v + d_2 V_{di}(p,v)u_{di}(p,v) = W_{di}(p,v)$, where

$$\begin{aligned} & W_{di} : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, 0] \\ & W_{di}(p,v) := -(n_{\sigma_v}(v))^{-1} \begin{bmatrix} \text{sat}_{\sigma_p}(p) \\ v \end{bmatrix}^T \underbrace{\begin{bmatrix} k_p \beta d\text{sat}_{\sigma_v}(v) & \frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v) \\ \frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v) & k_v n_{\sigma_v}(v) I_n - \beta d\text{sat}_{\sigma_p}(p) \end{bmatrix}}_{=:A \in \mathbb{R}^{(2n)\times(2n)}} \begin{bmatrix} \text{sat}_{\sigma_p}(p) \\ v \end{bmatrix} \quad . \end{aligned} \tag{47f}$$

Proposition 36: The function W_{di} is negative definite, that is, $W_{di}(p,v) < 0$ for any $(p,v) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash\{(0_n, 0_n)\}$, and $W_{di}(0_n, 0_n) = 0$.

Proof: First note that

$$d\text{sat}_\sigma(x) = \frac{1}{n_\sigma(x)} \left(I_n - \frac{\text{sat}_\sigma(x)}{\sigma} \frac{\text{sat}_\sigma(x)}{\sigma}^T \right) \in \mathbb{R}^{n\times n},$$

and, therefore,

$$\xi^T d\text{sat}_\sigma(x)\xi = \frac{1}{n_\sigma(x)} \left(\|\xi\|^2 - \left\langle \xi, \frac{\text{sat}_\sigma(x)}{\sigma} \right\rangle^2 \right) \Rightarrow 0 \overset{\because \|\text{sat}_\sigma(\cdot)\| < \sigma}{<} \xi^T d\text{sat}_\sigma(x)\xi \leq \frac{\|\xi\|^2}{n_\sigma(x)} \overset{\because \|n_\sigma(\cdot)\| \geq 1}{\leq} \|\xi\|^2, \tag{47g}$$

for any $\xi \in \mathbb{R}^n\backslash\{0_n\}$. Moreover, since $\beta < \beta^\star \overset{(47a)}{\Rightarrow} \beta < k_v$, it follows that the matrix $k_v I_n - \beta d\text{sat}_{\sigma_p}(p)$ is positive definite, since

$$\xi^T \left(k_v n_{\sigma_v}(v) I_n - \beta d\text{sat}_{\sigma_p}(p)\right)\xi = k_v n_{\sigma_v}(v)\|\xi\|^2 - \beta\xi^T d\text{sat}_\sigma(x)\xi \geq (k_v n_{\sigma_v}(v) - \beta)\|\xi\|^2 > 0, \tag{47h}$$

for any $\xi \in \mathbb{R}^n\backslash\{0_n\}$. Finally note that $d\text{sat}_\sigma(x)$ is positive definite (see (47g)), and therefore invertible.

Now, in order to prove that the function W_{di} is negative definite, it suffices to prove that the matrix $A \in \mathbb{R}^{(2n)\times(2n)}$ in (47f) is positive definite, i.e., positive for any $(p,v) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash\{(0_n, 0_n)\}$. Since both diagonal entries of A are positive definite (proved above), it suffices to check that its Schur complement is also positive definite:

$$\begin{aligned} & k_v n_{\sigma_v}(v) I_n - \beta d\text{sat}_{\sigma_p}(p) - \left(\frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v)\right)(k_p \beta d\text{sat}_{\sigma_v}(v))^{-1}\left(\frac{1}{2}\beta k_v d\text{sat}_{\sigma_v}(v)\right) > 0 \Leftrightarrow \\ \Leftrightarrow & k_v n_{\sigma_v}(v) I_n - \beta\left(d\text{sat}_{\sigma_p}(p) + \frac{k_v^2}{4k_p} d\text{sat}_{\sigma_v}(v)\right) > 0. \end{aligned}$$

Following the exact same steps as in (47h), we conclude that the Schur complement is positive definite if $\beta < \frac{k_v}{1+\frac{k_v^2}{4k_p}} n_{\sigma_v}(v)$, which is indeed satisfied since $\beta < \beta^\star$ (note that $n_\sigma(\cdot) \geq 1$). ∎

Remark 37: Let $[0,\infty) \ni t \mapsto (p(t), v(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ be a solution of $(\dot{p}(t), \dot{v}(t)) = (v(t), u_{di}(p(t), v(t)))$ with $(p(0), v(0)) = (p_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n$. That a complete solution exits (solution for all positive time instants) follows from the fact that u_{di} is continuous and the fact that the solution is contained in a sub-level set of V_{di} ($\{(p,v) \in \mathbb{R}^n \times \mathbb{R}^n : V_{di}(p,v) \leq V_{di}(p_0, v_0)\}$), which is compact.

Note that it can be the case that $\lim_{\|(p,v)\|\to\infty} W_{di}(p,v) = 0$, and therefore one cannot simply conclude that $\lim_{t\to\infty} W_{di}(p(t), v(t)) = 0$ implies that $\lim_{t\to\infty}(p(t), v(t)) = (0_n, 0_n)$. However, given that the solution is trapped in a sub-level set of V_{di}, which is compact, it follows that indeed $\lim_{t\to\infty} W_{di}(p(t), v(t)) = 0$ implies that $\lim_{t\to\infty}(p(t), v(t)) = (0_n, 0_n)$.

B. Step 2

Throughout this section, keep in mind the scheme illustrated in Fig. 6b. Consider then the vector field

$$\dot{x}_2 = X_{2,d,D}(x_2, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} = \begin{bmatrix} X_{1,d,D}(x_1, (T, n, g^1)) \\ \mathcal{S}(\omega)\, n \\ g^2 \end{bmatrix}, \tag{48}$$

where

- $x_2 \in \mathbb{X}_2 :\Leftrightarrow (x_1, n, g^1) \in \mathbb{X}_1 \times \mathbb{S}^2 \times \mathbb{R}^3$ is the state, composed of the state x_1 (described in step 1), the angular position n, and where g^1 is the time derivative of g^0 (i.e., $g^1 = \dot{g}$);
- T and ω (thrust and angular velocity) are the inputs to the vector field (and where the input angular velocity must be orthogonal to the angular position); and where g^2 is the time derivative of g^1 (i.e., $g^2 = \ddot{g}$);
- $X_{1,d,D}$ is the vector field described in step 1 (see (37));
- the disturbances d and D are assumed to be known.

Moreover, denote

$$\mathbb{X}^\star_{2,\pm} := \left\{ x_2 \in \mathbb{X}_2 : x_1 \in \mathbb{X}^\star_1 \text{ and } n = \pm \frac{g^0 - d}{\|g^0 - d\|} \right\} \tag{49}$$

as two sets, which turn out to be two disjoint equilibria sets: later we show that $\mathbb{X}^\star_{2,+}$ is stable, while $\mathbb{X}^\star_{2,-}$ is unstable.

The idea in every step is to attempt to reuse the control laws from the previous step (with some necessary adjustments). Because the angular position n is no longer an input (but rather part of the state), we then pick a control law for the thrust T such that we minimize the error between the desired vector field designed in the previous step (see X^{cl}_1 in (41)) and the current one: that is

$$\begin{aligned}
&\inf_{T\in\mathbb{R}} \|X_{1,d,D}(x_1, (T, n, g^1)) - X^{cl}_1(x_1, g^1)\|_{\mathbb{R}^9} = \\
&= \inf_{T\in\mathbb{R}} \|X_{1,d,D}(x_1, (T, n, g^1)) - X_{1,d,D}(x_1, (T^{cl}(x_1, d, D), n^{cl}(x_1, d), g^1))\|_{\mathbb{R}^9} && \because (41) \\
&= \inf_{T\in\mathbb{R}} \left\| \begin{bmatrix} v \\ (T + \langle \phi(n), D\rangle)\, n - g^0 + d \\ g^1 \end{bmatrix} - \begin{bmatrix} v \\ u(p, v) \\ g^1 \end{bmatrix} \right\|_{\mathbb{R}^9} && \because (41) \\
&= \inf_{T\in\mathbb{R}} \|(T + \langle \phi(n), D\rangle)\, n - (u_{di}(p, v) + g^0 - d)\|_{\mathbb{R}^3} \\
&= \inf_{T\in\mathbb{R}} \|(T + \langle \phi(n), D\rangle)\, n - T^{3d}(x_1, d)\|_{\mathbb{R}^3} && \because (39) \\
&= \inf_{T\in\mathbb{R}} \|(T - \langle n, T^{3d}(x_1, d)\rangle + \langle \phi(n), D\rangle)\, n - \Pi(n)\, T^{3d}(x_1, d)\|_{\mathbb{R}^3} && \because a = \langle n, a\rangle n + \Pi(n)\, a \\
&= \|\Pi(n)\, T^{3d}(x_1, d)\|_{\mathbb{R}^3} \text{ for } T = \langle n, T^{3d}(x_1, d)\rangle - \langle \phi(n), D\rangle.
\end{aligned} \tag{50}$$

Motivated by (50), we thus define the thrust control law T^{cl}_1 as[10]

$$\begin{aligned} &T^{cl}_1 : \mathbb{X}_2 \times \mathbb{B}^3_{\bar{d}} \times \mathbb{R}^3 \to \mathbb{R} \\ &T^{cl}_1(x_2, d, D) := \langle n, T^{3d}(x_1, d)\rangle - \langle \phi(n), D\rangle \end{aligned}. \tag{51}$$

It follows immediately that composing the vector field $X_{1,d,D}$ in (37) with the control law T^{cl}_1 in (51) yields

$$\dot{x}_1 = \underbrace{X_{1,d,D}(x_1, (T^{cl}_1(x_2, d, D), n, g^1))}_{\text{independent of } D} = \underbrace{X^{cl}_1(x_1, g^1) + \|T^{3d}(x_1, d)\|(e^3_2 \otimes \mathcal{S}(n))\mathcal{S}(n)\, n^{cl}(x_1, d)}_{=:\tilde{X}_1(x_2, d)}, \tag{52}$$

($e^3_2 = (0, 1, 0)$ is the second canonical basis vector in $\mathbb{R}^3$) since

$$\begin{aligned}
\dot{v} &= ((T + \langle \phi(n), D\rangle)\, n - g^0 + d)\,|_{T = T^{cl}_1(x_2, d, D)} && \because (37) \\
&= \langle n, T^{3d}(x_1, d)\rangle n - g^0 + d && \because (51) \\
&= u(p, v) - \Pi(n)\, T^{3d}(x_1, d) && \because (39) \\
&= u(p, v) + \|T^{3d}(x_1, d)\|\mathcal{S}(n)\, (\mathcal{S}(n)\, n^{cl}(x_1, d)) && \because (40b)
\end{aligned}$$

[10]The thrust control law goes through three iterations: that is, we define T^{cl}_1, T^{cl}_2, and T^{cl}_3; and where a successor control law is constructed on top of a predecessor control law.

and where we emphasize the independence of $\tilde{X}_1$ with respect to the disturbance D. Then, composing the vector field $X_{2,d,D}$ in (48) with the control law T_1^{cl} in (51) yields

$$\begin{aligned} &\dot{x}_2 = X_{2,d,D}(x_2, (T_1^{cl}(x_2, d, D), \omega, g^2)) \Leftrightarrow \\ \Leftrightarrow &\begin{bmatrix} \dot{x}_1 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} = \underbrace{\begin{bmatrix} X_1^{cl}(x_1, g^1) \\ 0_3 \\ 0_3 \end{bmatrix}}_{\text{step 1}} + \begin{bmatrix} \|T^{3d}(x_1, d)\|(e_2^3 \otimes \mathcal{S}(n))\mathcal{S}(n)\, n^{cl}(x_1, d) \\ \mathcal{S}(\omega)\, n \\ g^2 \end{bmatrix}. \end{aligned} \tag{53}$$

We are thus in conditions of applying a backsteepping step. Let k_θ, $\gamma_\theta, V_{1,0}$ be positive gains, and let us then choose the angular velocity control law (recall $\nabla_{\Gamma,V_{1,0}}$ in (36b))

$$\omega_1^{cl} : \mathbb{X}_2 \times \mathbb{B}_{\bar{d}} \times \mathbb{R}^3 \ni (x_2, d, \dot{d}) \mapsto \omega_1^{cl}(x_2, d, \dot{d}) \in T_n\mathbb{S}^2$$

$$\omega_1^{cl}(x_2, d, \dot{d}) := -k_\theta \mathcal{S}\left(n^{cl}(x_1, d)\right) n \tag{54a}$$

$$+ \Pi(n)\mathcal{S}\left(n^{cl}(x_1, d)\right) \frac{d_1 T^{3d}(x_1, d)\tilde{X}_1(x_2, d) + d_2 T^{3d}(x_1, d)\dot{d}}{\|T^{3d}(x_1, d)\|} \tag{54b}$$

$$+ \|T^{3d}(x_1, d)\|(e_2^3 \otimes \mathcal{S}(n))^T \nabla_{\Gamma,V_{1,0}} V_1(x_1) \tag{54c}$$

and where we included the term $\dot{d}$ (even though it is zero) just to emphasize its importance: indeed, once we replace d with its estimate (which is not constant and whose dynamics we design) that term cannot be neglected. We note that $d_2T^{3d}(x_1, \hat{d}_T) = -I_3$ (see (39)), but we kept it in (54b) just for the sake of clarity. Let us provide a brief description on the terms in (54) which, altogether, steer the angular position n to the angular position $n^{cl}(x_1, d)$:

- (54a) acts as a proportional feedback term, and k_θ is a proportional gain;
- (54b) is a feedforward term;
- and (54c) is a backsteeping term (see Remark 18), and γ_θ is a backstepping gain.

We then have that the closed-loop dynamics, at the end of step 2, are given by

$$\dot{x}_2 = \underbrace{X_{2,d,D}(x_2, (T_1^{cl}(x_2, d, D), \omega_1^{cl}(x_2, d, \dot{d}), g^2))|_{\dot{d}=0_3}}_{\text{independent of } D} =: X_{2,d}^{cl}(x_2, g^2), \tag{55}$$

where we emphasize that they depend on the disturbance d but not on the disturbance D (that is the case, because the thrust input cancels, and thus it suppresses, the effect of the disturbance D). We can then construct the Lyapunov function and its derivative along the vector field $X_{2,d}^{cl}$ in (55), as (see Section XIII)

$$\mathbb{X}_2 \ni x_2 \mapsto V_{2,d}(x_2) := \Gamma_{V_{1,0}}\left(V_1(x_1)\right) + \gamma_\theta\left(1 - \langle n, n^{cl}(x_1, d)\rangle\right), \tag{56a}$$

$$\begin{aligned} \mathbb{X}_2 \ni x_2 \mapsto W_{2,d}(x_2) &:= dV_{2,d}(x_2)X_{2,d}^{cl}(x_2, g^2) \quad \text{(independent from } g^2) \\ &= \underbrace{d\Gamma_{V_{1,0}}\left(V_1(x_1)\right)^{-1} W_1(x_1)}_{\because(42b)} \underbrace{-k_\theta\gamma_\theta \left\|\mathcal{S}(n)\, n^{cl}(x_1, d)\right\|^2}_{\leq 0}, \end{aligned} \tag{56b}$$

where we emphasize that the Lyapunov function $V_{2,d}$ (and its derivative) depends on the disturbance d (but not on the disturbance D); and, finally, we also emphasize that the objective of steering the position to the origin would be accomplished if the conditions set under step 2 were met.

Proposition 38: Consider the functions $V_{2,d}, W_{2,d}$ in (56a)–(56b), and the function ω_1^{cl} in (54). Consider also the sets $\mathbb{X}_{2,+}^\star, \mathbb{X}_{2,-}^\star$ in (49). It holds that

$$V_{2,d}(x_2) = 0 \text{ for all } x_2 \in \mathbb{X}_{2,+}^\star \tag{57a}$$

$$V_{2,d}(x_2) = 2\gamma_\theta =: V_{2,-}^\star \text{ for all } x_2 \in \mathbb{X}_{2,-}^\star, \tag{57b}$$

$$\mathbb{X}_{2,+}^\star \cup \mathbb{X}_{2,-}^\star = \{x_2 \in \mathbb{X}_2 : W_{2,d}(x_2) = 0\}, \tag{57c}$$

$$\omega_1^{cl}(x_2, d, 0_3) = \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right) \frac{g^1}{\|g^0 - d\|} \text{ for all } x_2 \in \mathbb{X}_{2,+}^\star \cup \mathbb{X}_{2,-}^\star. \tag{57d}$$

Verifying the identities in (57a)–(57d) is trivial, and thus omitted here.

Proposition 39: Recall Proposition 22. For similar reasons, define

$$U_2 := \left\{x_2 \in \mathbb{X}_2 : x_1 \in U_1 \text{ and } \|g^1\| \leq \sup_{t\in\mathbb{R}} \|g^{(1)}(t)\|\right\},$$

where g^1 is constrained to belong to a compact subset of $\mathbb{R}^3$ – see (25c), and where U_1 is found in Proposition 22. Consider also a sub-level set of V_2, i.e., for some non-negative constant $V_0 \geq 0$, consider $(V_2)_{\leq V_0} := \{x_2 \in \mathbb{X}_2 : V_2(x_2) \leq V_0\}$. Then $(V_2)_{\leq V_0} \cap U_2$ defines a compact subset of the state space $\mathbb{X}_2$.

Proof: Recall the definition of the state in step 2, namely $x_2 \in \mathbb{X}_2 :\Leftrightarrow (x_1, n, g^1) \in \mathbb{X}_1 \times \mathbb{S}^2 \times \mathbb{R}^3$. If $x_2 \in (V_{2,d})_{\leq V_0} \cap U_2$, then note that: (i) if $V_2(x_2) \leq V_0$, it then follows from (56a) that $V_1(x_1) \leq V_{1,0}(\exp(\frac{V_0}{V_{1,0}}) - 1) \in [0, \infty)$; one can then invoke Proposition 22, to conclude that x_1 is contained in a compact subset of $\mathbb{X}_1$. (ii) $n \in \mathbb{S}^2$ where $\mathbb{S}^2$ is already compact. (iii) g^1 belongs to a compact set as imposed by U_2. All together, it follows that $(V_2)_{\leq V_0} \cap U_2$ defines a compact subset of the state space $\mathbb{X}_2$. ■

Theorem 40: Consider the vector field $X_{2,d}^{cl}$ in (55), and the Lyapunov function $V_{2,d}$ in (56a). Moreover, consider the sets $\mathbb{X}_{2,+}^\star, \mathbb{X}_{2,-}^\star$ in (49); let the set U_2, as defined in Proposition 39, be invariant; and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). For brevity, denote $\tilde{\mathbb{X}}_2 := \mathbb{X}_2 \cap U_2$ and $\tilde{\mathbb{X}}_{2,\pm}^\star := \mathbb{X}_{2,\pm}^\star \cap U_2$. Finally, consider the differential equation

$$\dot{x}_2(t) = X_{2,d}^{cl}(x_2(t), g^{(2)}(t)) \text{ with } x_2(0) \in \tilde{\mathbb{X}}_2 \tag{58}$$

Then,

1) there exists a unique and complete solution $[0, \infty) \ni t \mapsto x_2(t) \in \tilde{\mathbb{X}}_2$ to (58);
2) the sets $\mathbb{X}_{2,+}^\star$ and $\mathbb{X}_{2,-}^\star$ are invariant;
3) the set $\tilde{\mathbb{X}}_{2,+}^\star \cup \tilde{\mathbb{X}}_{2,-}^\star$ is globally attractive, i.e.,

$$\lim_{t\to\infty} \text{dist}\left(x_2(t), \tilde{\mathbb{X}}_{2,+}^\star \cup \tilde{\mathbb{X}}_{2,-}^\star\right) = 0, \text{ for all } x_2(0) \in \tilde{\mathbb{X}}_2;$$

4) the set $\tilde{\mathbb{X}}_{2,+}^\star$ is stable, while the set $\tilde{\mathbb{X}}_{2,-}^\star$ is unstable;
5) the set $\tilde{\mathbb{X}}_{2,+}^\star$ is (locally) asymptotically stable and $\lim_{t\to\infty} \text{dist}\left(x_2(t), \tilde{\mathbb{X}}_{2,+}^\star\right) = 0$ if $x_2(0) \in \left\{x_2 \in \tilde{\mathbb{X}}_2 : V_{2,d}(x_2) < V_{2,-}^\star\right\}$, with $V_{2,-}^\star$ as defined in (57b).

Proof:

1) Define $V_0 := V_{2,d}(x_2(0)) \in [0, \infty)$. Since, $(V_{2,d})_{\leq V_0} \cap U_2$ defines a positively invariant compact subset of $\mathbb{X}_2$, since $g^{(2)}$ is also contained in a compact subset of $\mathbb{R}^3$, and since the vector field $X_{2,d}^{cl}$ is $\mathcal{C}^1$ continuous (and thus Lipschitz continuous) in $\mathbb{X}_2 \times \mathbb{R}^3$, the first conclusion follows immediately.
2) It is trivial to verify that

$$\begin{aligned} X_{2,d}^{cl}(x_2, g^2) &\in T_{x_2}\mathbb{X}_{2,+}^\star \text{ for all } x_2 \in \mathbb{X}_{2,+}^\star \text{ and } g^2 \in \mathbb{R}^3, \\ X_{2,d}^{cl}(x_2, g^2) &\in T_{x_2}\mathbb{X}_{2,-}^\star \text{ for all } x_2 \in \mathbb{X}_{2,-}^\star \text{ and } g^2 \in \mathbb{R}^3, \end{aligned}$$

which suffices to conclude that each of those sets is invariant. Just, for the purposes of completeness, the statements above are easily verified, once one notes that

$$T_{x_2}\mathbb{X}_{2,\pm}^\star = \left\{\dot{x}_2 \in T_{x_2}\mathbb{X}_2 : \dot{p} = 0_3, \dot{v} = 0_3, \dot{n} = \pm\Pi\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{\dot{g}^0}{\|g^0 - d\|}\right\} \text{ for any } x_2 \in \mathbb{X}_2,$$

and that, for any $x_2 \in \mathbb{X}_{2,\pm}^\star$,

$$\dot{x}_2 = X_{2,d}^{cl}(x_2, g^2) \Leftrightarrow \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{g}^0 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} = \begin{bmatrix} v \\ u(p, v) - \Pi(n) T^{3d}(x_1, d) \\ g^1 \\ \mathcal{S}(\omega_1^{cl}(x_2, d, 0_3)) n \\ g^2 \end{bmatrix} |_{x_2 \in \mathbb{X}_{2,\pm}^\star} = \begin{bmatrix} 0 \\ 0 \\ g^1 \\ \pm\Pi\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{g^1}{\|g^0 - d\|} \\ g^2 \end{bmatrix}.$$

3) To prove that the set $\tilde{\mathbb{X}}_{2,+}^\star \cup \tilde{\mathbb{X}}_{2,-}^\star$ is globally attractive, consider the solution $[0, \infty) \ni t \mapsto x_2(t) \in \tilde{\mathbb{X}}_2$, and note that
   a) $\tilde{\mathbb{X}}_{2,+}^\star \cup \tilde{\mathbb{X}}_{2,-}^\star = \{x_2 \in \tilde{\mathbb{X}}_2 : W_{2,d}(x_2) = 0\}$ – see Proposition 38;
   b) the solution is contained in $(V_{2,d})_{\leq V_0} \cap U_2$, which is a compact subset of $\mathbb{X}_2$ – see Proposition 39;
   c) since $V_{2,d}$ is lower bounded, and since $\dot{V}_{2,d}(x_2(t)) = W_{2,d}(x_2(t)) \leq 0$, it follows that $\lim_{t\to\infty} V_{2,d}(x_2(t))$ exists;
   d) finally,

$$\begin{aligned} \sup_{t\geq 0} |\ddot{V}_{2,d}(x_2(t))| &= \sup_{t\geq 0} |\dot{W}_{2,d}(x_2(t))| \\ &= \sup_{t\geq 0} |dW_{2,d}(x_2(t)) X_{2,d}^{cl}(x_2(t), g^{(2)}(t))| \\ &\leq \sup_{\substack{x_2 \in (V_{2,d})_{\leq V_0} \cap U_2 \\ g^2 \in \text{compact subset of } \mathbb{R}^3}} |dW_{2,d}(x_2) X_{2,d}^{cl}(x_2, g^2)| < \infty, \end{aligned}$$

   where the latter inequality follows since $W_{2,d}$ is $\mathcal{C}^1(\mathbb{X}_2)$ and $X_{2,d}^{cl}$ is $\mathcal{C}^0(\mathbb{X}_2 \times \mathbb{R}^3)$.

   Combining *c)* and *d)*, and by invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{V}_{2,d}(x_2(t)) = \lim_{t\to\infty} W_{2,d}(x_2(t)) = 0$; and, finally, combining the latter with *a)* and *b)*, it follows that $\lim_{t\to\infty} \text{dist}\left(x_2(t), \tilde{\mathbb{X}}_{2,+}^\star \cup \tilde{\mathbb{X}}_{2,-}^\star\right) = 0$.

4) To prove the set $\tilde{\mathbb{X}}^\star_{2,+}$ is stable, first note that $\tilde{\mathbb{X}}^\star_{2,+} = \{x_2 \in \tilde{\mathbb{X}}_2 : V_{2,d}(x_2) = 0\}$ and where we emphasize that $V_{2,d}$ is non-negative and continuous. We can then invoke Proposition 16, with $M = \tilde{\mathbb{X}}_2$ and $M^\star = \tilde{\mathbb{X}}^\star_{2,+}$ and $V = V_{2,d}|_{\tilde{\mathbb{X}}_2}$, to conclude that there exists $\alpha \in \mathcal{K}^\infty$ such that $\alpha\left(\mathrm{dist}_M(m, M^\star)\right) \leq V(m)$ for all $m \in M$. (That is, $\{x_2 \in \tilde{\mathbb{X}}_2 : V_{2,d}(x_2) \leq \epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}^\star_{2,+}$, which coincides with the latter iff $\epsilon = 0$, and which is positively invariant for any $\epsilon \geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}^\star_{2,+}$ is stable.
   We are now only left with studying the stability properties of $\tilde{\mathbb{X}}^\star_{2,-}$. Before we do so, let us however prove the fifth bullet first.
5) This conclusion follows immediately from the facts that
   - $V_{2,d}(x_2) = V^\star_{2,-}$ for all $x_2 \in \mathbb{X}^\star_{2,-}$ – see Proposition 38;
   - $W_{2,d}(x_2) \leq 0$ for all $x_2 \in \mathbb{X}_2$;
   - the set $\tilde{\mathbb{X}}^\star_{2,+}$ is stable and $\tilde{\mathbb{X}}^\star_{2,+} \cup \tilde{\mathbb{X}}^\star_{2,-}$ is globally attractive, where the sets $\tilde{\mathbb{X}}^\star_{2,+}$ and $\tilde{\mathbb{X}}^\star_{2,-}$ are disjoint.
4) (continuation) We are now only left with studying the stability properties of $\tilde{\mathbb{X}}^\star_{2,-}$. Before we do so, given a unit vector $n \in \mathbb{S}^2$ and a positive integer $k \in \mathbb{N}$, denote

$$\Gamma_k(n) := \frac{n^\perp - kn}{\|n^\perp - kn\|} = \frac{n^\perp - kn}{\sqrt{1+k^2}} \in \mathbb{S}^2, \text{ for some } n^\perp \in \mathbb{S}^2 \text{ orthogonal to } n.$$

   We note that a unit vector $n^\perp$ orthogonal to n (i.e., $\langle n^\perp, n\rangle = 0$) always exists; however, there are many such vectors, and, therefore, $\Gamma_k(n)$ depends on the particular choice that is made (which is not made explicit in the notation). Notwithstanding, the conclusions that follow next are agnostic to such a choice. Indeed, note that

$$1 - \langle n, \Gamma_k(n)\rangle = 1 + \frac{k}{\sqrt{k^2+1}} < 2 \text{ for all } k \in \mathbb{N}, \text{ and} \tag{59a}$$

$$\lim_{k\to\infty} \left(1 - \langle n, \Gamma_k(n)\rangle\right) = 2 \Leftrightarrow \lim_{k\to\infty} \Gamma_k(n) = -n. \tag{59b}$$

   To prove the set $\tilde{\mathbb{X}}^\star_{2,-}$ is unstable, consider then the sequence of initial conditions $\{x^k_{2,0}\}_{k\in\mathbb{N}} = \{x^1_{2,0}, x^2_{2,0}, \ldots\}$ where $x^k_{2,0} := \left(0_3, 0_3, g^0, \Gamma_k\left(\frac{g^0-d}{\|g^0-d\|}\right), g^1\right) \in U_2$. It then follows from (59a), that $x^k_{2,0} \in \left\{x_2 \in \tilde{\mathbb{X}}_2 : V_{2,d}(x_2) < V^\star_{2,-}\right\}$ and thus that $\lim_{t\to\infty} \mathrm{dist}\left(x_2(t), \tilde{\mathbb{X}}^\star_{2,+}\right) = 0$ when $x_2(0) = x^k_{2,0}$. It addition, it follows from (59b), that $\lim_{k\to\infty} x^k_{2,0} \in \tilde{\mathbb{X}}^\star_{2,-}$. Since we can make $\lim_{k\to\infty} x^k_{2,0}$ be any point in $\tilde{\mathbb{X}}^\star_{2,-}$, we conclude that the set $\tilde{\mathbb{X}}^\star_{2,-}$ is unstable since we can find initial conditions that are arbitrarily close to that set, but whose solutions do not remain arbitrarily close to the same set.

■

At this point we refer to Remark 51, which is only provided at the end of step 3, but whose conclusions apply at this step as well: that Remark explains why x_1 cannot approach the set where $T^{3d}(x_1, d)$ vanishes, despite the unit vector $n^{cl}(x_1, d) := \frac{T^{3d}(x_1,d)}{\|T^{3d}(x_1,d)\|}$ being well-defined.

C. Step 3

Throughout this section, keep in mind the scheme illustrated in Fig. 6c. Consider then the vector field

$$\dot{x}_3 = X_{3,d,D}(x_3, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{\hat{d}}_T \end{bmatrix} = \begin{bmatrix} X_{2,d,D}(x_2, (T, \omega, g^2)) \\ E_{\delta,T}(x_2, \hat{d}_T) \end{bmatrix}, \tag{60}$$

where

- $x_3 \in \mathbb{X}_3 :\Leftrightarrow (x_2, \hat{d}_T) \in \mathbb{X}_2 \times \mathbb{B}^3_{\bar{d}}$ is the state, composed of the state x_2 (described in step 2) and the estimate of the disturbance d;
- T and ω (thrust and angular velocity) are the inputs to the vector field; g^2 is the time derivative of g^1 (i.e., $g^2 = \ddot{g}$); and $E_{\delta,T}$ describes the estimator-dynamics/update-law that we wish to design for the estimate $\hat{d}_T$;
- $X_{2,d,D}$ is the vector field described in step 2 (see (48));
- the disturbance d is unknown, while the disturbance D is still assumed to be known.

Moreover, denote

$$\mathbb{X}^\star_{3,\pm} := \{x_3 \in \mathbb{X}_3 : x_2 \in \mathbb{X}^\star_{2,\pm} \text{ and } \hat{d}_T = d\} \tag{61}$$

as two sets, which turn out to be two disjoint equilibria sets: later we show that $\mathbb{X}^\star_{3,+}$ is stable, while $\mathbb{X}^\star_{3,-}$ is unstable.

We remind that the estimator dynamics $E_{\delta,T}$ must guarantee that the estimate $\hat{d}_T$ remains in the ball $\mathbb{B}^3_{\bar{d}}$, since we will replace d with $\hat{d}_T$ in (40b) (see domain of functions). For this purpose, we refer the reader to the projector update law described in Section VII-C, and to the discussion in Subsection VII-A.

Once again, the idea is to attempt to reuse the control laws from the previous steps (with some minor necessary adjustments). Because the disturbance d is no longer known, we must replace it by its estimate $\hat{d}_T$ in the thrust control law (see T_1^{cl} in (51)) and in the angular velocity control law (see ω_1^{cl} in (54)); i.e., we define the new thrust control law as

$$\begin{aligned} &T_2^{cl} : \mathbb{X}_3 \times \mathbb{R}^3 \to \mathbb{R} \\ &T_2^{cl}(x_3, D) := T_1^{cl}(x_2, \hat{d}_T, D) \end{aligned} \quad , \tag{62}$$

and we define the new angular velocity control law as

$$\begin{aligned} &\omega_2^{cl} : \mathbb{X}_3 \ni x_3 \mapsto \omega_2^{cl}(x_3) \in T_n\mathbb{S}^2 \\ &\omega_2^{cl}(x_3) := \omega_1^{cl}\left(x_2, \hat{d}_T, \dot{\hat{d}}_T\right)|_{\dot{\hat{d}}_T = E_{\delta,T}(x_2, \hat{d}_T)} \end{aligned} \quad . \tag{63}$$

Remark 41: The angular velocity control law ω_2^{cl} in (63) is well-defined in its domain: indeed note that $\hat{d}_T \in \mathbb{B}^3_{\bar{d}}$, which means that $T^{3d}(x_1, \hat{d}_T)$ does not vanish and thus that the unit vector $n^{cl}(x_1, \hat{d}_T) = \frac{T^{3d}(x_1, \hat{d}_T)}{\|T^{3d}(x_1, \hat{d}_T)\|}$ is well-defined – see (51). Moreover, the same control law is orthogonal to the angular position for all points in its domain, i.e., $\langle n, \omega^{cl}(x_3)\rangle = 0$ for all $x_3 \in \mathbb{X}_3$.

Remark 42: Note that the angular velocity control law ω_2^{cl} in (63) "depends" on $\dot{\hat{d}}_T = E_{\delta,T}(x_2, \hat{d}_T)$, and note that, ultimately, we would like that $\omega = \omega_2^{cl}(x_3)$. This is the reason why one should not design an update-law for $\hat{d}_T$ that depends on the angular velocity ω: if one does so, then one designs a control law for the angular velocity that depends on the angular velocity itself – this leads to an implicit equation ($\omega = \text{function}(\omega)$) which may or may not have a solution. This sheds some light into the difficulties in removing the disturbance d.

Composing the vector field $X_{1,d,D}$ in (37) with the control law T_2^{cl} in (62) yields

$$\dot{x}_1 = X_{1,d,D}(x_1, (T_2^{cl}(x_3, D), n, g^1)) \Rightarrow \dot{x}_1 = \tilde{X}_1(x_2, \hat{d}_T) + (e_2^3 \otimes I_3)(d - \hat{d}_T),$$

(with $\tilde{X}_1$ as defined in (52), and with $e_2^3 = (0,1,0)$ as the 2nd canonical basis vector in $\mathbb{R}^3$) since

$$\begin{aligned} \dot{v} &= ((T + \langle\phi(n), D\rangle)\, n - g^0 + d)\,|_{T = T_2^{cl}(x_3, D)} && \because (37) \\ &= \left(\langle n, u(p,v) + g^0 - \hat{d}_T\rangle\right) n - g^0 + d && \because (62) \\ &= u(p,v) - \Pi(n)\,(u(p,v) + g(t) - \hat{d}_T) + (d - \hat{d}_T) && \because (39) \\ &= \underbrace{u(p,v) - \Pi(n)\, T^{3d}(x_1, \hat{d}_T) + (d - \hat{d}_T)}_{=: v^1(x_3)}. \end{aligned} \tag{64}$$

Once again, we emphasize the independence of $\dot{x}_1$ with respect to the disturbance D. Therefore composing the vector field $X_{3,d,D}$ in (60) with the control laws (62)–(63) yields

$$\dot{x}_3 = \underbrace{X_{3,d,D}(x_3, (T_2^{cl}(x_3, D), \omega_2^{cl}(x_3), g^2))}_{=: X_{3,d}^{cl}(x_3, g^2) \text{ which is independent of } D} \Leftrightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{\hat{d}}_T \end{bmatrix} = \underbrace{\begin{bmatrix} X_{2,\hat{d}_T}^{cl}(x_2, g^2) \\ 0_3 \end{bmatrix}}_{\text{step 2 (independent of } d \text{ and } D)} + \underbrace{\begin{bmatrix} (e_2^5 \otimes I_3)(d - \hat{d}_T) \\ E_{\delta,T}(x_2, \hat{d}_T) \end{bmatrix}}_{\text{top is linear w.r.t. } (d - \hat{d}_T)}, \tag{65}$$

where $e_2^4 = (0,1,0,0,0,0)$ (2nd canonical basis in $\mathbb{R}^6$). Since the top part of the vector field in (65) is affine with respect to the estimation error $d - \hat{d}_T$, we can thus proceed with a standard estimator design (see Section VII-B). As such, given some positive $k_{\delta,T}$ and $V_{2,0}$, we then choose (recall (36b))

$$E_{\delta,T}(x_2, \hat{d}_T) := \text{Proj}_{\bar{d},2}\left(\tilde{E}_{\delta,T}(x_2, \hat{d}_T), \hat{d}_T\right), \tag{66a}$$

$$\tilde{E}_{\delta,T}(x_2, \hat{d}_T) := k_{\delta,T}(e_2^5 \otimes I_3)^T \nabla_{\Gamma, V_{2,0}} V_{2,\hat{d}_T}(x_2), \tag{66b}$$

for reasons that are made clear next (recall the properties of $\text{Proj}_{\bar{d}}$, in Section VII-C). As an observation,

- $k_{\delta,T}$ represents a (positive) integral gain, and
- $V_{2,0}$ represents a saturation quantity that determines when the estimator starts working ($\dot{\hat{d}}_T \approx 0_3$ when $V_{2,\hat{d}_T}(x_2) \gg V_{2,0}$).

We can then construct the Lyapunov function and its derivative along the vector field $X_{3,d}^{cl}$ in (65), as

$$\mathbb{X}_3 \ni x_3 \mapsto V_{3,d}(x_3) := \Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right) + \frac{1}{2k_{\delta,T}}\left\|d - \hat{d}_T\right\|^2 \in [0, \infty), \tag{67a}$$

$$\mathbb{X}_3 \ni x_3 \mapsto W_{3,d}(x_3) := \underbrace{dV_{3,d}(x_3) X_{3,d}^{cl}(x_3, g^2)}_{\text{(independent of } g^2)} \in (-\infty, 0], \tag{67b}$$

where (67b) is equivalently expressed as

$$
\begin{aligned}
&W_{3,d}(x_3) := dV_{3,d}(x_3)X^{cl}_{3,d}(x_3,g^2) = \\
&= d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)\left(W_{2,\hat{d}_T}(x_2) + dV_{2,\hat{d}_T}(x_2)\left(\left(e_2^5\otimes I_3\right)\left(d-\hat{d}_T\right)\right)\right) - \frac{1}{k_{\delta,T}}\left\langle d-\hat{d}_T, E_{\delta,T}(x_2,\hat{d}_T)\right\rangle \\
&= d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)W_{2,\hat{d}_T}(x_2) - \frac{1}{k_{\delta,T}}\left\langle d-\hat{d}_T, E_{\delta,T}(x_2,\hat{d}_T) - k_{\delta,T}d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)(e_2^5\otimes I_3)^T\nabla V_{2,\hat{d}_T}(x_2)\right\rangle \\
&= \underbrace{d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)W_{2,\hat{d}_T}(x_2)}_{\leq 0 \quad \because (56b)} \underbrace{- \frac{1}{k_{\delta,T}}\left\langle d-\hat{d}_T, E_{\delta,T}(x_2,\hat{d}_T) - \tilde{E}_{\delta,T}(x_2,\hat{d}_T)\right\rangle}_{\leq 0 \quad \because (30c)}.
\end{aligned} \tag{67c}
$$

Recall the properties of the projector function in Section VII-C: it then follows from (30b) that $\hat{d}_T$ remains in some closed subset of $\mathbb{B}^3_{\bar{d}}$, and therefore the control law in (63) is well-defined (that is, it is always well-defined along a solution of $\dot{x}_3 = X^{cl}_{3,d}(x_3,g^2)$). Finally, note that the objective of steering the position to the origin would be accomplished if the conditions set under step 3 were met, since $W_{2,\hat{d}_T}(x_2) = 0 \Rightarrow W_1(x_1) = 0 \Rightarrow p = 0_3$.

Remark 43: Note that the estimator dynamics $E_{\delta,T}$ in (66a) depends on the estimate $\hat{d}_T$ even if the projector function $\mathrm{Proj}_{\bar{d},2}$ were omitted. This is related with the fact that the disturbance d is not an input-affine disturbance.

Remark 44: Note that

$$(e_2^6\otimes I_3)^T\nabla V_{3,d}(x_3) \equiv \frac{\partial}{\partial v}V_{3,d}(x_3)|_{x_3=(p,v,g^0,n,g^1,\hat{d}_T)} \equiv d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)\frac{\partial}{\partial v}V_{2,\hat{d}_T}(x_2)|_{x_2=(p,v,g^0,n,g^1)}, \text{ and} \tag{68a}$$

$$(e_4^6\otimes I_3)^T\nabla V_{3,d}(x_3) \equiv \frac{\partial}{\partial n}V_{3,d}(x_3)|_{x_3=(p,v,g^0,n,g^1,\hat{d}_T)} \equiv d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)\frac{\partial}{\partial n}V_{2,\hat{d}_T}(x_2)|_{x_2=(p,v,g^0,n,g^1)}. \tag{68b}$$

We also note that the Lyapunov function $V_{3,d}$ in (67a) depends on the unknown disturbance d, so it cannot be used for the purposes of control. Nonetheless, it follows from above that

$$(e_2^6\otimes I_3)^T\nabla V_{3,d}(x_3) = d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)(e_2^5\otimes I_3)^T\nabla V_{2,\hat{d}_T}(x_2), \tag{68c}$$

$$(e_4^6\otimes I_3)^T\nabla V_{3,d}(x_3) = d\Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)(e_4^5\otimes I_3)^T\nabla V_{2,\hat{d}_T}(x_2), \tag{68d}$$

where the expressions on the right do not depend on the unknown disturbance d.

Remark 45: Note that the angular velocity control law ω_2^{cl} in (63) depends on the estimator dynamics, and thus on the projector function introduced in (66a). In step 5, we will require the partial derivatives of ω_2^{cl} which then motivates the necessity of the smoothness properties of the projector $\mathrm{Proj}_{\bar{d},2}$ in the update law in (66a) (which is twice continuous differentiable in this case). The next proposition is the basis for showing that the disturbance estimate $\hat{d}_T$ converges to the real disturbance d.

Proposition 46: Consider the functions $V_{3,d}, W_{3,d}$ in (67a), (67c), and the functions ω_2^{cl},$E_{\delta,T}$ in (63), (66a). Consider also the sets $\mathbb{X}^\star_{3,+}, \mathbb{X}^\star_{3,-}$ in (61). It holds that

$$V_{3,d}(x_2) = 0 \text{ for all } x_3 \in \mathbb{X}^\star_{3,+} \tag{69a}$$

$$V_{3,d}(x_3) = \Gamma_{V_{2,0}}(V^\star_{2,-}) =: V^\star_{3,-} \text{ for all } x_3 \in \mathbb{X}^\star_{3,-}, \tag{69b}$$

$$\mathbb{X}^\star_{3,+} \cup \mathbb{X}^\star_{3,-} = \{x_3 \in \mathbb{X}_3 : \text{“}\dot{V}_{3,d}\text{”} = W_{3,d}(x_3) = 0 \text{ and “}\dot{v}\text{”} = v^1(x_3) = 0_3\}, \tag{69c}$$

$$E_{\delta,T}(x_2,\hat{d}_T) = 0_3 \text{ for all } x_2 \in \mathbb{X}^\star_{2,+}\cup\mathbb{X}^\star_{2,-} \text{ and for all } \hat{d}_T \in \mathbb{B}^3_{\bar{d}}, \tag{69d}$$

$$\omega_2^{cl}(x_3) = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-d\|} \text{ for all } x_3 \in \mathbb{X}^\star_{3,+}\cup\mathbb{X}^\star_{3,-}, \tag{69e}$$

with $V^\star_{2,-}$ as defined in (57b).

Proof: Verifying (69a)–(69e) is simple, and, in this proof, we only verify (69c). Consider then (69c) (verifying that $\mathbb{X}^\star_{3,+}\cup\mathbb{X}^\star_{3,-} \subseteq \{x_3 \in \mathbb{X}_3 : \text{“}\dot{V}_{3,d}\text{”} = W_{3,d}(x_3) = 0 \text{ and “}\dot{v}\text{”} = v^1(x_3) = 0_3\}$ is very simple, so, next, we only verify that $\mathbb{X}^\star_{3,+}\cup\mathbb{X}^\star_{3,-} \supseteq \{x_3 \in \mathbb{X}_3 : \text{“}\dot{V}_{3,d}\text{”} = W_{3,d}(x_3) = 0 \text{ and “}\dot{v}\text{”} = v^1(x_3) = 0_3\}$)

1) from (67c), $W_{3,d}(x_3) = 0$ implies that $W_{2,d}(x_2) = 0$;
2) from (56b), $W_{2,d}(x_2) = 0$ implies that $p = 0_3$ and $v = 0_3$ and $n = \pm\frac{g^0-\hat{d}_T}{\|g^0-\hat{d}_T\|}$ (remark: ultimately, we wish to conclude that $n = \pm\frac{g^0-d}{\|g^0-d\|}$);
3) combining 1) and 2), it then follows that “$\dot{v}$” $= d - \hat{d}_T = 0_3$, which implies that $\hat{d}_T = d$;
4) combining 2) and 3), it then follows that $n = \pm\frac{g^0-d}{\|g^0-d\|}$. ■

Remark 47: Let us discuss the meaning of the results in Proposition 46.

- (69b) implies that if the Lyapuvov function is ever “below” the threshold $V^\star_{3,-}$, then a solution cannot converge to the set $\mathbb{X}^\star_{3,-}$, since the Lyapuvov function is non-increasing along any solution;
- (69c) provides a characterization of the equilibria set, and since it “depends” on *two* components – “$\dot{V}_{4,d}$” and “$\dot{v}$”, where “$V_{3,d}$” and “v” converge to some constants – we will be invoking Barbalat’s lemma *two* times;

- (69d) implies that if a solution approaches the equilibria set, then $\dot{\hat{d}}_T$ approaches 0_3 (which, by itself, does not imply that $\hat{d}_T$ approaches some constant in $\mathbb{R}^3$).

Proposition 48: Recall Proposition 39, and define

$$\begin{aligned} U_3 &:= \{x_3 \in \mathbb{X}_3 : x_2 \in U_2\} \text{ and },\\ T_3 &:= \left\{x_3 \in \mathbb{X}_3 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\bar{d}}\right\}, \end{aligned}$$

where U_2 is found in Proposition 39. Consider also a sub-level set of $V_{3,d}$, i.e., for some non-negative constant V_0, consider $(V_{3,d})_{\leq V_0} := \{x_3 \in \mathbb{X}_3 : V_{3,d}(x_3) \leq V_0\}$. Then $(V_{3,d})_{\leq V_0} \cap T_3 \cap U_3$ defines a compact subset of the state space $\mathbb{X}_3$.

Proof: Recall the definition of the state in step 3, namely $x_3 \in \mathbb{X}_3 :\Leftrightarrow (x_2, \hat{d}_T) \in \mathbb{X}_2 \times \mathbb{B}^3_{\bar{\bar{d}}}$. If $x_3 \in (V_{3,d})_{\leq V_0} \cap T_3 \cap U_3$, then note that: (i) if $V_{3,d}(x_3) \leq V_0$, it then follows from (67a) that $V_2(x_2) \leq V_{2,0}(\exp(\frac{V_0}{V_{2,0}}) - 1) < \infty$; one can then invoke Proposition 39, to conclude that x_2 is contained in a compact subset of $\mathbb{X}_2$. (ii) $\hat{d}_T$ belongs to some compact subset of $\mathbb{B}^3_{\bar{\bar{d}}}$ as specified in T_3. All together, it follows that $(V_{3,d})_{\leq V_0} \cap T_3 \cap U_3$ defines a compact subset of the state space $\mathbb{X}_3$. ■

Remark 49: In Proposition 48, we had to assume that $\hat{d}_T$ belongs to some compact subset of $\mathbb{B}^3_{\bar{\bar{d}}}$ because the proposition makes no reference (in order to be consistent with the other propositions) to the vector field. In the final theorem, however, we conclude that $\hat{d}_T$ belongs to some compact subset of $\mathbb{B}^3_{\bar{\bar{d}}}$ by noting that $\dot{\hat{d}}_T = E_{\delta,T}(x_2, \hat{d}_T)$ and by invoking the property (30b) of the projector function $\mathrm{Proj}_{\bar{\bar{d}},2}$.

Theorem 50: Consider the vector field $X^{cl}_{3,d}$ in (65) and the Lyapunov function $V_{3,d}$ in (67a). Moreover, consider the sets $\mathbb{X}^\star_{3,+}, \mathbb{X}^\star_{3,-}$ in (61); let the set U_3, as defined in (39), be invariant; and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). For brevity, denote $\tilde{\mathbb{X}}_3 := \mathbb{X}_3 \cap U_3$ and $\tilde{\mathbb{X}}^\star_{3,\pm} := \mathbb{X}^\star_{3,\pm} \cap U_3$. Finally, consider the differential equation

$$\dot{x}_3(t) = X^{cl}_{3,d}(x_3(t), g^{(2)}(t)) \text{ with } x_3(0) \in \tilde{\mathbb{X}}_3 \tag{70}$$

Then,

1) there exists a unique and complete solution $[0, \infty) \ni t \mapsto x_3(t) \in \tilde{\mathbb{X}}_3$ to (70);
2) the sets $\mathbb{X}^\star_{3,+}$ and $\mathbb{X}^\star_{3,-}$ are invariant;
3) the set $\tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-}$ is globally attractive, i.e.,

$$\lim_{t\to\infty} \mathrm{dist}\left(x_3(t), \tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-}\right) = 0, \text{ for all } x_3(0) \in \tilde{\mathbb{X}}_3;$$

4) the set $\tilde{\mathbb{X}}^\star_{3,+}$ is stable, while the set $\tilde{\mathbb{X}}^\star_{3,-}$ is unstable;
5) the set $\tilde{\mathbb{X}}^\star_{3,+}$ is (locally) asymptotically stable and $\lim_{t\to\infty} \mathrm{dist}\left(x_3(t), \tilde{\mathbb{X}}^\star_{3,+}\right) = 0$ if $x_3(0) \in \left\{x_3 \in \tilde{\mathbb{X}}_3 : V_{3,d}(x_3) < V^\star_{3,-}\right\}$, with $V^\star_{3,-}$ as defined in (69a).

Proof:

0) First note that $T_3 := \{x_3 \in \mathbb{X}_3 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\bar{d}}\}$ is a positively invariant set – see (30b).
1) Define $V_0 := V_{3,d}(x_3(0)) \in [0, \infty)$. Since, $(V_{3,d})_{\leq V_0} \cap T_3 \cap U_3$ defines a positively invariant compact subset of $\mathbb{X}_3$, since $g^{(2)}$ is also contained in a compact subset of $\mathbb{R}^3$, and since the vector field $X^{cl}_{3,d}$ is $\mathcal{C}^1$ continuous (and thus Lipschitz continuous) in $\mathbb{X}_3 \times \mathbb{R}^3$, the first conclusion follows immediately.
2) It is trivial to verify that

$$\begin{aligned} X^{cl}_{3,d}(x_3, g^2) &\in T_{x_3}\mathbb{X}^\star_{3,+} \text{ for all } x_3 \in \mathbb{X}^\star_{3,+} \text{ and } g^2 \in \mathbb{R}^3,\\ X^{cl}_{3,d}(x_3, g^2) &\in T_{x_3}\mathbb{X}^\star_{3,-} \text{ for all } x_3 \in \mathbb{X}^\star_{3,-} \text{ and } g^2 \in \mathbb{R}^3, \end{aligned}$$

which suffices to conclude that each of those sets is invariant.
3) To prove that the set $\tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-}$ is globally attractive, consider the solution $[0, \infty) \ni t \mapsto x_3(t) \in \tilde{\mathbb{X}}_3$, and note that
   a) $\tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-} = \{x_3 \in \mathbb{X}_3 : W_{3,d}(x_3) = 0 \text{ and "}\dot{v}\text{"} = v^1(x_3) = 0_3\}$ – see Proposition 46;
   b) the solution is contained in $(V_{3,d})_{\leq V_0} \cap T_3 \cap U_3$, which is a compact subset of $\mathbb{X}_3$ – see Proposition 48;
   c) since $V_{3,d}$ is lower bounded, and since $\dot{V}_{3,d}(x_3(t)) = W_{3,d}(x_3(t)) \leq 0$, it follows that $\lim_{t\to\infty} V_{3,d}(x_3(t))$ exists;
   d) finally,

$$\begin{aligned} \sup_{t\geq 0} |\ddot{V}_{3,d}(x_3(t))| &= \sup_{t\geq 0} |\dot{W}_{3,d}(x_3(t))|\\ &= \sup_{t\geq 0} |dW_{3,d}(x_3(t))X^{cl}_{3,d}(x_3(t), g^{(2)}(t))|\\ &\leq \sup_{\substack{x_3\in(V_{3,d})_{\leq V_0}\cap T_3\cap U_3\\ g^2\in\text{compact subset of } \mathbb{R}^3}} |dW_{3,d}(x_3)X^{cl}_{3,d}(x_3, g^2)| < \infty, \end{aligned}$$

   where the latter inequality follows since $W_{3,d}$ is $\mathcal{C}^1(\mathbb{X}_3)$ and $X^{cl}_{3,d}$ is $\mathcal{C}^0(\mathbb{X}_3 \times \mathbb{R}^3)$; thus, invoking *c)* and Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{V}_{3,d}(x_3(t)) = \lim_{t\to\infty} W_{3,d}(x_3(t)) = 0$;

e) it follows from *d)* that $\lim_{t\to\infty} v(t) = 0$ and, since

$$\begin{aligned}\sup_{t\geq 0} |\ddot{v}(t)| &= \sup_{t\geq 0} |\dot{v}^1(x_3(t))| \\ &= \sup_{t\geq 0} |dv^1(x_3(t))X^{cl}_{3,d}(x_3(t), g^{(2)}(t))| \\ &\leq \sup_{\substack{x_3\in(V_{3,d})_{\leq V_0}\cap T_3\cap U_3 \\ g^2\in\text{compact subset of } \mathbb{R}^3}} |dv^1(x_3)X^{cl}_{3,d}(x_3, g^2)| < \infty,\end{aligned}$$

where the latter inequality follows since v^1 is $\mathcal{C}^1(\mathbb{X}_3)$ and $X^{cl}_{3,d}$ is $\mathcal{C}^0(\mathbb{X}_3 \times \mathbb{R}^3)$; thus, invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{v}(t) = \lim_{t\to\infty} v^1(x_3(t)) = 0_3$;

finally, combining *a)*, *d)* and *e)*, it follows that $\lim_{t\to\infty} \text{dist}\left(x_3(t), \tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-}\right) = 0$.

4) To prove the set $\tilde{\mathbb{X}}^\star_{3,+}$ is stable, first note that $\tilde{\mathbb{X}}^\star_{3,+} = \{x_3 \in \tilde{\mathbb{X}}_3 : V_{3,d}(x_3) = 0\}$ and where we emphasize that $V_{3,d}$ is non-negative and continuous. We can then invoke Proposition 16, with $M = \tilde{\mathbb{X}}_3$ and $M^\star = \tilde{\mathbb{X}}^\star_{3,+}$ and $V = V_{3,d}|_{\tilde{\mathbb{X}}_3}$, to conclude that there exists $\alpha \in \mathcal{K}^\infty$ such that $\alpha\left(\text{dist}_M(m, M^\star)\right) \leq V(m)$ for all $m \in M$. (That is, $\{x_3 \in \tilde{\mathbb{X}}_3 : V_{3,d}(x_3) \leq \epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}^\star_{3,+}$, which coincides with the latter iff $\epsilon = 0$, and which is positively invariant for any $\epsilon \geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}^\star_{3,+}$ is stable.
   To conclude $\tilde{\mathbb{X}}^\star_{3,-}$ is unstable, it suffices to repeat the same reasoning as in the proof of Theorem 40.
5) This conclusion follows immediately from the facts that
   - $V_{3,d}(x_3) = V^\star_{3,-}$ for all $x_3 \in \mathbb{X}^\star_{3,-}$ – see Proposition 38;
   - $W_{3,d}(x_3) \leq 0$ for all $x_3 \in \mathbb{X}_3$;
   - the set $\tilde{\mathbb{X}}^\star_{3,+}$ is stable and $\tilde{\mathbb{X}}^\star_{3,+} \cup \tilde{\mathbb{X}}^\star_{3,-}$ is globally attractive, where the sets $\tilde{\mathbb{X}}^\star_{3,+}$ and $\tilde{\mathbb{X}}^\star_{3,-}$ are disjoint.

■

Remark 51: One issue that the reader may raise is why $(x_1, \hat{d}_T)$ cannot approach the set where $T^{3d}(x_1, \hat{d}_T)$ vanishes, since the unit vector $n^{cl}(x_1, \hat{d}_T) := \frac{T^{3d}(x_1,\hat{d}_T)}{\|T^{3d}(x_1,\hat{d}_T)\|}$ will be well-defined anyways. This remark focuses on this issue.

One concludes from the definition of the Lyapunov function $V_{3,d}$ in (67a) that, in sub-level sets of that Lyapunov function $\{V_{3,d}(x_3) \leq \epsilon\}$ (which are positively invariant), the unit vector n is "closer" to the unit vector $n^{cl}(x_1, \hat{d}_T)$, the closer ϵ is to 0. This, however, does not imply that the unit vector n is "closer" to the *equilibrium* unit vector $n^{cl}(x_1^\star, d) = \frac{g^0-d}{\|g^0-d\|}$, for $x_1^\star \in \mathbb{X}_1^\star$, the closer ϵ is to 0 (which is what necessary in order to guarantee that that equilibrium is stable).

Below, let $n \in \mathbb{S}^2$, $(x_1, \hat{d}_T) \in \mathbb{X}_1 \times \mathbb{B}^3_{\bar{d}}$, and $(x_1^\star, d) \in \mathbb{X}_1^\star \times \mathbb{B}^3_{\bar{d}}$. Note then that

$$\begin{aligned}\left\|n - \frac{g^0 - d}{\|g^0 - d\|}\right\| &= \|n - n^{cl}(x_1^\star, d)\| \\ &= \left\|\left(n - n^{cl}(x_1, \hat{d}_T)\right) - \left(n^{cl}(x_1^\star, d) - n^{cl}(x_1, \hat{d}_T)\right)\right\| \\ &\leq \left\|n - n^{cl}(x_1, \hat{d}_T)\right\| + \left\|n^{cl}(x_1, \hat{d}_T) - n^{cl}(x_1^\star, d)\right\|.\end{aligned}$$

It is then the case that

$$V_{3,d}(x_3) \leq V_0 \Rightarrow \left\|n - n^{cl}(x_1, \hat{d}_T)\right\| \leq \sqrt{V_{2,0}\left(\exp\left(\frac{\gamma_\theta^{-1}V_0}{V_{2,0}}\right) - 1\right)} =: \epsilon_1(V_0)$$

where $\epsilon_1(0) = 0$, $\epsilon_1(V_0) > 0$ for $V_0 > 0$, and ϵ_1 is non-decreasing. It is also the case that (below we assume that $x_1^\star + \sigma(x_1 - x_1^\star) \in \mathbb{X}_1$ for all $\sigma \in [0, 1]$; $d + \sigma(\hat{d}_T - d) \in \mathbb{B}^3_{\bar{d}}$ for all $\sigma \in [0, 1]$)

$$n^{cl}(x_1, \hat{d}_T) = n^{cl}(x_1^\star, d) + \left(\int_0^1 d_1 n^{cl}(x_1^\star + \sigma(x_1 - x_1^\star), d)d\sigma\right)(x_1 - x_1^\star) + \left(\int_0^1 d_2 n^{cl}(x_1^\star, d + \sigma(\hat{d}_T - d))d\sigma\right)(d - \hat{d}_T),$$

where

$$\begin{aligned}d_1 n^{cl}(x_1, d) &:= \Pi\left(n^{cl}(x_1, d)\right)\frac{\begin{bmatrix} d_1 u_{di}(p, v) & d_2 u_{di}(p, v) & I_3\end{bmatrix}}{\|T^{3d}(x_1, d)\|} \in \mathbb{R}^{3\times 12} \\ d_2 n^{cl}(x_1, d) &:= \Pi\left(n^{cl}(x_1, d)\right)\frac{-I_3}{\|T^{3d}(x_1, d)\|} \in \mathbb{R}^{3\times 3}\end{aligned}$$

which makes it clear that if $(x_1, \hat{d}_T)$ approaches the set where $T^{3d}(x_1, \hat{d}_T)$ vanishes, the unit vector $n^{cl}(x_1, \hat{d}_T) := \frac{T^{3d}(x_1,\hat{d}_T)}{\|T^{3d}(x_1,\hat{d}_T)\|}$ may be well-defined, but one cannot guarantee that the equilibrium unit vector $n^{cl}(x_1^\star, d) = \frac{g^0-d}{\|g^0-d\|}$ is stable.

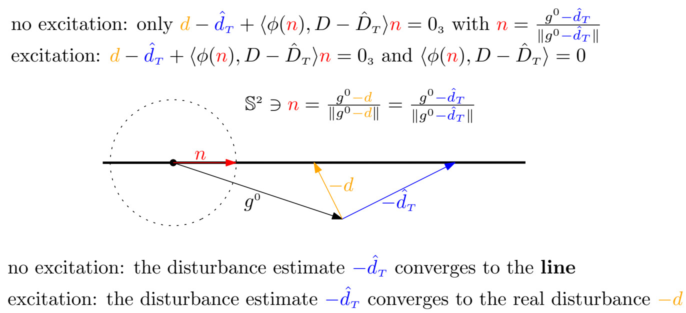


Fig. 8: Illustration of equilibria sets $\mathbb{X}^{\star}_{4,\pm}$ in (73) with and without excitation.

D. Step 4

Throughout this section, keep in mind the scheme illustrated in Fig. 6d. Consider then the vector field

$$\dot{x}_4 = X_{4,d,D}(x_4, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_3 \\ \dot{\hat{D}}_T \end{bmatrix} = \begin{bmatrix} X_{3,d,D}(x_3, (T, \omega, g^2)) \\ E_{\Delta,T}(x_3, \hat{D}_T) \end{bmatrix}, \tag{71}$$

where

- $x_4 \in \mathbb{X}_4 :\Leftrightarrow (x_3, \hat{D}_T) \in \mathbb{X}_3 \times \mathbb{R}^3$ is the state, composed of the state x_3 (described in step 3) and the estimate of the disturbance D;
- T and ω (thrust and angular velocity) are the inputs to the vector field; and $E_{\Delta,T}$ the estimator-dynamics/update-law that we wish to design for the estimate $\hat{D}_T$;
- $X_{3,d,D}$ is the vector field described in step 3 (see (60));
- both the disturbances d and D are unknown.

As in the previous steps, we need to define the equilibria sets, but, at this point, the equilibria sets depend on an excitation criterion. Let then the time-varying gravity acceleration $t \mapsto g(t)$ be given, and let us introduce the shorthand

$$E \equiv \lim_{t \to \infty} \mathcal{S}\left(\frac{g^{(0)}(t) - d}{\|g^{(0)}(t) - d\|}\right) \frac{g^{(1)}(t)}{\|g^{(0)}(t) - d\|}, \tag{72}$$

where the limit above may, or may not, exist. If $E = 0_3$, denote ($\mathbb{X}^{\star}_{3,\pm}$ defined in (61))

$$\mathbb{X}^{\star}_{4,\pm} := \left\{ x_4 \in \mathbb{X}_4 : p = 0_3 \text{ and } v = 0_3 \text{ and } n = \pm \frac{g^0 - d}{\|g^0 - d\|} \text{ and } (d - \hat{d}_T) + \langle \phi(n), D - \hat{D}_T \rangle n = 0_3 \right\}, \tag{73a}$$

otherwise, denote

$$\mathbb{X}^{\star}_{4,\pm} := \left\{ x_4 \in \mathbb{X}_4 : p = 0_3 \text{ and } v = 0_3 \text{ and } n = \pm \frac{g^0 - d}{\|g^0 - d\|} \text{ and } \hat{d}_T = d \text{ and } \left\langle \phi(n), D - \hat{D}_T \right\rangle = 0_3 \right\}. \tag{73b}$$

Finally, denote also

$$\mathbb{X}^{\star,\star}_{4,\pm} := \left\{ x_4 \in \mathbb{X}_4 : p = 0_3 \text{ and } v = 0_3 \text{ and } n = \pm \frac{g^0 - d}{\|g^0 - d\|} \text{ and } d = \hat{d}_T \text{ and } D = \hat{D}_T \right\}, \tag{73c}$$

where it is straightforward to verify that $\mathbb{X}^{\star,\star}_{4,\pm} \subset \mathbb{X}^{\star}_{4,\pm}$. The sets above are the equilibria sets, and later we show that

- $\mathbb{X}^{\star}_{4,+} \cup \mathbb{X}^{\star}_{4,-}$ is globally attractive, and $\mathbb{X}^{\star}_{4,+}$ is (locally) attractive (not necessarily stable though);
- $\mathbb{X}^{\star,\star}_{4,+}$ (which is a subset of $\mathbb{X}^{\star}_{4,+}$) is stable (not necessarily attractive though – we will only be able to show that $\mathbb{X}^{\star}_{4,+}$, which includes $\mathbb{X}^{\star,\star}_{4,+}$, is attractive);
- $\mathbb{X}^{\star,\star}_{4,-}$ is unstable (we will not be able to show that $\mathbb{X}^{\star}_{4,-}$ is unstable though).

The equilibria sets depend on the satisfaction of the condition $\lim_{t \to \infty} \mathcal{S}\left(\frac{g^{(0)}(t) - d}{\|g^{(0)}(t) - d\|}\right) g^{(1)}(t) = 0_3$. When this condition is satisfied, it requires the equilibria unit vectors $t \mapsto n^{\star}_{\pm}(t) = \pm \frac{g^{(0)}(t) - d}{\|g^{(0)}(t) - d\|}$ to be moving with an asymptotically vanishing angular velocity $t \mapsto \omega^{\star}(t) = \mathcal{S}\left(\frac{g^{(0)}(t) - d}{\|g^{(0)}(t) - d\|}\right) \frac{g^{(1)}(t)}{\|g^{(0)}(t) - d\|}$; this, however, does not imply that the equilibria unit vectors need to converge to some constant unit vector. On the other hand, when the criterion is not satisfied, it requires the equilibria unit vectors to be moving with an asymptotically non-vanishing angular velocity. The important point to keep in mind is that, when the condition is satisfied ("no excitation"), the disturbance estimate $\hat{d}_T$ does not necessarily have to converge to the real unknown disturbance d. On the other hand, when the condition is not satisfied ("excitation"), the disturbance estimate $\hat{d}_T$ does have to converge to the real unknown disturbance d. This is in contrast with the previous step, where the disturbance D was assumed know, and where the disturbance estimate $\hat{d}_T$ converges to the real unknown disturbance d regardless of any excitation criterion.

Once again, the idea is to attempt to reuse the control laws from the previous steps (with some minor necessary adjustments). As opposed to the previous step, the disturbance D is no longer known; moreover, recall that, in the latter step, neither the designed angular velocity ω_2^{cl}, nor the estimator dynamics $E_{\delta,T}$ depend on D. Thus, it suffices that we amend the thrust control law T_2^{cl} (see (62)) by replacing the disturbance D with its estimate $\hat{D}_T$, i.e., we define a new thrust control law as

$$\begin{aligned} &T_3^{cl}: \mathbb{X}_4 \to \mathbb{R} \\ &T_3^{cl}(x_4) := T_2^{cl}(x_3, \hat{D}_T) \end{aligned} \quad . \tag{74}$$

Composing the vector field $X_{1,d,D}$ in (37) with the control law T^{cl} (74) yields

$$\dot{x}_1 = X_{1,d,D}(x_1, (T^{cl}(x_4), n, g^1)) \Leftrightarrow \dot{x}_1 = \tilde{X}_1(x_2, \hat{d}_T) + (e_2^3 \otimes I_3)(d - \hat{d}_T) + \langle \phi(n), D - \hat{D}_T \rangle (e_2^3 \otimes I_3) n, \tag{75}$$

($\tilde{X}_1$ as defined in (52), and $e_2^3 = (0,1,0)$) since

$$\begin{aligned} \dot{v} &= \left((T + \langle \phi(n), D \rangle)\, n - g^0 + d \right)|_{T = T_3^{cl}(x_4)} && \because (37) \\ &= \underbrace{u(p,v) - \Pi\,(n)\, T^{3d}(x_1, \hat{d}_T) + (d - \hat{d}_T) + \langle \phi(n), D - \hat{D}_T \rangle n}_{=: v^1(x_4)}. \end{aligned} \tag{76}$$

(We note that v^1 in (76) is not the same as that in (64), as they have different domains – i.e., the former "depends" on x_4, while the latter "depends" on x_3). For reasons that will be made clear later, we also need to define the second time derivative of the linear velocity (a.ka. jerk), namely

$$\begin{aligned} \ddot{v} = &\underbrace{dv^1(x_4) X_{4,d,D}(x_3, (T, \omega, g^2))|_{\omega = \omega_2^{cl}(x_3) \text{ and } T = T_3^{cl}(x_4)}}_{\text{independent of } g^2 \text{ (because } v^1 \text{ does not depend on } g^1)} =: v^2(x_4) \\ = \quad & d_1 u(p,v) v + d_2 u(p,v) v^1(x_4) \\ & + (\mathcal{S}\,(\omega)\, n n^T - n n^T \mathcal{S}\,(\omega)) T^{3d}(x_1, \hat{d}_T) \\ & - \Pi\,(n) \left(d_1 T^{3d}(x_1, \hat{d}_T) \dot{x}_1 + d_2 T^{3d}(x_1, \hat{d}_T) E_{\delta,T}(x_2, \hat{d}_T) \right) |_{\dot{x}_1 \text{ as in (75)}} \\ & + (0_3 - E_{\delta,T}(x_2, \hat{d}_T)) \\ & + \langle d\phi(n) \mathcal{S}\,(\omega)\, n, D - \hat{D}_T \rangle n + \langle \phi(n), 0_3 - E_{\Delta,T}(x_3, \hat{D}_T) \rangle n + \langle \phi(n), D - \hat{D}_T \rangle \mathcal{S}\,(\omega)\, n|_{\omega = \omega_2^{cl}(x_3)}. \end{aligned} \tag{77}$$

Therefore composing the vector field $X_{4,d,D}$ in (71) with the thrust control law T_3^{cl} in (74) and the angular velocity control law ω_2^{cl} in (63) yields

$$\dot{x}_4 = X_{4,d,D}(x_4, (T_3^{cl}(x_4), \omega_2^{cl}(x_3), g^2)) =: X_{4,d,D}^{cl}(x_4, g^2) \Leftrightarrow \begin{bmatrix} \dot{x}_3 \\ \dot{\hat{D}}_T \end{bmatrix} = \underbrace{\begin{bmatrix} X_{3,d}^{cl}(x_3, g^2) \\ 0_3 \end{bmatrix}}_{\text{previous step}} + \underbrace{\begin{bmatrix} \langle \phi(n), D - \hat{D}_T \rangle (e_2^6 \otimes I_3) n \\ E_{\Delta,T}(x_3, \hat{D}_T) \end{bmatrix}}_{\text{top is linear w.r.t. } (D - \hat{D}_T)}, \tag{78}$$

where $e_2^6 = (0,1,0,0,0,0)$. Since the top part of the vector field in (78) is affine with respect to the estimation error $D - \hat{D}_T$, we can thus proceed with a standard estimator design (see Section VII-B). As such, given some positive $k_{\Delta,T}$ ($V_{2,0}$ introduced in the previous step), we then choose (recall (36b))

$$E_{\Delta,T}(x_3, \hat{D}_T) := \text{Proj}_{\bar{\hat{D}},2} \left(\tilde{E}_{\Delta,T}(x_3, \hat{D}_T), \hat{D}_T \right), \tag{79a}$$

$$\tilde{E}_{\Delta,T}(x_3, \hat{D}_T) := k_{\Delta,T} \left\langle n, (e_2^5 \otimes I_3)^T \nabla_{\Gamma, V_{2,0}} V_{2,\hat{d}_T}(x_2) \right\rangle \phi(n), \tag{79b}$$

for reasons that are made clear next (recall the properties of $\text{Proj}_{\bar{\hat{d}}}$, in Section VII-C). As an observation,

- $k_{\Delta,T}$ represents a (positive) integral gain, and
- $V_{2,0}$ represents a saturation quantity that determines when the estimator starts working ($\dot{\hat{D}}_T \approx 0_3$ when $V_{2,\hat{d}_T}(x_2) \gg V_{2,0}$).

We can then construct the Lyapunov function and its derivative along the vector field $X_{4,d,D}^{cl}$, described in (78), as

$$\mathbb{X}_4 \ni x_4 \mapsto V_{4,d,D}(x_4) := V_{3,d}(x_3) + \frac{1}{2k_{\Delta,T}} \left\| D - \hat{D}_T \right\|^2 \in [0, \infty), \tag{80a}$$

$$\mathbb{X}_4 \ni x_4 \mapsto W_{4,d,D}(x_4) := \underbrace{dV_{4,d,D}(x_4) X_{4,d,D}^{cl}(x_4, g^2)}_{\text{(independent of } g^2)} \in (-\infty, 0], \tag{80b}$$

where (80b) is equivalently expressed as

$$
\begin{aligned}
W_{4,d,D}(x_4) &:= dV_{4,d,D}(x_4)X^{cl}_{4,d,D}(x_4,g^2)\\
&= W_{3,d}(x_3) + dV_{3,d}(x_3)(\langle\phi(n), D-\hat{D}_T\rangle(e_2^6\otimes I_3)n) - \frac{1}{k_{\Delta,T}}\left\langle D-\hat{D}_T, E_{\Delta,T}(x_3,\hat{D}_T)\right\rangle\\
&= W_{3,d}(x_3) - \frac{1}{k_{\Delta,T}}\left\langle D-\hat{D}_T, E_{\Delta,T}(x_3,\hat{D}_T) - k_{\Delta,T}\langle n,(e_2^6\otimes I_3)^T\nabla V_{3,d}(x_3)\rangle\phi(n)\right\rangle\\
&= \underbrace{W_{3,d}(x_3)}_{\leq 0 \quad \because (67b)} \underbrace{-\frac{1}{k_{\Delta,T}}\left\langle D-\hat{D}_T, E_{\Delta,T}(x_3,\hat{D}_T) - \tilde{E}_{\Delta,T}(x_3,\hat{D}_T)\right\rangle}_{\leq 0 \quad \because (30c)} \qquad \because (68c). \qquad (80c)
\end{aligned}
$$

Remark 52: Notice that the angular velocity control law ω_2^{cl} does not depend on the estimate $\hat{D}_T$, only the thrust control law T_3^{cl}.

Remark 53: Notice that we could have designed an estimator dynamics $E_{\Delta,T}$ independent of the estimate, if we had chosen

$$E_{\Delta,T}(x_3,\hat{D}_T) := k_{\Delta,T}\left\langle n,(e_2^5\otimes I_3)^T\nabla_{\Gamma,V_{2,0}}V_{2,\hat{d}_T}(x_2)\right\rangle\phi(n).$$

The same cannot be accomplished with the estimator dynamics $E_{\delta,T}$, owing to the different nature of the disturbance d.
The next proposition is the basis for showing that the disturbance estimate $\hat{d}_T$ converges to the real disturbance d, and the basis for analyzing the stability and attractivity properties of the sets in (73).

Proposition 54: Consider the functions $V_{4,d,D}, W_{4,d,D}$ in (80a), (80b), and the functions $E_{\delta,T}$,$E_{\delta,T}$ in (66a), (79a). Consider also the sets $\mathbb{X}^{\star}_{4,\pm}$ and $\mathbb{X}^{\star,\star}_{4,\pm}$ in (73). It holds that

$$\begin{cases} V_{4,d,D}(x_4) = 0 \text{ for all } x_4\in\mathbb{X}^{\star,\star}_{4,+} \\ V_{4,d,D}(x_4) > 0 \text{ for all } x_4\in\mathbb{X}^{\star}_{4,+}\backslash\mathbb{X}^{\star,\star}_{4,+}\end{cases}, \qquad (81a)$$

$$\begin{cases} V_{4,d,D}(x_4) = V^{\star}_{3,-} =: V^{\star}_{4,-} \text{ for all } x_4\in\mathbb{X}^{\star,\star}_{4,-} \\ V_{4,d,D}(x_4) > V^{\star}_{3,-} =: V^{\star}_{4,-} \text{ for all } x_4\in\mathbb{X}^{\star}_{4,-}\backslash\mathbb{X}^{\star,\star}_{4,-}\end{cases}, \qquad (81b)$$

$$\mathbb{X}^{\star}_{4,+}\cup\mathbb{X}^{\star}_{4,-}\supset\{x_4\in\mathbb{X}_4 : \text{“}\dot{V}_{4,d}\text{”} = W_{4,d,D}(x_4)=0 \text{ and “}\dot{v}\text{”} = v^1(x_4)=0_3 \text{ and “}\ddot{v}\text{”} = v^2(x_4) = 0_3\}, \qquad (81c)$$

$$E_{\delta,T}(x_2,\hat{d}_T) = 0_3 \text{ for all } x_2\in\mathbb{X}^{\star}_{2,+}\cup\mathbb{X}^{\star}_{2,-} \text{ and for all } \hat{d}_T\in\mathbb{B}^3_{\bar{d}}, \qquad (81d)$$

$$E_{\Delta,T}(x_3,\hat{D}_T) = 0_3 \text{ for all } x_3\in\mathbb{X}^{\star}_{3,+}\cup\mathbb{X}^{\star}_{3,-} \text{ and for all } \hat{D}_T\in\mathbb{B}^3_{\bar{D}}, \qquad (81e)$$

with $V^{\star}_{3,-}$ as defined in (69b), v^1 as defined in (76), and v^2 as defined in (77).

Proof: Verifying (81a)–(81e) is simple, and, in this proof, we only verify (81b). Consider then (81b):

1) from (80c), $W_{4,d,D}(x_4)=0$ implies that $W_{2,d}(x_2)=0$;
2) from (56b), $W_{2,d}(x_2)=0$ implies that $p=0_3$ and $v=0_3$ and $n=\pm\frac{g^0-\hat{d}_T}{\|g^0-\hat{d}_T\|}$ (remark: ultimately, we wish to conclude that $n=\pm\frac{g^0-d}{\|g^0-d\|}$);
3) combining 1) and 2), it then follows that “$\dot{v}$” $= v^1(x_4) = 0_3 \Leftrightarrow (d-\hat{d}_T)+\langle\phi(n),D-\hat{D}_T\rangle n = 0_3$;
4) combining 2) and 3), it then follows that $n=\pm\frac{g^0-d}{\|g^0-d\|}$; that is indeed the case since

$$
\begin{aligned}
\text{“}\dot{v}\text{”} = v^1(x_4) = 0_3 &\Leftrightarrow (d-\hat{d}_T)+\langle\phi(n),D-\hat{D}_T\rangle n\big|_{n=\pm\frac{g^0-\hat{d}_T}{\|g^0-\hat{d}_T\|}} = 0_3\\
&\Leftrightarrow d-\hat{d}_T+\lambda(g^0-\hat{d}_T) = 0_3 \quad \text{for some } \lambda\in\mathbb{R}\\
&\Leftrightarrow (1+\lambda)(g^0-\hat{d}_T) = (g^0-d)\\
&\Leftrightarrow \frac{g^0-\hat{d}_T}{\|g^0-\hat{d}_T\|} = \frac{g^0-d}{\|g^0-d\|}.
\end{aligned}
$$

At this point the proof is over if the condition $\lim_{t\to\infty}\mathcal{S}\left(\frac{g^{(0)}(t)-d}{\|g^{(0)}(t)-d\|}\right)\frac{g^{(1)}(t)}{\|g^{(0)}(t)-d\|} = 0_3$ is satisfied (“no excitation condition”); and where we emphasize that we have not concluded that $\hat{d}_T = d$. Suppose now that the latter condition is not satisfied, and for that reason define $\omega := \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-d\|}\neq 0_3$.

5) combining 2)–4), it follows that $\omega_2^{cl}(x_3) = \mathcal{S}\left(\frac{g^0-\hat{d}_T}{\|g^0-\hat{d}_T\|}\right)\frac{g^1}{\|g^0-\hat{d}_T\|} = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-\hat{d}_T\|} = \omega\neq 0_3$.

6) combining 2)–5), it then follows that "$\ddot{v}$" $= v^2(x_4) = 0_3 \Rightarrow \langle \phi(n), D - \hat{D}_T\rangle = 0$. Indeed, since $\omega = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-\hat{d}_T\|}$ and $n = \frac{g^0-d}{\|g^0-d\|}$ and $T^{3d}(x_1, \hat{d}_T) = g^0 - \hat{d}_T$, then

$$
\begin{aligned}
&\ddot{v} = v^2(x_4) = 0_3 \overset{(77)}{\Leftrightarrow} \\
\Leftrightarrow\ &\underbrace{\langle n, T^{3d}(x_1, \hat{d}_T)\rangle \mathcal{S}(\omega)\, n - \Pi(n)\, g^1}_{\text{these terms cancel}} + \underbrace{\langle d\phi(n)\mathcal{S}(\omega)\, n, D - \hat{D}_T\rangle n + \langle \phi(n), D - \hat{D}_T\rangle \mathcal{S}(\omega)\, n}_{n \text{ and } \mathcal{S}(\omega)n \text{ are orthogonal to each other and } \omega \neq 0_3} = 0_3 \\
\Leftrightarrow\ &\langle d\phi(n)\mathcal{S}(\omega)\, n, D - \hat{D}_T\rangle = 0 \text{ and } \langle \phi(n), D - \hat{D}_T\rangle = 0.
\end{aligned} \tag{81f}
$$

6) Finally, combining 3) and 7), it follows that $\hat{d}_T = d$, and thus that $\omega = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-\hat{d}_T\|} = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-d\|}$, which completes the proof. ■

Remark 55: Let us discuss the meaning of the results in Proposition 54.

- (81b) implies that if the Lyapuvov function is ever "below" the threshold $V^\star_{4,-}$, then a solution cannot converge to the set $\mathbb{X}^\star_{4,-}$, since the Lyapuvov function is non-increasing along any solution;
- (81c) provides a mean to determine whether we are at the equilibria set, and since it "depends" on *three* components – "$\dot{V}_{4,d,D}$" and "$\dot{v}$" and "$\ddot{v}$", where "$V_{4,d,D}$" and "v" and "$\dot{v}$" all converge to some constants – we will be invoking Barbalat's lemma *three* times (in fact, $\mathbb{X}^\star_{4,+} \cup \mathbb{X}^\star_{4,-} \nsubseteq \{x_4 \in \mathbb{X}_4 : W_{4,d,D}(x_4) = 0 \text{ and } v^1(x_4) = 0_3 \text{ and } v^2(x_4) = 0_3\}$ because, as can be seen in (81f), the latter set requires that $\langle d\phi(n)\mathcal{S}(\omega)\, n, D - \hat{D}_T\rangle|_{n=\frac{g^0-d}{\|g^0-d\|} \text{ and } \omega = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-\hat{d}_T\|}} = 0$);
- (81d) and (81e) imply that if a solution approaches the equilibria set, then $\dot{\hat{d}}_T$ and $\dot{\hat{D}}_T$ approach 0_3 (which , by themselves, do not imply that $\hat{d}_T$ and $\hat{D}_T$ approach some constants in $\mathbb{R}^3$).

Remark 56: Similarly to Remark 44, note that the Lyapunov function $V_{4,d,D}$ in (80a) depends on the unknown disturbances d and D, so it cannot be used for the purposes of control. Nonetheless, we note that

$$(e_2^7 \otimes I_3)^T \nabla V_{4,d,D}(x_4) = \frac{\partial}{\partial v}\nabla V_{4,d,D}(x_4)|_{x4=(p,v,g^0,n,g^1,\hat{d}_T,\hat{D}_T)} = \Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)(e_2^5 \otimes I_3)^T \nabla V_{2,\hat{d}_T}(x_2), \tag{82a}$$

$$(e_4^7 \otimes I_3)^T \nabla V_{4,d,D}(x_4) = \frac{\partial}{\partial n}\nabla V_{4,d,D}(x_4)|_{x4=(p,v,g^0,n,g^1,\hat{d}_T,\hat{D}_T)} = \Gamma_{V_{2,0}}\left(V_{2,\hat{d}_T}(x_2)\right)(e_4^5 \otimes I_3)^T \nabla V_{2,\hat{d}_T}(x_2). \tag{82b}$$

Proposition 57: Recall Proposition 48, and define

$$
\begin{aligned}
&U_4 := \{x_4 \in \mathbb{X}_4 : x_3 \in U_3\},\, , \\
&T_4 := \left\{x_4 \in \mathbb{X}_4 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\hat{d}}\right\},
\end{aligned}
$$

where U_3 is found in Proposition 48. Consider also a sub-level set of $V_{4,d,D}$, i.e., for some non-negative constant V_0, consider $(V_{4,d,D})_{\leq V_0} := \{x_4 \in \mathbb{X}_4 : V_{4,d,D}(x_4) \leq V_0\}$. Then $(V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4$ defines a compact subset of the state space $\mathbb{X}_4$.

Proof: Recall the definition of the state in step 4, namely $x_4 \in \mathbb{X}_4 :\Leftrightarrow (x_3, \hat{D}_T) \in \mathbb{X}_3 \times \mathbb{R}^3$. If $x_4 \in (V_{4,d,D})_{\leq V_0} \cap U_4$, then note that: (i) if $V_{4,d,D}(x_3) \leq V_0$, it then follows from (80a) that $V_{3,d}(x_3) \leq V_0 < \infty$; one can then invoke Proposition 48, to conclude that x_3 is contained in a compact subset of $\mathbb{X}_3$. (ii) if $V_{4,d,D}(x_3) \leq V_0$, it then follows from (80a) that $\|D - \hat{D}_T\| \leq 2k_{\Delta,T}V_0$, which implies that $\hat{D}_T$ is trapped in some compact set of $\mathbb{R}^3$. All together, it follows that $(V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4$ defines a compact subset of the state space $\mathbb{X}_4$. ■

Theorem 58: Consider the vector field $X^{cl}_{4,d,D}$ in (78) and the Lyapunov function $V_{4,d,D}$ in (80a). Moreover, consider the sets $\mathbb{X}^\star_{4,\pm}$ and $\mathbb{X}^{\star,\star}_{4,\pm}$ in (73); let the set U_4, as defined in (57), be invariant; and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). For brevity, denote $\tilde{\mathbb{X}}_4 := \mathbb{X}_4 \cap U_4$, $\tilde{\mathbb{X}}^\star_{4,\pm} := \mathbb{X}^\star_{4,\pm} \cap U_4$, and $\tilde{\mathbb{X}}^{\star,\star}_{4,\pm} := \mathbb{X}^{\star,\star}_{4,\pm} \cap U_4$. Finally, consider the differential equation

$$\dot{x}_4(t) = X^{cl}_{4,d,D}(x_4(t), g^{(2)}(t)) \text{ with } x_4(0) \in \tilde{\mathbb{X}}_4 \tag{83}$$

Then,

1) there exists a unique and complete solution $[0, \infty) \ni t \mapsto x_4(t) \in \tilde{\mathbb{X}}_4$ to (83);
2) the sets $\mathbb{X}^\star_{4,+}$ and $\mathbb{X}^\star_{4,-}$ are invariant;
3) the set $\tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-}$ is globally attractive, i.e.,

$$\lim_{t\to\infty} \operatorname{dist}\left(x_4(t), \tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-}\right) = 0, \text{ for all } x_4(0) \in \tilde{\mathbb{X}}_4;$$

4) the set $\tilde{\mathbb{X}}^{\star,\star}_{4,+}$ is stable, while the set $\tilde{\mathbb{X}}^{\star,\star}_{4,-}$ is unstable;
5) the set $\tilde{\mathbb{X}}^\star_{4,+}$ is (locally) attractive and $\lim_{t\to\infty} \operatorname{dist}\left(x_4(t), \tilde{\mathbb{X}}^\star_{4,+}\right) = 0$ if $x_4(0) \in \left\{x_4 \in \tilde{\mathbb{X}}_4 : V_{4,d,D}(x_4) < V^\star_{4,-}\right\}$, with $V^\star_{4,-}$ as defined in (81b).

Proof:

0) First note that $T_4 := \{x_4 \in \mathbb{X}_4 : \hat{d}_T \in \bar{\mathbb{B}}^4_r \text{ for some } r < \bar{\hat{d}}\}$ is a positively invariant set – see (30b).

1) Define $V_0 := V_{4,d,D}(x_4(0)) \in [0, \infty)$. Since, $(V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4$ defines a positively invariant compact subset of $\mathbb{X}_4$, since $g^{(2)}$ is also contained in a compact subset of $\mathbb{R}^3$, and since the vector field $X^{cl}_{4,d,D}$ is $\mathcal{C}^1$ continuous (and thus Lipschitz continuous) in $\mathbb{X}_4 \times \mathbb{R}^3$, the first conclusion follows immediately.
2) It is trivial to verify that

$$
\begin{aligned}
X^{cl}_{4,d,D}(x_4, g^2) &\in T_{x_4}\mathbb{X}^\star_{4,+} \text{ for all } x_4 \in \mathbb{X}^\star_{4,+} \text{ and } g^2 \in \mathbb{R}^3, \\
X^{cl}_{4,d,D}(x_4, g^2) &\in T_{x_4}\mathbb{X}^\star_{4,-} \text{ for all } x_4 \in \mathbb{X}^\star_{4,-} \text{ and } g^2 \in \mathbb{R}^3,
\end{aligned}
$$

which suffices to conclude that each of those sets is invariant.
3) To prove that the set $\tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-}$ is globally attractive, consider the solution $[0, \infty) \ni t \mapsto x_4(t) \in \tilde{\mathbb{X}}_4$, and note that
   a) $\tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-} \supseteq \{x_4 \in \mathbb{X}_4 : W_{4,d,D}(x_4) = 0$ and "$\dot{v}$" $= v^1(x_4) = 0_3$ and "$\ddot{v}$" $= v^2(x_4) = 0_3\}$ – see Proposition 54;
   b) the solution is contained in $(V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4$, which is a compact subset of $\mathbb{X}_4$ – see Proposition 57;
   c) since $V_{4,d,D}$ is lower bounded, and since $\dot{V}_{4,d,D}(x_4(t)) = W_{4,d,D}(x_4(t)) \leq 0$, it follows that $\lim_{t\to\infty} V_{4,d,D}(x_4(t))$ exists;
   d) finally,

$$
\begin{aligned}
\sup_{t\geq 0} |\ddot{V}_{4,d,,D}(x_4(t))| &= \sup_{t\geq 0} |\dot{W}_{4,d,D}(x_4(t))| \\
&= \sup_{t\geq 0} |dW_{4,d,D}(x_4(t)) X^{cl}_{4,d,D}(x_4(t), g^{(2)}(t))| \\
&\leq \sup_{\substack{x_4 \in (V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4 \\ g^2 \in \text{compact subset of } \mathbb{R}^3}} |dW_{4,d,D}(x_4) X^{cl}_{4,d,D}(x_4, g^2)| < \infty,
\end{aligned}
$$

   where the latter inequality follows since $W_{4,d,D}$ is $\mathcal{C}^1(\mathbb{X}_4)$ and $X^{cl}_{4,d,D}$ is $\mathcal{C}^0(\mathbb{X}_4 \times \mathbb{R}^3)$; thus, invoking *c)* and Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{V}_{4,d}(x_4(t)) = \lim_{t\to\infty} W_{4,d,D}(x_4(t)) = 0$;
   e) it follows from *d)* that $\lim_{t\to\infty} v(t) = 0$ and, since

$$
\begin{aligned}
\sup_{t\geq 0} |\ddot{v}(t)| &= \sup_{t\geq 0} |\dot{v}^1(x_4(t))| \\
&= \sup_{t\geq 0} |dv^1(x_4(t)) X^{cl}_{4,d,D}(x_4(t), g^{(2)}(t))| \\
&\leq \sup_{\substack{x_4 \in (V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4 \\ g^2 \in \text{compact subset of } \mathbb{R}^3}} |dv^1(x_4) X^{cl}_{4,d,D}(x_4, g^2)| < \infty,
\end{aligned}
$$

   where the latter inequality follows since v^1 is $\mathcal{C}^1(\mathbb{X}_4)$ and $X^{cl}_{4,d,D}$ is $\mathcal{C}^0(\mathbb{X}_4 \times \mathbb{R}^3)$; thus, invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{v}(t) = \lim_{t\to\infty} v^1(x_4(t)) = 0_3$;
   f) it follows from *e)* that $\lim_{t\to\infty} \dot{v}(t) = 0$ and, since

$$
\begin{aligned}
\sup_{t\geq 0} |\dddot{v}(t)| &= \sup_{t\geq 0} |\dot{v}^2(x_4(t))| \\
&= \sup_{t\geq 0} |dv^2(x_4(t)) X^{cl}_{4,d,D}(x_4(t), g^{(2)}(t))| \\
&\leq \sup_{\substack{x_4 \in (V_{4,d,D})_{\leq V_0} \cap T_4 \cap U_4 \\ g^2 \in \text{compact subset of } \mathbb{R}^3}} |dv^2(x_4) X^{cl}_{4,d,D}(x_4, g^2)| < \infty,
\end{aligned}
$$

   where the latter inequality follows since v^2 is $\mathcal{C}^1(\mathbb{X}_4)$ and $X^{cl}_{4,d,D}$ is $\mathcal{C}^0(\mathbb{X}_4 \times \mathbb{R}^3)$; thus, invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty} \ddot{v}(t) = \lim_{t\to\infty} v^2(x_4(t)) = 0_3$;

   finally, combining *a)*, *b)*, *d)*, *e)* and *f)*, it follows that $\lim_{t\to\infty} \text{dist}\left(x_4(t), \tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-}\right) = 0$.
4) To prove the set $\tilde{\mathbb{X}}^{\star,\star}_{4,+}$ is stable, first note that $\tilde{\mathbb{X}}^\star_{4,+} = \{x_4 \in \tilde{\mathbb{X}}_4 : V_{4,d,D}(x_4) = 0\}$ and where we emphasize that $V_{4,d,D}$ is non-negative and continuous. We can then invoke Proposition 16, with $M = \tilde{\mathbb{X}}_4$ and $M^\star = \tilde{\mathbb{X}}^{\star,\star}_{4,+}$ and $V = V_{4,d,D}|_{\tilde{\mathbb{X}}_4}$, to conclude that there exists $\alpha \in \mathcal{K}^\infty$ such that $\alpha\left(\text{dist}_M(m, M^\star)\right) \leq V(m)$ for all $m \in M$. (That is, $\{x_4 \in \tilde{\mathbb{X}}_4 : V_{4,d,D}(x_4) \leq \epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}^{\star,\star}_{4,+}$, which coincides with the latter iff $\epsilon = 0$, and which is positively invariant for any $\epsilon \geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}^{\star,\star}_{4,+}$ is stable.
   To conclude $\tilde{\mathbb{X}}^{\star,\star}_{4,-}$ is unstable, it suffices to repeat the same reasoning as in the proof of Theorem 40 (note that $V_{4,d,D}(x_4) = V^\star_{4,-}$ for all $x_4 \in \tilde{\mathbb{X}}^{\star,\star}_{4,-}$).
5) This conclusion follows immediately from the facts that
   - $V_{4,d,D}(x_4) \geq V^\star_{4,-}$ for all $x_4 \in \mathbb{X}^\star_{4,-}$ – see Proposition 38;
   - $W_{4,d,D}(x_4) \leq 0$ for all $x_4 \in \mathbb{X}_4$;
   - the set $\tilde{\mathbb{X}}^\star_{4,+} \cup \tilde{\mathbb{X}}^\star_{4,-}$ is globally attractive, where the sets $\tilde{\mathbb{X}}^\star_{4,+}$ and $\tilde{\mathbb{X}}^\star_{4,-}$ are disjoint. ■

Remark 59: We cannot prove stability of the set $\tilde{\mathbb{X}}^\star_{4,+}$ because the Lyapunov function $V_{4,d,D}$ we constructed does not vanish at all points of that set; and, as such, sub-level sets of that $V_{4,d,D}$ (which would be positively invariant) do not define neighborhoods around the set $\tilde{\mathbb{X}}^\star_{4,+}$.

In a different vein, we cannot prove instability of the set $\tilde{\mathbb{X}}^{\star}_{4,-}\backslash\tilde{\mathbb{X}}^{\star,\star}_{4,-}$ because the Lyapunov function $V_{4,d,D}$ is strictly greater than $V^{\star}_{4,-}$ for all points in that set; and, as such, arbitrarily small perturbations around that set will not make $V_{4,d,D}$ cross below the threshold $V^{\star}_{4,-}$.

E. Step 5

Throughout this section, keep in mind the scheme illustrated in Fig. 6e. Consider then the vector field

$$\dot{x}_5 = X_{5,d,D}(x_5,(T,\tau,g^3)) :\Leftrightarrow \begin{bmatrix}\dot{x}_4\\ \dot{\omega}\\ \dot{g}^2\end{bmatrix} = \begin{bmatrix}X_{4,d,D}(x_4,(T,\omega,g^2))\\ \Pi\left(n\right)\left(\tau+\Phi(n)D\right)\\ g^3\end{bmatrix}, \tag{84}$$

where

- $x_5 \in \mathbb{X}_5 :\Leftrightarrow (x_4,\omega,g^2) \in \mathbb{X}_4 \times T_n\mathbb{S}^2 \times \mathbb{R}^3$ is the state[11], composed of the state x_4 (described in step 4), the angular velocity ω (which is orthogonal to the angular position n), and where g^2 is the second time derivative of the time-varying acceleration (i.e., $g^2 = \ddot{g}$)
- T and τ (thrust and angular acceleration) are the inputs to the vector field; and where g^3 is the third time derivative of the time-varying gravity acceleration (i.e., $g^3 = \dddot{g}$);
- $X_{4,d,D}$ is the vector field described in step 4 (see (71));
- both the disturbances d and D are assumed known by the control law we design for the angular acceleration τ (but unknown by the control law for the linear acceleration T);

As in the previous steps, we need to define the equilibria sets, which depend on the same excitation criterion as before, namely (72). To be specific, denote

- if $E = 0$, denote
$$\mathbb{X}^{\star}_{5,\pm} := \left\{x_5 \in \mathbb{X}_5 : x_4 \in \mathbb{X}^{\star}_{4,\pm} \text{ and } \omega = 0_3\right\}; \tag{85a}$$
- otherwise, denote
$$\mathbb{X}^{\star}_{5,\pm} := \left\{x_5 \in \mathbb{X}_5 : x_4 \in \mathbb{X}^{\star}_{4,\pm} \text{ and } \omega = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-d\|}\right\}. \tag{85b}$$

Finally, denote also

$$\mathbb{X}^{\star,\star}_{5,\pm} := \left\{x_5 \in \mathbb{X}_5 : x_4 \in \mathbb{X}^{\star,\star}_{4,\pm} \text{ and } \omega = \mathcal{S}\left(\frac{g^0-d}{\|g^0-d\|}\right)\frac{g^1}{\|g^0-d\|}\right\}, \tag{85c}$$

where it is straightforward to verify that $\mathbb{X}^{\star,\star}_{5,\pm} \subset \mathbb{X}^{\star}_{5,\pm}$. The sets in (85) are the equilibria sets: later we show that $\mathbb{X}^{\star}_{5,+}$ is (locally) attractive, while $\mathbb{X}^{\star,\star}_{5,+}$ is stable. The equilibria sets depend on the satisfaction of the condition $\lim_{t\to\infty}\mathcal{S}\left(\frac{g^{(0)}(t)-d}{\|g^{(0)}(t)-d\|}\right)\frac{g^{(1)}(t)}{\|g^{(0)}(t)-d\|} = 0_3$, whose meaning and intuition was discussed at the beginning of the previous step.

Once again, the idea is to attempt to reuse the control laws from the previous steps (with some minor necessary adjustments). Compared with the previous step, we have now lifted the assumption that we have control over the angular velocity. However note that the vector field in (84) is linear with respect to the angular velocity, i.e.,

$$X_{4,d,D}(x_4,(T,\omega,g^2)) = X_{4,d,D}(x_2,(T,0_3,g^2)) - (e^7_4 \otimes \mathcal{S}\left(n\right))\omega.$$

As such, if we reuse the previous control laws, T^{cl}_3 for the thrust and ω^{cl}_2 for the angular velocity, we have that

$$\dot{x}_5 = X_{5,d,D}(x_5,(T^{cl}_3(x_4),\tau,g^3)) \Leftrightarrow \begin{bmatrix}\dot{x}_4\\ \dot{\omega}\\ \dot{g}^2\end{bmatrix} = \underbrace{\begin{bmatrix}X^{cl}_{4,d,D}(x_4,g^2)\\ 0_3\\ 0_3\end{bmatrix}}_{\text{step 4}} + \begin{bmatrix}-(e^7_4\otimes\mathcal{S}\left(n\right))(\omega-\omega^{cl}_2(x_3))\\ \Pi\left(n\right)\left(\tau+\Phi(n)D\right)\\ g^3\end{bmatrix}, \tag{86}$$

and where we recall that

$$\dot{x}_4 = X^{cl}_{4,d,D}(x_4,g^2) :\Leftrightarrow \begin{bmatrix}\dot{x}_1\\ \dot{n}\\ \dot{g}^1\\ \dot{\hat{d}}_T\\ \dot{\hat{D}}_T\end{bmatrix} = \underbrace{\begin{bmatrix}\tilde{X}_1(x_2,\hat{d}_T)\\ \mathcal{S}\left(\omega^{cl}(x_3)\right)n\\ g^2\\ E_{\delta,T}(x_2,\hat{d}_T)\\ E_{\Delta,T}(x_3,\hat{D}_T)\end{bmatrix}}_{\text{does not depend on any unknown disturbance}} + \underbrace{\begin{bmatrix}(e^3_2\otimes I_3)\left(d-\hat{d}_T\right)+\langle\phi(n),D-\hat{D}_T\rangle\left(e^3_2\otimes I_3\right)n\\ 0_3\\ 0_3\\ 0_3\\ 0_3\end{bmatrix}}_{\text{top is linear w.r.t. } d-\hat{d}_T \text{ and } D-\hat{D}_T}. \tag{87}$$

[11]Technically speaking, the set $\mathbb{X}_5$ in (34e) cannot be expressed as a Cartesian product $\mathbb{X}_4 \times T_n\mathbb{S}^2 \times \mathbb{R}^3$. The correct formulation is $\{(p,v,g^0,n,g^1,\hat{d}_T,\hat{D}_T,\omega,g^2) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 : g^0 \in \mathbb{C}^3_{\underline{g}} \text{ and } n \in \mathbb{S}^2 \Leftrightarrow \langle n,n\rangle = 1 \text{ and } \omega \in T_n\mathbb{S}^2 \Leftrightarrow \langle\omega,n\rangle = 0\}$.

We are thus in conditions of applying a backsteepping step (see Section XIV.). Let k_ω, γ_ω be positive gains ($V_{2,0}$ already introduced in previous sections), and let us then choose the angular acceleration control law (recall from Remark 56 that $\frac{\partial}{\partial n}V_{4,d,D}(x_4) = \frac{\partial}{\partial n}\Gamma_{V_{2,0}}(V_{2,\hat{d}_T}(x_2)) = (-e_4^5 \otimes I_3)^T \nabla_{\Gamma,V_{2,0}} V_{2,\hat{d}_T}(x_2))$

$$
\begin{aligned}
&\tau_1^{cl} : \mathbb{X}_5 \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (x_5, d, D) \mapsto \tau_1^{cl}(x_5, d, D) \in T_n\mathbb{S}^2 \\
&\tau_1^{cl}(x_5, d, D) := - k_\omega(\omega - \omega^{cl}(x_3)) && \text{(88a)}\\
&\qquad + \Pi(n)\, d\omega^{cl}(x_3) X_{3,d,D}(x_3, (T_3^{cl}(x_4, \omega, g^2))) && \text{(88b)}\\
&\qquad + \Pi(n)\, (-e_4^5 \otimes \mathcal{S}(n))^T \nabla_{\Gamma,V_{2,0}} V_{2,\hat{d}_T}(x_2) && \text{(88c)}\\
&\qquad - \Phi(n) D, && \text{(88d)}
\end{aligned}
$$

where we emphasize that the control law τ_1^{cl} depends on both disturbances d and D: it depends on d because of the feed-forward term in (88b) (the feed-forward term depends on the vector field $X_{3,d,D}^{cl}$); and it depends on D because of the same feed-forward term and because of the input disturbance in (88d) that needs to be canceled. Let us provide a brief description on the terms in (88) which, altogether, steer the angular velocity ω to the angular velocity $\omega_2^{cl}(x_3)$:

- (88a) acts as a proportional feedback term, and k_ω is a proportional gain;
- (88b) is a feedforward term;
- (88c) is a backsteeping term (see Remark 18), and γ_ω is a backstepping gain;
- and (88d) is a disturbance removal term term (to cancel the term $\Phi(n)D$ in (86)).

We then have that the closed-loop dynamics, at the end of step 5, are given by

$$
\dot{x}_5 = X_{5,d,D}(x_5, (T_3^{cl}(x_4), \tau_1^{cl}(x_5, d, D), g^3)) =: X_{5,d,D}^{cl}(x_5, g^3), \tag{89}
$$

where we emphasize that they depend on both d and D. We can then construct the Lyapunov function and its derivative along the vector field $X_{5,d,D}^{cl}$ in (89), as (see Section XIV)

$$
\begin{aligned}
&\mathbb{X}_5 \ni x_5 \mapsto V_{5,d,D}(x_5) := V_{4,d,D}(x_4) + \gamma_\omega \frac{1}{2}\|\omega - \omega_2^{cl}(x_3)\|^2, && \text{(90a)}\\
&\mathbb{X}_5 \ni x_5 \mapsto W_{5,d,D}(x_5) := \underbrace{dV_{5,d,D}(x_5) X_{5,d,D}^{cl}(x_5, g^3)}_{\text{independent of } g^3} = W_{4,d,D}(x_4) - k_\omega\gamma_\omega\|\omega - \omega_2^{cl}(x_3)\|^2 \leq 0. && \text{(90b)}
\end{aligned}
$$

Remark 60: In (88a), (90a) and (90b), $\omega - \omega_2^{cl}(x_3)$ makes sense, since ω and $\omega_2^{cl}(x_3)$ belong to same vector space (namely $T_n\mathbb{S}^2$).

Remark 61: Note that in (90a) one could, for the reasons discussed in Remark 18, replace $V_{4,d,D}(x_4)$ with $\Gamma_{V_{4,0}}(V_{4,d,D}(x_4))$ for some positive $V_{4,0}$. This however is not a feasible option, because $V_{4,d,D}$ depends on unknown disturbances, and, thus, it cannot be used for the purposes of control (only for the purposes of analysis).

Remark 62: The angular acceleration control law τ^{cl} in (88) requires $(e_4^7 \otimes I_3)^T \nabla V_{4,d,D}(x_4) = \frac{\partial}{\partial n}V_{4,d,D}(x_4)$ which is equivalent to (see Remark 56) $(e_4^5 \otimes I_3)^T d\Gamma_{V_{2,0}}(V_{2,\hat{d}_T}(x_2))\nabla V_{2,\hat{d}_T}(x_2) = \frac{\partial}{\partial n}\Gamma_{V_{2,0}}(V_{2,\hat{d}_T}(x_2))$: the former makes explicit reference to the unknown disturbances d and D, while the latter does not.

Remark 63: Similarly to Remark 56, note that the Lyapunov function $V_{5,d,D}$ in (90a) depends on the unknown disturbances d and D (because of $V_{4,d,D}$), so it cannot be used for the purposes of control. Nonetheless, we note that

$$
(e_8^9 \otimes I_3)^T \nabla V_{5,d,D}(x_5) \equiv \frac{\partial}{\partial \omega} V_{5,d,D}(x_5)|_{x_5=(p,v,g^0,n,g^1,\hat{d}_T,\hat{D}_T,\omega)} = \gamma_\omega(\omega - \omega^{cl}(x_3)), \tag{91a}
$$

and since both ω and $\omega^{cl}(x_4)$ are orthogonal to n, it follows that

$$
(e_8^9 \otimes \Pi(n))^T \nabla V_{5,d,D}(x_5) \equiv \Pi(n) \frac{\partial}{\partial \omega} V_{5,d,D}(x_5)|_{x_5=(p,v,g^0,n,g^1,\hat{d}_T,\hat{D}_T,\omega)} = \gamma_\omega(\omega - \omega^{cl}(x_3)). \tag{91b}
$$

The next proposition is the basis for analyzing the stability and attractivity properties of the sets in (85).

Proposition 64: Consider the functions $V_{5,d,D}, W_{5,d,D}$ in (90a), (90b). Consider also the sets $\mathbb{X}_{5,\pm}^\star$ and $\mathbb{X}_{5,\pm}^{\star,\star}$ in (85). It holds that

$$
\begin{cases} V_{5,d,D}(x_5) = 0 \text{ for all } x_5 \in \mathbb{X}_{5,+}^{\star,\star} \\ V_{5,d,D}(x_5) > 0 \text{ for all } x_5 \in \mathbb{X}_{5,+}^{\star} \backslash \mathbb{X}_{5,+}^{\star,\star} \end{cases}, \tag{92a}
$$

$$
\begin{cases} V_{5,d,D}(x_5) = V_{4,-}^\star =: V_{5,-}^\star \text{ for all } x_5 \in \mathbb{X}_{5,-}^{\star,\star} \\ V_{5,d,D}(x_5) > V_{4,-}^\star =: V_{5,-}^\star \text{ for all } x_5 \in \mathbb{X}_{5,-}^{\star} \backslash \mathbb{X}_{5,-}^{\star,\star} \end{cases}, \tag{92b}
$$

$$
\mathbb{X}_{5,+}^\star \cup \mathbb{X}_{5,-}^\star \supset \{x_5 \in \mathbb{X}_5 : \text{“}\dot{V}_{5,d}\text{”} = W_{5,d,D}(x_5) = 0 \text{ and “}\dot{v}\text{”} = v^1(x_4) = 0_3 \text{ and “}\ddot{v}\text{”} = v^2(x_5) = 0_3\}, \tag{92c}
$$

with $V_{4,-}^\star$ as defined in (81b), with v^1 (the first time derivative of the linear velocity) is as defined in (76), and where v^2 (the second time derivative of the linear velocity) is as defined in (77) except that ω is not replaced with $\omega(x_3)$.

Proof: Verifying (92a)–(92c) is simple, and, in this proof, we only verify (92b). Consider then (92b):

1) from (90b), $W_{5,d,D}(x_5) = 0$ implies that $W_{4,d,D}(x_4) = 0$ and that $\omega = \omega_2(x_3)$;
2) from 1) it follows that $p = 0_3$ and $v = 0_3$ and $n = \pm\frac{g^0 - \hat{d}_T}{\|g^0 - \hat{d}_T\|}$, and therefore $\omega = \omega_2(x_3) = \mathcal{S}\left(\frac{g^0 - \hat{d}_T}{\|g^0 - \hat{d}_T\|}\right)\frac{g^1}{\|g^0 - \hat{d}_T\|}$.
3) combining 1) and 2), and following the same steps as in Proposition 54, it follows that $x_4 \in \mathbb{X}^\star_{4,+} \cap \mathbb{X}^\star_{4,-}$;
4) combining 1) and 2) and 3), it then follows that if $\lim_{t\to\infty}\mathcal{S}\left(\frac{g^{(0)}(t) - d}{\|g^{(0)}(t) - d\|}\right)\frac{g^{(1)}(t)}{\|g^{(0)}(t) - d\|} = 0_3$ then $\omega = 0_3$; otherwise, $\omega = \omega_2(x_3) = \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{g^1}{\|g^0 - d\|}$; this completes the proof. ■

Remark 65: Let us discuss the meaning of the results in Proposition 64.

- (92b) implies that if the Lyapuvov function is ever "below" the threshold $V^\star_{5,-}$, then a solution cannot converge to the set $\mathbb{X}^\star_{5,-}$, since the Lyapuvov function is non-increasing along any solution;
- (92c) provides a mean to determine whether we are at the equilibria set, and since it "depends" on *three* components – "$\dot{V}_{5,d,D}$" and "$\dot{v}$" and "$\ddot{v}$", where "$V_{5,d,D}$" and "v" and "$\dot{v}$" all converge to some constants – we will be invoking Barbalat's lemma *three* times.

Proposition 66: Recall Proposition 57, and define

$$U_5 := \left\{x_5 \in \mathbb{X}_5 : x_4 \in U_4 \text{ and } \|g^2\| \leq \sup_{t\in\mathbb{R}}\|g^{(2)}(t)\|\right\},$$
$$T_5 := \left\{x_5 \in \mathbb{X}_5 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\hat{d}}\right\},$$

where g^2 is constrained to belong to a compact subset of $\mathbb{R}^3$ – see (25c), and where U_4 is found in Proposition 57. Consider also a sub-level set of the Lyapunov function $V_{5,d,D}$, i.e., for some non-negative constant V_0, consider $(V_{5,d,D})_{\leq V_0} := \{x_5 \in \mathbb{X}_5 : V_{5,d,D}(x_5) \leq V_0\}$. Then $(V_5)_{\leq V_0} \cap T_5 \cap U_5$ defines a compact subset of the state space $\mathbb{X}_5$.

Proof: It suffices to combine Proposition 17 with Proposition 57. ■

Theorem 67: Consider the vector field $X^{cl}_{5,d,D}$ in (89) and the Lyapunov function $V_{5,d,D}$ in (90a). Moreover, consider the sets $\mathbb{X}^\star_{5,\pm}$ and $\mathbb{X}^{\star,\star}_{5,\pm}$ in (85); let the set U_5, as defined in Proposition 66, be invariant; and let $t \mapsto g^{(3)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). For brevity, denote $\tilde{\mathbb{X}}_5 := \mathbb{X}_5 \cap U_5$, $\tilde{\mathbb{X}}^\star_{5,\pm} := \mathbb{X}^\star_{5,\pm} \cap U_5$ and $\tilde{\mathbb{X}}^{\star,\star}_{5,\pm} := \mathbb{X}^{\star,\star}_{5,\pm} \cap U_5$. Finally, consider the differential equation

$$\dot{x}_5(t) = X^{cl}_{5,d,D}(x_5(t), g^{(3)}(t)) \text{ with } x_5(0) \in \tilde{\mathbb{X}}_5 \tag{93}$$

Then,

1) there exists a unique and complete solution $[0,\infty) \ni t \mapsto x_5(t) \in \tilde{\mathbb{X}}_5$ to (93);
2) the sets $\mathbb{X}^\star_{5,+}$ and $\mathbb{X}^\star_{5,-}$ are invariant;
3) the set $\tilde{\mathbb{X}}^\star_{5,+} \cup \tilde{\mathbb{X}}^\star_{5,-}$ is globally attractive, i.e.,
$$\lim_{t\to\infty}\text{dist}\left(x_5(t), \tilde{\mathbb{X}}^\star_{5,+} \cup \tilde{\mathbb{X}}^\star_{5,-}\right) = 0, \text{ for all } x_5(0) \in \tilde{\mathbb{X}}_5;$$
4) the set $\tilde{\mathbb{X}}^{\star,\star}_{5,+}$ is stable, while the set $\tilde{\mathbb{X}}^{\star,\star}_{5,-}$ is unstable;
5) the set $\tilde{\mathbb{X}}^\star_{5,+}$ is (locally) attractive and $\lim_{t\to\infty}\text{dist}\left(x_5(t), \tilde{\mathbb{X}}^\star_{5,+}\right) = 0$ if $x_5(0) \in \left\{x_5 \in \tilde{\mathbb{X}}_5 : V_{5,d,D}(x_5) < V^\star_{5,-}\right\}$, with $V^\star_{5,-}$ as defined in (92b).

Proof:

0) First note that $T_5 := \{x_5 \in \mathbb{X}_5 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\hat{d}}\}$ is a positively invariant set – see (30b).
1) Define $V_0 := V_{5,d,D}(x_5(0)) \in [0,\infty)$. Since, $(V_{5,d,D})_{\leq V_0} \cap T_5 \cap U_5$ defines a positively invariant compact subset of $\mathbb{X}_5$, since $g^{(3)}$ is also contained in a compact subset of $\mathbb{R}^3$, and since the vector field $X^{cl}_{5,d,D}$ is $\mathcal{C}^1$ continuous (and thus Lipschitz continuous) in $\mathbb{X}_5 \times \mathbb{R}^3$, the first conclusion follows immediately.
2) It is trivial to verify that
$$X^{cl}_{5,d,D}(x_5, g^2) \in T_{x_5}\mathbb{X}^\star_{5,+} \text{ for all } x_5 \in \mathbb{X}^\star_{5,+} \text{ and } g^2 \in \mathbb{R}^3,$$
$$X^{cl}_{5,d,D}(x_5, g^2) \in T_{x_5}\mathbb{X}^\star_{5,-} \text{ for all } x_5 \in \mathbb{X}^\star_{5,-} \text{ and } g^2 \in \mathbb{R}^3,$$
which suffices to conclude that each of those sets is invariant.
3) To prove that the set $\tilde{\mathbb{X}}^\star_{5,+} \cup \tilde{\mathbb{X}}^\star_{5,-}$ is globally attractive, consider the solution $[0,\infty) \ni t \mapsto x_5(t) \in \tilde{\mathbb{X}}_5$, and note that
   a) $\tilde{\mathbb{X}}^\star_{5,+} \cup \tilde{\mathbb{X}}^\star_{5,-} \supseteq \{x_5 \in \mathbb{X}_5 : W_{5,d,D}(x_5) = 0 \text{ and } \text{"}\dot{v}\text{"} = v^1(x_4) = 0_3 \text{ and } \text{"}\ddot{v}\text{"} = v^2(x_5) = 0_3\}$ – see Proposition 64;
   b) the solution is contained in $(V_{5,d,D})_{\leq V_0} \cap T_5 \cap U_5$, which is a compact subset of $\mathbb{X}_5$ – see Proposition 66;
   c) since $V_{5,d,D}$ is lower bounded, and since $\dot{V}_{5,d,D}(x_5(t)) = W_{5,d,D}(x_5(t)) \leq 0$, it follows that $\lim_{t\to\infty}V_{5,d,D}(x_5(t))$ exists;

d) finally,

$$\begin{aligned}\sup_{t\geq 0}|\ddot{V}_{5,d,D}(x_5(t))| &= \sup_{t\geq 0}|\dot{W}_{5,d,D}(x_5(t))| \\ &= \sup_{t\geq 0}|dW_{5,d,D}(x_5(t))X^{cl}_{5,d,D}(x_5(t),g^{(2)}(t))| \\ &\leq \sup_{\substack{x_5\in(V_{5,d,D})_{\leq V_0}\cap T_5\cap U_5\\ g^2\in\text{compact subset of }\mathbb{R}^3}}|dW_{5,d,D}(x_5)X^{cl}_{5,d,D}(x_5,g^2)|<\infty,\end{aligned}$$

where the latter inequality follows since $W_{5,d,D}$ is $\mathcal{C}^1(\mathbb{X}_5)$ and $X^{cl}_{5,d,D}$ is $\mathcal{C}^0(\mathbb{X}_5\times\mathbb{R}^3)$; thus, invoking *c)* and Barbalat's lemma, one concludes that $\lim_{t\to\infty}\dot{V}_{5,d}(x_5(t))=\lim_{t\to\infty}W_{5,d,D}(x_5(t))=0$;

e) following the same steps as in Theorem 67 (we only need to replace $X^{cl}_{4,d,D}(x_4,g^2)$ with $X^{cl}_{4,d,D}(x_4,g^2)-(e_4^7\otimes\mathcal{S}(n))(\omega-\omega_2^{cl}(x_3))$), we conclude that $\lim_{t\to\infty}\dot{v}(t)=\lim_{t\to\infty}v^1(x_4(t))=0_3$ and that $\lim_{t\to\infty}\ddot{v}(t)=\lim_{t\to\infty}v^2(x_5(t))=0_3$;

f) finally, combining *a)*, *b)*, *d)*, and *e)*, it follows that $\lim_{t\to\infty}\text{dist}\left(x_5(t),\tilde{\mathbb{X}}^\star_{5,+}\cup\tilde{\mathbb{X}}^\star_{5,-}\right)=0$.

4) To prove the set $\tilde{\mathbb{X}}^{\star,\star}_{5,+}$ is stable, first note that $\tilde{\mathbb{X}}^{\star,\star}_{5,+}=\{x_5\in\tilde{\mathbb{X}}_5:V_{5,d,D}(x_5)=0\}$ and where we emphasize that $V_{5,d,D}$ is non-negative and continuous. We can then invoke Proposition 16, with $M=\tilde{\mathbb{X}}_5$ and $M^\star=\tilde{\mathbb{X}}^{\star,\star}_{5,+}$ and $V=V_{5,d,D}|_{\tilde{\mathbb{X}}_5}$, to conclude that there exists $\alpha\in\mathcal{K}^\infty$ such that $\alpha\left(\text{dist}_M(m,M^\star)\right)\leq V(m)$ for all $m\in M$. (That is, $\{x_5\in\tilde{\mathbb{X}}_5:V_{5,d,D}(x_5)\leq\epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}^{\star,\star}_{5,+}$, which coincides with the latter iff $\epsilon=0$, and which is positively invariant for any $\epsilon\geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}^{\star,\star}_{5,+}$ is stable.
To conclude $\tilde{\mathbb{X}}^{\star,\star}_{5,-}$ is unstable, it suffices to repeat the same reasoning as in the proof of Theorem 40 (note that $V_{5,d,D}(x_5)=V^\star_{5,-}$ for all $x_5\in\tilde{\mathbb{X}}^{\star,\star}_{5,-}$).

5) This conclusion follows immediately from the facts that
- $V_{5,d,D}(x_5)\geq V^\star_{5,-}$ for all $x_5\in\mathbb{X}^\star_{5,-}$ – see Proposition 38;
- $W_{5,d,D}(x_5)\leq 0$ for all $x_5\in\mathbb{X}_5$;
- the set $\tilde{\mathbb{X}}^\star_{5,+}\cup\tilde{\mathbb{X}}^\star_{5,-}$ is globally attractive, where the sets $\tilde{\mathbb{X}}^\star_{5,+}$ and $\tilde{\mathbb{X}}^\star_{5,-}$ are disjoint. ■

At this point, a similar remark to that in Remark 59 could be made.

F. Step 6

Throughout this section, keep in mind the scheme illustrated in Fig. 6f. Consider then the vector field

$$\dot{x}_6=X_{6,d,D}(x_6,(T,\tau,g^3)):\Leftrightarrow\begin{bmatrix}\dot{x}_5\\ \dot{\hat{d}}_\tau\\ \dot{\hat{D}}_\tau\end{bmatrix}=\begin{bmatrix}X_{5,d,D}(x_2,(T,\tau,g^3))\\ E_{\delta,\tau}(x_5,\hat{d}_\tau)\\ E_{\Delta,\tau}(x_5,\hat{D}_\tau)\end{bmatrix},\tag{94}$$

where
- $x_6\in\mathbb{X}_6:\Leftrightarrow(x_5,\hat{d}_\tau,\hat{D}_\tau)\in\mathbb{X}_5\times\mathbb{R}^3\times\mathbb{R}^3$ is the state, composed of the state x_5 (described in step 5) and the estimates $\hat{d}_\tau$ and $\hat{D}_\tau$ of d and D, respectively;
- T and τ (thrust and angular acceleration) are the inputs to the vector field; while, $E_{\delta,\tau}$ and $E_{\Delta,\tau}$ describe the estimators dynamics (of $\hat{d}_\tau$ and $\hat{D}_\tau$, respectively) that we wish to design;
- $X_{5,d,D}$ is the vector field described in step 5 (see (84));
- both the disturbances d and D are assumed unknown.

As in the previous steps, we need to define the equilibria sets. To be specific, denote

$$\mathbb{X}^\star_{6,\pm}:=\left\{x_6\in\mathbb{X}_6:x_5\in\mathbb{X}^\star_{5,\pm}\right\},\tag{95a}$$

$$\mathbb{X}^{\star,\star}_{6,\pm}:=\left\{x_6\in\mathbb{X}_6:x_5\in\mathbb{X}^{\star,\star}_{5,\pm}\text{ and }\hat{d}_\tau=d\text{ and }\hat{D}_\tau=D\right\},\tag{95b}$$

where $\mathbb{X}^\star_{5,\pm}$ and $\mathbb{X}^{\star,\star}_{5,\pm}$ are defined in (85). The sets in (95) are the equilibria sets: later we show that $\mathbb{X}^\star_{6,+}$ is (locally) attractive, while $\mathbb{X}^{\star,\star}_{6,+}$ is stable. Note that the convergence of the disturbance estimate $\hat{d}_T$ to the real disturbance d, as well as the convergence of the disturbance $\hat{D}_T$ to the set $\{\hat{D}_T\in\mathbb{R}^3:\left\langle\phi\left(\pm\frac{g^0-d}{\|g^0-d\|}\right),D-\hat{D}_T\right\rangle=0_3\}$, were paramount in guaranteeing that "$x(t)\to x^\star(t)\Leftrightarrow(p(t),v(t),n(t),\omega(t))\to(p_\star(t),v_\star(t),n_{\star,\pm}(t),\omega_\star(t))$" ($x_{\star,\pm}$ in (26a)). However, the same does not apply for the disturbance estimates $\hat{d}_\tau$ and $\hat{D}_\tau$ which is why where they "converge to" is not discriminated in (95a).

Remark 68: The estimate $\hat{d}_\tau$, unlike the estimate $\hat{d}_T$ does not to need to satisfy the requirement that $\hat{d}_\tau\in\mathbb{B}^3_{\bar{d}}$. That is the case since the disturbance estimate $\hat{D}_\tau$ is used to replace the disturbance D in (88b) and (88d), which is well defined for all $D\in\mathbb{R}^3$; while the disturbance estimate $\hat{d}_T$ is used to replace the disturbance d is (88b), which is well-defined for all $d\in\mathbb{R}^3$.

Once again, the idea is to attempt to reuse the control laws from the previous steps (with some minor necessary adjustments). As opposed to the previous step, the disturbances d and D are no longer known, and their knowledge was required when implementing the angular acceleration control law τ_1^{cl} defined in (88). As such, we replace those by their estimates, i.e., we define the new angular acceleration control law

$$\begin{aligned} &\tau_2^{cl} : \mathbb{X}_6 \ni x_6 \mapsto \tau^{cl}(x_6) \in T_n\mathbb{S}^2 \\ &\tau_2^{cl}(x_6) := \tau_1^{cl}(x_5, \hat{d}_T, \hat{D}_T) \end{aligned} . \tag{96}$$

Notice that τ_1^{cl} in (88) is (simultaneously) affine in both disturbances (i.e., affine in its second and third entries), and therefore

$$\begin{aligned} \tau_2^{cl}(x_6) &= \tau_1^{cl}(x_5, d, D) + \tau^{cl}(x_5, \hat{d}_T - d, \hat{D}_T + D) \\ &= \tau_1^{cl}(x_5, d, D) + \Pi(n)\left(d\omega^{cl}(x_3)(X_{3,\hat{d}_\tau,\hat{D}_\tau}(x_3, (T_3^{cl}(x_4, \omega, g^2))) - X_{3,d,D}(x_3, (T_3^{cl}(x_4, \omega, g^2)))) + \Phi(n)(D - \hat{D}_\tau)\right) \\ &= \underbrace{\tau_1^{cl}(x_5, d, D)}_{\text{previous control law}} + \underbrace{\Pi(n)\left(d\omega^{cl}(x_3)\left((e_2^6 \otimes I_3)(d - \hat{d}_\tau) + \langle \phi(n), D - \hat{D}_\tau \rangle (e_2^6 \otimes I_3) n\right) + \Phi(n)(D - \hat{D}_\tau)\right)}_{\text{error linear in } (d - \hat{d}_T) \text{ and } (D - \hat{D}_T)} \\ &=: \tau_1^{cl}(x_5, d, D) + E_\delta(x_3)(d - \hat{d}_\tau) + E_\Delta(x_3)(D - \hat{D}_\tau), \end{aligned} \tag{97}$$

for some $E_\delta(x_3), E_\Delta(x_3) \in \mathbb{R}^{3\times 3}$[12]. [13] As such, the control law τ_2^{cl} in (96) leads to the closed-loop vector field (explore linearity)

$$\dot{x}_6 = X_{6,d,D}(x_6, (T^{cl}(x_4), \tau^{cl}(x_6), g^3)) =: X_{6,d,D}^{cl}(x_6, g^3) \Leftrightarrow \tag{98}$$

$$\Leftrightarrow \begin{bmatrix} \dot{x}_5 \\ \dot{\hat{d}}_\tau \\ \dot{\hat{D}}_\tau \end{bmatrix} = \underbrace{\begin{bmatrix} X_{5,d,D}^{cl}(x_5, g^3) \\ 0_3 \\ 0_3 \end{bmatrix}}_{\text{step 5}} + \underbrace{\begin{bmatrix} (e_8^9 \otimes I_3)\left(E_\delta(x_3)(d - \hat{d}_\tau) + E_\Delta(x_3)(D - \hat{D}_\tau)\right) \\ E_{\delta,\tau}(x_5, \hat{d}_\tau) \\ E_{\Delta,\tau}(x_5, \hat{D}_\tau) \end{bmatrix}}_{\text{top is linear w.r.t. } (d - \hat{d}_T) \text{ and } (D - \hat{D}_T)} . \tag{99}$$

Since the $\dot{x}_5$-part of the vector field in (99) is affine with respect to the estimation errors $d - \hat{d}_T$ and $D - \hat{D}_T$, we can thus proceed with a standard estimator design (see Section VII-B). As such, given some positive $k_{\delta,\tau}, k_{\Delta,\tau}$, we then choose (recall from (91a) that $(e_8^9 \otimes I_3)^T \nabla V_{5,d,D}(x_5) = \gamma_\omega(\omega - \omega^{cl}(x_3))$)

$$\begin{aligned} E_{\delta,\tau}(x_5, \hat{d}_\tau) &:= \mathrm{Proj}_{\bar{d},1}\left(\tilde{E}_{\delta,\tau}(x_5, \hat{d}_\tau), \hat{d}_\tau\right), \\ \tilde{E}_{\delta,\tau}(x_5, \hat{d}_\tau) &:= k_{\delta,\tau}\gamma_\omega(\omega - \omega^{cl}(x_3)), \end{aligned} \tag{100a}$$

and

$$\begin{aligned} E_{\Delta,\tau}(x_5, \hat{D}_\tau) &:= \mathrm{Proj}_{\bar{D},1}\left(\tilde{E}_{\Delta,\tau}(x_5, \hat{D}_\tau), \hat{D}_\tau\right), \\ \tilde{E}_{\Delta,\tau}(x_5, \hat{D}_\tau) &:= k_{\Delta,\tau}\gamma_\omega(\omega - \omega^{cl}(x_3)), \end{aligned} \tag{100b}$$

for reasons that are made clear next (recall the properties of $\mathrm{Proj}_{\bar{d}}$, in Section VII-C). As an observation, note that $k_{\delta,\tau}, k_{\Delta,\tau}$ represent (positive) integral gains.

We can then construct the Lyapunov function and its derivative along the vector field $X_{4,d,D}^{cl}$, described in (78), as

$$\mathbb{X}_6 \ni x_6 \mapsto V_{6,d,D}(x_6) := V_{5,d,D}(x_5) + \frac{1}{2k_{\delta,\tau}}\left\|d - \hat{d}_\tau\right\|^2 + \frac{1}{2k_{\Delta,\tau}}\left\|D - \hat{D}_\tau\right\|^2 \in [0, \infty), \tag{101a}$$

$$\mathbb{X}_6 \ni x_6 \mapsto W_{6,d,D}(x_6) := \underbrace{dV_{6,d,D}(x_6)X_{6,d,D}^{cl}(x_6, g^3)}_{\text{(independent of } g^3)} \in (-\infty, 0], \tag{101b}$$

[12] $E_\delta(x_3) := \Pi(n)\, d\omega^{cl}(x_3)(e_2^6 \otimes I_3)$ and $E_\Delta(x_3) := \Pi(n)\,\Phi(n) + E_\delta(x_3) n \phi(n)^T$

[13] In (95a), we could have defined the equilibria set $\mathbb{X}_{6,\pm}^\star$ as $\mathbb{X}_{6,\pm}^\star := \{x_6 \in \mathbb{X}_6 : x_5 \in \mathbb{X}_{5,\pm}^\star$ and $E_\delta(x_3)(d - \hat{d}_\tau) + E_\Delta(x_3)(D - \hat{D}_\tau) = 0\}$.

where (101b) is equivalently expressed as

$$\begin{aligned}
W_{6,d,D}(x_6) &:= dV_{6,d,D}(x_6)X^{cl}_{6,d,D}(x_6,g^3)\\
&= W_{5,d,D}(x_5)+dV_{5,d,D}(x_5)(e^9_8\otimes I_3)\left(E_\delta(x_4)(d-\hat{d}_\tau)+E_\Delta(x_4)(D-\hat{D}_\tau)\right) && \because (99)\\
&\quad -\frac{1}{k_{\delta,\tau}}\left\langle d-\hat{d}_\tau, E_{\delta,\tau}(x_5,\hat{d}_\tau)\right\rangle-\frac{1}{k_{\Delta,\tau}}\left\langle D-\hat{D}_\tau, E_{\Delta,\tau}(x_5,\hat{D}_\tau)\right\rangle\\
&= W_{5,d,D}(x_5)+ && \because (91a)\\
&\quad -\frac{1}{k_{\delta,\tau}}\left\langle d-\hat{d}_\tau, E_{\delta,\tau}(x_5,\hat{d}_\tau)-k_{\delta,\tau}E_\delta(x_3)^T\gamma_\omega(\omega-\omega^{cl}(x_3))\right\rangle\\
&\quad -\frac{1}{k_{\Delta,\tau}}\left\langle D-\hat{D}_\tau, E_{\Delta,\tau}(x_5,\hat{D}_\tau)-k_{\Delta,\tau}E_\Delta(x_3)^T\gamma_\omega(\omega-\omega^{cl}(x_3))\right\rangle\\
&= \underbrace{W_{5,d,D}(x_5)}_{\leq 0\quad \because (90b)}\underbrace{-\frac{1}{k_{\delta,\tau}}\left\langle d-\hat{d}_\tau, E_{\delta,\tau}(x_5,\hat{d}_\tau)-\tilde{E}_{\delta,\tau}(x_5,\hat{d}_\tau)\right\rangle}_{\leq 0\quad \because (30c)}\underbrace{-\frac{1}{k_{\Delta,\tau}}\left\langle D-\hat{D}_\tau, E_{\Delta,\tau}(x_5,\hat{D}_\tau)-\tilde{E}_{\Delta,\tau}(x_5,\hat{D}_\tau)\right\rangle}_{\leq 0\quad \because (30c)}. && (101c)
\end{aligned}$$

Remark 69: From a well-posedness perspective, there is no necessity of having a bounded estimate $\hat{d}_\tau$ and $\hat{D}_\tau$. If one had chosen a standard update-law for $\hat{d}_\tau$ and $\hat{D}_\tau$, those estimates would nevertheless be bounded, with their upper bound being set by initial condition: i.e., since $\dot{V}_6(x_6(t)) = W_{6,d,D}(x_6(t)) \leq 0$ it follows from the definition of the Lyapunov function $V_{6,d,D}$ in (101a) that $\|d-\hat{d}_\tau(t)\|^2 \leq 2k_{\delta,\tau}V_{6,d,D}(t_0, x_6(t_0))$ and, similarly, that that $\|D-\hat{D}_\tau(t)\|^2 \leq 2k_{\Delta,\tau}V_{6,d,D}(t_0, x_6(t_0))$ for all $t \geq t_0$. With the chosen strategy their upper bounded is pre-specified, and it does not depend on the initial condition. The caveat is that we need to know that the disturbances d and D are contained in the balls $\mathbb{B}^3_{\bar{d}}$ and $\mathbb{B}^3_{\bar{D}}$ for some known $\bar{d}$ and $\bar{D}$. Thus we only need to know an upper bound on the norm of the disturbances, which is not very restrictive.
The next proposition is the basis for analyzing the stability and attractivity properties of the sets in (95).

Proposition 70: Consider the functions $V_{6,d,D}, W_{6,d,D}$ in (101a), (101b), the sets $\mathbb{X}^\star_{6,\pm}$ and $\mathbb{X}^\star_{6,\pm}$ in (95), and the functions $E_{\delta,\tau}$ and $E_{\Delta,\tau}$ in (100a) and (100b). It holds that

$$\begin{cases} V_{6,d,D}(x_6) = 0 \text{ for all } x_6 \in \mathbb{X}^{\star,\star}_{6,+} \\ V_{6,d,D}(x_6) > 0 \text{ for all } x_6 \in \mathbb{X}^{\star}_{6,+}\backslash\mathbb{X}^{\star,\star}_{6,+} \end{cases}, \tag{102a}$$

$$\begin{cases} V_{6,d,D}(x_6) = V^\star_{5,-} =: V^\star_{6,-} \text{ for all } x_6 \in \mathbb{X}^{\star,\star}_{6,-} \\ V_{6,d,D}(x_6) > V^\star_{5,-} =: V^\star_{6,-} \text{ for all } x_6 \in \mathbb{X}^{\star}_{6,-}\backslash\mathbb{X}^{\star,\star}_{6,-} \end{cases}, \tag{102b}$$

$$\mathbb{X}^\star_{6,+}\cup\mathbb{X}^\star_{6,-} \supset \{x_6\in\mathbb{X}_6 : \text{“}\dot{V}_{6,d}\text{”} = W_{6,d,D}(x_6) = 0 \text{ and “}\dot{v}\text{”} = v^1(x_4) = 0_3 \text{ and “}\ddot{v}\text{”} = v^2(x_5) = 0_3\}, \tag{102c}$$

$$E_{\delta,\tau}(x_5,\hat{d}_\tau) = 0_3 \text{ for all } x_5 \in \mathbb{X}^\star_{5,+}\cup\mathbb{X}^\star_{5,-} \text{ and for all } \hat{d}_\tau \in \mathbb{B}^3_{\bar{\bar{d}}}, \tag{102d}$$

$$E_{\Delta,\tau}(x_5,\hat{D}_\tau) = 0_3 \text{ for all } x_5 \in \mathbb{X}^\star_{5,+}\cup\mathbb{X}^\star_{5,-} \text{ and for all } \hat{D}_\tau \in \mathbb{B}^3_{\bar{\bar{D}}}, \tag{102e}$$

with $V^\star_{5,-}$ as defined in (92b), with v^1 (the first time derivative of the linear velocity) is as defined in (76), and where v^2 (the second time derivative of the linear velocity) is as defined in (77) except that ω is not replaced with $\omega(x_3)$.

Proof: Verifying (102a)–(102e) is simple, and, in this proof, we only verify (102b). From (101c), $W_{6,d,D}(x_6) = 0$ implies that $W_{5,d,D}(x_4) = 0$. At this point we can repeat the same steps as in the proof of Proposition 64, which completes the proof. ■

Remark 71: Let us discuss the meaning of the results in Proposition 70.

- (102b) implies that if the Lyapuvov function is ever “below” the threshold $V^\star_{6,-}$, then a solution cannot converge to the set $\mathbb{X}^\star_{6,-}$, since the Lyapuvov function is non-increasing along any solution;
- (102c) provides a mean to determine whether we are at the equilibria set, and since it “depends” on *three* components – “$\dot{V}_{6,d,D}$” and “$\dot{v}$” and “$\ddot{v}$”, where “$V_{6,d,D}$” and “v” and “$\dot{v}$” all converge to some constants – we will be invoking Barbalat's lemma *three* times;
- (102d) and (102e) imply that if a solution approaches the equilibria set, then $\dot{\hat{d}}_\tau$ and $\dot{\hat{D}}_\tau$ approach 0_3 (which , by themselves, do not imply that $\hat{d}_\tau$ and $\hat{D}_\tau$ approach some constants in $\mathbb{R}^3$).

Proposition 72: Recall Proposition 66, and define

$$\begin{aligned}
U_6 &:= \{x_6 \in \mathbb{X}_6 : x_5 \in U_5\},\\
T_6 &:= \left\{x_6 \in \mathbb{X}_6 : \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for some } r < \bar{\bar{d}}\right\},
\end{aligned}$$

where U_5 is found in Proposition 66. Consider also a sub-level set of the Lyapunov function $V_{6,d,D}$, i.e., for some non-negative constant V_0, consider $(V_{6,d,D})_{\leq V_0} := \{x_6 \in \mathbb{X}_6 : V_{6,d,D}(x_6) \leq V_0\}$. Then $(V_6)_{\leq V_0} \cap T_6 \cap U_6$ defines a compact subset of the state space $\mathbb{X}_6$.

Proof: It suffices to combine Proposition 17 with Proposition 66. ■

Theorem 73: Consider the vector field $X^{cl}_{6,d,D}$ in (99) and the Lyapunov function $V_{6,d,D}$ in (101a). Moreover, consider the sets $\mathbb{X}^{\star}_{6,\pm}$ and $\mathbb{X}^{\star,\star}_{6,\pm}$ in (95a) and in (95b); let the set U_6, as defined in Proposition 72, be invariant; and let $t \mapsto g^{(3)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). For brevity, denote $\tilde{\mathbb{X}}_6 := \mathbb{X}_6 \cap U_6$, $\tilde{\mathbb{X}}^{\star}_{6,\pm} := \mathbb{X}^{\star}_{6,\pm} \cap U_6$, and $\tilde{\mathbb{X}}^{\star,\star}_{6,\pm} := \mathbb{X}^{\star,\star}_{6,\pm} \cap U_6$. Finally, consider the differential equation

$$\dot{x}_6(t) = X^{cl}_{6,d,D}(x_6(t), g^{(3)}(t)) \text{ with } x_6(0) \in \tilde{\mathbb{X}}_6 \tag{103}$$

Then,

1) there exists a unique and complete solution $[0, \infty) \ni t \mapsto x_6(t) \in \tilde{\mathbb{X}}_6$ to (103);
2) the sets $\mathbb{X}^{\star}_{6,+}$ and $\mathbb{X}^{\star}_{6,-}$ are invariant;
3) the set $\tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-}$ is globally attractive, i.e.,

$$\lim_{t\to\infty} \operatorname{dist}\left(x_6(t), \tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-}\right) = 0, \text{ for all } x_6(0) \in \tilde{\mathbb{X}}_6;$$

4) the set $\tilde{\mathbb{X}}^{\star,\star}_{6,+}$ is stable, while the set $\tilde{\mathbb{X}}^{\star,\star}_{6,-}$ is unstable;
5) the set $\tilde{\mathbb{X}}^{\star}_{6,+}$ is (locally) attractive and $\lim_{t\to\infty} \operatorname{dist}\left(x_6(t), \tilde{\mathbb{X}}^{\star}_{6,+}\right) = 0$ if $x_6(0) \in \left\{x_6 \in \tilde{\mathbb{X}}_6 : V_{6,d,D}(x_6) < V^{\star}_{6,-}\right\}$, with $V^{\star}_{6,-}$ as defined in (102b).

Proof:

0) First note that $T_6 := \{x_6 \in \mathbb{X}_6 : \hat{d}_T \in \bar{\mathbb{B}}^6_r \text{ for some } r < \bar{\hat{d}}\}$ is a positively invariant set – see (30b).
1) Define $V_0 := V_{6,d,D}(x_6(0)) \in [0, \infty)$. Since, $(V_{6,d,D})_{\leq V_0} \cap T_6 \cap U_6$ defines a positively invariant compact subset of $\mathbb{X}_6$, since $g^{(3)}$ is also contained in a compact subset of $\mathbb{R}^3$, and since the vector field $X^{cl}_{6,d,D}$ is $\mathcal{C}^1$ continuous (and thus Lipschitz continuous) in $\mathbb{X}_6 \times \mathbb{R}^3$, the first conclusion follows immediately.
2) It is trivial to verify that

$$X^{cl}_{6,d,D}(x_6, g^3) \in T_{x_6}\mathbb{X}^{\star}_{6,+} \text{ for all } x_6 \in \mathbb{X}^{\star}_{6,+} \text{ and } g^3 \in \mathbb{R}^3,$$
$$X^{cl}_{6,d,D}(x_6, g^3) \in T_{x_6}\mathbb{X}^{\star}_{6,-} \text{ for all } x_6 \in \mathbb{X}^{\star}_{6,-} \text{ and } g^3 \in \mathbb{R}^3,$$

which suffices to conclude that each of those sets is invariant.
3) To prove that the set $\tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-}$ is globally attractive, consider the solution $[0, \infty) \ni t \mapsto x_6(t) \in \tilde{\mathbb{X}}_6$, and note that
   a) $\tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-} \supseteq \{x_6 \in \mathbb{X}_6 : W_{6,d,D}(x_6) = 0 \text{ and } \text{“}\dot{v}\text{”} = v^1(x_4) = 0_3 \text{ and } \text{“}\ddot{v}\text{”} = v^2(x_5) = 0_3\}$ – see Proposition 70;
   b) the solution is contained in $(V_{6,d,D})_{\leq V_0} \cap T_6 \cap U_6$, which is a compact subset of $\mathbb{X}_6$ – see Proposition 72;
   c) since $V_{6,d,D}$ is lower bounded, and since $\dot{V}_{6,d,D}(x_6(t)) = W_{6,d,D}(x_6(t)) \leq 0$, it follows that $\lim_{t\to\infty} V_{6,d,D}(x_6(t))$ exists;
   d) finally,

$$\begin{aligned}
\sup_{t\geq 0} |\ddot{V}_{6,d,D}(x_6(t))| &= \sup_{t\geq 0} |\dot{W}_{6,d,D}(x_6(t))| \\
&= \sup_{t\geq 0} |dW_{6,d,D}(x_6(t))X^{cl}_{6,d,D}(x_6(t), g^{(3)}(t))| \\
&\leq \sup_{\substack{x_6 \in (V_{6,d,D})_{\leq V_0} \cap T_6 \cap U_6 \\ g^3 \in \text{compact subset of } \mathbb{R}^3}} |dW_{6,d,D}(x_6)X^{cl}_{6,d,D}(x_6, g^3)| < \infty,
\end{aligned}$$

   where the latter inequality follows since $W_{6,d,D}$ is $\mathcal{C}^1(\mathbb{X}_6)$ and $X^{cl}_{6,d,D}$ is $\mathcal{C}^0(\mathbb{X}_6 \times \mathbb{R}^3)$; thus, invoking *c)* and Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{V}_{6,d}(x_6(t)) = \lim_{t\to\infty} W_{6,d,D}(x_6(t)) = 0$;
   e) following the same steps as in Theorem 58 (we only need to replace $X^{cl}_{4,d,D}(x_4, g^2)$ with $X^{cl}_{4,d,D}(x_4, g^2) - (e^7_4 \otimes \mathcal{S}(n))(\omega - \omega^{cl}_2(x_3))$), we conclude that $\lim_{t\to\infty} \dot{v}(t) = \lim_{t\to\infty} v^1(x_4(t)) = 0_3$ and that $\lim_{t\to\infty} \ddot{v}(t) = \lim_{t\to\infty} v^2(x_5(t)) = 0_3$;
   f) finally, combining *a)*, *b)*, *d)*, and *e)*, it follows that $\lim_{t\to\infty} \operatorname{dist}\left(x_6(t), \tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-}\right) = 0$.
4) To prove the set $\tilde{\mathbb{X}}^{\star,\star}_{6,+}$ is stable, first note that $\tilde{\mathbb{X}}^{\star,\star}_{6,+} = \{x_6 \in \tilde{\mathbb{X}}_6 : V_{6,d,D}(x_6) = 0\}$ and where we emphasize that $V_{6,d,D}$ is non-negative and continuous. We can then invoke Proposition 16, with $M = \tilde{\mathbb{X}}_6$ and $M^\star = \tilde{\mathbb{X}}^{\star,\star}_{6,+}$ and $V = V_{6,d,D}|_{\tilde{\mathbb{X}}_6}$, to conclude that there exists $\alpha \in \mathcal{K}^\infty$ such that $\alpha\left(\operatorname{dist}_M(m, M^\star)\right) \leq V(m)$ for all $m \in M$. (That is, $\{x_6 \in \tilde{\mathbb{X}}_6 : V_{6,d,D}(x_6) \leq \epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}^{\star,\star}_{6,+}$, which coincides with the latter iff $\epsilon = 0$, and which is positively invariant for any $\epsilon \geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}^{\star,\star}_{6,+}$ is stable.
   To conclude $\tilde{\mathbb{X}}^{\star,\star}_{6,-}$ is unstable, it suffices to repeat the same reasoning as in the proof of Theorem 40 (note that $V_{6,d,D}(x_6) = V^{\star}_{6,-}$ for all $x_6 \in \tilde{\mathbb{X}}^{\star,\star}_{6,-}$).
5) This conclusion follows immediately from the facts that
   - $V_{6,d,D}(x_6) \geq V^{\star}_{6,-}$ for all $x_6 \in \mathbb{X}^{\star}_{6,-}$ – see Proposition 38;
   - $W_{6,d,D}(x_6) \leq 0$ for all $x_6 \in \mathbb{X}_6$;
   - the set $\tilde{\mathbb{X}}^{\star}_{6,+} \cup \tilde{\mathbb{X}}^{\star}_{6,-}$ is globally attractive, where the sets $\tilde{\mathbb{X}}^{\star}_{6,+}$ and $\tilde{\mathbb{X}}^{\star}_{6,-}$ are disjoint. ■

At this point, a similar remark to that in Remark 59 could be made. The complete control strategy is shown in Fig. 9.

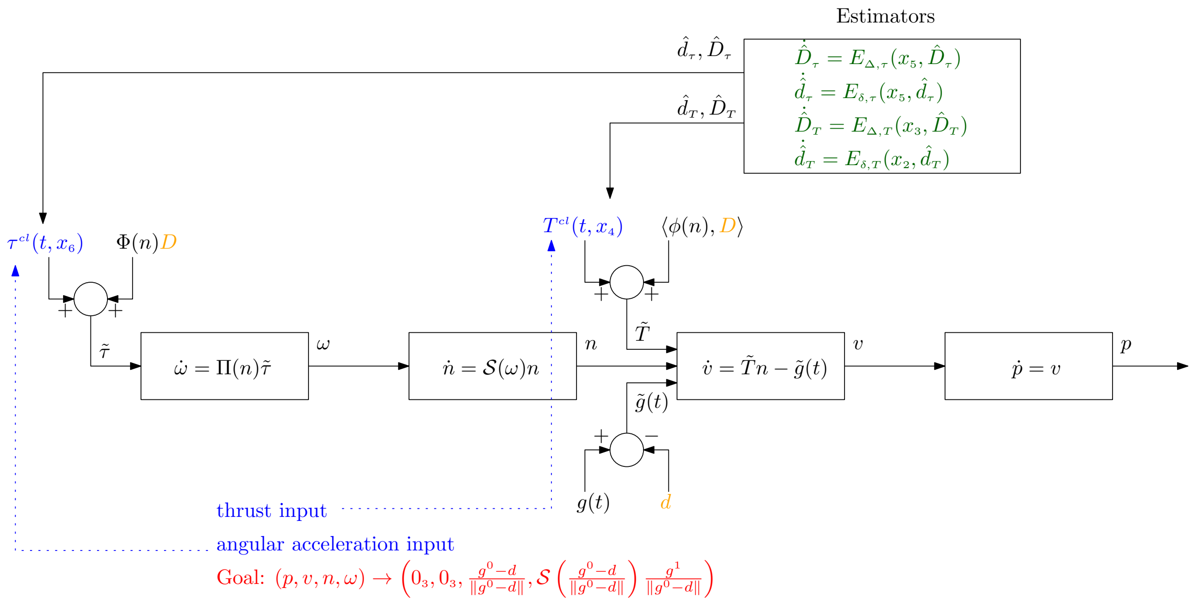


Fig. 9: Complete control strategy.

G. Complete strategy

We note the controller we constructed is a dynamic one (not a static one), and as such it has internal states of its own. With the latter in mind, and for convenience, denote

$$\begin{aligned}\Xi &:= \mathbb{B}^3_{\bar{d}} \times \mathbb{B}^3_{\bar{D}} \times \mathbb{B}^3_{\bar{d}} \times \mathbb{B}^3_{\bar{D}} \\ \xi &:= (\hat{d}_T, \hat{D}_T, \hat{d}_\tau, \hat{D}_\tau)\end{aligned}$$

with ξ as the collection of all the internal states of the controller, and Ξ as the domain where ξ belongs to. Given any time instant $t \in \mathbb{R}$, consider then the map that constructs the state x_6 (see (34)), namely

$$\begin{aligned}&f_{6,t} : \mathbb{Z} \times \Xi \to \mathbb{X}_6 \\ &f_{6,t}(z, \xi) := \underbrace{(p, v, g^0, n, g^1, \hat{d}_T, \hat{D}_T, \omega, g^2, \hat{d}_\tau, \hat{D}_\tau,)}_{\text{state } x_6}\Big|_{\substack{(p,v,n,\omega)=g_t(z) \\ (\hat{d}_T,\hat{D}_T,\hat{d}_\tau,\hat{D}_\tau)=\xi \\ g^i = g^{(i)}(t) \equiv \frac{d^i g(t)}{dt^i} \text{ for } i\in\{0,1,2\}}}\end{aligned} \tag{104}$$

where the map $f_{6,t}$ takes a physical state z and a controller internal state ξ, and it constructs a state $x_6 = f_{6,t}(z, \xi)$ (we emphasize that this map is a bijection, and thus it is invertible). We then combine the thrust control law T_3^{cl} in (74) with the angular acceleration control law τ_2^{cl} in (96) to define

$$\begin{aligned}&\nu^{cl} : \mathbb{X}_6 \ni x_6 \to \mathbb{R} \times T_n\mathbb{S}^2 \\ &\nu^{cl}(x_6) := (T_3^{cl}(x_3), \tau_2^{cl}(x_6)),\end{aligned} \tag{105}$$

which let us finally define the complete control law to be applied on the physical system as

$$\begin{aligned}&u^{cl} : \mathbb{R} \times \mathbb{Z} \times \Xi \to \mathbb{R}^3 \\ &u^{cl}(t, z, \xi) := \nu_x(\nu^{cl}(x_6))|_{x=g_t(z)}|_{x_6=f_{6,t}(z,\xi)},\end{aligned} \tag{106}$$

where the map ν_x was defined in (20b), the change of coordinates map g_t was defined in (18c), and the map $f_{6,t}$ is that presented in (104). That is, the map u^{cl} takes a time instant t, a physical state z and a controller internal state ξ, and it constructs a three dimensional force $u = u^{cl}(t, z, \xi)$ that the UAV must apply (see u in Fig. 1).

Let us define a constant which we use next to construct a sub-level where the tension in the cable is guaranteed to be positive, namely

$$V_6^+ \in \{V_0 \geq 0 : \epsilon b_1(V_0) - b_2(V_0) = 0\}, \text{ where} \tag{107a}$$

$$\begin{aligned}b_1(V_0) &:= 1 - \gamma_\theta^{-1}\Gamma^{-1}_{V_{2,0}}(V_0) \\ b_2(V_0) &:= \sqrt{2k_{\Delta,T}V_0},\end{aligned}$$

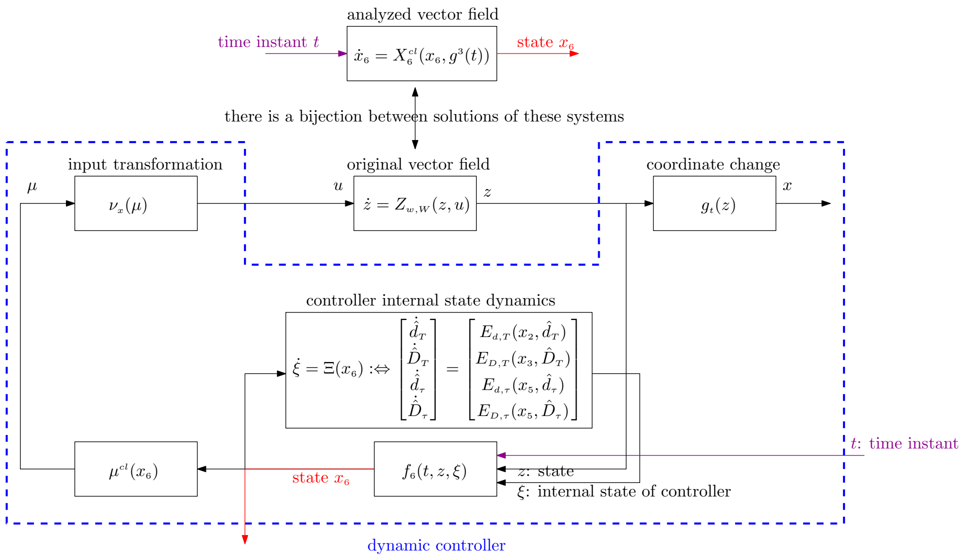


Fig. 10: Dynamic controller in "original coordinates".

and where $\epsilon = \underline{g} - (\bar{u} + \bar{\hat{d}}) > 0$ (see (39)). We note that V_6^+ is well defined, since b_1 and b_2 are continuous, since $-b_1$ and $+b_2$ are increasing in $(0, \infty)$, and since $\epsilon b_1(0) - b_2(0) = \epsilon > 0$. The next Theorem provides a solution to the Problem 3 (which is a refinement of Problems 2 and 1).

Theorem 74: Given some desired position trajectory $p_\star : \mathbb{R} \to \mathbb{R}^3$ (such that the conditions in (2) are satisfied), consider: *(1)* the slung-load vector field $Z_{w,W}$ as defined in (9) for some, *unknown by the controller*, winds $w, W \in \mathbb{R}^3$; *(2)* the control law u^{cl} as defined in (106). *(3)* and the estimator dynamics Ξ as defined in the diagram in Fig 10. Then, consider the system

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} Z_{w,W}(z(t), u^{cl}(t, z(t), \xi(t))) \\ \Xi(t, \xi(t), z(t)) \end{bmatrix}, \begin{bmatrix} z(0) \in \mathbb{Z} \\ \xi(0) \in \Xi \end{bmatrix}. \tag{107b}$$

Consider finally the map $f_{6,t}$ in (104), the Lyapunov function V_6 in (101a) and the constant $V_6^\star$ in (107a). Then, for all initial conditions

$$x_6(0) := f_{6,0}(z(0), \xi(0)) \in (V_6)_{<V_6^\star}, \tag{107c}$$

it follows that (recall the equilibrium solution $z_{\star,+}$ in (14a), and the equilibria set $X_{6,+}^{\star,\star}$ in (95))

- there exists a unique and complete solution $[0, \infty) \in t \to (z(t), \xi(t)) \in \mathbb{Z} \times \Xi$ to (107b);
- $\lim_{\text{dist}(x_6(0), X_{6,+}^{\star,\star}) \to 0} \sup_{t \geq 0} \|z(t) - z_{\star,+}(t)\| = 0$ (stability of $z_{\star,+}$);
- $\lim_{t \to \infty} \|z(t) - z_{\star,+}(t)\| = 0$ (attractivity of $z_{\star,+}$);
- $\inf_{t \geq 0} T(z(t), u^{cl}(t, z(t), \xi(t))) > 0$ (i.e, cable always taut along solution).

Proof: Studying the solution to (107b) is equivalent to studying a solution to (99), where $x_6(t) := f_{6,t}(z(t), \xi(t))$ (for any time instant t, there is a bijection between $(z(t), \xi(t))$ and $x_6(t)$). Define then $V_0 := V_6(x_6(0))$, and note that, by assumption, $V_0 < V_6^\star$. We know the sub-level set $(V_6)_{\leq V_0}$, which is a subset of $(V_6)_{<V_6^\star}$, is positively invariant, and therefore $x_6(t) \in (V_6)_{\leq V_0} \subset (V_6)_{<V_6^\star}$ for all $t \geq 0$. Now in order for a solution of (107b) to be physically valid, the tension in the cable, as defined in (11), must be positive:

(a) note that $T(z_{\star,+}(t), u_{\star,+}(t)) = \|mp_\star^{(2)}(t) + ge_3 - w\|$ and that $\inf_{t \geq 0} T(z_{\star,+}(t), u_{\star,+}(t)) \geq m(\underline{g} - \bar{\hat{d}}) > 0$ (this follows from the constraints imposed on the desired trajectory $p_\star$ and on the disturbance $d := \frac{w}{m}$);

(b) the tension, when composed with the proposed control law in (106), is given by

$$T(z, u^{cl}(t, z, \xi)) = m \left(\|T^{3d}(x_1, \hat{d}_T)\| \langle n, n^{cl}(x_1, \hat{d}_T) \rangle + \langle \phi(n), D - \hat{D}_T \rangle \right),$$

where we recall from (39) that $\|T^{3d}(x_1, \hat{d}_T) > \epsilon$.

(c) $x_6 \in (V_6)_{\leq V_0}$ implies that (recall the functions b_1 and b_2 defined in (107a))

$$V_6(x_6) \leq V_0 \overset{(101a)}{\Rightarrow} \|D - \hat{D}_T\| \leq b_2(V_0) \qquad \text{for } b_2(V_0) := \sqrt{2k_{\Delta,T} V_0}$$

$$V_6(x_6) \leq V_0 \overset{(101a)}{\Rightarrow} V_{5,d,D}(x_5) < V_0 \overset{(90a)}{\Rightarrow} \cdots \overset{(67a)}{\Rightarrow} \Gamma_{V_{2,0}}(V_{2,\hat{d}_T}(x_2)) < V_0$$

$$\overset{(56a)}{\Rightarrow} \langle n, n^{cl}(x_1, \hat{d}_T) \rangle \geq b_1(V_0) \qquad \text{for } b_1(V_0) := 1 - \gamma_\theta^{-1} \Gamma_{V_{2,0}}^{-1}(V_0);$$

because $V_0 < V_6^\star$, and since $-b_1$ and $+b_2$ are increasing, it follows that

$$\|D - \hat{D}_T\| < b_2(V_6^\star)$$
$$\langle n, n^{cl}(x_1, \hat{d}_T)\rangle > b_1(V_6^\star).$$

(d) combing *(b)* and *(c)* and the definition of $V_6^\star$ in (107a), it follows that $\inf_{t\geq 0} T(z(t), u^{cl}(t, z(t), \xi(t))) > 0$, that is, the tension in the cable is strictly positive for all positive time instants t.

After proving the cable remains taut, items *(1)–(3)* in the Theorem follow from the results in Theorem 73. ■

IX. Simulation

We now provide simulations which validate our main results. We also provide simulations which shed light into the robustness of the proposed strategy.

In the simulaton, the system has physical constants $M = 1.1$ kg, $m = 0.4$ kg, $l = 1.1$ m, and $g = 9.81$ m/s/s; and the wind forces are $W = 0.1Mg\frac{s}{\|s\|}$ N with $s = (2, 2, 1)$, and $w = 0.1mg\frac{s}{\|s\|}$ N with $s = (1, 0, -1)$ (wind forces corresponding to 10% of bodies' weights). The load is required to track the trajectory $p_\star : \mathbb{R} \to \mathbb{R}^3$ defined as

$$p_\star(t) := R_1(25^\circ)R_2(25^\circ)\left(\frac{r\,(\sin(2\omega t), \cos(\omega t), 0)}{\sin(\omega t)^2 + 1} + he_3\right)$$

where $R_i(\alpha)$ stands for a positive rotation around the ith axis by an angle α, $r = 2$ m, $h = 0.5$ m, and $\omega = \frac{2\pi}{12}$ Hz (period of 12 s), which corresponds to an eight like path in a tilted plane – in particular, it follows that $\inf_{t\in\mathbb{R}} \|ge_3 + p_\star^{(2)}(t)\| \approx 9.0$ m/s/s. For the simulations, we let the initial condition be $(p(0), P(0), v(0), V(0)) = (7\,1_3, 7\,1_3 + le_3, 0_3, 0_3)$ and $(\hat{d}_T(0), \hat{D}_T(0), \hat{d}_\tau(0), \hat{D}_\tau(0)) = (0_3, 0_3, 0_3, 0_3)$. For the controller parameters we take $\bar{u} \approx 0.72$ m/s/s, $\bar{\hat{d}}_T = \bar{\hat{d}}_\tau = 1.5 > \bar{d} = 1.2 \geq \|d\| \approx 0.98$ m/s/s, $\bar{\hat{D}}_T, = \bar{\hat{D}}_\tau = 1.8 > \bar{D} = 1.5 \geq \|D\| \approx 1.21$ m/s/s, $V_{1,0} = 1.5$ and $V_{2,0} = 1.5$. In particular, note that the condition $\inf_{t\in\mathbb{R}} \|g(t)\| - (\bar{u} + \bar{\hat{d}}_T) > 0$ is satisfied.

We present several simulations: *(1)* default/baseline simulation, where all conditions assumed in the paper are respected. *(2)* we assume the model is incorrectly known by the controller (model mismatch), and we implement the controller with $m_{\text{controller}} = 1.1m_{\text{model}}$ and $l_{\text{controller}} = 1.1l_{\text{model}}$; because the model is not known we increase (by a factor of ≈ 2) the norms on the estimators, i.e., $\bar{\hat{d}}_T = \bar{\hat{d}}_\tau = 2.7 > \bar{d} = 2.4$ m/s/s, $\bar{\hat{D}}_T, = \bar{\hat{D}}_\tau = 3.3 > \bar{D} = 3.0$ m/s/s. *(3)* we take $\omega \in \{\frac{2\pi}{8}, \frac{2\pi}{6}\}$ Hz (period of 8 or 6 s), and we investigate the effect of the excitation criterion on the convergence of the estimators. *(4)* we take $(p(0), P(0), v(0), V(0)) = (20\,1_3, 20\,1_3 + le_3, 5\,1_3, 5\,1_3)$, we take $V_{1,0} = V_{2,0} \in \{2, 20, 200\}$, and we observe the effect on the disturbance estimators. *(5)* we let the winds be non-constant, i.e., $W = 0.1Mg\frac{s}{\|s\|}$ N with $s = (2 + 0.1\cos(t), 2 + 0.1\sin(t), 1)$, and $w = 0.1mg\frac{s}{\|s\|}$ N with $s = (1 + 0.1\cos(t), 0 + 0.1\sin(t), -1)$, and we investigate the tracking error.

Let us now comment on these simulations, starting with *(1)*, the baseline simulation. In Figs. 11a and 11b, a visual inspection of convergence is shown, while Figs. 11c and 11d show that the state and input (z, u) converge to the equilibrium state and input $(z_{\star,+}, u_{\star,+})$ defined in (14). In Fig. 11e, the inputs, coming from the control law in (106), are shown, as well as the tension on the cable, where the latter indicates that the cable remains taut. Finally, Figs. 11f and 11g show the estimators $\hat{d}_T, \hat{D}_T, \hat{d}_\tau, \hat{D}_\tau$, which do not converge to the values of the real disturbances.

Regarding *(2)*: Fig. 11h shows that, despite the model mismatch, the state and input still converge to their desired trajectories, owing to the robustness added by the estimators. Regarding *(3)*: Fig. 11i shows that, the bigger the excitation is, the faster the estimators (we show only $\hat{d}_T$ and $\hat{D}_T$) converge to the real disturbances (the excitation criterion is understood as the average of $t \mapsto E(t) = \mathcal{S}\left(\frac{g^{(0)}(t)-d}{\|g^{(0)}(t)-d\|}\right)\frac{g^{(1)}(t)}{\|g^{(0)}(t)-d\|}$, and $\frac{1}{T}\int_0^T \|E(t)\|dt \in \{0.06, 0.19, 0.45\}$ Hz for $T \in \{12, 8, 6\}$ s). Regarding *(4)*: Fig. 11j shows that, the bigger $V_{1,0}, V_{2,0}$ are, the quicker the estimators tend to saturate for "large" initial conditions – this illustrates the importance of the function $\Gamma_\cdot$ in (36a) and that $V_{1,0}, V_{2,0}$ determine when the estimators (integral-action) should start working. Regarding *(5)*: Fig. 11k shows that, despite the winds being non-constant, tracking still takes places, owing to the robustness added by the estimators, which rather than settling down, try to estimate the time-varying winds.

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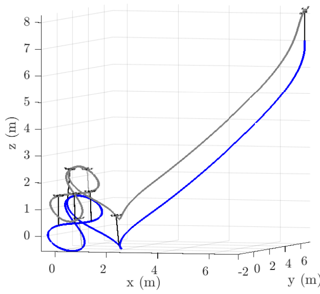
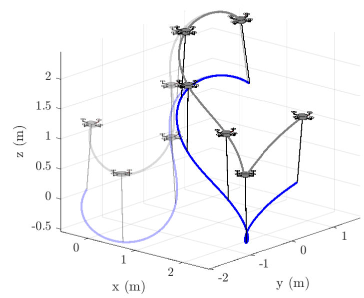
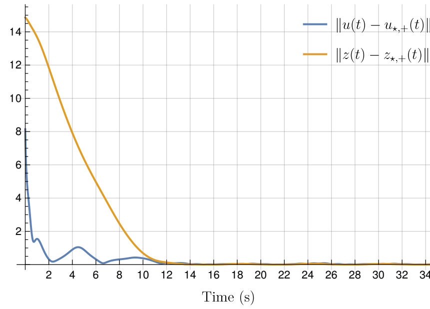


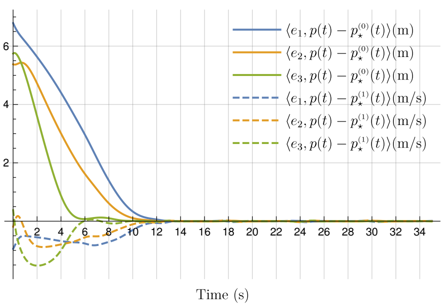
(a) Complete trajectory: load in blue; UAV in gray.



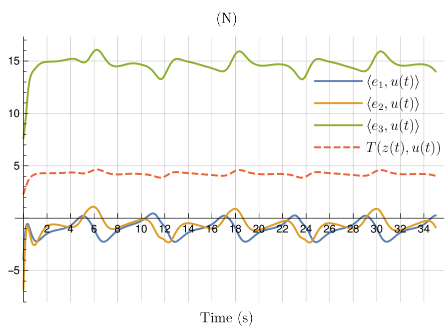
(b) 5 sec – 12 sec: real system in opaque; desired system in transparent.



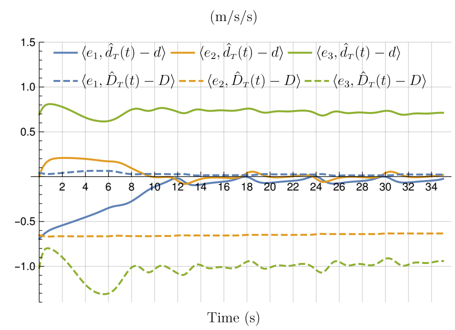
(c) Errors to equilibrium state and input trajectories



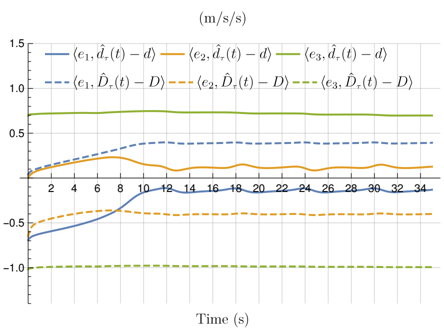
(d) Position and velocity tracking errors



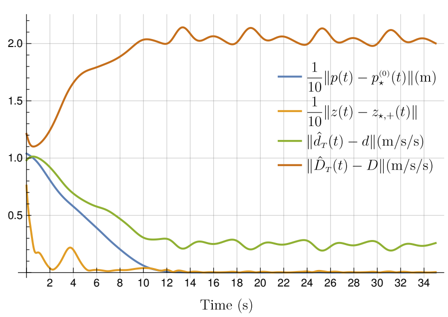
(e) Inputs to slung-load system, and cable tension.



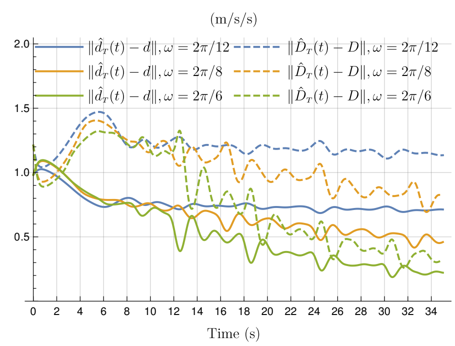
(f) Disturbance estimates $\hat{d}_T$ and $\hat{D}_T$



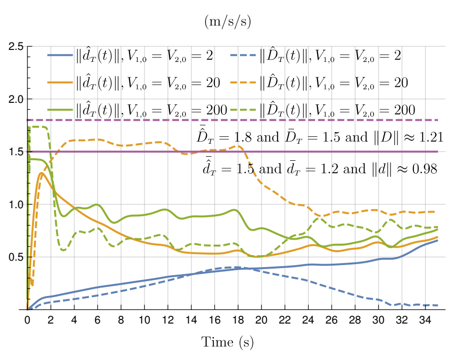
(g) Disturbance estimates $\hat{d}_\tau$ and $\hat{D}_\tau$



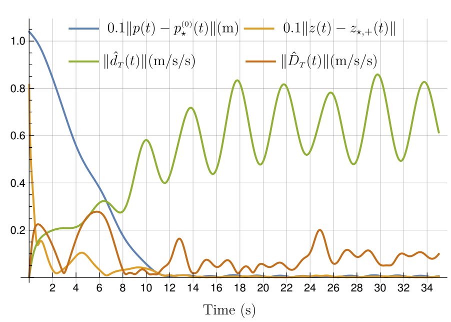
(h) Sim *(2)* (model mismatch): position and state tracking errors, and disturbance estimates errors.



(i) Sim *(3)* (excitation vs converge of estimators): disturbance estimates errors.



(j) Sim *(4)* (purpose of Γ_{V_0}): Disturbance estimate $\hat{d}_T$



(k) Sim *(5)* (time-varying winds): position and state tracking errors, and disturbance estimates $\hat{d}_T$ and $\hat{D}_T$.

Fig. 11: Simulations (baseline simulation and simulations under conditions *(2)* to *(5)*).

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X. UAV ATTITUDE INNER LOOP

We assumed thus far that the UAV, tethered to the load, is fully actuated. In this section, we lift this assumption. In an under-actuated UAV

- the UAV has a body direction, hereafter called thrust body direction, ($Re_3 \in \mathbb{S}^2$, with R as the UAV's rotation matrix) along which a thrust input ($U \in \mathbb{R}$) can be provided;
- the UAV has an angular velocity ($\omega \in \mathbb{R}^3$, expressed in the body frame) which determines how the UAV's rotation matrix evolves ($\dot{R} = R\mathcal{S}(\omega)$);
- and, finally, the UAV can be provided a torque input ($\tau \in \mathbb{R}^3$, expressed in the body frame) which can be used to control the UAV's rotation matrix, and, in particular, to control the UAV's thrust direction.

Recall then the vector field $Z_{w,W}$ in (9), where the input $u \in \mathbb{R}^3$ is an input force we assumed was provided by UAV. This input must now be replaced by $URe_3 \in \mathbb{R}^3$; it is worth emphasizing that if one had control over the UAV attitude, then one would chose the thrust input $U = \|u\| \in \mathbb{R}$ and the attitude $Re_3 = \frac{u}{\|u\|} \in \mathbb{S}^2$, in which case $URe_3 = u$, for any desired $u \in \mathbb{R}^3 \backslash \{0_3\}$. The vector field with the UAV attitude dynamics can then be described by

$$\begin{bmatrix} \dot{z} \\ \dot{R} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} Z_{w,W}(z, URe_3) \\ R\mathcal{S}(\omega) \\ J^{-1}(\tau - \mathcal{S}(\omega) J\omega) \end{bmatrix} \tag{108}$$

where $J \in \mathbb{R}^{3\times3}$ is the UAV's moment of inertia matrix, and τ is the UAV's torque input. Loosely speaking, if one picks the trust input as $U = \langle u, Re_3 \rangle$, one can then state that the goal is to design a torque τ such that the thrust body direction Re_3 is steered to $\frac{u}{\|u\|}$, in which case $URe_3 = \langle u, Re_3 \rangle Re_3$ is steered to u.

In order to simplify the solution, define

$$\begin{bmatrix} r \\ \varpi \end{bmatrix} := \begin{bmatrix} Re_3 \\ \Pi(Re_3) R\omega \end{bmatrix} \in \begin{bmatrix} \mathbb{S}^2 \\ T_{Re_3}\mathbb{S}^2 \end{bmatrix}$$

and let the torque τ in (108) be chosen as

$$\tau := \underbrace{J(R^T \bar{\tau} - \langle e_3, \omega \rangle \mathcal{S}(e_3)\omega)}_{\substack{\text{control of } \frac{d}{dt}\Pi(Re_3)R\omega \\ \text{where } \bar{\tau} \text{ is designed later}}} \quad \underbrace{+\mathcal{S}(\omega) J\omega}_{\substack{\text{cancel acceleration term} \\ \text{in (108)}}} \quad \underbrace{-k_{\dot{\psi}}^{\text{uav}} \langle e_3, \omega \rangle e_3}_{\substack{\text{steer yaw rate to zero} \\ \text{for some positive gain } k_{\dot{\psi}}^{\text{uav}}}} \quad . \tag{109}$$

(As a remark, we emphasize that the third term in (109) can be chosen differently such that, for example, the UAV's "yaw position" is steered to some desired "yaw position".) Composing the dynamics in (108) with the torque control law in (109) yields

$$\begin{bmatrix} \dot{z} \\ \dot{r} \\ \dot{\varpi} \end{bmatrix} = \begin{bmatrix} Z_{w,W}(z, Ur) \\ \mathcal{S}(\varpi) \\ \Pi(r)\,\bar{\tau} \end{bmatrix}. \tag{110}$$

Once again, and loosely speaking, let $u \in \mathbb{R}^3$ be the desired three dimensional force we wish the UAV to provide; then, if we pick the trust input as $U = \langle u, r \rangle$, one can then state that the goal is to design an angular acceleration $\bar{\tau}$[14] such that the thrust body direction $r \in \mathbb{S}^2$ is steered to $\frac{u}{\|u\|} \in \mathbb{S}^2$, in which case the real force $Ur = \langle u, r \rangle r \in \mathbb{R}^3$ is steered to the desired force $u \in \mathbb{R}^3$.

The next proposition describes a possible controller for the angular acceleration $\bar{\tau}$ in (110).

Proposition 75: Let

$$\mathbb{R} \ni t \mapsto u(t) \in \{u \in \mathbb{R}^3 : \|u\| \geq \epsilon \text{ for some } \epsilon > 0\}, \text{ with } \sup_{t \in \mathbb{R}} \|u^{(i)}(t)\| < \infty \text{ for } i \in \{0, 1, 2\},$$

be given and define

$$\begin{aligned} t &\mapsto r^\star(t) := \frac{u^{(0)}(t)}{\|u^{(0)}(t)\|} \in \mathbb{S}^2, \\ t &\mapsto \varpi^\star(t) := \mathcal{S}(r^\star(t))\,\dot{r}^\star(t) = \mathcal{S}\left(\frac{u^{(0)}(t)}{\|u^{(0)}(t)\|}\right)\frac{u^{(1)}(t)}{\|u^{(0)}(t)\|} \in T_{r^\star(t)}\mathbb{S}^2, \\ t &\mapsto \tau^\star(t) := \dot{\varpi}^\star(t) = \mathcal{S}\left(\frac{u^{(0)}(t)}{\|u^{(0)}(t)\|}\right)\left(\frac{u^{(2)}(t)}{\|u^{(0)}(t)\|} - 2\frac{\langle u^{(0)}(t), u^{(1)}(t)\rangle}{\langle u^{(0)}(t), u^{(0)}(t)\rangle}\frac{u^{(1)}(t)}{\|u^{(0)}(t)\|}\right) \in T_{r^\star(t)}\mathbb{S}^2. \end{aligned}$$

Consider then the system with state $(r, \varpi) \in \{(r, \varpi) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle r, r \rangle = 1 \text{ and } \langle r, \varpi \rangle = 0\}$ and evolving according to

$$\begin{bmatrix} \dot{r} \\ \dot{\varpi} \end{bmatrix} = \begin{bmatrix} \mathcal{S}(\varpi)\, r \\ \Pi(r)\,\tau \end{bmatrix}, \tag{111a}$$

and where τ is an angular acceleration input (note that $\{(r, \varpi) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle r, r \rangle = 1 \text{ and } \langle r, \varpi \rangle = 0\}$ is indeed positively invariant with respect to the vector field above). Consider then the angular acceleration control law

$$\begin{aligned} &\tau^{cl} : \mathbb{R} \times \{(r, \varpi) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle r, r \rangle = 1 \text{ and } \langle r, \varpi \rangle = 0\} \to [0, \infty) \\ &\tau^{cl}(t, (r, \varpi)) := \underbrace{\Pi(r)\,\tau^\star(t) + \langle r, \varpi^\star(t) \rangle}_{\text{feed-forward term}} - \underbrace{k_\varpi^{\text{uav}}(\varpi - \Pi(r)\,\varpi^\star(t))}_{\text{derivative term}} - \underbrace{k_\theta^{\text{uav}}\mathcal{S}(r^\star(t))\, r}_{\text{proportional term}}, \end{aligned} \tag{111b}$$

for some positive gains k_ϖ^{uav} and k_θ^{uav}. If $\tau = \tau^{cl}(t, (r, \varpi))$ in (111a), then $\lim_{t \to \infty} \mathcal{S}(r(t))\, r^\star(t) = 0_3$ and $\lim_{t \to \infty} \varpi(t) - \varpi^\star(t) = 0_3$. If $k_\theta^{\text{uav}}(1 - \langle r(0), r^\star(0) \rangle) + \frac{1}{2}\|\varpi(0) - \Pi(r(0))\,\varpi^\star(0)\|^2 < 2k_\theta^{\text{uav}}$, then $\lim_{t \to \infty} r(t) - r^\star(t) = 0_3$ and $\lim_{t \to \infty} \varpi(t) - \varpi^\star(t) = 0_3$.

Proof: If we consider the Lyapunov function

$$\begin{aligned} &V : \mathbb{R} \times \{(r, \varpi) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle r, r \rangle = 1 \text{ and } \langle r, \varpi \rangle = 0\} \to [0, \infty) \\ &V(t, (r, \varpi)) := k_\theta^{\text{uav}}(1 - \langle r, r^\star(t) \rangle) + \frac{1}{2}\|\varpi - \Pi(r)\,\varpi^\star(t)\|^2 \end{aligned},$$

it then follows that

$$\begin{aligned} d_1 V(t, (r, \varpi)) + d_2 V(t, (r, \varpi))(\mathcal{S}(\varpi)\, r, \Pi(r)\,\tau) &= k_\theta^{\text{uav}}\langle \mathcal{S}(r)\, r^\star(t), \varpi - \Pi(r)\,\varpi^\star(t) \rangle + \\ &\quad \langle \varpi - \Pi(r)\,\varpi^\star(t), \tau - (\Pi(r)\,\tau^\star(t) + (-\dot{r}\langle r, \varpi^\star(t) \rangle - r\langle \dot{r}, \varpi^\star(t) \rangle)) \rangle \\ &= k_\theta^{\text{uav}}\langle \mathcal{S}(r)\, r^\star(t), \varpi - \Pi(r)\,\varpi^\star(t) \rangle + \\ &\quad \langle \varpi - \Pi(r)\,\varpi^\star(t), \tau - (\Pi(r)\,\tau^\star(t) - \mathcal{S}(\varpi)\, r\langle r, \varpi^\star(t) \rangle) \rangle. \end{aligned}$$

Thus, if $\tau = \tau(t, (r, \varpi))$, it follows that

$$d_1 V(t, (r, \varpi)) + d_2 V(t, (r, \varpi))(\mathcal{S}(\varpi)\, r, \Pi(r)\,\tau)|_{\tau = \tau(t, (r, \varpi))} = -k_\varpi^{\text{uav}}\|\varpi - \Pi(r)\,\varpi^\star(t)\|^2 \leq 0.$$

1) By invoking Barbalat's lemma on $t \mapsto \frac{d}{dt}V(t, (r(t), \omega(t)))$, we conclude that $\lim_{t \to \infty} V(t, (r(t), \omega(t))) = 0 \Rightarrow \lim_{t \to \infty} \varpi(t) - \Pi(r(t))\,\varpi^\star(t) = 0_3$ (note that we wish to conclude that $\lim_{t \to \infty} \varpi(t) - \varpi^\star(t) = 0_3$).
2) By invoking Barbalat's lemma on $t \mapsto \varpi(t) - \Pi(r(t))\,\varpi^\star(t)$, we conclude that $\lim_{t \to \infty} \frac{d}{dt}(\varpi(t) - \Pi(r(t))\,\varpi^\star(t)) = 0_3 \Rightarrow \lim_{t \to \infty} \mathcal{S}(r(t))\, r^\star(t) = 0_3$.

[14] Dimensionally, the quantity τ in (108) is different than the quantity $\bar{\tau}$ in (110) (let [placeholder] stand for the dimensions of "placeholder"): the former has the dimensions of a torque (i.e., [force][distance]), and the latter has the dimensions of an angular acceleration (i.e., [force]/([mass][distance])).

3) By combining 1) and 2), it follows that $\lim_{t\to\infty} \varpi(t)-\varpi^\star(t) = 0$, since $\Pi\left(\pm r^\star(t)\right)\varpi^\star(t) = \varpi^\star(t)$; and that $\lim_{t\to\infty} \mathcal{S}\left(r(t)\right) r^\star(t) = 0_3$.
4) Finally, from 3), it follows that $\lim_{t\to\infty} V(t,(r(t),\varpi(t))) = 0$ if $\lim_{t\to\infty}\langle r(t), r^\star(t)\rangle = 1$ and $\lim_{t\to\infty} V(t,(r(t),\varpi(t))) = 2k_\theta^{\text{uav}}$ if $\lim_{t\to\infty}\langle r(t), r^\star(t)\rangle = -1$; thus if $V(0,(r(0),\varpi(0))) < 2k_\theta^{\text{uav}}$, then $\lim_{t\to\infty}\langle r(t), r^\star(t)\rangle = 1$ (given that $t \mapsto V(t,(r(t),\varpi(t)))$ is non-increasing).

■

Implementing the angular acceleration control law in (111b) requires the knowledge of $t \mapsto u^{(0)}(t)$, $t \mapsto u^{(1)}(t)$ and $t \mapsto u^{(2)}(t)$. In our particular case, the input $u^{(0)}$ is described by

$$t \mapsto u^{(0)}(t) := u^{cl}(t, z(t), \xi(t)) \tag{112}$$

where u^{cl} is found in (106) (this control law was described in Subsection VIII-G), where the first entry encapsulates the time-dependency coming from the desired position trajectory; the second entry corresponds to the state of the physical system; and the third entry corresponds to the internal state of the controller (the disturbance estimators); and where we note that

- if $r = \frac{u^{cl}(x_6)}{\|u^{cl}(x_6)\|}$, then $\dot{x}_6 = X^{cl}_{6,d,D}(x_6, g^3)$ since $\nu_x^{-1}(u^{cl}(x_6)) = (T_3^{cl}(x_4), \tau_2^{cl}(x_6))$;
- $\dot{z}$ cannot be exactly known by the controller since it depends on the unknown winds w and W;
- it does not make sense to estimate $\dot{z}$ by using the disturbance estimators $\hat{d}_T$ and $\hat{D}_T$ (or $\hat{d}_\tau$ and $\hat{D}_\tau$), since these estimators are not guaranteed to converge to the real disturbances – as such, the design of new estimators would be necessary.

Then the two time derivatives of the input $u^{(0)}$ in (112) are then given by

$$t \mapsto u^{(1)}(t) := \underbrace{d_1 u^{cl}(t,z,\xi) + d_2 u^{cl}(t,z,\xi)\dot{z} + d_3 u^{cl}(t,z,\xi)\dot{\xi}}_{u^1(t,z,\xi,r):=\text{function}(t,z,\xi,r)} \Big|_{\substack{\dot{z}=Z_{w,W}(z,Ur) \text{ with } U=\langle u^{cl}(t,z,\xi),r\rangle \\ \dot{\xi}=\text{some-function}(t,z,\xi)}} \Big|_{\substack{z=z(t), \\ \xi=\xi(t) \\ r=r(t)}}$$

$$t \mapsto u^{(2)}(t) := \underbrace{d_1 u^1(t,z,\xi,r) + d_2 u^1(t,z,\xi,r)\dot{z} + d_3 u^1(t,z,\xi,r)\dot{\xi} + d_4 u^1(t,z,\xi,r)\dot{r}}_{u^1(t,z,\xi,r,\varpi):=\text{another-function}(t,z,\xi,r,\varpi)} \Big|_{\substack{\dot{z}=Z_{w,W}(z,Ur) \text{ with } U=\langle u^{cl}(t,z,\xi),r\rangle \\ \dot{\xi}=\text{some-function}(t,z,\xi) \\ \dot{r}=\mathcal{S}(\varpi)r}} \Big|_{\substack{z=z(t) \\ \xi=\xi(t) \\ r=r(t) \\ \varpi=\varpi(t)}} .$$

Note however that the derivative $u^{(1)}$ and $u^{(2)}$ depend on the unknown wind forces w and W, and, for the reasons pointed out above, the design of new estimators would be necessary by following identical steps as those in Section VIII-F. For the purposes of the simulation, however, we compute only an approximation of those derivatives by numeric differentiation of $u^{(0)}$.

As a final remark, we note that $Z_{w,W}$ is affine in its second entry (the vector field is input-affine) and thus

$$\begin{aligned} Z_{w,W}(z, Ur) &= Z_{w,W}(z, u^{cl}(t,z,\xi) + Ur - u^{cl}(t,z,\xi)) \\ &= \underbrace{Z_{w,W}(z, u^{cl}(t,z,\xi))}_{\text{vector field we studied before}} + \underbrace{E(z)(Ur - u^{cl}(t,z,\xi))}_{\text{error coming from underactuation}}, \end{aligned}$$

for some $E(z)$ which is a linear map from $\mathbb{R}^3$ to $T_z\mathbb{Z}$: however, the cascaded strucutre we exploited during the backstepping procedure is not preserved.

XI. State Augmentation

Let us provide an example that motivates the benefits of redefining/augmenting the state of a non-autonomous system. Consider the following is given

- a reference signal $a : \mathbb{R} \ni t \mapsto a(t) \in \mathbb{R}^3$, $a \in \mathcal{C}^1$ which satisfies

$$\inf_{t\in\mathbb{R}} \|a^{(0)}(t)\| > 0, \tag{113a}$$

$$\sup_{t\in\mathbb{R}} \|a^{(0)}(t)\| < \infty, \tag{113b}$$

$$\sup_{t\in\mathbb{R}} \|a^{(1)}(t)\| < \infty; \tag{113c}$$

- and define unit vector $x^\star$ given by (it is well defined, given the conditions above)

$$x^\star : \mathbb{R} \ni t \mapsto x^\star(t) := \frac{x(t)}{\|x(t)\|} \in \mathbb{S}^2,$$

Consider then

1) the state space $\mathbb{X} := \mathbb{S}^2$;
2) the time-varying vector field X is given by (k is some positive proportional gain)

$$X : \mathbb{R} \times \mathbb{X} \ni (t,x) \mapsto X(t,x) \in T_n\mathbb{X} := T_n\mathbb{S}^2$$

$$X(t,x) := \mathcal{S}\left(\mathcal{S}\left(x^\star(t)\right)\dot{x}^\star(t) - k\mathcal{S}\left(x^\star(t)\right)x\right)x = \mathcal{S}\left(\mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|}\right)\frac{a^{(1)}(t)}{\|a^{(0)}(t)\|} - k\mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|}\right)x\right)x;$$

3) the unit vector solution $t \mapsto x(t)$ to the differential equation

$$\dot{x}(t) = X(t, x(t)), \text{ with } x(0) \in \mathbb{X}; \tag{114}$$

4) define the Lyapunov function

$$V : \mathbb{R} \times \mathbb{X} \ni (t, x) \mapsto V(t, x) := 1 - \langle x, x^\star(t)\rangle = 1 - \left\langle x, \frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right\rangle \in [0, 2]; \tag{115}$$

5) define the derivative of the Lyapunov function V along the vector field X as

$$\begin{aligned} W : \mathbb{R} \times \mathbb{X} \ni (t, x) \mapsto W(t, x) :=& d_1 V(t, x) + d_2 V(t, x) X(t, x) \\ =& -k \|\mathcal{S}(x)\, x^\star(t)\|^2 = -k \left\| \mathcal{S}(x) \frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right\|^2 \\ \leq& 0 \text{ for all } (t, n) \in \mathbb{R} \times \mathbb{S}^2 \\ <& 0 \text{ for all } (t, n) \in \{(t, n) \in \mathbb{R} \times \mathbb{S}^2 : n \neq \pm n^\star(t)\}. \end{aligned}$$

6) $V \in \mathcal{C}^1(\mathbb{R} \times \mathbb{X})$ and $W \in \mathcal{C}^1(\mathbb{R} \times \mathbb{X})$.

We wish to conclude that the $t \mapsto x(t)$ asymptotically tracks either $t \mapsto +x^\star(t)$ or $t \mapsto -x^\star(t)$ (i.e., either $\lim_{t\to+\infty}\langle x(t), x^\star(t)\rangle = +1$ or $\lim_{t\to+\infty}\langle x(t), x^\star(t)\rangle = -1$).

In order to reach that conclusion, we follow the logical steps:

1) since $\mathbb{X} := \mathbb{S}^2$ is compact, and the vector field X is locally Lipschitz in its second entry, given that

$$\begin{aligned} &\sup_{t\geq 0}\sup_{x\in\mathbb{X}} \|d_2 X(t, x)\| = \\ &= \sup_{t\geq 0}\sup_{x\in\mathbb{X}} \left\| \mathcal{S}\left(\mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) \frac{a^{(1)}(t)}{\|a^{(0)}(t)\|} - k\mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) x \right) - k\mathcal{S}(x)\, \mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) \right\| \\ &\leq \sup_{t\geq 0}\sup_{x\in\mathbb{X}} \left\| \mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) \frac{a^{(1)}(t)}{\|a^{(0)}(t)\|} - k\mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) x \right\| + k \left\| \mathcal{S}(x)\, \mathcal{S}\left(\frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right) \right\| \quad \because \|\mathcal{S}(\cdot)\| = \|\cdot\| \\ &\leq \sup_{t\geq 0}\sup_{x\in\mathbb{X}} \frac{\|a^{(1)}(t)\|}{\|a^{(0)}(t)\|} + 2k \qquad \because \|x\| = 1 \text{ and } \left\| \frac{a^{(0)}(t)}{\|a^{(0)}(t)\|} \right\| = 1 \\ &< \infty, \end{aligned}$$

where the latter inequality follows from the conditions in (113). As such, a unique and complete solution $[0, \infty) \ni t \mapsto x(t) \in \mathbb{S}^2$ to (114) exists;

2) the Lyapunov function V in (115) is lower bounded (by 0), and since its derivative along a solution is non-positive, it follows that the limit $\lim_{t\to\infty} V(t, x(t))$ exists and it lies in set $[0, V(0, x(0))] \subset [0, 2]$;

3) one cannot invoke LaSalle's invariance principle to conlude that $\lim_{t\to\infty} \dot{V}(t, x(t)) = \lim_{t\to\infty} W(t, x(t)) = 0$, since the system (114) is non-autonomous;

4) we invoke Barbalat's Lemma instead, by verifying that $\sup_{t\geq 0} \|\ddot{V}(t, x(t))\| < \infty$, which implies that $t \mapsto \dot{V}(t, x(t))$ is uniformly continuous in $(0, \infty)$; indeed,

$$\begin{aligned} \sup_{t\geq 0} |\ddot{V}(t, x(t))| &= \sup_{t\geq 0} \left| \dot{W}(t, x(t)) \right| \\ &= \sup_{t\geq 0} \left| (d_1 W(t, x) + d_2 W(t, x) X(t, x)) \,|_{x=x(t)} \right| \\ &= \sup_{t\geq 0} |2k \langle \mathcal{S}(x)\, x^\star(t), \mathcal{S}(x)\, \dot{x}^\star(t) - \mathcal{S}(x^\star(t))\, X(t, x) \rangle| \,|_{x=x(t)} \\ &\leq \sup_{t\geq 0} 2k \|\mathcal{S}(x)\, \dot{x}^\star(t) - \mathcal{S}(x^\star(t))\, X(t, x)\| \,|_{x=x(t)} \qquad \because \|\mathcal{S}(x)\, x^\star(t)\| \leq 1 \\ &\leq \sup_{t\geq 0} 2k (\|\dot{x}^\star(t)\| + \|X(t, x(t))\|) \qquad \because \|x^\star(t)\| = 1 \\ &\leq \sup_{t\geq 0} 2k \left(\frac{\|a^{(1)}(t)\|}{\|a^{(0)}(t)\|} + \left(\frac{\|a^{(1)}(t)\|}{\|a^{(0)}(t)\|} + k \right) \right) \\ &\leq \sup_{t\geq 0} 2k \left(2\frac{\|a^{(1)}(t)\|}{\|a^{(0)}(t)\|} + k \right) < \infty \end{aligned}$$

where the latter inequality follows from the conditions in (113);

5) by invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty} \dot{V}(t, x(t)) = \lim_{t\to\infty} W(t, x(t)) = 0 \Rightarrow \lim_{t\to\infty} \|\mathcal{S}(x)\, x^\star(t)\| = 0$;

6) continuity of $t \mapsto x(t)$, continuity of $t \mapsto x^\star(t)$, and boundedness of $t \mapsto \dot{x}(t)$ then imply that either $\lim_{t\to\infty}\langle x(t), x^\star(t)\rangle = 1$ or $\lim_{t\to\infty}\langle x(t), -x^\star(t)\rangle = 1$.

Let us then consider now a state augmentation strategy, which will simplify the derivation of the same conclusions as before. Consider then

1) the augmented state space $\mathbb{X} := \mathbb{S}^2 \times (\mathbb{R}^3 \backslash \{0_3\})$, and the state decomposition
$$x \in \mathbb{X} :\Leftrightarrow (n, a^0) \in \mathbb{S}^2 \times (\mathbb{R}^3 \backslash \{0_3\});$$
2) the time-varying augmented vector field X is given by (k is some positive proportional gain)
$$\begin{aligned} &X : \mathbb{X} \times \mathbb{R}^3 \ni (x, a^1) \mapsto X(x, a^1) \in T_x\mathbb{X} \\ &\dot{x} = X(x, a^1) :\Leftrightarrow \begin{bmatrix} \dot{n} \\ \dot{a}^0 \end{bmatrix} = \begin{bmatrix} \mathcal{S}\left(\mathcal{S}\left(\frac{a^0}{\|a^0\|}\right)\frac{a^1}{\|a^0\|} - k\mathcal{S}\left(\frac{a^0}{\|a^0\|}\right)n\right)n \\ a^1 \end{bmatrix}. \end{aligned} \tag{116}$$
3) the solution $t \mapsto x(t)$ to the differential equation
$$\dot{x}(t) = X(x(t), a^{(1)}(t)), \text{ with } x(0) \in \mathbb{X}; \tag{117}$$
4) define the Lyapunov function
$$\mathbb{X} \ni x \mapsto V(x) := 1 - \left\langle n, \frac{a^0}{\|a^0\|} \right\rangle \in [0, 2], \tag{118}$$
5) define the derivative of the Lyapunov function V along the vector field X as
$$\begin{aligned} \mathbb{X} \ni x \mapsto W(x) &:= dV(x)X(x, a^1) \quad \text{(independent of } a^1\text{)} \\ &= -k\left\|\mathcal{S}(n)\frac{a^0}{\|a^0\|}\right\|^2 \leq 0. \end{aligned}$$
6) $X \in \mathcal{C}^\omega(\mathbb{X} \times \mathbb{R}^3)$, $V \in \mathcal{C}^\omega(\mathbb{X})$ and $W \in \mathcal{C}^\omega(\mathbb{X})$ (analytic functions).

We wish to conclude that the $t \mapsto n(t)$ (as part of the $t \mapsto x(t)$) asymptotically tracks either $t \mapsto +n^\star(t)$ or $t \mapsto -n^\star(t)$ (i.e., either $\lim_{t\to+\infty}\langle n(t), n^\star(t)\rangle = +1$ or $\lim_{t\to+\infty}\langle n(t), n^\star(t)\rangle = -1$).

Before we proceed, we make the following assumption: we assume that
$$U = \left\{ x \in \mathbb{X} : 0 < \inf_{t\geq 0}\|a^{(0)}(t)\| \leq \|a^0\| \leq \sup_{t\geq 0}\|a^{(0)}(t)\| < \infty \right\}$$
is positively invariant, and given the conditions in (113), it follows that a^0 is in a compact subset of $\mathbb{R}^3 \backslash \{0_3\}$.

In order to reach the desired conclusion, we follow the logical steps:

1) a solution $t \mapsto x(t)$ of (116) is trapped in $V_{\leq V(x(0))} \cap U$, which is a compact subset of $\mathbb{X}$; moreover, $t \to a^{(1)}(t)$ is also in a compact subset of $\mathbb{R}^3$; since the vector field X is $\mathcal{C}^1(\mathbb{X} \times \mathbb{R}^3)$, it follows that a unique and complete solution $[0, \infty) \ni t \mapsto x(t) \in \mathbb{X}$ to (117) exists;
2) the Lyapunov function V in (118) is lower bounded (by 0), and since its derivative along a solution is non-positive, it follows that the limit $\lim_{t\to\infty} V(x(t))$ exists and it lies in set $[0, V(x(0))] \subset [0, 2]$;
3) one cannot invoke LaSalle's invariance principle to conlude that $\lim_{t\to\infty}\dot{V}(x(t)) = \lim_{t\to\infty} W(x(t)) = 0$, since the system (117) is non-autonomous;
4) we invoke Barbalat's Lemma instead, by verifying that $\sup_{t\geq 0}\|\ddot{V}(x(t))\| < \infty$, which implies that $t \mapsto \dot{V}(x(t))$ is uniformly continuous in $(0, \infty)$; indeed,
$$\begin{aligned} \sup_{t\geq 0}|\ddot{V}(x(t))| &= \sup_{t\geq 0}|\dot{W}(x(t))| \\ &= \sup_{t\geq 0}|dW(x(t))X(x(t), a^{(1)}(t))| \\ &= \sup_{\substack{x\in V_{\leq V(x(0))}\cap U \\ a^1 \in \text{compact subset of } \mathbb{R}^3}} |dW(x)X(x, a^1)| < \infty, \end{aligned}$$
where the latter inequality follows because $X \in \mathcal{C}^0(\mathbb{X} \times \mathbb{R}^3)$ and $W \in \mathcal{C}^1(\mathbb{X})$ (and since $V_{\leq V(x(0))} \cap U$ is a compact subset of $\mathbb{X}$);
5) by invoking Barbalat's lemma, one concludes that $\lim_{t\to\infty}\dot{V}(x(t)) = \lim_{t\to\infty} W(x(t)) = 0 \Rightarrow \lim_{t\to\infty}\|\mathcal{S}(n)\,n^\star(t)\| = 0$;
6) continuity of $t \mapsto n(t)$, continuity of $t \mapsto n^\star(t)$, and boundedness of $t \mapsto \dot{n}(t)$ then imply that either $\lim_{t\to\infty}\langle n(t), n^\star(t)\rangle = 1$ or $\lim_{t\to\infty}\langle n(t), -n^\star(t)\rangle = 1$.

As exemplified, augmenting the state significantly simplifies the steps one must follow in order to reach the desired conclusion.

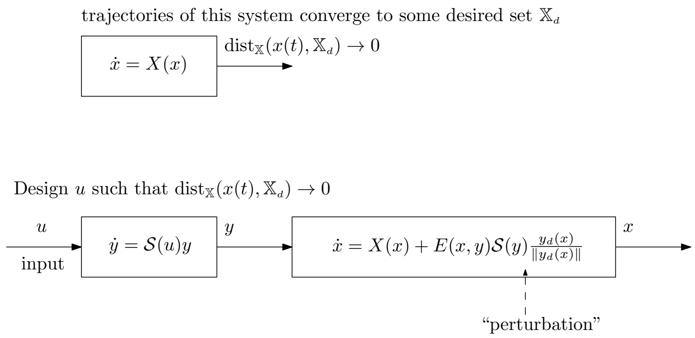

Fig. 12: Cascaded control with angular velocity.

XII. Auxiliary Propositions

Proposition 76: Consider the following:

- a disturbance $d \in \mathbb{R}^3$, where $\|d\|_{\mathbb{R}^3} \leq \bar{d}$;
- an estimate $\hat{d} \in \mathbb{R}^3$ of the disturbance $d \in \mathbb{R}^3$, where $\|\hat{d}\|_{\mathbb{R}^3} < \bar{\bar{d}}$ and $\bar{d} < \bar{\bar{d}}$;
- a gravity acceleration $g \in \{g \in \mathbb{R}^3 : \|g\| > \bar{\bar{d}}\}$.

Then

$$\left\langle \frac{g-d}{\|g-d\|}, \frac{g-\hat{d}}{\|g-\hat{d}\|} \right\rangle = \pm 1 \Leftrightarrow \Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)\left(d-\hat{d}\right) = 0_3. \tag{119a}$$

Proof: Necessity of (119a) ($\Rightarrow$):

$$\begin{aligned}
\left\langle \frac{g-d}{\|g-d\|}, \frac{g-\hat{d}}{\|g-\hat{d}\|} \right\rangle = \pm 1 &\Leftrightarrow \frac{g-d}{\|g-d\|} = \pm\frac{g-\hat{d}}{\|g-\hat{d}\|} \\
&\Leftrightarrow \frac{(g-\hat{d})-(d-\hat{d})}{\|g-d\|} = \pm\frac{g-\hat{d}}{\|g-\hat{d}\|} \\
&\Leftrightarrow d-\hat{d} = \left(\|g-\hat{d}\| \mp \|g-d\|\right)\frac{g-\hat{d}}{\|g-\hat{d}\|} \\
&\Rightarrow \Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)\left(d-\hat{d}\right) = 0_3.
\end{aligned}$$

Sufficiency of (119a) ($\Leftarrow$):

$$\begin{aligned}
\Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)\left(d-\hat{d}\right) = 0_3 &\Leftrightarrow \Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)\left((g-\hat{d})-(g-d)\right) = 0_3 \\
&\Leftrightarrow \Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)(g-d) = 0_3 \\
&\Leftrightarrow \Pi\left(\frac{g-\hat{d}}{\|g-\hat{d}\|}\right)\frac{g-d}{\|g-d\|} = 0_3 \\
&\Rightarrow \left\langle \frac{g-d}{\|g-d\|}, \frac{g-\hat{d}}{\|g-\hat{d}\|} \right\rangle = \pm 1.
\end{aligned}$$

■

XIII. Cascaded control with angular velocity

Consider the state space $\mathbb{X} \times \mathbb{S}^2$, and the consider then system evolving in that state space, namely

$$\dot{x} = \mathcal{X}(x, y) := X(x) + E(x, y)\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|} \tag{120a}$$

$$\dot{y} = \mathcal{Y}(y, u) := \mathcal{S}(u)\, y \tag{120b}$$

as illustrated in Fig. 13, and where

- $X : \mathbb{X} \ni x \mapsto X(x) \in T_x\mathbb{X}$ is a vector field;
- $V : \mathbb{X} \ni x \mapsto V(x) \in [0, \infty)$ is a Lyapunov function

- such that $dV(x)X(x) =: W(x) \leq 0$ for all $x \in \mathbb{X}$;
- such that $V(x_d) = 0$ and $W(x_d) = 0$ for all $x_d \in \mathbb{X}_d$, for some $\mathbb{X}_d \subset \mathbb{X}$ ($\mathbb{X}_d$ is an equilibria set);
- such that $\{x \in \mathbb{X} : V(x) \leq \epsilon$, for a positive ϵ, defines a neighborhood of $\mathbb{X}_d$ (stability of $\mathbb{X}_d$);
- such that it guarantees that $\lim_{t\to\infty} d_{\mathbb{X}}(x(t), \mathbb{X}_d) = 0$ for solutions of $\dot{x}(t) = X(x(t))$ with $x(0) \in \mathbb{X}_0$ for some $\mathbb{X}_0 \subset \mathbb{X}$ (attractivity of $\mathbb{X}_d$);
- there is some positive constant $\bar{V}$ such that the set $\mathbb{X}_0$ can be underestimated as $\mathbb{X}_0 \supseteq \{x \in \mathbb{X} : V(x) < \bar{V}\}$.

- $y_d : \mathbb{X} \ni x \mapsto y_d(x) \in \{y \in \mathbb{R}^3 : \|y\| > \epsilon$ for some positive $\epsilon\}$ is the desired solution for y (since, if $y \equiv y_d(x)$ then $\dot{x} = X(x)$). Since $y_d(x) \neq 0$, the unit vector $\frac{y_d(x)}{\|y_d(x)\|}$, that appears in (120a), is always well defined.
- $E(x, y)$ is a linear map from $\mathbb{R}^3$ to $T_x\mathbb{X}$;
- $u \in \mathbb{R}^3$ is the input to the system.

The idea is to design a control law for the input u such that $\lim_{t\to\infty} d_{\mathbb{X}}(x(t), \mathbb{X}_d) = 0$ for solutions of (120). Because the dynamics of x in (120a) come disturbed by the error $\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}$, it follows that along solutions of (120)

$$\dot{V}(x) = W(x) + dV(x)E(x, y)\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}. \tag{121}$$

Let then the following tuple be given, $\mathcal{T} = (\mathcal{X}, E, y_d, V, \gamma, k)$, where

- $\mathcal{X}, E, y_d, V$ are the maps described above;
- $k > 0$ is a positive feedback gain;
- and $\gamma > 0$ is a positive backstepping gain.

Consider then the following control law

$$\begin{aligned} u^{cl} &: \mathbb{X} \times \mathbb{S}^2 \ni (x, y) \mapsto u^{cl}(x, y) \in T_y\mathbb{S}^2 \subset \mathbb{R}^3 \\ u^{cl}(x, y) &:= -k\mathcal{S}\left(\frac{y_d(x)}{\|y_d(x)\|}\right)y && \text{(122a)} \\ &\quad + \Pi(y)\mathcal{S}\left(\frac{y_d(x)}{\|y_d(x)\|}\right)\frac{dy_d(x)\mathcal{X}(x, y)}{\|y_d(x)\|} && \text{(122b)} \\ &\quad + \gamma\Pi(y)E(x, y)^T\nabla V(x) && \text{(122c)} \end{aligned}$$

where

- (122a) is a feedback term, which guarantees that y gets closer to $y_d(x)$;
- (122b) is a feedforward term, which guarantees that y tracks $y_d(x)$ as it changes;
- (122c) is a backstepping term, which guarantees that the combined motion of (x, y) gets attracted to the set $\{(x, y) \in \mathbb{X} \times \mathbb{S}^2 : x \in \mathbb{X}_d$ and $y = y_d(x)\}$.

The first and second comments above are obvious, if one defines

$$\begin{aligned} &V_2 : \mathbb{X} \times \mathbb{S}^2 \to [0, \infty) \\ &V_2(x, y) := \frac{1}{2}\left\|y - \frac{y_d(x)}{\|y_d(x)\|}\right\|^2 = 1 - \left\langle y, \frac{y_d(x)}{\|y_d(x)\|}\right\rangle, \end{aligned}$$

for which it follows that (fact (i) $\mathcal{S}(a)b = -\mathcal{S}(b)a$ for $a, b \in \mathbb{R}^3$; fact (ii) $\Pi(a) = -\mathcal{S}(a)\mathcal{S}(a)$ for $a \in \mathbb{S}^2$)

$$\begin{aligned} \dot{V}_2(x, y) &:= d_1V_2(x, y)\mathcal{X}(x, y) + d_2V_2(x, y)\mathcal{Y}(x, y, u)|_{u=u^{cl}(x,y)} \\ &= -\left\langle \mathcal{S}(u)y, \frac{y_d(x)}{\|y_d(x)\|}\right\rangle - \left\langle y, \Pi\left(\frac{y_d(x)}{\|y_d(x)\|}\right)\frac{d_1y_d(x) + d_2y_d(x)\mathcal{X}(x, y)}{\|y_d(x)\|}\right\rangle|_{u=u^{cl}(x,y)} \\ &= \left\langle \mathcal{S}\left(\frac{y_d(x)}{\|y_d(x)\|}\right)y, u\right\rangle - \left\langle \mathcal{S}\left(\frac{y_d(x)}{\|y_d(x)\|}\right)y, \mathcal{S}\left(\frac{y_d(x)}{\|y_d(x)\|}\right)\frac{d_1y_d(x) + d_2y_d(x)\mathcal{X}(x, y)}{\|y_d(x)\|}\right\rangle|_{u=u^{cl}(x,y)} \because \text{(i) and (ii)} \\ &= -k\left\|\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}\right\|^2 + \gamma\left\langle \mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}, \Pi(y)E(x, y)^T\nabla V(x)\right\rangle \qquad \because (122) \\ &= -k\left\|\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}\right\|^2 - \gamma dV(x)E(x, y)\mathcal{S}(y)\frac{y_d(x)}{\|y_d(x)\|}. \end{aligned} \tag{123}$$

Theorem 77: Consider the system (120), composed with the control law $u = u^{cl}(x, y)$ in (122). Then the set $\{(x, y) \in \mathbb{X} \times \mathbb{S}^2 : x \in \mathbb{X}_d$ and $y = y_d(x)\}$ is locally attractive.

Proof: It sufficies to consider the Lyapunov function

$$\begin{aligned} &V_3 : \mathbb{X} \times \mathbb{S}^2 \to [0, \infty) \\ &V_3(x, y) := \gamma V(x) + V_2(x, y), \end{aligned}$$

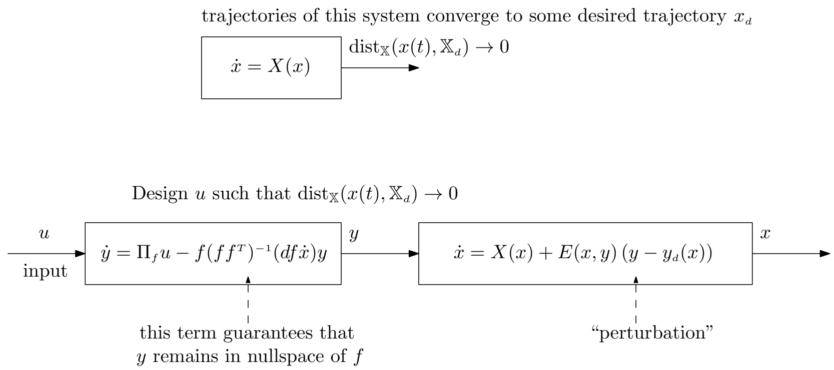

Fig. 13: Cascaded control with angular acceleration

whose derivative along the vector field in (120), composed with the control law $u = u^{cl}(x, y)$ in (122), is given by

$$\begin{aligned}\dot{V}_3(x, y) &= d_1 V_3(x, y)\mathcal{X}(x, y) + d_2 V_3(x, y)\mathcal{Y}(x, y, u)|_{u=u^{cl}(x,y)} \\ &= \gamma\left(W(x) + dV(x)E(x, y)\mathcal{S}\,(y)\,\frac{y_d(x)}{\|y_d(x)\|}\right) + \dot{V}_2(x, y) \quad \because (121) \\ &= \underbrace{\gamma W(x)}_{\leq 0} \underbrace{-k\left\|\mathcal{S}\,(y)\,\frac{y_d(x)}{\|y_d(x)\|}\right\|^2}_{\leq 0}. \quad \because (123)\end{aligned}$$

■

XIV. Cascaded control with angular acceleration

Consider the state space ($\mathcal{N}(A)$ denotes the null-space of some matrix $A \in \mathbb{R}^{p\times m}$)

$$\{(x, y) \in \mathbb{X} \times \mathbb{R}^m : f(x)y = 0_p\} = \bigcup_{x\in\mathbb{X}} \{x\} \times \mathcal{N}(f(x)), \tag{124}$$

where x lies in some manifold $\mathbb{X}$, and $y \in \mathbb{R}^m$ is constrained to be in the null-space of $f(x) \in \mathbb{R}^{p\times m}$ for each $x \in \mathbb{X}$ (and where $f(x)$ is assumed full rank for every $x \in \mathbb{X}$).

Consider then following system, evolving in the manifold in (124),

$$\dot{x} = \mathcal{X}(x, y) := X(x) + E(x, y)(y - y_d(x)) \tag{125a}$$

$$\dot{y} = \mathcal{Y}(x, y, u) := \Pi_{f(x)}u - f(x)^T(f(x)f(x)^T)^{-1}(df(x)\mathcal{X}(x, y))y, \tag{125b}$$

as illustrated in Fig. 13, and where

- $X : \mathbb{X} \ni x \mapsto X(x) \in T_x\mathbb{X}$ is a vector field;
- $V : \mathbb{X} \ni x \mapsto V(x) \in [0, \infty)$ is a Lyapunov function
  - such that $dV(x)X(x) =: W(x) \leq 0$ for all $x \in \mathbb{X}$;
  - such that $V(x_d) = 0$ and $W(x_d) = 0$ for all $x_d \in \mathbb{X}_d$, for some $\mathbb{X}_d \subset \mathbb{X}$ ($\mathbb{X}_d$ is an equilibria set);
  - such that $\{x \in \mathbb{X} : V(x) \leq \epsilon$, for a positive ϵ, defines a neighborhood of $\mathbb{X}_d$ (stability of $\mathbb{X}_d$);
  - such that it guarantees that $\lim_{t\to\infty} d_{\mathbb{X}}(x(t), \mathbb{X}_d) = 0$ for solutions of $\dot{x}(t) = X(x(t))$ with $x(0) \in \mathbb{X}_0$ for some $\mathbb{X}_0 \subset \mathbb{X}$ (attractivity of $\mathbb{X}_d$);
  - there is some positive constant $\bar{V}$ such that the set $\mathbb{X}_0$ can be underestimated as $\mathbb{X}_0 \supseteq \{x \in \mathbb{X} : V(x) < \bar{V}\}$.
- $y_d : \mathbb{X} \ni x \mapsto y_d(x) \in \mathcal{N}(f(x))$ is the desired solution for y (since, if $y \equiv y_d(x)$ then $\dot{x} = X(x)$).
- $E(x, y)$ is a linear map from $\mathcal{N}(f(x)) \subset \mathbb{R}^m$ to $T_x\mathbb{X}$;
- $\Pi_f = I_n - f^T(ff^T)^{-1}f$ is a projection matrix into the null-space of $f \in \mathbb{R}^{p\times n}$;
- $u \in \mathbb{R}^m$ is the input to the system.

The idea is to design a control law for the input u such that $\lim_{t\to\infty} d_{\mathbb{X}}(x(t), \mathbb{X}_d) = 0$ for solutions of (125). Because the dynamics of x in (125) come disturbed by some additive error $y - y_d(x)$, it follows that along solutions of (125)

$$\dot{V}(x) = W(x) + dV(x)E(x, y)(y - y_d(x)). \tag{126}$$

Let then the following tuple be given, $\mathcal{T} = (\mathcal{X}, E, y_d, f, V, \gamma, k)$, where

- $\mathcal{X}, E, y_d, f, V$ are the maps described above;
- $k > 0$ is a positive feedback gain;
- and $\gamma > 0$ is a positive backstepping gain.

Consider then the following control law

$$
\begin{aligned}
&u^{cl} : \bigcup_{\bar{x}\in\mathbb{X}} \{\bar{x}\} \times \mathcal{N}(f(\bar{x})) \ni (x, y) \mapsto u^{cl}(x, y) \in \mathcal{N}(f(x)) \subset \mathbb{R}^m \\
&u^{cl}(x, y) := -k(y - y_d(x)) && \text{(127a)}\\
&\qquad\qquad + \Pi_{f(x)} dy_d(x)\mathcal{X}(x, y) && \text{(127b)}\\
&\qquad\qquad + \gamma\Pi_{f(x)} E(x, y)^T \nabla V(x) && \text{(127c)}
\end{aligned}
$$

where

- (127a) is a feedback term, which guarantees that y gets closer to $y_d(x)$;
- (127b) is a feedforward term, which guarantees that y tracks $y_d(x)$ as it changes;
- (127c) is a backstepping term, which guarantees that the combined motion of (x, y) gets attracted to the set $\{(x, y) \in \mathbb{X} \times \mathbb{S}^2 : x \in \mathbb{X}_d \text{ and } y = y_d(x)\}$.

The first and second comments above are obvious, if one defines

$$
\begin{aligned}
&V_2 : \bigcup_{x\in\mathbb{X}} \{x\} \times \mathcal{N}(f(x)) \to [0, \infty)\\
&V_2(x, y) := \frac{1}{2}\|y - y_d(x)\|^2,
\end{aligned}
$$

for which it follows that (fact (i): $y - y_d(x) \in \mathcal{N}(f(x)) \Rightarrow \langle y - y_d(x), \cdot\rangle = \langle y - y_d(x), \Pi_{f(x)}\cdot\rangle$)

$$
\begin{aligned}
\dot{V}_2(x, y) &:= d_1V_2(x, y)\mathcal{X}(x, y) + d_2V_2(x, y)\mathcal{Y}(x, y, u)|_{u=u^{cl}(x,y)}\\
&= \langle y - y_d(x), \mathcal{Y}(x, y, u) - dy_d(x)\mathcal{X}(x, y)\rangle|_{u=u^{cl}(x,y)}\\
&= \langle y - y_d(x), \mathcal{Y}(x, y, u) - \Pi_{f(x)}dy_d(x)\mathcal{X}(x, y)\rangle|_{u=u^{cl}(x,y)} && \because \text{(i)}\\
&= \langle y - y_d(x), u - \Pi_{f(x)}dy_d(x)\mathcal{X}(x, y)\rangle|_{u=u^{cl}(x,y)} && \because \text{(125) and (i)}\\
&= -k\|y - y_d(x)\|^2 - \gamma\langle y - y_d(x), \Pi_{f(x)}E(x, y)^T\nabla V(x)\rangle && \because \text{(127)}\\
&= -k\|y - y_d(x)\|^2 - \gamma dV(x)E(x, y)(y - y_d(x)). && \because \text{(i)} \qquad \text{(128)}
\end{aligned}
$$

Theorem 78: Consider the system (125), composed with the control law $u = u^{cl}(x, y)$ in (127). Then the set $\{(x, y) \in \mathbb{X} \times \mathbb{S}^2 : x \in \mathbb{X}_d \text{ and } y = y_d(x)\}$ is locally attractive.

Proof: It sufficies to consider the Lyapunov function

$$
\begin{aligned}
&V_3 : \bigcup_{x\in\mathbb{X}} \{x\} \times \mathcal{N}(f(x)) \to [0, \infty)\\
&V_3(x, y) := \gamma V(x) + V_2(x, y),
\end{aligned}
$$

whose derivative along the vector field in (125), composed with the control law $u = u^{cl}(x, y)$ in (127), is given by

$$
\begin{aligned}
\dot{V}_3(x, y) &= d_1V_3(x, y)\mathcal{X}(x, y) + d_2V_3(x, y)\mathcal{Y}(x, y, u)|_{u=u^{cl}(x,y)}\\
&= \gamma\left(W(x) + dV(x)E(x, y)(y - y_d(x))\right) + \dot{V}_2(x, y) && \because \text{(126)}\\
&= \underbrace{\gamma W(x)}_{\leq 0} \underbrace{-k\left\|y - y_d(x)\right\|^2}_{\leq 0}. && \because \text{(128)}
\end{aligned}
$$

■

Remark 79: The manifold in (124) is invariant with respect to the dynamics in (125). Indeed, by assumption $\mathcal{X}(x, y) \in T_x\mathbb{X}$ which guarantees that x remains in $\mathbb{X}$; on the other hand, y remains in the null-space of $f(x)$ because

$$
\begin{aligned}
f(x)y = 0_p &\Rightarrow f(x)\dot{y} + (df(x)\dot{x})y = 0_p\\
&\Leftrightarrow f(x)\mathcal{Y}(x, y, u) + (df(x)\mathcal{X}(x, y))y = 0_p && \because \text{(125)}\\
&\Leftrightarrow f(x)\big(\Pi_{f(x)}u - f(x)^T(f(x)f(x)^T)^{-1}(df(x)\mathcal{X}(x, y))y\big) + (df(x)\mathcal{X}(x, y))y = 0_p && \because \text{(125)}\\
&\Leftrightarrow f(x)\Pi_{f(x)}u = 0_p\\
&\Leftrightarrow 0_p = 0_p.
\end{aligned}
$$