Family of Controllers for Attitude Synchronization in $\mathcal{S}^2$

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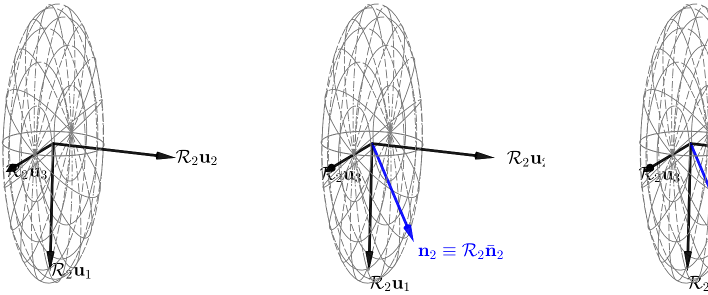
Summary

Attitude Synchronization in $\mathcal{S}^2$

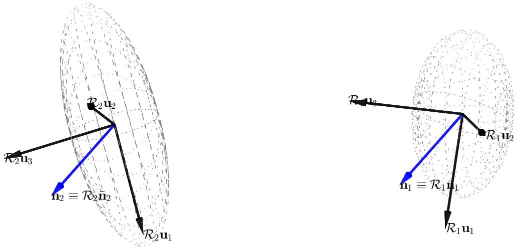
1. Torque control (constrained torque)
2. Distributed control laws
3. Only local information
4. Multiple equilibria

Objective


$\mathcal{R}_2\mathbf{u}_2$
$\mathcal{R}_2\mathbf{u}_3$
$\mathcal{R}_2\mathbf{u}_1$
$\mathbf{n}_2 \equiv \mathcal{R}_2\bar{\mathbf{n}}_2$

(a)

Objective



Objective

Given fixed $\bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2 \in \mathcal{S}^2$, denote $\mathbf{n}_1 = \mathcal{R}_1\bar{\mathbf{n}}_1$ and $\mathbf{n}_2 = \mathcal{R}_2\bar{\mathbf{n}}_2$.
Goal: $\lim_{t\to\infty} \mathbf{n}_1(t) = \lim_{t\to\infty} \mathbf{n}_2(t)$

Problem Statement

Definition

(Incomplete) synchronization: when $\mathbf{n}_1 = \cdots = \mathbf{n}_N$

Problem Statement

Given $\{\bar{\mathbf{n}}_i\}_{i=1,\cdots,N}$,

$$\dot{\mathcal{R}}_i = \mathcal{R}_i \mathcal{S}(\boldsymbol{\omega}_i),$$
$$\frac{d}{dt}\left(\mathcal{R}_i J_i \boldsymbol{\omega}_i\right) = \mathcal{R}_i \mathbf{T}_i,$$

design distributed torques $\{\mathbf{T}_i\}_{i=1,\cdots,N}$ that guarantee that synchronization is asymptotically reached.

W. Song, X. Hu, et al. Distributed Control for Intrinsic Reduced Attitude Formation with Ring Inter-Agent Graph, CDC 2015

Distance function in $\mathcal{S}^2$

Distance function $d(\mathbf{n}_1, \mathbf{n}_2)$

If

- $d(\mathbf{n}_1, \mathbf{n}_2) > 0$, $\mathbf{n}_1 \neq \mathbf{n}_2$
- $d(\mathbf{n}_1, \mathbf{n}_2) = 0$, $\mathbf{n}_1 = \mathbf{n}_2$
- $\mathcal{S}(\mathbf{n}_1)\frac{\partial d(\mathbf{n}_1,\mathbf{n}_2)}{\partial \mathbf{n}_1} = -\mathcal{S}(\mathbf{n}_2)\frac{\partial d(\mathbf{n}_1,\mathbf{n}_2)}{\partial \mathbf{n}_2}$,

Then

$$\dot{d}(\mathbf{n}_1, \mathbf{n}_2) = \underbrace{\begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{bmatrix}^T}_{\boldsymbol{\omega}} \underbrace{\begin{bmatrix} \mathcal{R}_1^T & \mathbf{0} \\ \mathbf{0} & \mathcal{R}_2^T \end{bmatrix}}_{\boldsymbol{\mathcal{R}}^T} \underbrace{\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \mathbf{I} \right)}_{B \otimes \mathbf{I}} \underbrace{\mathcal{S}(\mathbf{n}_1)\frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1}}_{\mathbf{e}}$$

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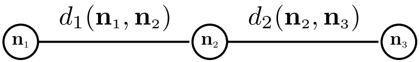
Then

$$\sum_{k=1}^{k=M} \dot{d}_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}) = \boldsymbol{\omega}^T \boldsymbol{\mathcal{R}}^T \left(B \otimes \mathbf{I}\right) \mathbf{e}$$



Distance function in $\mathcal{S}^2$

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If

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- $\mathcal{S}(\mathbf{n}_1)\frac{\partial d(\mathbf{n}_1,\mathbf{n}_2)}{\partial \mathbf{n}_1} = -\mathcal{S}(\mathbf{n}_2)\frac{\partial d(\mathbf{n}_1,\mathbf{n}_2)}{\partial \mathbf{n}_2}$,

Then, solution of the PDE is

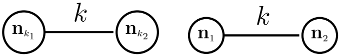
$$d(\mathbf{n}_1, \mathbf{n}_2) = f(1 - \mathbf{n}_1^T \mathbf{n}_2),$$

where $f : [0, 2] \mapsto \mathbb{R}_{\geq 0}$ and $f(0) = 0$.

Distance function in $\mathcal{S}^2$

Distance function $d(\mathbf{n}_1, \mathbf{n}_2)$:

$$d(\mathbf{n}_1, \mathbf{n}_2) = f(1 - \mathbf{n}_1^T \mathbf{n}_2).$$



Edge k error ($\mathbf{e} = \begin{bmatrix} \mathbf{e}_1^T & \cdots & \mathbf{e}_M^T \end{bmatrix}^T$)

- $\mathbf{e}_k = \mathcal{S}(\mathbf{n}_1) \frac{\partial d_k(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1} = -f_k'(1 - \mathbf{n}_1^T \mathbf{n}_2) \mathcal{S}(\mathbf{n}_1) \mathbf{n}_2.$
- $\mathbf{e}_k = \mathbf{0} \Rightarrow \mathbf{n}_1 || \mathbf{n}_2$

Proposed control law

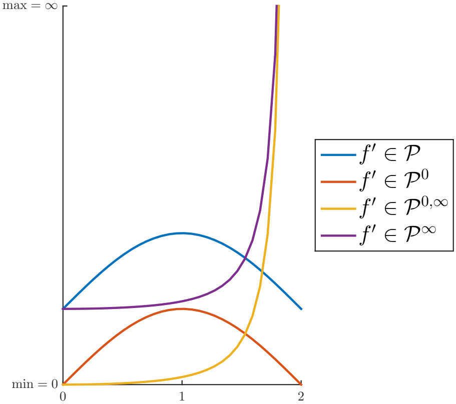


Figure : Classes of $f' : [0,2] \mapsto \mathbb{R}_{\geq 0}$ $(d^{\max} = \max_{s\in[0,2]} f(s))$

Proposed control law

Proposed control law

$$\mathbf{T}_i = -\boldsymbol{\sigma}(\boldsymbol{\omega}_i) - \sum_{j\in\mathcal{N}_i} f'_{ij}(1-\bar{\mathbf{n}}_i^T\mathcal{R}_i^T\mathbf{n}_j)\mathcal{S}(\bar{\mathbf{n}}_i)\mathcal{R}_i^T\mathbf{n}_j,$$

$$\mathbf{T} = -\boldsymbol{\sigma}(\boldsymbol{\omega}) - \boldsymbol{\mathcal{R}}^T(B\otimes\mathbf{I})\mathbf{e},$$

Proposed control law

$$\mathbf{T}_i = -\boldsymbol{\sigma}(\boldsymbol{\omega}_i) - \sum_{j\in\mathcal{N}_i} f'_{ij}(1-\bar{\mathbf{n}}_i^T\mathcal{R}_i^T\mathbf{n}_j)\mathcal{S}(\bar{\mathbf{n}}_i)\mathcal{R}_i^T\mathbf{n}_j,$$

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Lyapunov function

Proposed control law

Analysis

$$\dot{V}(\omega) \to 0 \Rightarrow \omega \to 0 \Rightarrow \dot{\omega} \to \mathbf{0} \Rightarrow \mathbf{T} \to \mathbf{0} \Rightarrow (B \otimes \mathbf{I})\mathbf{e} \to \mathbf{0}$$

Analysis

- Tree graph: $\mathbf{e} \to \mathbf{0}$
- In general: $\mathbf{e} \to \mathcal{N}(B \otimes \mathbf{I})$

Proposed control law

$$\mathbf{T}_i = -\cancelto{\boldsymbol{\sigma}(\Pi(\bar{\mathbf{n}}_i)\boldsymbol{\omega}_i)}{\boldsymbol{\sigma}(\boldsymbol{\omega}_i)} - \sum_{j \in \mathcal{N}_i} f'_{ij}(\cdot)\mathcal{S}(\bar{\mathbf{n}}_i)\mathcal{R}_i^T \mathbf{n}_j \perp \bar{\mathbf{n}}_i$$

Lyapunov function & Analysis: $\bar{\mathbf{n}}_i$ principal axis

$$\dot{V}(\omega) = -\sum_{l=1}^{l=N} \omega_l^T \cancelto{\boldsymbol{\sigma}(\Pi(\bar{\mathbf{n}}_l)\boldsymbol{\omega}_l)}{\boldsymbol{\sigma}(\boldsymbol{\omega}_l)}$$

$$\dot{V} \to 0 \Rightarrow \Pi(\bar{\mathbf{n}}_i)\omega_i \to \mathbf{0} \Rightarrow \Pi(\bar{\mathbf{n}}_i)\dot{\omega}_i \to \mathbf{0} \Rightarrow$$
$$\Rightarrow \mathbf{T}_i \to (\bar{\mathbf{n}}_i^T \boldsymbol{\omega}_i)^2 \mathcal{S}(\bar{\mathbf{n}}_i) J_i \bar{\mathbf{n}}_i \Rightarrow (B \otimes \mathbf{I})\mathbf{e} \to \mathbf{0}$$

Result: Tree Graphs

$$d_{\min} := \min_{k \in \{1, \cdots, M\}} (d_k^{\max})$$

Result (Tree Graph)

If $H(0) \leq d_{\min}$ and if, for all $k \in \{1, \cdots, M\}$,

- $f_k' \in \bar{\mathcal{P}}$
- $d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})|_{t=0} \leq \frac{1}{M}(d_{\min} - H(0))$

then synchronization is asymptotically reached.

Result

In tree graph, if $f_k' \in \mathcal{P}^\infty \cup \mathcal{P}^{0,\infty}$, then synchronization is asymptotically reached for almost all initial conditions.

Result

$$d^\star := \min_k f_k\left(\min_k f_k^{-1}\left(\min_k f_k\left(\frac{\pi}{2}\frac{1}{N-1}\right)\right)\right)$$

Result

If $H(0) \leq d^\star$ and, for all $k \in \{1, \cdots, M\}$,

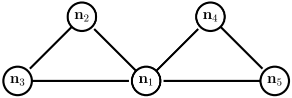
- $f_k' \in \bar{\mathcal{P}}$
- $d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})|_{t=0} \leq \frac{1}{M}(d^\star - H(0))$

then synchronization is asymptotically reached for almost all initial conditions.

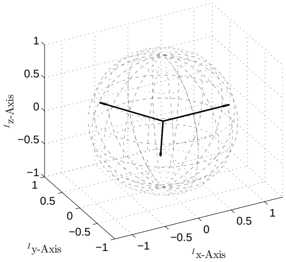
Remark:

$f_k\left(\frac{\pi}{2}\frac{1}{N-1}\right) \leq d_k^{\max} \Rightarrow d^\star \leq d_{\min}$

Equilibria solutions: $\mathbf{e} \in \mathcal{N}(B \otimes \mathbf{I})$

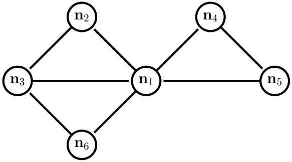


(a) No shared edges between cycles

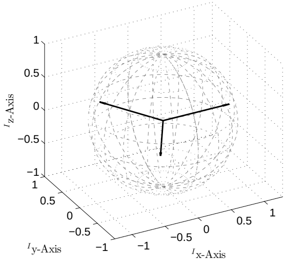


(b) Planar unit vectors for each cycle

Equilibria solutions: $\mathbf{e} \in \mathcal{N}(B \otimes \mathbf{I})$



(c) Cycles that share only one edge



(d) Planar unit vectors

Simulations

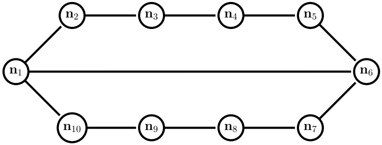
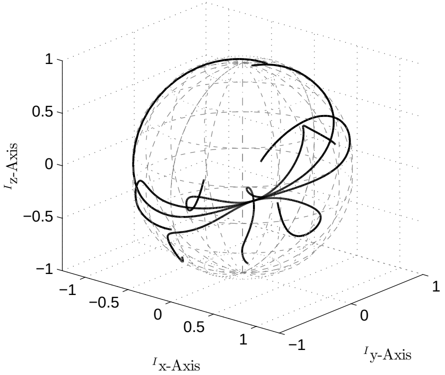


Figure : $N = 10$, $M = 10$

Simulations



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Thank you! Questions?