A Common Framework for Attitude Synchronization of Unit Vectors in Networks with Switching Topology

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Problem formulation

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Design angular velocity control laws that guarantee total or partial orientation synchronization among group of agents.

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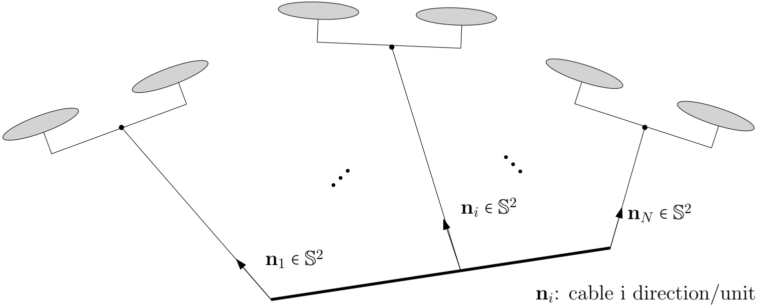

$\mathbf{n}_i$: cable i direction/unit vector

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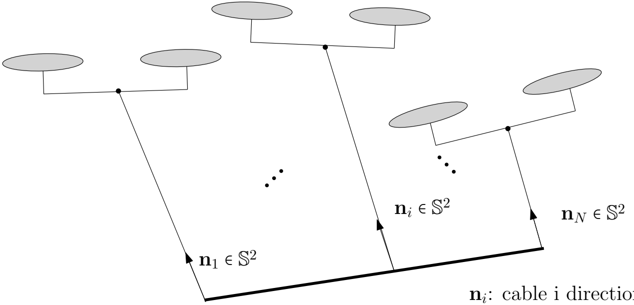
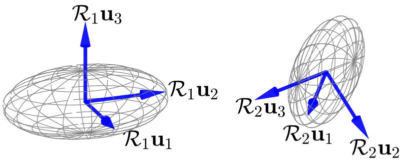
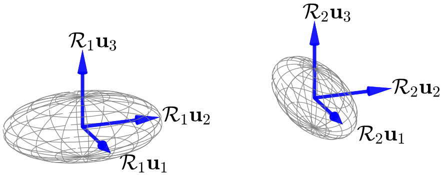


$\mathbf{n}_i$: cable i direction/unit vector

Complete Synchronization



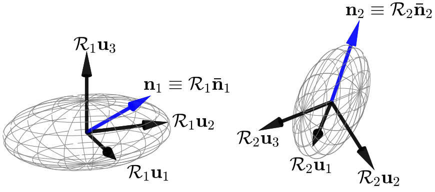
(a) Two rotation matrices not synchronized, i.e., $\mathcal{R}_1 \neq \mathcal{R}_2$



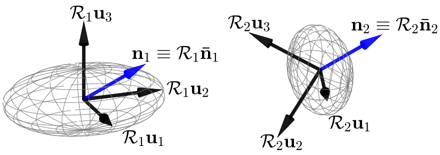
(b) Two rotation matrices synchronized, i.e., $\mathcal{R}_1 = \mathcal{R}_2$

- In complete synchronization, $\mathcal{R}_1 = \cdots = \mathcal{R}_N \in \mathbb{SO}^3$

Incomplete Synchronization ($\bar{\mathbf{n}}_1 \neq \bar{\mathbf{n}}_2$)



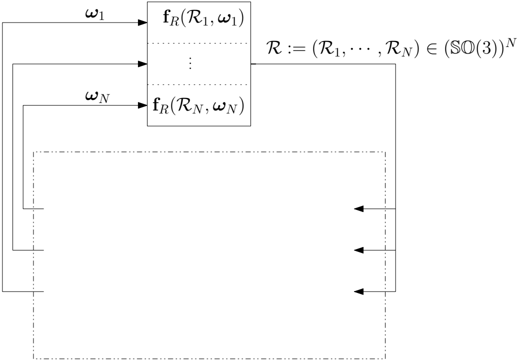
(a) Not synchronized, $\mathbf{n}_1 \neq \mathbf{n}_2$.



(b) Synchronized, $\mathbf{n}_1 = \mathbf{n}_2$.

- Body axis to be synchronized, $\bar{\mathbf{n}}_1, \cdots, \bar{\mathbf{n}}_N \in \mathbb{S}^2$.
- In incomplete synchronization, $\mathcal{R}_1\bar{\mathbf{n}}_1 = \cdots = \mathcal{R}_N\bar{\mathbf{n}}_N \in \mathbb{S}^2$.
- Complete synchronization $\not\Leftarrow$ Incomplete synchronization

Output Feedback



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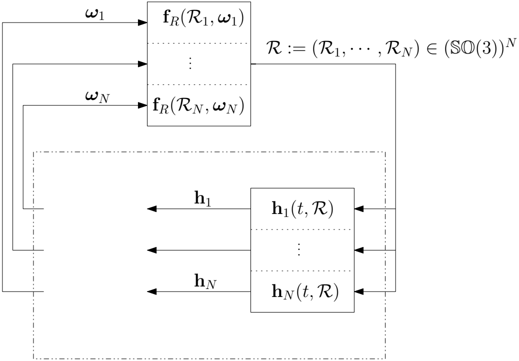

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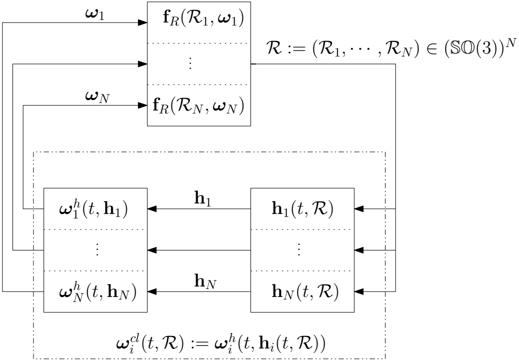

Complete Synchronization: Agents dynamics

Agent $i \in \{1, \cdots, N\}$

- State: rotation matrix $\mathcal{R}_i \in \mathbb{SO}^3$
- Input: angular velocity $\boldsymbol{\omega}_i \in \mathbb{R}^3$

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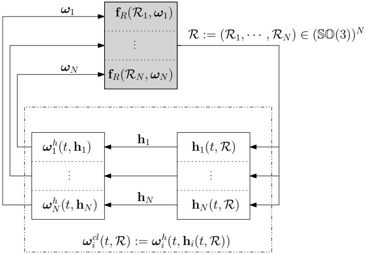
Agent dynamics

$$\dot{\mathcal{R}}_i = \mathbf{f}_R(\mathcal{R}_i, \boldsymbol{\omega}_i) := \mathcal{R}_i \mathcal{S}(\boldsymbol{\omega}_i)$$



Agents output functions

At each time instant t, agent $i \in \{1, \cdots, N\}$ makes $|\mathcal{N}_i(t)|$ measurements

- neighbors rotation matrices projected in body framed

Agent output function

Denote $\mathcal{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_N) \in (\mathbb{SO}^3)^N$ and $\mathcal{N}_i =: \{i_1, \cdots, i_{|\mathcal{N}_i|}\}$,

$$\mathbf{h}_i(t, \mathcal{R}) := (\mathcal{R}_i^T \mathcal{R}_{i_1}, \cdots, \mathcal{R}_i^T \mathcal{R}_{i_{|\mathcal{N}_i(t)|}}) \in (\mathbb{SO}^3)^{|\mathcal{N}_i(t)|}$$

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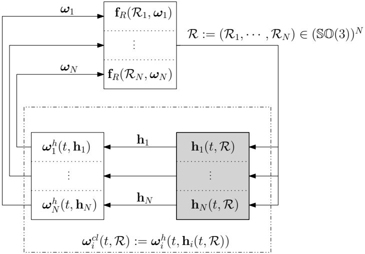
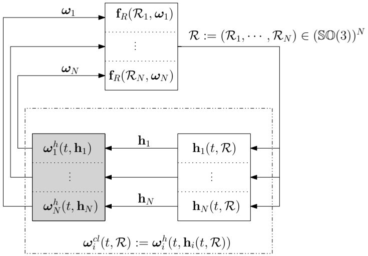


Agents output-feedback control law

Output-feedback control law

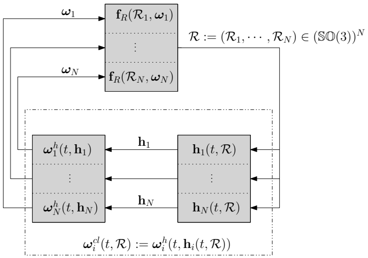
$$\boldsymbol{\omega}_i^h(t, \mathbf{h}_i) := -k \sum_{l=1}^{l=|\mathcal{N}_i(t)|} \mathcal{S}^{-1}(\mathbf{h}_{i,l} - \mathbf{h}_{i,l}^T) \in \mathbb{R}^3.$$



Closed loop dynamics: Complete synchronization

Closed loop dynamics

$$\dot{\mathcal{R}}_i = \mathbf{f}_R^{cl}(t, \mathcal{R}) := \mathbf{f}_R(\mathcal{R}_i, \boldsymbol{\omega}_i^h \circ \mathbf{h}_i).$$



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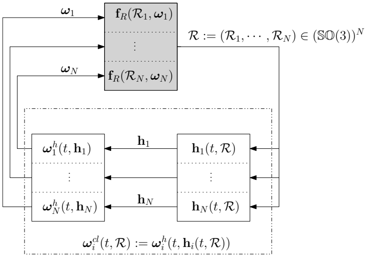
- $\boxed{\textbf{Body axis to synchronize: } \bar{\mathbf{n}}_i}$



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At each time instant t, agent $i \in \{1, \cdots, N\}$ makes $|\mathcal{N}_i(t)|$ measurements

- neighbors axis projected in body frame

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Denote $\mathcal{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_N) \in (\mathbb{SO}^3)^N$ and $\mathcal{N}_i =: \{i_1, \cdots, i_{|\mathcal{N}_i|}\}$,

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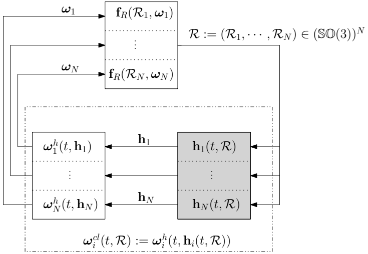

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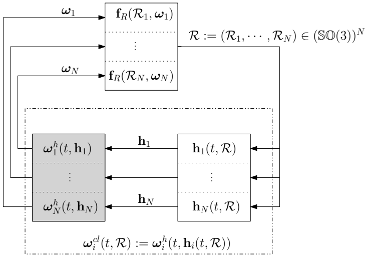
$$\boldsymbol{\omega}_i^h(t,\mathbf{h}_i) := -k\sum_{l=1}^{l=|\mathcal{N}_i(t)|} \mathcal{S}(\bar{\mathbf{n}}_i)\mathbf{h}_{i,l} \in T_{\bar{\mathbf{n}}_i}\mathbb{S}^2.$$



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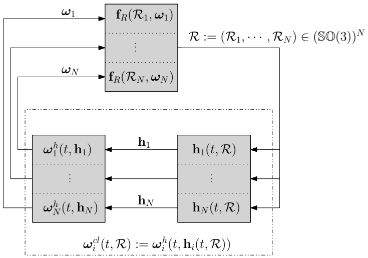


Common Framework

Complete synchronization as synchronization in $\mathbb{S}^3$

$$\begin{bmatrix} \mathbb{SO}^3 \ni \mathcal{R}_1 \\ \vdots \\ \mathbb{SO}^3 \ni \mathcal{R}_N \end{bmatrix} \Longrightarrow \begin{bmatrix} \frac{1}{2}\left(\frac{\mathcal{S}^{-1}(\mathcal{R}_1-\mathcal{R}_1^T)}{\sqrt{1+\mathrm{tr}(\mathcal{R}_1)}}, \sqrt{1+\mathrm{tr}(\mathcal{R}_1)}\right) =: \boldsymbol{\nu}_1 \in \mathbb{S}^3 \\ \vdots \\ \frac{1}{2}\left(\frac{\mathcal{S}^{-1}(\mathcal{R}_N-\mathcal{R}_N^T)}{\sqrt{1+\mathrm{tr}(\mathcal{R}_N)}}, \sqrt{1+\mathrm{tr}(\mathcal{R}_N)}\right) =: \boldsymbol{\nu}_N \in \mathbb{S}^3 \end{bmatrix}$$

$$\dot{\mathcal{R}}_i = \mathbf{f}_R^{cl}(t,\mathcal{R}) \Longrightarrow \dot{\boldsymbol{\nu}}_i = \sum_{j\in\mathcal{N}_i(t)} w(\boldsymbol{\nu}_i,\boldsymbol{\nu}_j)\Pi(\boldsymbol{\nu}_i)\boldsymbol{\nu}_j$$

$$w(\boldsymbol{\nu}_i,\boldsymbol{\nu}_j) = \boldsymbol{\nu}_i^T\boldsymbol{\nu}_j$$

Common Framework

Incomplete synchronization as synchronization in $\mathbb{S}^2$

$$\begin{bmatrix} \mathbb{SO}^3 \ni \mathcal{R}_1 \\ \vdots \\ \mathbb{SO}^3 \ni \mathcal{R}_N \end{bmatrix} \Longrightarrow \begin{bmatrix} \mathcal{R}_1 \bar{\mathbf{n}}_1 =: \boldsymbol{\nu}_1 \in \mathbb{S}^2 \\ \vdots \\ \mathcal{R}_N \bar{\mathbf{n}}_N =: \boldsymbol{\nu}_N \in \mathbb{S}^2 \end{bmatrix}$$

$$\dot{\mathcal{R}}_i = \mathbf{f}_R^{cl}(t, \mathcal{R}) \Longrightarrow \dot{\boldsymbol{\nu}}_i = \sum_{j \in \mathcal{N}_i(t)} w(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j$$

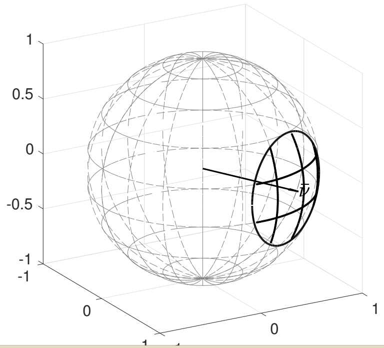
$$w(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) = 1$$

Cone

Cone

Notion of neighborhood in sphere:

$$\mathcal{C}(\alpha, \bar{\boldsymbol{\nu}}) = \{\boldsymbol{\nu} \in \mathbb{S}^n : \boldsymbol{\nu}^T \bar{\boldsymbol{\nu}} > \cos(\alpha)\}$$



Cone

Cone

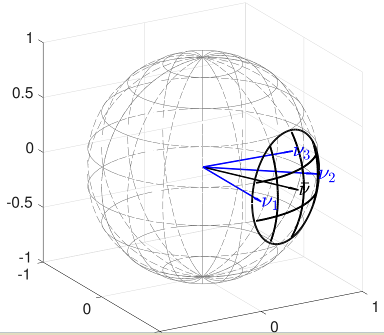
Consider a group of unit vectors $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) \in (\mathbb{S}^n)^N$.

Cone

Cone

Consider a group of unit vectors $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) \in (\mathbb{S}^n)^N$.
$\boldsymbol{\nu}$ belongs to α-cone $:\Leftrightarrow \boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) \in \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^N$ for some $\bar{\boldsymbol{\nu}} \in \mathbb{S}^n$

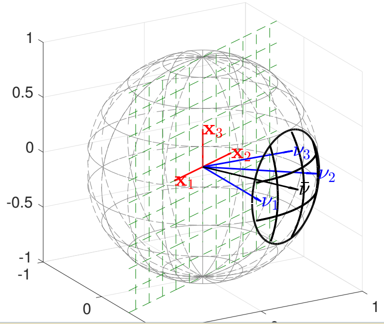
$$\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3) \in \mathcal{C}(30^\circ, \bar{\boldsymbol{\nu}})^N$$



Synchronization in $\mathbb{S}^n$

$$\begin{bmatrix} \mathbb{S}^n \ni \boldsymbol{\nu}_1 \\ \vdots \\ \mathbb{S}^n \ni \boldsymbol{\nu}_N \end{bmatrix} \Leftrightarrow \begin{bmatrix} Q_{\bar{\boldsymbol{\nu}}}\boldsymbol{\nu}_1 =: \mathbf{x}_1 \in \mathbb{R}^n \\ \vdots \\ Q_{\bar{\boldsymbol{\nu}}}\boldsymbol{\nu}_N =: \mathbf{x}_N \in \mathbb{R}^n \end{bmatrix} =: \mathbf{x}$$

$$\dot{\boldsymbol{\nu}}_i = \sum_{j \in \mathcal{N}_i(t)} w(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j)\Pi(\boldsymbol{\nu}_i)\boldsymbol{\nu}_j \Leftrightarrow \dot{\mathbf{x}}_i = \mathbf{f}_i(t, \mathbf{x})$$



Main result

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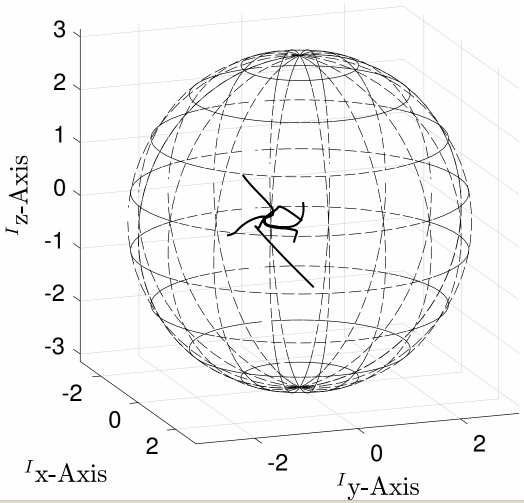
If

- $\boldsymbol{\nu}(0)$ belongs to α-cone,
- and the network graph is connected at all times,

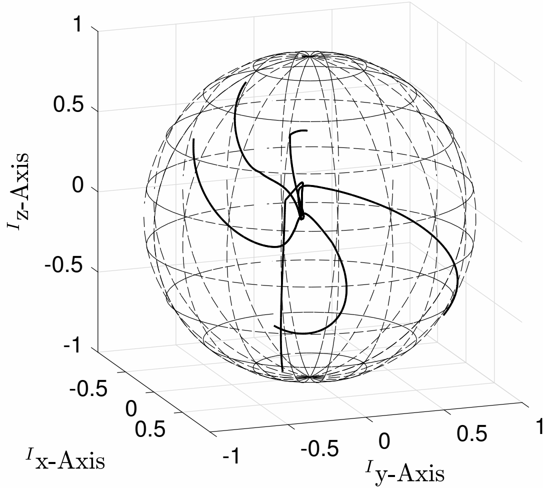
then

- the group of unit vectors remains in the cone it stars on;
- all unit vectors converges to one another;
- all unit vectors converge to a constant unit vector.

Simulation: Complete synchronization



Simulation: Incomplete synchronization



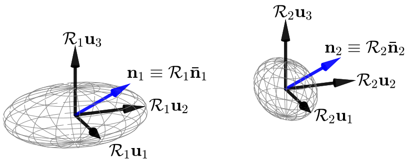
Future research directions

Research directions

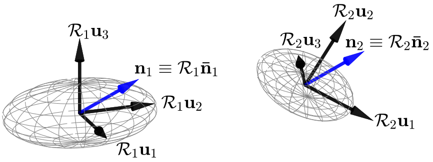
- Develop strategies for disturbance removal: $\dot{\mathcal{R}} = \mathcal{R}\mathcal{S}(\boldsymbol{\omega} + \mathbf{d})$.
- Determine and study the stability of other equilibria.
- Determine constant unit vector that network converges to.

Thank you! Questions?

Incomplete Synchronization ($\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2$)



(a) $\mathbf{n}_1 = \mathbf{n}_2$ and $\mathcal{R}_1 = \mathcal{R}_2$



(b) $\mathbf{n}_1 = \mathbf{n}_2$ and $\mathcal{R}_1 \neq \mathcal{R}_2$

- Body axis to be synchronized, $\bar{\mathbf{n}}_1, \cdots, \bar{\mathbf{n}}_N \in \mathbb{S}^2$.
- In incomplete synchronization, $\mathcal{R}_1\bar{\mathbf{n}}_1 = \cdots = \mathcal{R}_N\bar{\mathbf{n}}_N \in \mathbb{S}^2$.
- Incomplete synchronization $\not\Rightarrow$ Complete synchronization