Slung Load Transportation with a Single Aerial Vehicle and Disturbance Removal

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***Abstract*— We present a trajectory tracking controller for a quadrotor-load system, composed of a single load and a single unmanned aerial vehicle connected by a cable or rope. The load is modeled as a point mass while the aerial vehicle is assumed to be fully actuated, with thrust and attitude of the quadrotor as inputs to the system quadrotor-load. We assume there is a constant input disturbance at the thrust input, and a disturbance estimator is presented that guarantees that asymptotic tracking is guaranteed in the presence of such a disturbance. The load and the aerial vehicle are connected by a cable of fixed length that behaves as a rigid link under tensile forces, and as a non-rigid link when under compressive forces. The proposed controller guarantees that the cable is always under tensile forces, provided that the position trajectory to be tracked satisfies some mild conditions. The system quadrotor-load can be transformed into a form that resembles that of systems describing underactuated aerial vehicles, and for which a variety of control strategies have been proposed. In particular, we propose a controller based on a backstepping procedure in conjunction with a bounded double integrator controller. We present simulations validating the proposed control algorithm, and some preliminary experimental results are also presented.**

I. Introduction

Control of under-actuated systems is an active topic of research with many practical applications. Slung load transportation by aerial vehicles forms a class of underactuated systems for which trajectory tracking control strategies are necessary [1]. The dynamics of an n-dimensional generalized coordinate of an under-actuated system cannot be reduced to those of n decoupled double integrator systems, which poses specific challenges in the control design.

Quadrotors are a class of UAV's (Unmanned Aerial Vehicles) whose popularity stems from their ability to fly in relatively small spaces, their reduced mechanical complexity, and the fact that they are composed of inexpensive components. Apart from research on stabilization and on trajectory tracking of quadrotors [2], [3], [4], there is also research on using quadrotors to perform specific tasks [5], [6], [7], [8], [9], [10], [11], [12]; coverage, vision-based navigation, interaction with the environment by means of a mobile manipulator and construction of three-dimensional structures are among examples of such tasks.

Slung load transportation by a UAV is a challenging control problem, since the load sways with respect to the UAV and therefore it is desirable to reduce the relative motion between the UAV and the load. Different control strategies have been proposed for slung loads attached to one or several UAV's by cables. In [13], a flatness based controller for trajectory tracking with a pendulum load is presented. [14] uses an adaptive controller to compensate for a changing center of gravity, and open-loop dynamic programming is used to plan a trajectory that minimizes oscillations of the load. A closed loop approach, where the motion of the load is tracked visually from the transporting helicopter, is found in [15], where the information from the visual tracking system is used to determine the frequency of the load sway and thereby the length of the rope; this information is used in the real-time design of an input shaper for the feed-forward part of the control law and the design of a "delayed resonator"-controller for damping the oscillations. In [16], several quadrotors lift a single load, and the relations in static equilibrium between three quadrotors and a load are analyzed. It also discusses how to find positions for the quadrotors, given a desired position and orientation for the load; and, conversely, how to find the position and orientation of the load, for known positions of the quadrotors, such that the load is stable in the sense that it has minimal potential energy. In [17], differential flatness of the quadrotor-load system is explored for the purposes of trajectory planning.

Many controller strategies for position trajectory tracking of quadrotors have been proposed, and a comprehensive description of control laws is found in [18]. Notice that the system quadrotor-load dynamics change according to whether the cable connecting the load and the quadrotor is under tension or compression. In particular, in this manuscript we describe and impose conditions on the desired trajectory that guarantee that the cable remains under tension. Also, in this manuscript, the system quadrotor-load is shown to be similar to that of an under-actuated aerial vehicle such as a quadrotor, when the cable in under tension; this means the previous control strategies are applicable under some appropriate changes, and provided that it is guaranteed that the cable remains under tension, as is done in this manuscript. Another contribution concerns a disturbance estimator that guarantees that the load tracks a desired trajectory under a constant thrust input disturbance.

The remainder of this paper is structured as follows. In Section III, we model the quadrotor-load system. In Section IV, we provide a coordinate transformation and, in Sections V and VI, a control solution for the transformed system is presented. Finally, in Section VII, we present illustrative simulations and preliminary experimental results.

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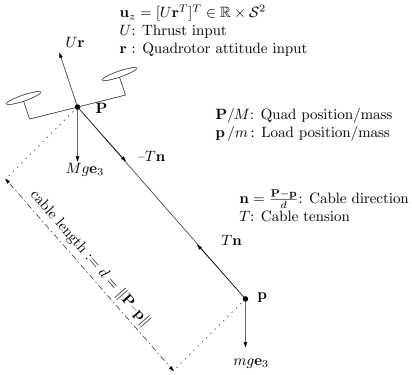


Fig. 1: Modeling of quadrotor-load system

II. Notation

Given two sets $\Omega_1 \in \mathbb{R}^n$ and $\Omega_2 \in \mathbb{R}^m$, and a function $\mathbf{f} : \Omega_1 \mapsto \Omega_2$, we denote $\mathbf{f} \in \mathcal{C}^n(\Omega_1, \Omega_2)$ if all its partial derivatives, up to order n inclusive, are continuous in Ω_1. Moreover, if $\Omega_1 = \mathbb{R}_{\geq 0}$, we denote $\mathbf{f} \in \mathcal{C}^n(\Omega_2)$ for brevity. Consider a system with state $\mathbf{x} \in \mathcal{C}(\Omega_\mathbf{x} \subseteq \mathbb{R}^n)$, a control input $\mathbf{u}_\mathbf{x} \in \mathcal{C}(\Omega^\mathbf{u}_\mathbf{x} \subseteq \mathbb{R}^m)$, and denote $\Delta_\mathbf{x} \subseteq \mathbb{R}_{\geq 0} \times \Omega_\mathbf{x}$. We denote $\mathbf{f}_\mathbf{x} \in \mathcal{C}(\Delta_\mathbf{x} \times \Omega^\mathbf{u}_\mathbf{x}, \mathbb{R}^n)$ as the open loop vector field, i.e., given $\mathbf{u}_\mathbf{x} \in \mathcal{C}(\Omega^\mathbf{u}_\mathbf{x})$, $\dot{\mathbf{x}}(t) = \mathbf{f}_\mathbf{x}(t, \mathbf{x}(t), \mathbf{u}_\mathbf{x}(t))$. Given a control law $\mathbf{u}^{cl}_\mathbf{x} \in \mathcal{C}(\Delta_\mathbf{x}, \Omega^\mathbf{u}_\mathbf{x})$, we denote $\mathbf{f}^{cl}_\mathbf{x}(t, \mathbf{x}) := \mathbf{f}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}^{cl}_\mathbf{x}(t, \mathbf{x}))$ as the closed loop vector field. Moreover, given $V_\mathbf{x} \in \mathcal{C}^1(\Delta_\mathbf{x}, \mathbb{R}_{\geq 0})$, we denote $\tilde{W}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x}) := -\frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial t} - \frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial \mathbf{x}}^T \mathbf{f}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x})$ and $W_\mathbf{x}(t, \mathbf{x}) := \tilde{W}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}^{cl}_\mathbf{x}(t, \mathbf{x})) = -\frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial t} - \frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial \mathbf{x}}^T \mathbf{f}^{cl}_\mathbf{x}(t, \mathbf{x})$. Finally, we say $\mathbf{x}^\star \in \mathcal{C}^1(\Omega_\mathbf{x})$ is an equilibrium trajectory of $\dot{\mathbf{x}}(t) = \mathbf{f}^{cl}_\mathbf{x}(t, \mathbf{x}(t))$ if $\dot{\mathbf{x}}^\star(t) = \mathbf{f}^{cl}_\mathbf{x}(t, \mathbf{x}^\star(t))$ and if $\mathbf{0}$ is an equilibrium of $\dot{\mathbf{e}}(t) = \mathbf{f}^{cl}_\mathbf{e}(t, \mathbf{e}(t))$, where $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^\star(t)$ and $\mathbf{f}^{cl}_\mathbf{e}(t, \mathbf{e}) := \mathbf{f}^{cl}_\mathbf{x}(t, \mathbf{e} + \mathbf{x}^\star(t)) - \dot{\mathbf{x}}^\star(t)$. Given an equilibrium trajectory $\mathbf{x}^\star(t)$ we denote $\mathbf{u}^{cl,\star}_\mathbf{x}(t) := \mathbf{u}^{cl}_\mathbf{x}(t, \mathbf{x}^\star(t))$ as the equilibrium control input.

III. Modeling

Consider a quadrotor vehicle and a point mass load attached to each other by a cable, as illustrated in Fig. 1. Also, the cable end-points coincide with the quadrotor's and the load's center of mass. When the cable is not under tension, the load behaves as a free falling (un-actuated) point mass, while the quadrotor vehicle behaves as a *standard* quadrotor. On the other hand, when the cable is under tension, it imposes a kinematic constraint: specifically, it enforces the distance between the quadrotor and the load to be identical to the cable length, and for as long as the cable remains under tension. This kinematic constraint *links* the quadrotor and the load, and the load is no longer un-actuated. It thus follows that the system quadrotor-load can be modeled as a hybrid system, with its (open-loop) vector field switching according to some function of the state and the input. Such modeling is performed in [17], where differential flatness of the system with respect to the load's position is verified, and exploited so as to plan a trajectory for the quadrotor vehicle. In this manuscript, the focus is on providing a closed loop control law that guarantees that the load tracks a desired position trajectory, while guaranteeing that the cable is always under tension. If the later is satisfied, the (open-loop) vector field never switches, and, in fact, the cable behaves as a rigid link connecting the quadrotor's and the load's center of mass.

We denote by $\mathbf{P} \in \mathcal{C}(\mathbb{R}^3)$ and by $\mathbf{p} \in \mathcal{C}(\mathbb{R}^3)$ the quadrotor's and the load's center of mass positions, respectively; by $\mathbf{V} \in \mathcal{C}(\mathbb{R}^3)$ and by $\mathbf{v} \in \mathcal{C}(\mathbb{R}^3)$ the quadrotor's and the load's center of mass velocity vectors; and by $M > 0$ and by $m > 0$ the quadrotor's and load's masses, respectively. We denote by $\mathbf{n} \in \mathcal{C}(\mathcal{S}^2)$ the cable's unit vector, pointing from the load to the quadrotor; and by $T \in \mathcal{C}(\mathbb{R}_{\geq 0})$ the tension on the cable; and by $d > 0$ the cable length. Finally, we denote by $U \in \mathcal{C}(\mathbb{R})$ the quadrotor's thrust and by $\mathcal{R} \in \mathcal{C}(\mathcal{SO}(3))$ the quadrotor's rotation matrix, where $\mathbf{r} := \mathcal{R}\mathbf{e}_3 \in \mathcal{C}(\mathcal{S}^2)$ is the quadrotor's direction where input thrust is provided. We assume $U \in \mathcal{C}(\mathbb{R})$ and $\mathbf{r} \in \mathcal{C}(\mathcal{S}^2)$ are inputs to the system quadrotor-load. Let us then denote $[\mathbf{p}^T\, \mathbf{v}^T\, \mathbf{P}^T\, \mathbf{V}^T] =: \mathbf{z} \in \mathcal{C}(\Omega_\mathbf{z})$ as the state of the quadrotor-load system, where $\Omega_\mathbf{z} = \{\mathbf{z} \in \mathbb{R}^{12} : \|\mathbf{P} - \mathbf{p}\| = d, (\mathbf{P} - \mathbf{p})^T(\mathbf{V} - \mathbf{v}) = 0\}$; and $\mathbf{u}_\mathbf{z} := [U\, \mathbf{r}^T]^T \in \mathcal{C}(\mathbb{R} \times \mathcal{S}^2 =: \Omega^\mathbf{u}_\mathbf{z})$ as the control input. The state $\mathbf{z}(\cdot)$ evolves according to the dynamics

$$\dot{\mathbf{z}}(t) = \mathbf{f}_\mathbf{z}(\mathbf{z}(t), \mathbf{u}_\mathbf{z}(t)), \tag{1}$$

where

$$\mathbf{f}_\mathbf{z}(\mathbf{z}, \mathbf{u}_\mathbf{z}) = \begin{bmatrix} \mathbf{v} \\ \frac{\bar{T}(\mathbf{z}, \mathbf{u}_\mathbf{z} + b\mathbf{e}_1)}{m} \bar{\mathbf{n}}(\mathbf{z}) - g\mathbf{e}_3 \\ \mathbf{V} \\ \frac{U+b}{M}\mathbf{r} - \frac{\bar{T}(\mathbf{z}, \mathbf{u}_\mathbf{z} + b\mathbf{e}_1)}{M} \bar{\mathbf{n}}(\mathbf{z}) - g\mathbf{e}_3 \end{bmatrix} \tag{2}$$

where g is the acceleration due to gravity, and where $\bar{\mathbf{n}} : \Omega_\mathbf{z} \mapsto \mathcal{S}^2$ and $\bar{T} : \Omega_\mathbf{z} \times \Omega^\mathbf{u}_\mathbf{z} \mapsto \mathbb{R}$ are defined as

$$\begin{aligned} \bar{\mathbf{n}}(\mathbf{z}) &= \frac{\mathbf{P} - \mathbf{p}}{\|\mathbf{P} - \mathbf{p}\|} = \frac{\mathbf{P} - \mathbf{p}}{d} \\ \bar{T}(\mathbf{z}, \mathbf{u}_\mathbf{z}) &= \frac{m}{M + m} \left(U\mathbf{r}^T \bar{\mathbf{n}}(\mathbf{z}) + Md\|\mathbf{V} - \mathbf{v}\|^2 \right), \end{aligned} \tag{3}$$

and where $b \in \mathbb{R}$ is a constant unknown disturbance acting on the thrust input (and $\mathbf{e}_1 = [1\,0\,0\,0]^T$). Physically, the functions $\bar{\mathbf{n}}(\cdot)$ and $\bar{T}(\cdot, \cdot)$ are related to the cable's unit vector and to the tension on the cable, respectively; i.e., given $\mathbf{u}_z \in \mathcal{C}(\Omega^\mathbf{u}_\mathbf{z})$ and along a solution $\mathbf{z}(\cdot)$ of (1), it follows that $\mathbf{n}(t) = \bar{\mathbf{n}}(\mathbf{z}(t))$ and that $T(t) = \bar{T}(\mathbf{z}(t), \mathbf{u}_z(t) + b\mathbf{e}_1)$, for all positive time instants t when the tension is positive, i.e., $T(t) > 0$. Let us provide some insight on how the vector field (2) can be derived. One alternative is by means of the Euler-Lagrange formalism, as done in [17]. Another alternative is to explore Newton's second law, based on the net forces applied on each point mass (see Fig. 1), which yields the equations in (2), except for $\bar{\mathbf{n}}(\cdot)$ and $\bar{T}(\cdot, \cdot)$; the later are obtained by differentiating the constraint $\|\mathbf{P}(t) - \mathbf{p}(t)\| = d\ \forall t \geq 0$ once and twice with respect to time.

Problem 1: Given the system (1) and a desired trajectory $\mathbf{p}^\star \in \mathcal{C}^4(\mathbb{R}_{\geq 0})$, design $U : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R}_{\geq 0} \mapsto \mathcal{S}^2$ such that $\lim_{t \to \infty}(\mathbf{p}(t) - \mathbf{p}^\star(t)) = \mathbf{0}$.

Notice that if $U : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R}_{\geq 0} \mapsto \mathcal{S}^2$ are control inputs then the quadrotor system itself is fully-actuated; however, for the same inputs, the system quadrotor-load is under-actuated.

IV. Change of Coordinates

In this section, we provide a diffeomorphism $\boldsymbol{\phi}_{\mathbf{x}} : \mathbb{R}_{\geq 0} \times \Omega_{\mathbf{z}} \mapsto \Omega_{\mathbf{x}}$ from the state $\mathbf{z}$, described in the previous section to a transformed state $\mathbf{x}$ for all positive time instants, i.e., $\mathbf{x} = \boldsymbol{\phi}_{\mathbf{x}}(t, \mathbf{z}) \Leftrightarrow \mathbf{z} = \boldsymbol{\phi}_{\mathbf{x}}^{-1}(t, \mathbf{x}) \, \forall t \geq 0$. Intuitively, $\boldsymbol{\phi}_{\mathbf{x}}(t, \cdot)$ corresponds to a change of coordinates for each time instant $t \geq 0$, and, for convenience, we denote $\boldsymbol{\phi}_{\mathbf{z}}(\cdot, \cdot) := \boldsymbol{\phi}_{\mathbf{x}}^{-1}(\cdot, \cdot) \Leftrightarrow \boldsymbol{\phi}_{\mathbf{x}}(\cdot, \cdot) := \boldsymbol{\phi}_{\mathbf{z}}^{-1}(\cdot, \cdot)$ Moreover, we also provide a change of inputs $\mathbf{u}_{\mathbf{z}} = \boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})$, whose motivation is provided later. The transformed state trajectory $\mathbf{x} \in \mathcal{C}^1(\mathbb{R}_{\geq 0})$ evolves according to the dynamics $\dot{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}_{\mathbf{x}}(t))$, where

$$\mathbf{f}_{\mathbf{x}}(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}) := \left(\frac{\partial \boldsymbol{\phi}_{\mathbf{x}}(t, \mathbf{z})}{\partial t} + \frac{\partial \boldsymbol{\phi}_{\mathbf{x}}(t, \mathbf{z})}{\partial \mathbf{z}} \mathbf{f}_{\mathbf{z}}(\mathbf{z}, \mathbf{u}_{\mathbf{z}}) \right) \Big|_{\substack{\mathbf{z} = \boldsymbol{\phi}_{\mathbf{z}}(t, \mathbf{x}) \\ \mathbf{u}_{\mathbf{z}} = \boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}}, \quad (4)$$

and where the (open-loop) vector field $\mathbf{f}_{\mathbf{z}}(\cdot, \cdot)$ is that in (2). The motivation for designing $\boldsymbol{\phi}_{\mathbf{x}}(\cdot, \cdot)$ and $\boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\cdot, \cdot)$ is that $\mathbf{f}_{\mathbf{x}}(\cdot, \cdot, \cdot)$, in (4), is in a form for which controllers are found in the literature. Specifically, for the proposed mappings, the (open-loop) vector field $\mathbf{f}_{\mathbf{x}}(\cdot, \cdot, \cdot)$ is similar to the (open-loop) vector field of a quadrotor.

With the help of Proposition 4 in the Appendix, it follows that $\dot{\mathbf{n}}(t) = \mathcal{S}(\boldsymbol{\omega}(t)) \mathbf{n}(t)$, and that $\dot{\boldsymbol{\omega}}(t) = \mathcal{S}(\mathbf{n}(t)) \boldsymbol{\tau}(t)$, where $\boldsymbol{\omega}(t) = \bar{\boldsymbol{\omega}}(\mathbf{z}(t))$ and $\boldsymbol{\tau}(t) = \frac{1}{d} \Pi(\mathbf{n}(t)) (\ddot{\mathbf{P}}(t) - \ddot{\mathbf{p}}(t)) = \bar{\boldsymbol{\tau}}(\mathbf{z}(t), \mathbf{u}_{\mathbf{z}}(t) + b\mathbf{e}_1)$ with $\bar{\boldsymbol{\omega}} \in \mathcal{C}^{\infty}(\Omega_{\mathbf{z}}, \mathbb{R}^3)$ and $\bar{\boldsymbol{\tau}} \in \mathcal{C}^{\infty}(\Omega_{\mathbf{z}} \times \Omega_{\mathbf{z}}^{\mathbf{u}}, \mathbb{R}^3)$ defined as

$$\bar{\boldsymbol{\omega}}(\mathbf{z}) = \mathcal{S}(\bar{\mathbf{n}}(\mathbf{z})) \frac{\mathbf{V} - \mathbf{v}}{d}, \bar{\boldsymbol{\tau}}(\mathbf{z}, \mathbf{u}_{\mathbf{z}}) = \frac{1}{Md} \Pi(\bar{\mathbf{n}}(\mathbf{z})) U\mathbf{r}. \quad (5)$$

Note that $U\mathbf{r}^T \bar{\mathbf{n}}(\mathbf{z})$ acts on the tension function in (3), while $\Pi(\bar{\mathbf{n}}(\mathbf{z}))U\mathbf{r}$ acts on the angular acceleration (torque) of the cable's unit vector in (5). This insight suggests a path for designing the control law, namely, $U\mathbf{r}^T \bar{\mathbf{n}}(\mathbf{z})$ is designed so as to control the cable tension, and guaranteeing it remains positive; while $\Pi(\bar{\mathbf{n}}(\mathbf{z}))U\mathbf{r}$ is designed so as to control the cable's unit vector.

For convenience, let us define the transformed state as

$$\mathbf{x} := [\mathbf{e}^T \; \boldsymbol{\upsilon}^T \; \mathbf{n}^T \; \boldsymbol{\omega}^T]^T \in \Omega_{\mathbf{x}}, \quad (6)$$

where $\Omega_{\mathbf{x}} = \{\mathbf{x} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}^2 \times \mathbb{R}^3 : \mathbf{n}^T \boldsymbol{\omega} = 0\}$, and with a physical interpretation for the state components that we provide later. We can now define the coordinate transformation mappings, namely

$$\boldsymbol{\phi}_{\mathbf{x}}(t, \mathbf{z}) = \begin{bmatrix} \mathbf{p} - \mathbf{p}^\star(t) \\ \mathbf{v} - \dot{\mathbf{p}}^\star(t) \\ \bar{\mathbf{n}}(\mathbf{z}) \\ \bar{\boldsymbol{\omega}}(\mathbf{z}) \end{bmatrix}, \boldsymbol{\phi}_{\mathbf{z}}(t, \mathbf{x}) = \begin{bmatrix} \mathbf{e} + \mathbf{p}^\star(t) \\ \boldsymbol{\upsilon} + \dot{\mathbf{p}}^\star(t) \\ L\mathbf{n} + \mathbf{e} + \mathbf{p}^\star(t) \\ L\mathcal{S}(\boldsymbol{\omega}) \mathbf{n} + \boldsymbol{\upsilon} + \dot{\mathbf{p}}^\star(t) \end{bmatrix}, \quad (7)$$

where we emphasize that $\mathbf{x} = \boldsymbol{\phi}_{\mathbf{x}}(\cdot, \boldsymbol{\phi}_{\mathbf{z}}(\cdot, \mathbf{x})) \forall \mathbf{x} \in \Omega_{\mathbf{x}}$ and that $\mathbf{z} = \boldsymbol{\phi}_{\mathbf{z}}(\cdot, \boldsymbol{\phi}_{\mathbf{x}}(\cdot, \mathbf{z})) \forall \mathbf{z} \in \Omega_{\mathbf{z}}$. Let us now provide a physical interpretation for the previous variables. Along a solution $\mathbf{z}(\cdot)$ of (1), $\mathbf{e}(\cdot) = \mathbf{p}(\cdot) - \mathbf{p}^\star(\cdot)$ and it corresponds to the position tracking error (as such, the goal of Problem 1 can be restated as $\lim_{t \to \infty} \mathbf{e}(t) = \mathbf{0}$); $\boldsymbol{\upsilon}(\cdot) = \mathbf{v}(\cdot) - \dot{\mathbf{p}}^\star(\cdot)$ corresponds to the velocity tracking error; $\mathbf{n}(\cdot) = \bar{\mathbf{n}}(\mathbf{z}(\cdot))$ corresponds to the unit vector associated to the cable direction, as illustrated in Fig. 1; and $\boldsymbol{\omega}(\cdot) = \bar{\boldsymbol{\omega}}(\mathbf{z}(\cdot))$ corresponds to the angular velocity of $\mathbf{n}(\cdot)$.

For convenience, denote $\mathbf{u}_{\mathbf{x}}^T := [T \, \boldsymbol{\tau}^T] \in \mathbb{R}^4$, where $T \in \mathbb{R}$ stands for the tension in the cable, and $\boldsymbol{\tau} \in \mathbb{R}^3$ stands for the torque input to control the cable direction. Additionally, denote $\mathbf{U}_{\mathbf{x}} \in \mathcal{C}^{\infty}(\Omega_{\mathbf{x}} \times \mathbb{R}^4, \mathbb{R}^3)$ defined as

$$\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) = \mathbf{n}((M + m)T - md\|\boldsymbol{\omega}\|^2) + Md\Pi(\mathbf{n}) \boldsymbol{\tau}, \quad (8)$$

and $\Gamma = \{(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) \in \Omega_{\mathbf{x}} \times \mathbb{R}^4 : \mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) \neq \mathbf{0}\}$. Let us now provide the control input transformation mapping, namely $\boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}} : \Gamma \mapsto \mathbb{R}_{>0} \times \mathcal{S}^2$ defined as

$$\boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) := \begin{bmatrix} \|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\| & \frac{\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}{\|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\|}^T \end{bmatrix}^T. \quad (9)$$

Recall that the mapping $\boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\cdot, \cdot)$ provides a change of inputs, namely from $\mathbf{u}_{\mathbf{x}}$ to $\mathbf{u}_{\mathbf{z}}$, which will be clear next. Indeed, from the tension in the cable, given in (3), it follows that

$$\bar{T}(\mathbf{z}, \mathbf{u}_{\mathbf{z}} + b\mathbf{e}_1)|_{\substack{\mathbf{z} = \boldsymbol{\phi}_{\mathbf{z}}(t, \mathbf{x}) \\ \mathbf{u}_{\mathbf{z}} = \boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}} \overset{(7),(8),(9)}{=} mT + \frac{m}{M+m} b \frac{\mathbf{n}^T \mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}{\|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\|}, \quad (10)$$

while from the torque in the cable, given in (5), it follows that (with the help of (7), (8) and (9))

$$\bar{\boldsymbol{\tau}}(\mathbf{z}, \mathbf{u}_{\mathbf{z}} + b\mathbf{e}_1)|_{\substack{\mathbf{z} = \boldsymbol{\phi}_{\mathbf{z}}(t, \mathbf{x}) \\ \mathbf{u}_{\mathbf{z}} = \boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}} = \Pi(\mathbf{n}) \boldsymbol{\tau} + \frac{b}{Md} \Pi(\mathbf{n}) \frac{\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}{\|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\|}. \quad (11)$$

An interpretation for $\mathbf{u}_{\mathbf{x}}^T := [T \, \boldsymbol{\tau}^T] \in \mathbb{R}^4$ is now clearer from (10) and (11). Indeed, in the absence of a disturbance, i.e., $b = 0$, T yields the tension in the cable, apart from a positive multiplicative constant, namely the load's mass; while $\boldsymbol{\tau}$ yields the torque in the cable.

We now provide the vector field (4). Given the mappings (7) and (9), and along a solution $\mathbf{z}(\cdot)$ of (1), it follows that $\dot{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}_{\mathbf{x}}(t))$ where (denote $\mathbf{g}(t) := g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)$)

$$\begin{aligned} \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}) &:= \begin{bmatrix} \boldsymbol{\upsilon} \\ T\mathbf{n} - \mathbf{g}(t) \\ \mathcal{S}(\boldsymbol{\omega}) \mathbf{n} \\ \mathcal{S}(\mathbf{n}) \boldsymbol{\tau} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \frac{1}{M+m} \mathbf{n}\mathbf{n}^T \frac{\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}{\|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\|} \\ \mathbf{0} \\ \frac{1}{Md} \Pi(\mathbf{n}) \frac{\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})}{\|\mathbf{U}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})\|} \end{bmatrix} b \\ &:= \mathbf{f}_{\mathbf{x}}^u(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}) + \Phi(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) b \end{aligned} \quad (12)$$

and note that one equilibrium trajectory can be found for this system (there is, however, one more trajectory). In fact, if we denote

$$\mathbf{n}^\star(t) := \frac{g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)}{\|g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)\|}, \boldsymbol{\omega}^\star(t) := \mathcal{S}(\mathbf{n}^\star(t)) \frac{\mathbf{p}^{\star(3)}(t)}{\|g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)\|}, \quad (13)$$

and $\mathbf{x}^\star(t) := [\mathbf{0}_3^T \, \mathbf{0}_3^T \, \mathbf{n}^{\star T}(t) \, \boldsymbol{\omega}^{\star T}(t)]^T$, it follows that $\mathbf{x}^\star(\cdot)$ is an equilibrium trajectory of (12). If $b = 0$, then $T^\star(t) := \|g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)\|$, $\boldsymbol{\tau}^\star(t) := \dot{\boldsymbol{\omega}}^\star(t)$, and $\mathbf{u}_{\mathbf{x}}^\star(t) := [T^\star(t) \, \boldsymbol{\tau}^{\star T}(t)]^T$, where $\mathbf{u}_{\mathbf{x}}^\star(\cdot)$ is the equilibrium control input of (12). Regarding the original control input $\mathbf{u}_{\mathbf{z}}$, it follows that its equilibrium can be found with the help of (9), specifically, $\mathbf{u}_{\mathbf{z}}^\star(t) = \boldsymbol{\phi}_{\mathbf{z}}^{\mathbf{u}}(\mathbf{x}^\star(t), \mathbf{u}_{\mathbf{x}}^\star(t))$.

V. Control of $\mathbf{f}_{\mathbf{x}}^u(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}})$

For convenience, denote $\Delta_{\mathbf{x}} = \mathbb{R}_{\geq 0} \times \Omega_{\mathbf{x}}$ and $\tilde{\Delta}_{\mathbf{x}} \subseteq \Delta_{\mathbf{x}}$. The vector field $\mathbf{f}_{\mathbf{x}}^u \in \mathcal{C}(\Delta_{\mathbf{x}} \times \mathbb{R}^4, \mathbb{R}^{12})$ is that of a quadrotor-like system, and we assume that there exist functions $\mathbf{u}_{\mathbf{x}}^{cl} \in \mathcal{C}(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}^4)$, $V_{\mathbf{x}} \in \mathcal{C}^1(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}_{\geq 0})$ and $W_{\mathbf{x}} \in \mathcal{C}(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}_{\geq 0})$, such that along a solution of $\dot{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}^u(t, \mathbf{x}(t), \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x}(t)))$ it is guaranteed that $\dot{V}_{\mathbf{x}}(t, \mathbf{x}(t)) = -W_{\mathbf{x}}(t, \mathbf{x}(t)) \leq 0$ and that $\lim_{t \to \infty} \mathbf{e}(t) = \mathbf{0}$. Moreover, we also assume that, for $V_0 := V(0, \mathbf{x}(0))$ sufficiently small (i.e., $V_0 \leq \epsilon$ for some $\epsilon > 0$), there exists a strictly increasing function $\alpha \in \mathcal{C}([0\,\epsilon], \mathbb{R}_{\geq 0})$, such that, along a trajectory $\mathbf{x}(\cdot)$ of $\dot{\mathbf{x}}(t) = \mathbf{f}^u_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t))$, it is guaranteed that $\|\mathbf{x}(t) - \mathbf{x}^\star(t)\| \leq \alpha(V_0)\,\forall t \geq 0$.

We emphasize that the final (to be presented) control law is a function of $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$ and $\frac{\partial V_{\mathbf{x}}(\cdot,\mathbf{x})}{\partial \mathbf{x}}$; and, for the purposes of analysis, explicit functions $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$, $V_{\mathbf{x}}(\cdot,\cdot)$, $W_{\mathbf{x}}(\cdot,\cdot)$ and $\alpha(\cdot)$ are not necessary, i.e., it suffices to assume their existence. However, we emphasize that, when implementing the final control law, explicit functions $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$ and $\frac{\partial V_{\mathbf{x}}(\cdot,\mathbf{x})}{\partial \mathbf{x}}$ are necessary, while explicit functions $W_{\mathbf{x}}(\cdot,\cdot)$ and $\alpha(\cdot)$ are not necessary. Possible $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$ and $V_{\mathbf{x}}(\cdot,\cdot)$ are provided in Section V-B.

The control law $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$ must satisfy the following properties. First $\mathbf{e}_1^T \mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot) = T^{cl}(\cdot,\cdot) \geq T_{\min} > 0$. Secondly, for all closed subsets $\Omega'_{\mathbf{x}} \subset \Omega_{\mathbf{x}}$, $\exists \bar{T} \in \mathbb{R}_{>0}$: $\sup_{t\geq 0} \max_{\mathbf{x}\in\Omega'_{\mathbf{x}}} \left\| \frac{\partial T^{cl}(t,\mathbf{x})}{\partial \mathbf{x}} \right\| \leq \bar{T}$ ($\bar{T}$ need not be the same for every $\Omega'_{\mathbf{x}}$ though). This property is important since it guarantees that if $\mathbf{x}^\star(t)$ is in a bounded set $\Omega'_{\mathbf{x}}$ for all $t \geq 0$, and $\mathbf{x}(\cdot)$ remains in an ϵ-neighborhood of $\mathbf{x}^\star(t)$ for all $t \geq 0$, then

$$\|T^{cl}(t, \mathbf{x}(t))) - T^\star(t)\| \leq \max_{\|\mathbf{x}-\mathbf{x}^\star(t)\|\leq\epsilon} \left\| \frac{\partial T^{cl}(t,\mathbf{x})}{\partial \mathbf{x}} \right\| \|\mathbf{x}(t) - \mathbf{x}^\star(t)\|$$
$$\Rightarrow \sup_{t\geq 0} \|T^{cl}(t, \mathbf{x}(t))) - T^\star(t)\| \leq \bar{T}\epsilon. \tag{14}$$

Given a control law $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$, it follows that the control law for the original input can be found with the help of (9),

$$\mathbf{u}^{cl}_{\mathbf{z}}(t, \mathbf{z}) = \boldsymbol{\phi}^{\mathbf{u}}_{\mathbf{z}}(\mathbf{x}, \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}))|_{\mathbf{x}=\boldsymbol{\phi}_{\mathbf{x}}(t,\mathbf{z})}. \tag{15}$$

Definition 1: We say $\mathbf{p}^\star \in \mathcal{C}^4(\mathbb{R}_{\geq 0})$ is a feasible trajectory if *i)* $\sup_{t\geq 0} \|\mathbf{p}^{\star(i)}(t)\| < \infty$ for $i \in \{2,3,4\}$, *ii)* $\sup_{t\geq 0} \mathbf{e}_3^T \mathbf{p}^{\star(2)}(t) > -g$, and *iii)*

$$\inf_{t\geq 0} \frac{M}{M+m} \frac{d\|\mathcal{S}\left(g\mathbf{e}_3 + \mathbf{p}^{\star(2)}(t)\right)\mathbf{p}^{\star(3)}(t)\|^2}{\|g\mathbf{e}_3 + \mathbf{p}^{\star(2)}(t)\|^5} < 1. \tag{16}$$

Remark 1: Notice that $\mathbf{n}^\star(t)\mathbf{U}(\mathbf{x}^\star(t), \mathbf{u}^\star_{\mathbf{x}}(t, \mathbf{x}^\star(t))) = (M+m)\|g\mathbf{e}_3 + \mathbf{p}^{\star(2)}(t)\| \left(1 - \frac{M}{M+m} \frac{d\|\mathcal{S}(\mathbf{n}^\star(t))\mathbf{p}^{\star(3)}(t)\|^2}{\|g\mathbf{e}_3+\mathbf{p}^{\star(2)}(t)\|^3}\right)$. Therefore, if $\mathbf{p}^\star(\cdot)$ is a feasible trajectory, it follows that $\inf_{t\geq 0} \|\mathbf{U}^{cl}_{\mathbf{x}}(t, \mathbf{x}^\star(t))\| > 0$ and therefore, at the equilibrium, (9) is well defined.

Proposition 2: Consider a feasible trajectory $\mathbf{p}^\star(\cdot)$, and a trajectory $\mathbf{x}(\cdot)$ of $\dot{\mathbf{x}}(t) = \mathbf{f}^u_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t)))$. If $\exists \epsilon > 0$: $\|\mathbf{x}(t) - \mathbf{x}^\star(t)\| \leq \epsilon\,\forall t \geq 0$, then $\inf_{t\geq 0} \|\mathbf{U}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t))\| > 0$, along the trajectory $\mathbf{x}(\cdot)$.

Proof: First, notice that since $\mathbf{p}^\star(\cdot)$ is a feasible trajectory, it follows that $\bar{\omega}^\star := \sup_{t\geq 0} \|\boldsymbol{\omega}^\star(t)\| \stackrel{(13)}{=} \sup_{t\geq 0} \|\mathcal{S}\left(\mathbf{n}^\star(t)\right) \frac{\mathbf{p}^{\star(3)}(t)}{\|g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t)\|}\| < \infty$. On the other hand,

$$\|\mathbf{U}(\mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t)))\| \geq |(M+m)T^{cl}(t, \mathbf{x}(t)) - md\|\boldsymbol{\omega}(t)\|^2|$$
$$= |(M+m)(T^{cl}(t, \mathbf{x}(t)) - T^\star(t) + T^\star(t)) - md\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}^\star(t) + \boldsymbol{\omega}^\star(t)\|^2|$$
$$\stackrel{(14)}{\geq} |(M+m)T^\star(t) - md\|\boldsymbol{\omega}^\star(t)\|^2 - \epsilon\left(md\left(\epsilon + 2\bar{\omega}^\star\right) + (M+m)\bar{T}\right)|. \tag{17}$$

Therefore, for ϵ sufficiently small, it follows from (16) and (17) that, along a trajectory $\mathbf{x}(\cdot)$ of $\dot{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t)))$, $\inf_{t\geq 0} \|\mathbf{U}(\mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t)))\| > 0$ and that $\boldsymbol{\phi}^{\mathbf{u}}_{\mathbf{z}}(\mathbf{x}(\cdot), \mathbf{u}^{cl}_{\mathbf{x}}(\cdot, \mathbf{x}(\cdot)))$ is well defined (see (9)). ∎

Proposition 2 is important in guaranteeing that the timed control law $[U(t)\,\mathbf{r}^T(t)] = \mathbf{u}^{cl}_{\mathbf{z}}(t, \mathbf{z}(t))$ is well-defined for all $t \geq 0$; in particular, that $\mathbf{r}(\cdot)$ is always well defined, i.e., that the desired attitude for the quadrotor is always well defined.

Theorem 3: Consider a control law $\mathbf{u}^{cl}_{\mathbf{x}} \in \mathcal{C}(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}^4)$, a feasible trajectory $\mathbf{p}^\star \in \mathcal{C}^4(\mathbb{R}_{\geq 0})$, and a trajectory $\mathbf{z}(\cdot)$ of $\dot{\mathbf{z}}(t) = \mathbf{f}_{\mathbf{x}}(\mathbf{z}(t), \mathbf{u}^{cl}_{\mathbf{z}}(t, \mathbf{z}(t)))$, with $\mathbf{u}^{cl}_{\mathbf{z}}(\cdot,\cdot)$ in (15) and $b = 0$. Then, for $\mathbf{z}(0)$ sufficiently close to $\mathbf{z}^\star(0) = \boldsymbol{\phi}_{\mathbf{z}}(0, \mathbf{x}^\star(0))$, the control law $[U(\cdot)\,\mathbf{r}^T(\cdot)] = \mathbf{u}^{cl}_{\mathbf{z}}(\cdot, \mathbf{z}(\cdot))$ is always well defined and $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$. Moreover, the cable connecting the quadrotor and load is always under tension, i.e., $\inf_{t\geq 0} \bar{T}(\mathbf{z}(t), \mathbf{u}^{cl}_{\mathbf{z}}(t, \mathbf{z}(t))) = mT_{\min} > 0$.

Proof: By assumption, for the system $\dot{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t)))$, $\dot{V}_{\mathbf{x}}(t, \mathbf{x}(t)) = -W_{\mathbf{x}}(t, \mathbf{x}(t)) \leq 0$; $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$; and $\|\mathbf{x}(t) - \mathbf{x}^\star(t)\| \leq \alpha(V_0 := V_{\mathbf{x}}(0, \mathbf{x}(0)))\,\forall t \geq 0$. Then, by invoking Proposition 2, $\inf_{t\geq 0} \|\mathbf{U}^{cl}_{\mathbf{x}}(t, \mathbf{x}(t))\| > 0$ for some sufficiently small V_0, in which case the control law $[U(\cdot)\,\mathbf{r}^T(\cdot)] = \mathbf{u}^{cl}_{\mathbf{z}}(\cdot, \mathbf{z}(\cdot))$ is always well defined. Finally, since by assumption, $T^{cl}(\cdot,\cdot) \geq T_{\min}$, it follows that $\inf_{t\geq 0} \bar{T}(\mathbf{z}(t), \mathbf{u}^{cl}_{\mathbf{z}}(t, \mathbf{z}(t))) = \inf_{t\geq 0} mT^{cl}(t, \mathbf{x}(t)) \geq mT_{\min} > 0$, which means the cable connecting the quadrotor and load is always under tension. ∎

In Subsection V-B, we provide a possible control law $\mathbf{u}^{cl}_{\mathbf{x}} \in \mathcal{C}(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}^4)$. This control depends implicitly on a bounded sufficiently smooth controller for a double integrator that renders the origin asymptotically stable, and for which there exists a sufficiently smooth Lyapunov function. These functions are required to be sufficiently smooth since the control law $\mathbf{u}^{cl}_{\mathbf{x}}(\cdot,\cdot)$ depends on some of the gradients of these functions. A possible controller for a double integrator satisfying the conditions above is provided next.

A. Bounded Control of Double Integrator

The proposed control law for the double integrator is inspired by the strategy proposed in [19]. Also, the same control law depends on functions whose properties we describe in the next definition.

Definition 2: We say $\sigma \in \Sigma$, if $\sigma \in \mathcal{C}^3(\mathbb{R}, [-\bar{\sigma}, \bar{\sigma}])$ for some $\bar{\sigma} \in \mathbb{R}_{>0}$, $\sigma(s)s > 0$ for all $s \in \mathbb{R}$, $\sigma'(s) > 0$ for all $s \in \mathbb{R}$, and $\bar{\sigma}' := \sup_{s\in\mathbb{R}} |\sigma'(s)| < \infty$.

Consider now the double integrator system $\dot{\boldsymbol{\xi}}(t) = \mathbf{f}_{\boldsymbol{\xi}}(\boldsymbol{\xi}(t), u_{\boldsymbol{\xi}}(t))$, with state $\boldsymbol{\xi} = [p\,v]^T \in \mathcal{C}(\mathbb{R}^2)$, input $u_{\boldsymbol{\xi}} \in \mathcal{C}(\mathbb{R})$, and where $\mathbf{f}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, u) = [v\,u]^T$. A bounded control law $u^{cl}_{\boldsymbol{\xi}} \in \mathcal{C}^2(\mathbb{R}^2, [-\bar{B}, \bar{B}])$ can be found that guarantees that $\lim_{t\to\infty} \boldsymbol{\xi}(t) = \mathbf{0}$ along a trajectory of $\dot{\boldsymbol{\xi}}(t) = \mathbf{f}_{\boldsymbol{\xi}}(\boldsymbol{\xi}(t), u^{cl}_{\boldsymbol{\xi}}(\boldsymbol{\xi}(t)))$. Indeed, consider

$$u^{cl}_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = -\rho(\Omega(v) + \sigma(p)) - k\frac{v+\sigma(p)}{\Omega(v)+\sigma(p)}\frac{\sigma(p)}{\Omega'(v)} - \sigma'(p)\frac{v}{\Omega'(v)}, \tag{18}$$

where $\sigma, \rho \in \Sigma$, and $k > 0$, and $\Omega \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ satisfies $\Omega(v) = v, \forall v \in \Omega_v, [-\bar{\sigma}, \bar{\sigma}] \subset \Omega_v \subset \mathbb{R}$ and $|\Omega(v)| > |v| \wedge \Omega'(v) \geq |v|, \forall v \notin \Omega_v$. It follows that $\sup_{v\in\mathbb{R}} \left|\frac{v}{\Omega'(v)}\right| =: M < \infty$, where $M > \bar{\sigma}$ necessarily. Therefore $\bar{B} = \bar{\rho} + k\bar{\sigma} + \bar{\sigma}'M < \infty$. The equilibrium $\mathbf{0}$ of $\mathbf{f}^{cl}_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = \mathbf{f}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, u^{cl}_{\boldsymbol{\xi}}(\boldsymbol{\xi}))$ is asymptotically stable, and there exists a Lyapunov function $V_{\boldsymbol{\xi}} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}_{\geq 0})$ defined as $V_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = k\int_0^p \sigma(s)\mathrm{d}s + \frac{1}{2}(\Omega(v) + \sigma(p))^2$, for which it follows that $W_{\boldsymbol{\xi}} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}_{\geq 0})$ defined as $W_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = -\frac{\partial V_{\boldsymbol{\xi}}^T}{\partial \boldsymbol{\xi}}\mathbf{f}^{cl}_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = k\sigma^2(p) + \Omega'(v)(\Omega(v) + \sigma(p))\rho(\Omega(v) + \sigma(p))$.

For a three dimensional double integrator $\dot{\bar{\boldsymbol{\xi}}}(t) = \mathbf{f}_{\bar{\xi}}(\bar{\boldsymbol{\xi}}(t), \mathbf{u}_{\bar{\xi}}(t))$, with state $\bar{\boldsymbol{\xi}} = [\boldsymbol{\xi}_1^T\, \boldsymbol{\xi}_2^T\, \boldsymbol{\xi}_3^T]^T \in \mathcal{C}(\mathbb{R}^6)$, and input $\mathbf{u}_{\bar{\xi}} = [u_{\boldsymbol{\xi}_1}\, u_{\boldsymbol{\xi}_2}\, u_{\boldsymbol{\xi}_3}]^T \in \mathcal{C}(\mathbb{R}^3)$, and where $\mathbf{f}_{\bar{\xi}}(\bar{\boldsymbol{\xi}}, \mathbf{u}_{\bar{\xi}}) = [\mathbf{f}_{\boldsymbol{\xi}_1}^T(\boldsymbol{\xi}_1, u_{\boldsymbol{\xi}_1})\, \mathbf{f}_{\boldsymbol{\xi}_2}^T(\boldsymbol{\xi}_2, u_{\boldsymbol{\xi}_2})\, \mathbf{f}_{\boldsymbol{\xi}_3}^T(\boldsymbol{\xi}_3, u_{\boldsymbol{\xi}_3})]^T$, choosing the control law $\mathbf{u}_{\bar{\xi}}^{cl}(\bar{\boldsymbol{\xi}}) = [u_{\boldsymbol{\xi}_1}^{cl}(\boldsymbol{\xi}_1)\, u_{\boldsymbol{\xi}_2}^{cl}(\boldsymbol{\xi}_2)\, u_{\boldsymbol{\xi}_3}^{cl}(\boldsymbol{\xi}_3)]^T$ it follows that the origin is an asymptotic stable equilibrium of $\mathbf{f}_{\bar{\xi}}^{cl}(\bar{\boldsymbol{\xi}}) = \mathbf{f}_{\bar{\xi}}(\bar{\boldsymbol{\xi}}, \mathbf{u}_{\bar{\xi}}^{cl}(\bar{\boldsymbol{\xi}}))$; also, for the Lypuanov function $V_{\bar{\xi}} \in \mathcal{C}^2(\mathbb{R}^6, \mathbb{R}_{\geq 0})$, defined as $V_{\bar{\xi}}(\bar{\boldsymbol{\xi}}) = V_{\boldsymbol{\xi}_1}(\boldsymbol{\xi}_1) + V_{\boldsymbol{\xi}_2}(\boldsymbol{\xi}_2) + V_{\boldsymbol{\xi}_3}(\boldsymbol{\xi}_3)$, it follows that $W_{\bar{\xi}}(\bar{\boldsymbol{\xi}}) = W_{\boldsymbol{\xi}_1}(\boldsymbol{\xi}_1) + W_{\boldsymbol{\xi}_2}(\boldsymbol{\xi}_2) + W_{\boldsymbol{\xi}_3}(\boldsymbol{\xi}_3)$, with $W_{\bar{\xi}} \in \mathcal{C}^1(\mathbb{R}^6, \mathbb{R}_{\geq 0})$.

B. Controller $\mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x})$

Let us now present a possible control law $\mathbf{u}_{\mathbf{x}}^{cl} \in \mathcal{C}(\tilde{\Delta}_{\mathbf{x}}, \mathbb{R}^4)$ satisfying the conditions described in Section V. This is inspired on the control strategy described in [20]. Consider the state $\mathbf{x}$, as defined in (6), and denote $\bar{\boldsymbol{\xi}} = [\mathbf{e}^T\, \boldsymbol{v}^T]^T$ and $\bar{\mathbf{x}} = [\bar{\boldsymbol{\xi}}^T\, \mathbf{n}^T]^T$. Consider the functions defined in [20], specifically $T^{cl}(\cdot, \cdot)$ in (19) and $\boldsymbol{\tau}^{cl}(\cdot, \cdot)$ in (31), where $\mathbf{u}_{di}(\mathbf{e}, \boldsymbol{v}) = \mathbf{u}_{\bar{\xi}}(\bar{\boldsymbol{\xi}})$ and $V_{di}(\mathbf{e}, \boldsymbol{v}) = V_{\bar{\xi}}(\bar{\boldsymbol{\xi}})$ can be those proposed Subsection V-A (we emphasize here that $T^{cl}(t, \mathbf{x}) = \mathbf{n}^T \mathbf{T}^{cl}(t, \mathbf{e}, \boldsymbol{v})$ with $\mathbf{T}^{cl}(t, \mathbf{e}, \boldsymbol{v}) = \mathbf{g}(t) + \mathbf{u}_{\bar{\xi}}(\bar{\boldsymbol{\xi}})$). From the above, we define the following control law

$$\mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x}) := \begin{bmatrix} T^{cl}(t, \bar{\mathbf{x}}) & \boldsymbol{\tau}^{cl,T}(t, \mathbf{x}) \end{bmatrix}^T. \tag{19}$$

For this control law $\mathbf{e}_1^T \mathbf{u}_{\mathbf{x}}^{cl}(\cdot, \cdot) = T^{cl}(\cdot, \cdot) \geq T_{\min}$, where $T_{\min} = g + \inf_{t \geq 0} \ddot{\mathbf{p}}^\star(t) - \sup_{\bar{\xi} \in \mathbb{R}^6} |\mathbf{e}_3^T \mathbf{u}_{\bar{\xi}}(\bar{\boldsymbol{\xi}})| > 0$, where the inequality is satisfied by properly tunning the controller $\mathbf{u}_{\bar{\xi}}(\cdot)$. Moreover,

$$\sup_{t \geq 0} \max_{\mathbf{x} \in \Omega'_{\mathbf{x}}} \left\| \frac{\partial T^{cl}(t, \bar{\mathbf{x}})}{\partial \mathbf{x}} \right\| = \sup_{t \geq 0} \max_{\mathbf{x} \in \Omega'_{\mathbf{x}}} \left\| \begin{bmatrix} \frac{\partial \mathbf{u}_{\bar{\xi}}(\bar{\boldsymbol{\xi}})^T}{\partial \bar{\boldsymbol{\xi}}} \mathbf{n} \\ g\mathbf{e}_3 + \ddot{\mathbf{p}}^\star(t) + \mathbf{u}_{\bar{\xi}}(\bar{\boldsymbol{\xi}}) \\ \mathbf{0} \end{bmatrix} \right\|$$

which is bounded provided that $\mathbf{e}$ and $\boldsymbol{v}$ are bounded, and that $\mathbf{p}^\star(\cdot)$ is a feasible trajectory (notice that $\mathbf{n}$ belongs to a compact set). Given $V_{\mathbf{x}} \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times \Omega_{\mathbf{x}}, \mathbb{R}_{\geq 0})$ defined as $V_{\mathbf{x}}(t, \mathbf{x}) = V_{\bar{\xi}}(\bar{\boldsymbol{\xi}}) + v_\theta \xi(t, \bar{\mathbf{x}}) + v_\omega \frac{1}{2} \|\mathbf{e}_\omega(t, \mathbf{x})\|^2$ then $W_{\mathbf{x}} \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \times \Omega_{\mathbf{x}}, \mathbb{R}_{\geq 0})$ is given by $W_{\mathbf{x}}(t, \mathbf{x}) = -\frac{\partial V_{\mathbf{x}}(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_{\mathbf{x}}^d(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x})) = W_{\bar{\xi}}(\bar{\boldsymbol{\xi}}) + v_\theta k_\theta \xi(t, \bar{\mathbf{x}})(2 - \xi(t, \bar{\mathbf{x}})) + v_\omega k_\omega \|\mathbf{e}_\omega(t, \mathbf{x})\|^2$ (details are provided in [20]).

VI. Disturbance Estimator

In this section, we provide a solution that accomplishes the goal described in Problem 1, when a constant unknown disturbance $b \in \{\beta \in \mathbb{R}^3 : |\beta| \leq b^{\max}\} =: \Omega_b$ exists, for some known $b^{\max} \geq 0$. Denote $\hat{b} \in \mathcal{C}(\mathbb{R})$ as a disturbance estimate whose dynamics are designed next such that the goal described in Problem 1 is accomplished. For convenience, and since the disturbance estimate is dynamic, denote $\tilde{\mathbf{x}} = [\mathbf{x}^T\, \hat{b}]^T \in \mathcal{C}(\Omega_{\mathbf{x}} \times \mathbb{R} =: \Omega_{\tilde{\mathbf{x}}})$ as an extended state. Also, $\dot{\hat{b}}(t) = f_b(t, \tilde{\mathbf{x}}(t))$, where $f_b \in \mathcal{C}(\mathbb{R}_{\geq 0} \times \Omega_{\tilde{\mathbf{x}}}, \mathbb{R})$ is a vector field that is constructed later in this section. With the previous notions in mind, it follows that $\dot{\tilde{\mathbf{x}}}(t) = \mathbf{f}_{\tilde{\mathbf{x}}}(t, \tilde{\mathbf{x}}(t), \mathbf{u}_{\mathbf{x}}(t))$, where $\mathbf{f}_{\tilde{\mathbf{x}}}(t, \tilde{\mathbf{x}}, \mathbf{u}_{\mathbf{x}}) = \begin{bmatrix} \mathbf{f}_{\mathbf{x}}^u(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}) + \Phi(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) b \\ f_b(t, \tilde{\mathbf{x}}) \end{bmatrix}$ and moreover

$$\mathbf{f}_{\tilde{\mathbf{x}}}(t, \tilde{\mathbf{x}}, \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x}) - \hat{b}\mathbf{e}_1) = \begin{bmatrix} \mathbf{f}_{\mathbf{x}}^u(t, \mathbf{x}, \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x})) + \Phi(\mathbf{x}, \mathbf{u}_{\mathbf{x}}^{cl})(b - \hat{b}) \\ f_b(t, \tilde{\mathbf{x}}) \end{bmatrix} \tag{20}$$

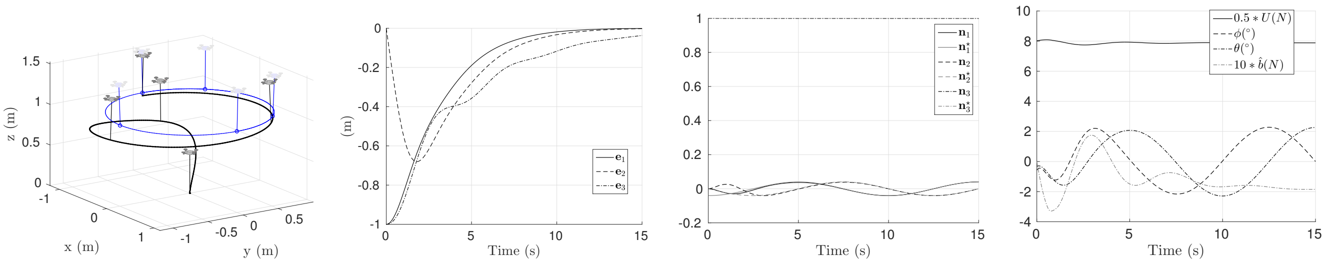
If the disturbance b were known, it would suffice to choose $\hat{b}(0) = b$, and $f_b(t, \tilde{\mathbf{x}}) = 0$, in order to accomplish the goal in Problem 1. Since b is unknown, a different strategy is pursed, namely the disturbance estimate is updated with a projector operator that guarantees that the disturbance estimate remains in $\Omega_{\hat{b}} := \{\beta \in \mathbb{R}^3 : |\beta| \leq b^{\max} + \epsilon\}$, where $\epsilon > 0$ is a design parameter that can be chosen as small as desired; and provided that $\hat{b}(0) \in \Omega_{\hat{b}}$, which is satisfied if $\hat{b}(0) = 0$. Consider then the vector field

$$f_b(t, \tilde{\mathbf{x}}) = \text{Proj}\left(\Phi^T(\mathbf{x}, \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x})) \frac{\partial V_{\mathbf{x}}(t, \mathbf{x})}{\partial \mathbf{x}}, \hat{b} \right), \tag{21}$$

whose choice will be clear next (Proj$(\cdot, \cdot)$ as defined in [21]). Consider the Lyapunov function $V_{\tilde{\mathbf{x}}} \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \times \Omega_{\tilde{\mathbf{x}}}, \mathbb{R}_{\geq 0})$, defined as $V_{\tilde{\mathbf{x}}}(t, \tilde{\mathbf{x}}) = V_{\mathbf{x}}(t, \mathbf{x}) + \frac{1}{k_b} \frac{(b - \hat{b})}{2}$, where $k_b > 0$. Given (20) and (21), it follows that $W_{\tilde{\mathbf{x}}}(t, \tilde{\mathbf{x}}) = W_{\mathbf{x}}(t, \mathbf{x}) - k_b(b - \hat{b})(f_b(t, \tilde{\mathbf{x}}) - \Phi^T(\mathbf{x}, \mathbf{u}_{\mathbf{x}}^{cl}(t, \mathbf{x})) \frac{\partial V_{\mathbf{x}}(t, \mathbf{x})}{\partial \mathbf{x}}) \leq 0$. In the presence of a disturbance b, the concept of feasible trajectory in Definition 1 needs to be redefined. Indeed, note that, from (10), *big* disturbances can lead to a *negative* tension in the cable, at which point the dynamics (2) do not hold. Also, a similar result to Theorem 3 can be proved when a disturbance estimator with vector field (21) is implemented.

VII. Simulations

Consider a quadrotor with mass $M = 1.442$ kg, a load with mass $m = 0.144$ kg, a cable with length $d = 0.5$ m, and a disturbance $b = 0.2$ N. Consider the control law (18) with $\sigma(s) = 0.25 \frac{s}{\sqrt{1 + s^2}}$, $\rho(s) = 0.70 \frac{s}{\sqrt{1 + s^2}}$, $k = 1$ and $\Omega(\cdot)$ as an odd function and as the solution to the differential equation $\Omega'''(s) = 0$ for $s \in [0, 1)$, $\Omega'''(s) = s - 1$ for $s \in [1, 2]$ and $\Omega'''(s) = 1$ for $s > 2$ and initial conditions $\Omega(0) = 0$, $\Omega'(0) = 1$ and $\Omega''(0) = 0$. Consider the control law $\mathbf{u}_{\mathbf{x}}^{cl}(\cdot, \cdot)$, in (19), with gains $v_\theta = 50$, $k_\theta = 1$, $v_\omega = 50$, $k_\omega = 1$; and the estimator vector field, in (20), with $k_b = 5$ and $\epsilon = 0.3$. For these choices, we provide a simulation in Fig. 2, as a solution of (1) with $\mathbf{z}(0) = [\mathbf{0}^T\, \mathbf{0}^T\, 0.5\mathbf{e}_3^T\, \mathbf{0}^T]^T$. In Fig. 2(a), one can visualize in blue the desired trajectory, namely one with the load describing a circular motion of 1 m of radius, and an angular velocity of 0.1 rev/sec; and in black, the actual trajectory of the quadrotor-load system, where convergence to the desired trajectory is verified. In Fig. 2(b), the position tracking error is presented, and its convergence to $\mathbf{0}$ indicates convergence of the system's trajectory to the desired trajectory. In Fig. 2(c), the cable's unit vector and its equilibrium are presented, and $\mathbf{n}(\cdot)$ converges to its equilibrium $\mathbf{n}^\star(\cdot)$ as defined in (13). Finally, in Fig. 2(d), the thrust input, $U(\cdot)$, is presented, and the attitude input, $\mathbf{r}(\cdot)$, is also presented, where the attitude unit vector has been parametrized in pitch and roll angles. The thrust input stabilizes around a value that cancels the accumulated weight of the quadrotor and the load; while the pitch and roll oscillate around zero, since the load is describing a circle, and therefore the cable must rotate so as to point inwards the circular trajectory. In Fig. 2(d), the disturbance estimate is also presented, and it converges to the real unknown disturbance, thus canceling its effect. Preliminary experimental results are presented in Fig.3, where a quadrotor-load system is first commanded



(a) Trajectories

(b) Position tracking error $\mathbf{e}(t)$

(c) Unit vectors $\mathbf{n}(t)$ in state and $\mathbf{n}^*(t)$ in (13)

(d) Inputs $\mathbf{u}_{\mathbf{z}}(t) = \lfloor U(t)\,\mathbf{r}^T(t) \rfloor^T = \mathbf{u}^{cl}(t, \mathbf{z}(t))$ (see (15)) and $\hat{b}(t)$

Fig. 2: Simulation for $\mathbf{z}(0) = [\mathbf{0}^T\ \mathbf{0}^T\ 0.5\mathbf{e}_3^T\ \mathbf{0}^T]^T$ (In Fig. 2(d), $\mathbf{r} = \lfloor \mathrm{c}(\phi)\mathrm{s}(\theta)\ -\mathrm{s}(\phi)\ \mathrm{c}(\phi)\mathrm{c}(\phi) \rfloor^T$).

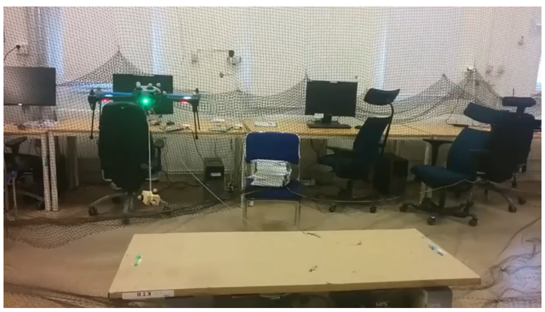


Fig. 3: Preliminary experimental result: load hovering over green pen.

to hover over a green pen, and afterwards is commanded to hover over a blue pen. The quadrotor was a commercial one, namely an IRIS+ from 3D Robotics; the load weighted about 144 g; the cable had a length of approximately 0.5 m; and position measurements were obtained from a Qualisys motion capture system. A video of this experiment is found in [22].

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Appendix

Proposition 4: Consider the time-varying vector $\mathbf{a} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^3 \backslash \mathbf{0})$, and the unit vector $\mathbf{n}(t) = \frac{\mathbf{a}(t)}{\|\mathbf{a}(t)\|}$, which is always well defined. It follows that $\dot{\mathbf{n}}(t) = \mathcal{S}(\boldsymbol{\omega}(t))\,\mathbf{n}(t)$ and $\dot{\boldsymbol{\omega}}(t) = \mathcal{S}(\mathbf{n}(t))\,\boldsymbol{\tau}(t)$, where $\boldsymbol{\omega}(t) = \mathcal{S}(\mathbf{n}(t))\,\frac{\dot{\mathbf{a}}(t)}{\|\mathbf{a}(t)\|}$, and $\boldsymbol{\tau}(t) = \Pi(\mathbf{n}(t))\left(\frac{\ddot{\mathbf{a}}(t)}{\|\mathbf{a}(t)\|} - 2\frac{\dot{\mathbf{a}}(t)}{\|\mathbf{a}(t)\|}\frac{\dot{\mathbf{a}}^T(t)}{\|\mathbf{a}(t)\|}\mathbf{n}(t)\right)$.
The proof follows from straightforward calculations.