Collaborative Transportation of a Bar by Two Aerial Vehicles with Attitude Inner Loop and Experimental Validation

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Department of Automatic Control,
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December 12th

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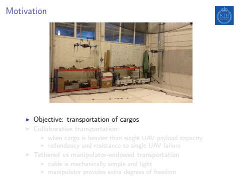
- Submission number: 1202
- Schedule Code: TuC17. Room: 218
- https://css.paperplaza.net/conferences/conferences/CDC17/program/CDC17_ContentListWeb_1.html
- Session: **Cooperative Control V**
- Time of presentation: Tuesday December 12, 2017, 17:00-17:20 (GMT+11)
- Chair: Jorge Finke http://jfinke.org/
- Co-chair: Pedro Pereira http://people.kth.se/~ppereira/index.html

Motivation

- Objective: transportation of cargos
- Collaborative transportation:
  - when cargo is heavier than single UAV payload capacity
  - redundancy and resistance to single UAV failure
- Tethered *vs* manipulator-endowed transportation
  - cable is mechanically simple and light
  - manipulator provides extra degrees of freedom



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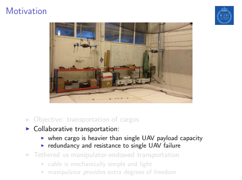

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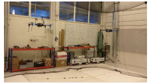

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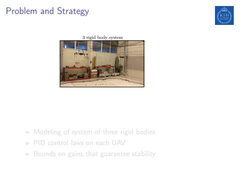
Problem and Strategy

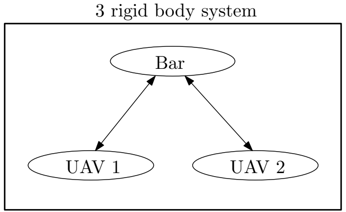
3 rigid body system

- Modeling of system of three rigid bodies
- PID control laws on each UAV
- Bounds on gains that guarantee stability



- system of rigid bodies: chicken and egg problem
- Is the UAV 1 tethered to the bar, or the bar tethered to the UAV 1?
- Physical coupling is a bidirectional link
- We will not be able to apply the Routh-Hurwitz criterion immediately, we will need to do some prep work first.
- Infer stability from linearization procedure
- Routh-Hurwitz criterion to establish precise bounds on gains that guarantee stability

Problem and Strategy



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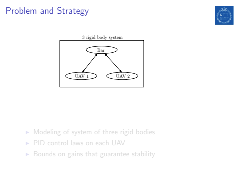

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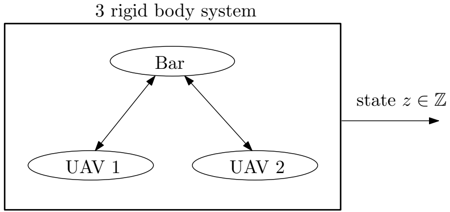

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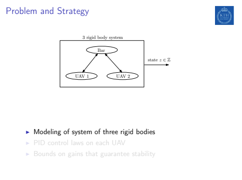

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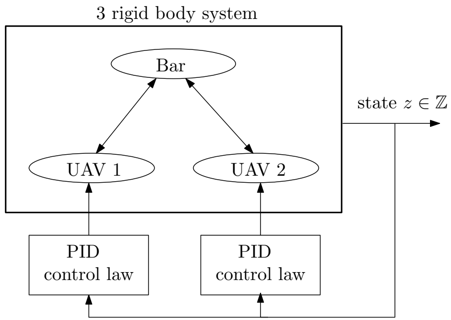

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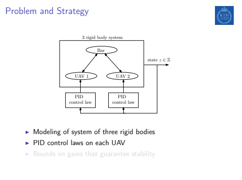

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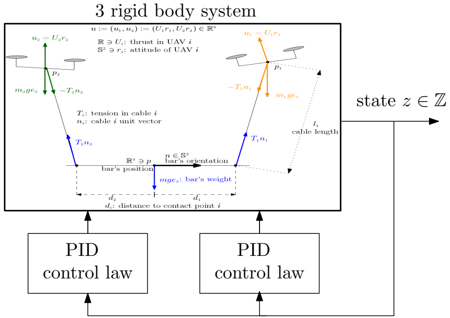

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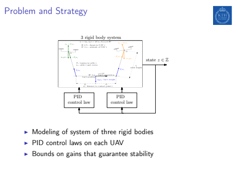
2017-12-12

Collaborative Transportation of a Bar
└Motivation
└Problem and Strategy



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Modeling

$$z = \left[\begin{array}{c} p \\ n \\ p_1 \\ p_2 \\ \hline v \\ \omega \\ v_1 \\ v_2 \\ \hline r_1 \\ r_2 \\ \hline \xi_1 \\ \xi_2 \end{array}\right] \in \mathbb{Z}$$

- State decomposition
- Linear and angular positions of bar + linear and angular velocities of bar
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- Integral states for the PID control laws

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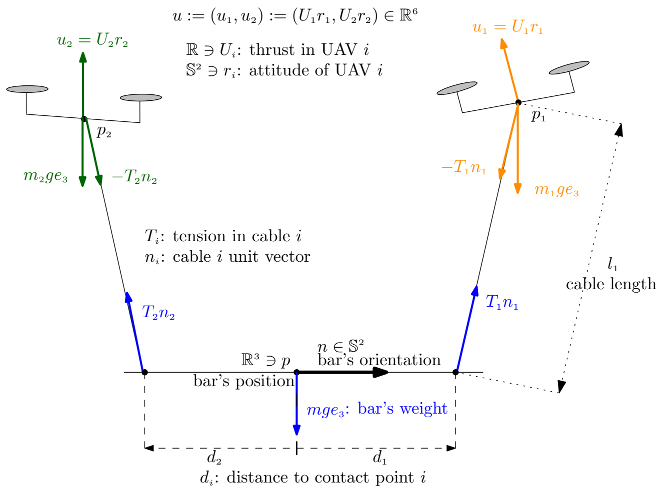

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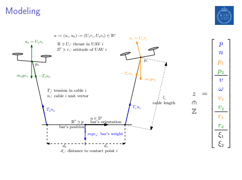

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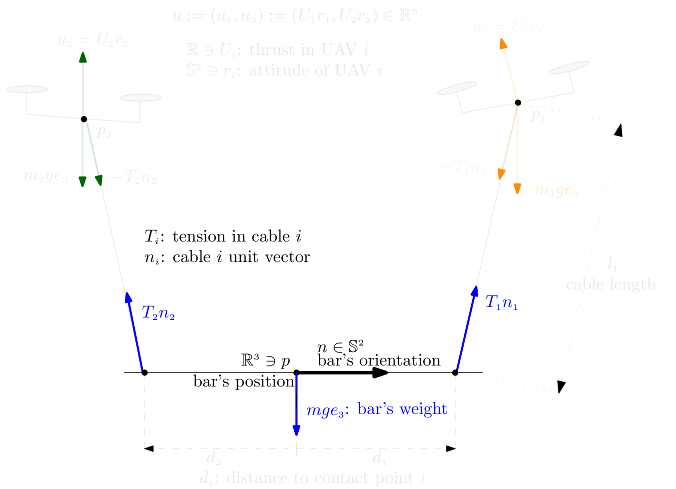

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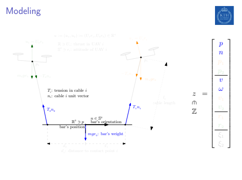

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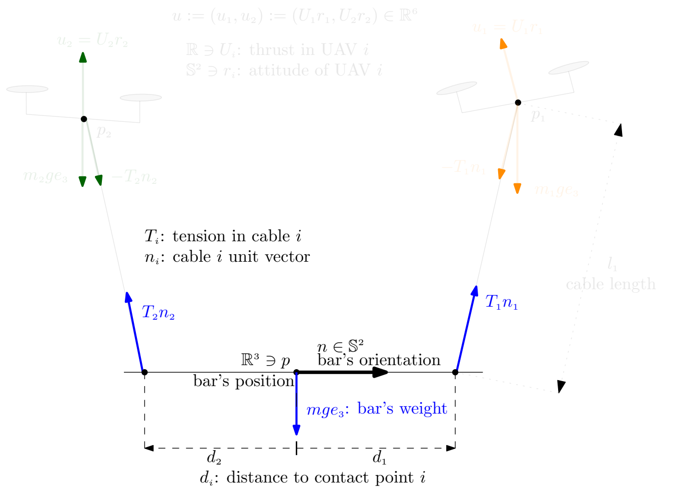

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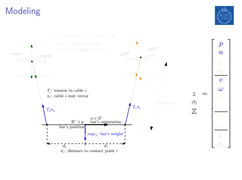

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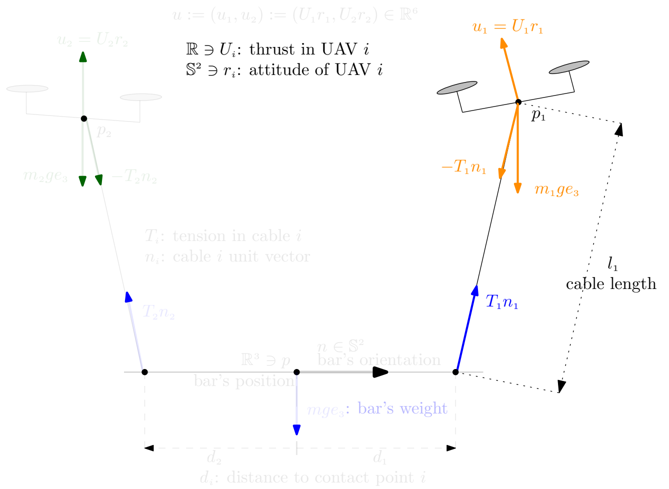

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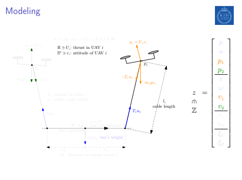

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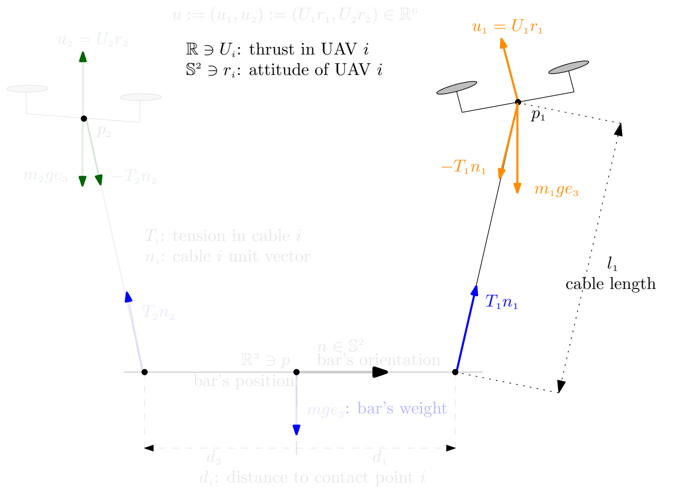

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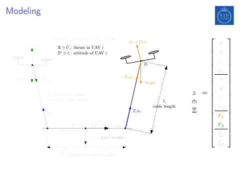
$z \in \mathbb{Z}$



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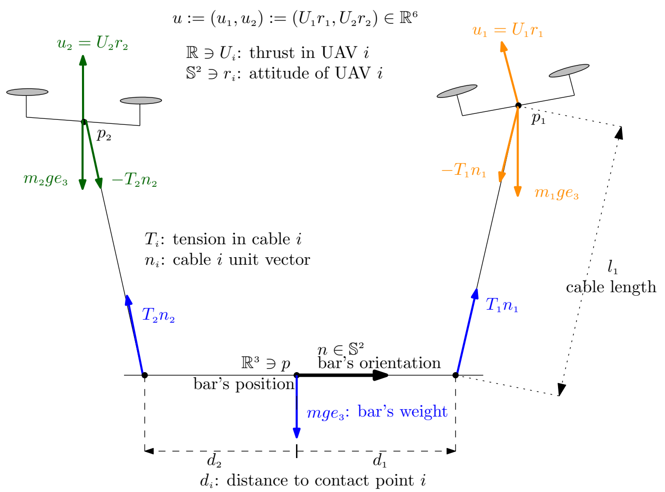

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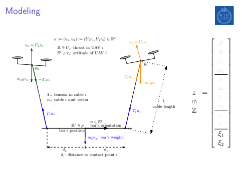

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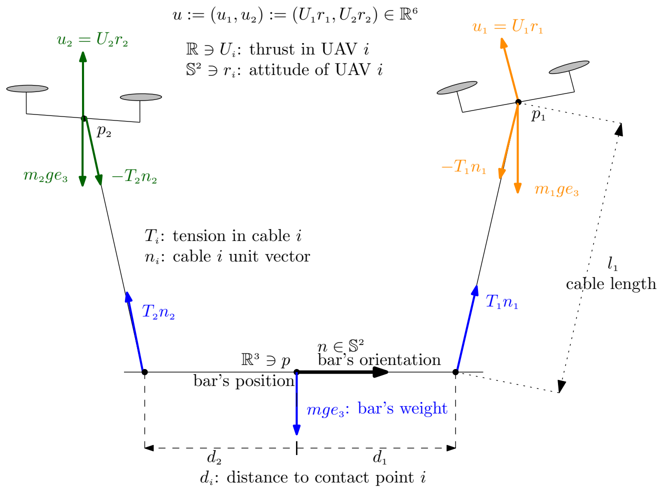

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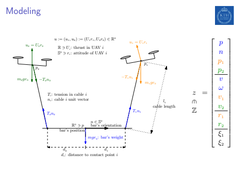

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State nomenclature

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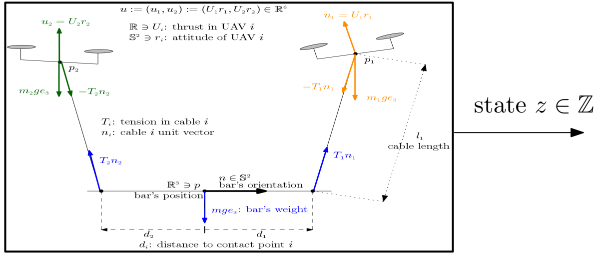
$$

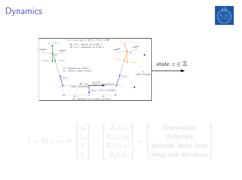
State nomenclature

- A bit more of nomenclature (for convenience)
- z_k: positions
- z_d: velocities
- r: UAVs attitude
- ξ Integral states

Dynamics



$$\dot{z} = Z(z, u) \Leftrightarrow \begin{bmatrix} \dot{z}_k \\ \dot{z}_d \\ \dot{r} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} Z_k(z) \\ Z_d(z,u) \\ Z_r(z,u) \\ Z_\xi(z) \end{bmatrix} = \begin{bmatrix} \text{kinematics} \\ \text{dynamics} \\ \text{attitude inner loop} \\ \text{integrator dynamics} \end{bmatrix}$$



- Replace model with “open-loop vector field”
- Explain four components individually next

Dynamics

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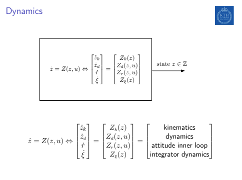

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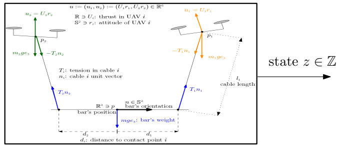
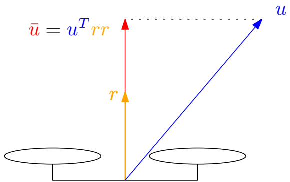
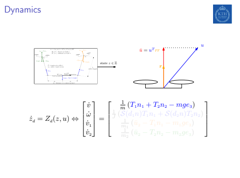
- kinematic equations
- standard equations

Dynamics



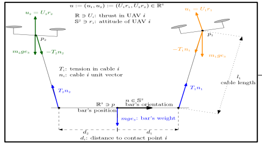


$$\dot{z}_d = Z_d(z,u) \Leftrightarrow \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\left(T_1 n_1 + T_2 n_2 - m g e_3\right) \\ \frac{1}{J}\left(\mathcal{S}(d_1 n) T_1 n_1 + \mathcal{S}(d_2 n) T_2 n_2\right) \\ \frac{1}{m_1}\left(\bar{u}_2 - T_1 n_1 - m_1 g e_3\right) \\ \frac{1}{m_2}\left(\bar{u}_2 - T_2 n_2 - m_2 g e_3\right) \end{bmatrix}$$



- g stands for the acceleration due to gravity
- T_1, T_2 stand for the tensions on the cables
- tensions on the cables are functions of the state and the input
- Cable direction: $n_1 \equiv n_1(z) = \frac{p_1 - (p + d_1 n)}{l_1}$
- Input force: $\bar{u}_1 = u_1^T r_1 r_1$
- Tension on cable: $T_1 = T_1(z, \bar{u}) = \ldots$

Dynamics



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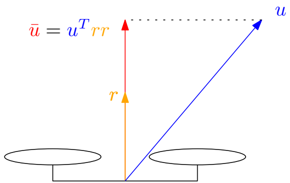

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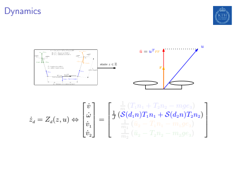

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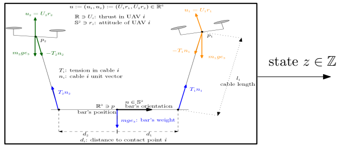
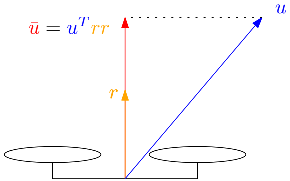
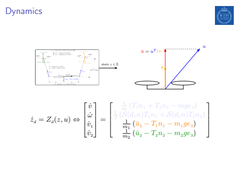
- Input force: $\bar{u}_1 = u_1^T r_1 r_1$
- Tension on cable: $T_1 = T_1(z, \bar{u}) = \ldots$

Dynamics





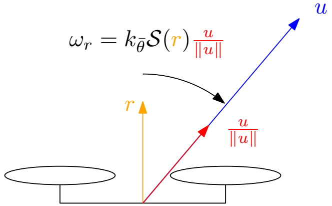
$$\dot{z}_d = Z_d(z,u) \Leftrightarrow \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\left(T_1 n_1 + T_2 n_2 - mge_3\right) \\ \frac{1}{J}\left(\mathcal{S}(d_1 n) T_1 n_1 + \mathcal{S}(d_2 n) T_2 n_2\right) \\ \frac{1}{m_1}\left(\bar{u}_2 - T_1 n_1 - m_1 g e_3\right) \\ \frac{1}{m_2}\left(\bar{u}_2 - T_2 n_2 - m_2 g e_3\right) \end{bmatrix}$$



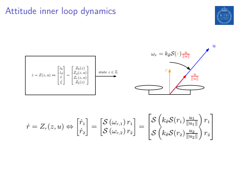
- g stands for the acceleration due to gravity
- T_1, T_2 stand for the tensions on the cables
- tensions on the cables are functions of the state and the input
- Cable direction: $n_1 \equiv n_1(z) = \frac{p_1 - (p + d_1 n)}{l_1}$
- Input force: $\bar{u}_1 = u_1^T r_1 r_1$
- Tension on cable: $T_1 = T_1(z, \bar{u}) = \dots$

Attitude inner loop dynamics

$$\dot{z} = Z(z,u) \Leftrightarrow \begin{bmatrix} \dot{z}_k \\ \dot{z}_d \\ \dot{r} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} Z_k(z) \\ Z_d(z,u) \\ Z_r(z,u) \\ Z_\xi(z) \end{bmatrix} \xrightarrow{\text{state } z \in \mathbb{Z}}$$



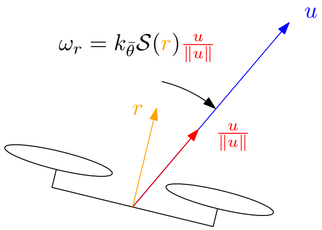
$$\dot{r} = Z_r(z,u) \Leftrightarrow \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{S}\left(\omega_{r,1}\right) r_1 \\ \mathcal{S}\left(\omega_{r,2}\right) r_2 \end{bmatrix} = \begin{bmatrix} \mathcal{S}\left(k_{\bar{\theta}} \mathcal{S}(r_1) \frac{u_1}{\|u_1\|}\right) r_1 \\ \mathcal{S}\left(k_{\bar{\theta}} \mathcal{S}(r_2) \frac{u_2}{\|u_2\|}\right) r_2 \end{bmatrix}$$



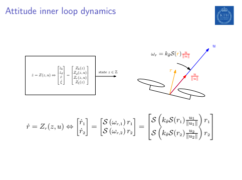
- Attitude inner loop: makes the UAV track the desired attitude
- Desired attitude is given by the $\frac{u}{\|u\|}$

Attitude inner loop dynamics

$$\dot z = Z(z,u) \Leftrightarrow \begin{bmatrix}\dot z_k\\ \dot z_d\\ \dot r\\ \dot \xi\end{bmatrix} = \begin{bmatrix}Z_k(z)\\ Z_d(z,u)\\ Z_r(z,u)\\ Z_\xi(z)\end{bmatrix} \xrightarrow{\text{state } z \in \mathbb{Z}}$$



$$\dot r = Z_r(z,u) \Leftrightarrow \begin{bmatrix}\dot r_1\\ \dot r_2\end{bmatrix} = \begin{bmatrix}\mathcal{S}\left(\omega_{r,1}\right) r_1\\ \mathcal{S}\left(\omega_{r,2}\right) r_2\end{bmatrix} = \begin{bmatrix}\mathcal{S}\left(k_{\bar\theta}\mathcal{S}(r_1)\frac{u_1}{\|u_1\|}\right) r_1\\ \mathcal{S}\left(k_{\bar\theta}\mathcal{S}(r_2)\frac{u_2}{\|u_2\|}\right) r_2\end{bmatrix}$$



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Attitude inner loop dynamics

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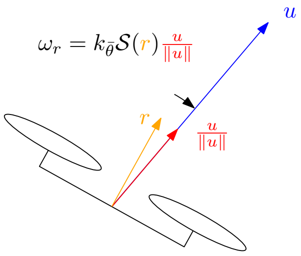

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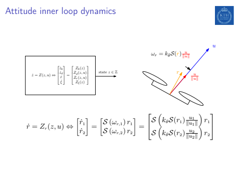

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Attitude inner loop dynamics

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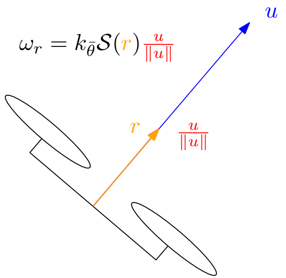
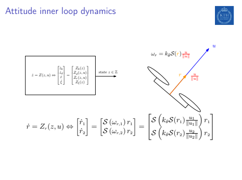
$$\omega_r = k_{\bar{\theta}} \mathcal{S}(r) \frac{u}{\|u\|}$$



$$\dot{r} = Z_r(z,u) \Leftrightarrow \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{S}\left(\omega_{r,1}\right) r_1 \\ \mathcal{S}\left(\omega_{r,2}\right) r_2 \end{bmatrix} = \begin{bmatrix} \mathcal{S}\left(k_{\bar{\theta}} \mathcal{S}(r_1) \frac{u_1}{\|u_1\|}\right) r_1 \\ \mathcal{S}\left(k_{\bar{\theta}} \mathcal{S}(r_2) \frac{u_2}{\|u_2\|}\right) r_2 \end{bmatrix}$$



- Attitude inner loop: makes the UAV track the desired attitude
- Desired attitude is given by the $\frac{u}{\|u\|}$

Integrator dynamics

$$\dot{z} = Z(z,u) \Leftrightarrow \begin{bmatrix} \dot{z}_k \\ \dot{z}_d \\ \dot{r} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} Z_k(z) \\ Z_d(z,u) \\ Z_r(z,u) \\ Z_\xi(z) \end{bmatrix} \xrightarrow{\text{state } z \in \mathbb{Z}}$$

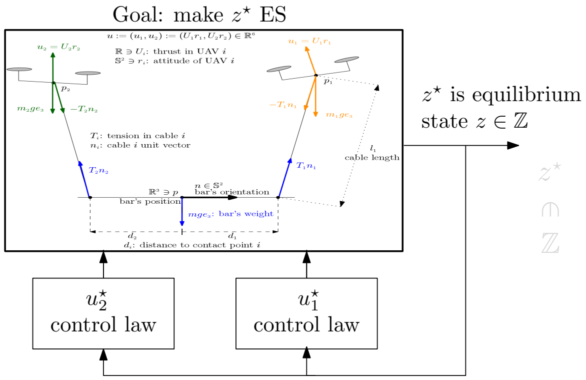
- Integral action adds robustness against model uncertainties

$$\dot{\xi} = Z_\xi(z,u) \Leftrightarrow \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} e_3^T p_1 - l_1 \\ e_3^T p_2 - l_2 \end{bmatrix}$$

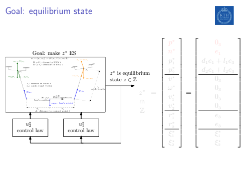
Integrator dynamics
Integral action adds robustness against model uncertainties

- At equilibrium, input must be the "equilibrium input"
- However, this equilibrium input depends on model parameters which are not exactly know (and so that input cannot be exactly provided)
- Robustness against model uncertainties: unknown bar mass for example
- Equilibrium input depends on model parameters

Goal: equilibrium state

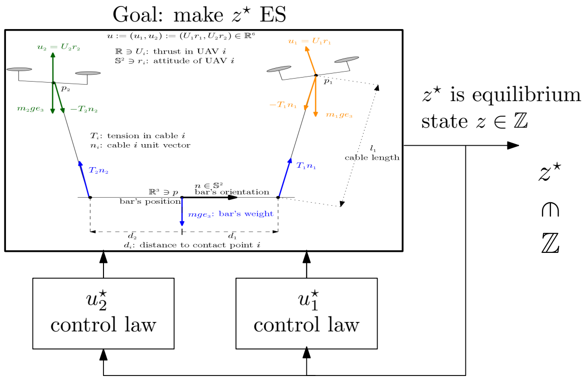


$$z^\star \in \mathbb{Z} = \begin{bmatrix} p^\star \\ n^\star \\ p_1^\star \\ p_2^\star \\ \hline v^\star \\ \omega^\star \\ v_1^\star \\ v_2^\star \\ \hline r_1^\star \\ r_2^\star \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix} = \begin{bmatrix} 0_3 \\ e_1 \\ d_1 e_1 + l_1 e_3 \\ d_2 e_1 + l_2 e_3 \\ \hline 0_3 \\ 0_3 \\ 0_3 \\ 0_3 \\ \hline e_3 \\ e_3 \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix}$$

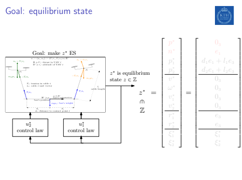


- In order to talk about stability, we need to talk about equilibria points
- Any other $p^\star \in \mathbb{R}^3$ and $n^\star \in \mathbb{S}^2$ (provided that $e_3^T n^\star = 0$) would work as well (without loss of generality, we can take $p^\star = 0_3$ and $n^\star = e_1$)

Goal: equilibrium state



$$\begin{array}{c} z^\star \\ \Cap \\ \mathbb{Z} \end{array} = \begin{bmatrix} p^\star \\ n^\star \\ p_1^\star \\ p_2^\star \\ \hline v^\star \\ \omega^\star \\ v_1^\star \\ v_2^\star \\ \hline r_1^\star \\ r_2^\star \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix} = \begin{bmatrix} 0_3 \\ e_1 \\ d_1 e_1 + l_1 e_3 \\ d_2 e_1 + l_2 e_3 \\ \hline 0_3 \\ 0_3 \\ 0_3 \\ 0_3 \\ \hline e_3 \\ e_3 \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix}$$



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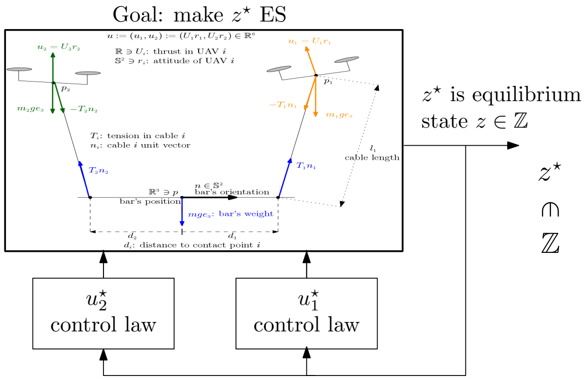
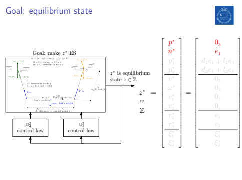
Goal: equilibrium state



$$z^\star \in \mathbb{Z} \qquad z^\star = \begin{bmatrix} p^\star \\ n^\star \\ p_1^\star \\ p_2^\star \\ \hline v^\star \\ \omega^\star \\ v_1^\star \\ v_2^\star \\ \hline r_1^\star \\ r_2^\star \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix} = \begin{bmatrix} 0_3 \\ e_1 \\ d_1 e_1 + l_1 e_3 \\ d_2 e_1 + l_2 e_3 \\ \hline 0_3 \\ 0_3 \\ 0_3 \\ 0_3 \\ \hline e_3 \\ e_3 \\ \hline \xi_1^\star \\ \xi_2^\star \end{bmatrix}$$



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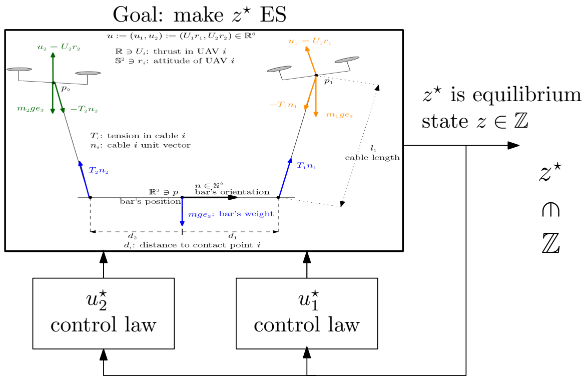
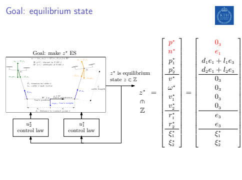
Goal: equilibrium state



$$
\begin{array}{c} z^\star \\ \Cap \\ \mathbb{Z} \end{array}
=
\left[\begin{array}{c}
\color{red}{p^\star} \\ \color{red}{n^\star} \\ p_1^\star \\ p_2^\star \\ \hline v^\star \\ \omega^\star \\ v_1^\star \\ v_2^\star \\ \hline r_1^\star \\ r_2^\star \\ \hline \xi_1^\star \\ \xi_2^\star
\end{array}\right]
=
\left[\begin{array}{c}
\color{red}{0_3} \\ \color{red}{e_1} \\ d_1 e_1 + l_1 e_3 \\ d_2 e_1 + l_2 e_3 \\ \hline 0_3 \\ 0_3 \\ 0_3 \\ 0_3 \\ \hline e_3 \\ e_3 \\ \hline \xi_1^\star \\ \xi_2^\star
\end{array}\right]
$$



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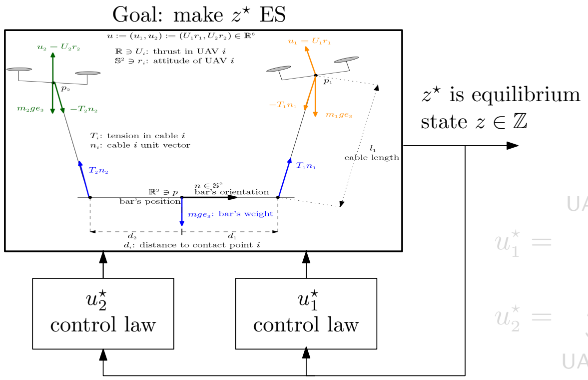
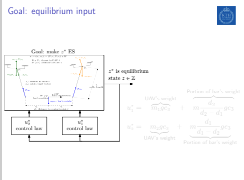
Goal: equilibrium input



$$u_1^\star = \overbrace{m_1 g e_3}^{\text{UAV's weight}} + \overbrace{m \frac{d_2}{d_2 - d_1} g e_3}^{\text{Portion of bar's weight}}$$

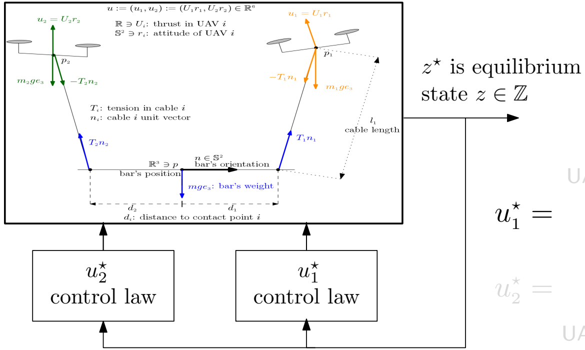
$$u_2^\star = \underbrace{m_2 g e_3}_{\text{UAV's weight}} + \underbrace{m \frac{d_1}{d_1 - d_2} g e_3}_{\text{Portion of bar's weight}}$$



- Equilibrium input depends on bar's weight and the weight of the UAV itself, which are not precisely known.
- Motivation for including an integral action term (only in vertical direction).

Goal: equilibrium input

Goal: make $z^\star$ ES



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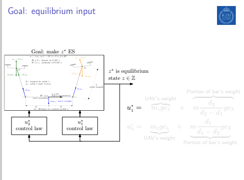

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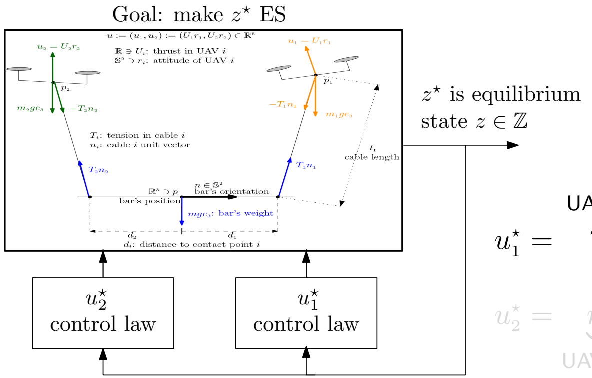
Goal: equilibrium input



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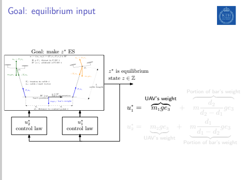

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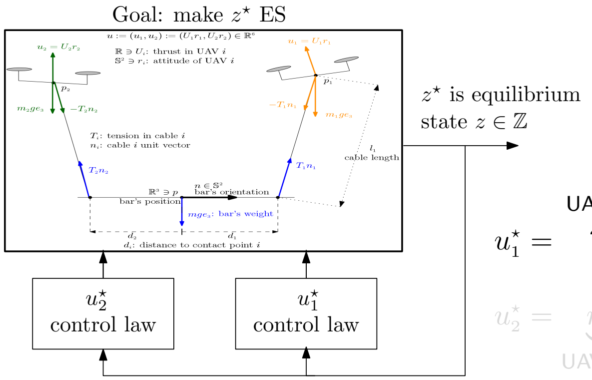
Goal: equilibrium input



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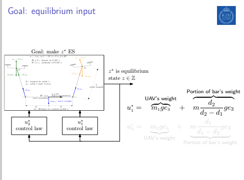

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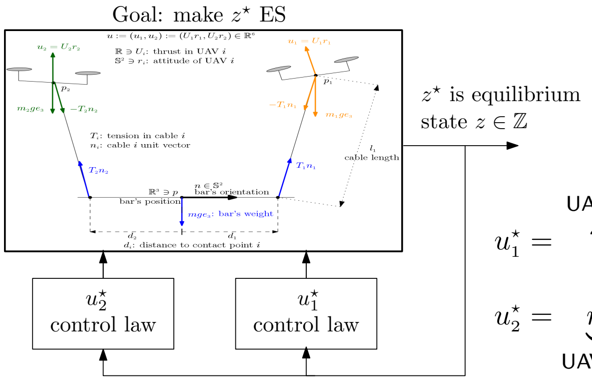

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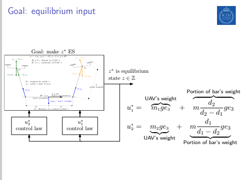

- Equilibrium input depends on bar's weight and the weight of the UAV itself, which are not precisely known.
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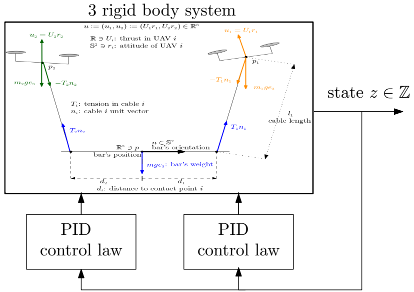
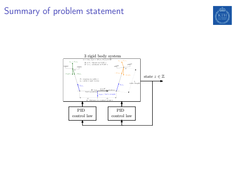
Summary of problem statement





- Close the loop
- Analyze stability by linearizing

Control laws applied on each UAV

PID control law

For each vehicle $i \in \{1,2\}$,

$$u_i^{pid}(z) = \begin{bmatrix} -m_i(k_{p,x}e_1^T(p_i - p_i^\star) + k_{d,x}e_1^T v_i) \\ -m_i(k_{p,y}e_2^T(p_i - p_i^\star) + k_{d,y}e_2^T v_i) \\ u_i^\star - \left(m_i + \frac{m}{2}\right)\left(k_{p,z}e_3^T(p_i - p_i^\star) + k_{d,z}e_3^T v_i + k_{i,z}\xi_i\right) \end{bmatrix}$$

Equilibrium integral terms: corrupted control law $u_i^{pid}|_{m=\hat{m}}$

$$\begin{bmatrix} \xi_1^\star \\ \xi_2^\star \end{bmatrix} = \begin{bmatrix} \frac{g}{k_{i,z}} \frac{m-\hat{m}}{m_1+\frac{\hat{m}}{2}} \frac{d_2}{d_2-d_1} \\ \frac{g}{k_{i,z}} \frac{m-\hat{m}}{m_2+\frac{\hat{m}}{2}} \frac{d_1}{d_1-d_2} \end{bmatrix} \Rightarrow u_i^{pid}|_{m=\hat{m}}(z^\star) = u_i^\star$$

- Integral term makes sure that equilibrium input is attained
- If $\hat{m} = m$, then $\xi_1^\star = \xi_2^\star = 0$.

Control laws applied on each UAV

PID control law

For each vehicle $i \in \{1, 2\}$,

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Equilibrium integral terms: corrupted control law $u_i^{pid}|_{m=\hat{m}}$

$$\begin{bmatrix} \xi_1^\star \\ \xi_2^\star \end{bmatrix} = \begin{bmatrix} \frac{g}{k_{i,z}} \frac{m-\hat{m}}{m_1+\frac{\hat{m}}{2}} \frac{d_2}{d_2-d_1} \\ \frac{g}{k_{i,z}} \frac{m-\hat{m}}{m_2+\frac{\hat{m}}{2}} \frac{d_1}{d_1-d_2} \end{bmatrix} \Rightarrow u_i^{pid}|_{m=\hat{m}}(z^\star) = u_i^\star$$

Control laws applied on each UAV

- Integral term makes sure that equilibrium input is attained
- If $\hat{m} = m$, then $\xi_1^\star = \xi_2^\star = 0$.

Control laws applied on each UAV

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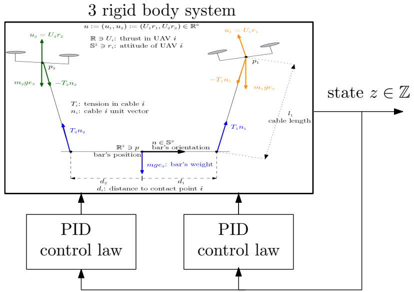
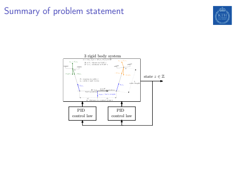
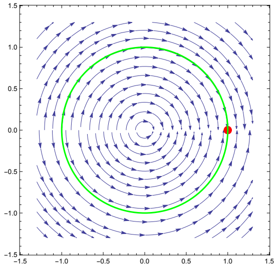
Summary of problem statement





- Close the loop
- Analyze stability by linearizing

Complication when linearizing



Θ

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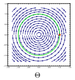
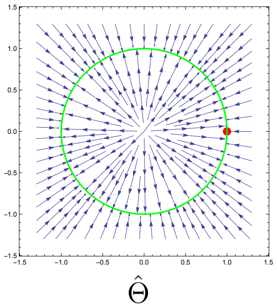


- Our vector field is 6th dimensional, so we cannot visualize it: we show a 2nd dimensional system that pictures the idea described in the next/previous slide
- On the circle, vector fiels are exactly the same
- Outside the circle, they are not
- Another option: parametrize circle with angle, and study stability of equilibrium with the vector field in the "angle coordinate" (however, more complicated sets, require more complicated parametrizations)
- Another option: center manifold theorem

Complication when linearizing



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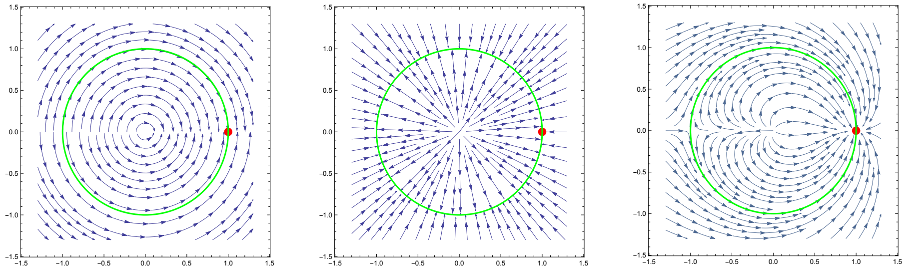
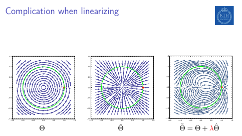


Θ $\qquad$ $\hat{\Theta}$ $\qquad$ $\tilde{\Theta} = \Theta + \lambda\hat{\Theta}$



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Coordinate change to unit vectors

$$
z = \begin{bmatrix} p \\ n \\ p_1 \\ p_2 \\ v \\ \omega \\ v_1 \\ v_2 \\ r_1 \\ r_2 \\ \xi_1 \\ \xi_2 \end{bmatrix} \mapsto g(z) = \left[\begin{array}{c} p \\ v \\ \hdashline n \\ \omega \\ \hdashline \frac{p_1-(p+d_1 n)}{l_1} \\ S\left(\frac{p_1-(p+d_1 n)}{l_1}\right)\frac{v_1-(v+d_1 S(\omega) n)}{l_1} \\ \hdashline \frac{p_2-(p+d_2 n)}{l_2} \\ S\left(\frac{p_2-(p+d_2 n)}{l_2}\right)\frac{v_2-(v+d_2 S(\omega) n)}{l_2} \\ \hdashline r_1 \\ r_2 \\ \hdashline \xi_1 \\ \xi_2 \end{array}\right] = \left[\begin{array}{c} \\ \hdashline \\ \hdashline n_1 \\ \omega_1 \\ \hdashline n_2 \\ \omega_2 \\ \hdashline \\ \hdashline \\ \end{array}\right]
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Inverse of g is needed.

Coordinate change to unit vectors

Inverse of g is needed.

- The inverse of g is found in the article
- The inverse of g is necessary to compute the dynamics in the new coordinates

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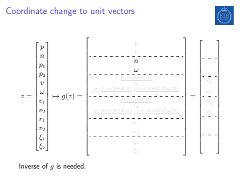

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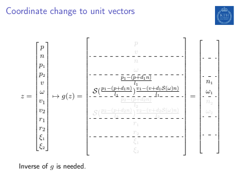

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Dynamics in new coordinates

$$x = g(z) \Rightarrow \dot{x} = \underbrace{Dg(z)Z(z, u^{cl}(z))|_{z=g^{-1}(x)}}_{=:X(x)} \Leftrightarrow \dot{x} = X(x)$$

$$\tilde{X}(x) := X(x) + \lambda \hat{X}(x) \text{ where } \hat{X}(x) = \begin{bmatrix} 0_6 \\ \hat{\Theta}\left(\begin{bmatrix} n \\ \omega \end{bmatrix}\right) \\ \hat{\Theta}\left(\begin{bmatrix} n_1 \\ \omega_1 \end{bmatrix}\right) \\ \hat{\Theta}\left(\begin{bmatrix} n_2 \\ \omega_2 \end{bmatrix}\right) \\ \hat{\Theta}(r_1) \\ \hat{\Theta}(r_2) \\ 0_2 \end{bmatrix}$$

- Need to add “stability” vector fields, whose purpose we illustrated before
- angular motion of bar
- angular motions of cables
- angular motion of UAVs

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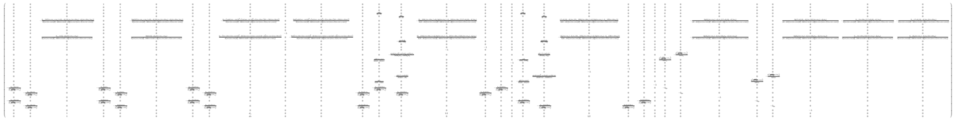
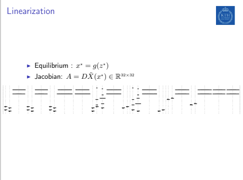
Linearization

- Equilibrium : $x^\star = g(z^\star)$
- Jacobian: $A = D\tilde{X}(x^\star) \in \mathbb{R}^{32\times 32}$





- Objective: determine whether A is Hurwitz
- As is, Jacobian is unstructured: it is sparse though (it has many zeros).
- Find similarity transformation that breaks Jacobian into block diagonal matrix
- Block diagonal entries in controllable form
- Jacobian presented in this slide has the simplifications presented later: without simplifications, if

Similarity transformation

$$P := \begin{bmatrix} P_z & P_\theta & P_x & P_\delta & P_y & P_\psi & P_\perp \end{bmatrix}^T \in \mathbb{R}^{32\times 32}$$

where

$$P_z := \begin{bmatrix} e_{31}+e_{32} & A(e_{31}+e_{32}) & A^2(e_{31}+e_{32}) \end{bmatrix} \in \mathbb{R}^{32\times 3}$$
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$$P_\perp := \begin{bmatrix} e_7 & e_{10} & e_{15} & e_{18} & e_{21} & e_{24} & e_{27} & e_{30} \end{bmatrix} \in \mathbb{R}^{32\times 8}$$

- Idea is that there are decouple motions, and the similarity matrix P is what describes these motions
- Similarity transformation highligths structure of the state-matrix/Jacobian
- A is the Jacobian matrix in the previous slide
- e_i is the the ith canonical basis vector in $\mathbb{R}^{32}$

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$$P_\perp := \begin{bmatrix} e_7 & e_{10} & e_{15} & e_{18} & e_{21} & e_{24} & e_{27} & e_{30} \end{bmatrix} \in \mathbb{R}^{32\times 8}$$

- Idea is that there are decouple motions, and the similarity matrix P is what describes these motions
- Similarity transformation highligths structure of the state-matrix/Jacobian
- A is the Jacobian matrix in the previous slide
- e_i is the the ith canonical basis vector in $\mathbb{R}^{32}$

Similarity transformation

$$P := \begin{bmatrix} P_z & P_\theta & P_x & P_\delta & P_y & P_\psi & P_\perp \end{bmatrix}^T \in \mathbb{R}^{32\times 32}$$

where

$$P_z := \begin{bmatrix} e_{31}+e_{32} & A(e_{31}+e_{32}) & A^2(e_{31}+e_{32}) \end{bmatrix} \in \mathbb{R}^{32\times 3}$$
$$P_\theta := \begin{bmatrix} e_{31}-e_{32} & A(e_{31}-e_{32}) & A^2(e_{31}-e_{32}) \end{bmatrix} \in \mathbb{R}^{32\times 3}$$
$$P_y := \begin{bmatrix} e_2 & Ae_2 & A^2e_2 & A^3e_2 & A^4e_2 \end{bmatrix} \in \mathbb{R}^{32\times 5}$$
$$P_\psi := \begin{bmatrix} e_8 & Ae_8 & A^2e_8 & A^3e_8 & A^4e_8 \end{bmatrix} \in \mathbb{R}^{32\times 5}$$
$$P_x := \begin{bmatrix} e_1 & Ae_1 & A^2e_1 & A^3e_1 & A^4e_1 \end{bmatrix} \in \mathbb{R}^{32\times 5}$$
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$$P_\perp := \begin{bmatrix} e_7 & e_{10} & e_{15} & e_{18} & e_{21} & e_{24} & e_{27} & e_{30} \end{bmatrix} \in \mathbb{R}^{32\times 8}$$

Similarity transformation

$P := [P_z \; P_\theta \; P_x \; P_\delta \; P_y \; P_\psi \; P_\perp]^T \in \mathbb{R}^{32\times 32}$

where

$P_x := [e_1 \; Ae_1 \; A^2e_1 \; A^3e_1 \; A^4e_1] \in \mathbb{R}^{32\times 5}$
$P_\delta := [(e_{13}-e_{19}) \; A(e_{13}-e_{19}) \; A^2(e_{13}-e_{19})] \in \mathbb{R}^{32\times 3}$

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Similarity transformation

$$P := \begin{bmatrix} P_z & P_\theta & P_x & P_\delta & P_y & P_\psi & P_\perp \end{bmatrix}^T \in \mathbb{R}^{32\times 32}$$

where

$$P_z := \begin{bmatrix} e_{31}+e_{32} & A(e_{31}+e_{32}) & A^2(e_{31}+e_{32}) \end{bmatrix} \in \mathbb{R}^{32\times 3}$$

$$P_\theta := \begin{bmatrix} e_{31}-e_{32} & A(e_{31}-e_{32}) & A^2(e_{31}-e_{32}) \end{bmatrix} \in \mathbb{R}^{32\times 3}$$

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$$P_\psi := \begin{bmatrix} e_8 & Ae_8 & A^2e_8 & A^3e_8 & A^4e_8 \end{bmatrix} \in \mathbb{R}^{32\times 5}$$

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Similarity transformation

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- Idea is that there are decouple motions, and the similarity matrix P is what describes these motions
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- A is the Jacobian matrix in the previous slide
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Similar state matrix

$$
PAP^{-1} = \left[\begin{array}{ccc:c}
\begin{bmatrix} A_z & \star \\ \star & A_\theta \end{bmatrix} & & & \\
& \begin{bmatrix} A_x & \star \\ \star & A_\delta \end{bmatrix} & & \star \\
& & \begin{bmatrix} A_y & \star \\ \star & A_\psi \end{bmatrix} & \\
\hdashline
& 0_{8\times 32} & & -\lambda I_{8\times 8}
\end{array}\right]
$$

Assumptions

- Identical UAVs: $m_1 = m_2 = M$
- Identical cables: $l_1 = l_2 = l$
- Opposite contact points: $d_1 = -d_2 = d$

- Block triangular matrix, where bigger block is a block diagonal matrix.
- **3 smaller problems to check** (the block diagonal matrices need to be Hurwitz).
- Decoupled motions: vertical, lateral and longitudinal

$$
PAP^{-1} = \left[\begin{array}{c:c}
\begin{bmatrix} A_z & \star \\ \star & A_\theta \end{bmatrix} \oplus \begin{bmatrix} A_x & \star \\ \star & A_\delta \end{bmatrix} \oplus \begin{bmatrix} A_y & \star \\ \star & A_\psi \end{bmatrix} & \star \\
\hdashline
0_{8\times 32} & -\lambda I_{8\times 8}
\end{array}\right]
$$

Similar state matrix

$$PAP^{-1} = \left[\begin{array}{cccccc:c} A_z & \star & & & & & \\ \star & A_\theta & & & & & \\ & & A_x & \star & & & \\ & & \star & A_\delta & & & \star \\ & & & & A_y & \star & \\ & & & & \star & A_\psi & \\ \hdashline & & & 0_{8\times32} & & & -\lambda I_{8\times8} \end{array}\right]$$

Assumptions

- Identical UAVs: $m_1 = m_2 = M$
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Similar state matrix

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- Identical UAVs: $m_1 = m_2 = M$
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- Block triangular matrix, where bigger block is a block diagonal matrix.
- **3 smaller problems to check** (the block diagonal matrices need to be Hurwitz).
- Decoupled motions: vertical, lateral and longitudinal

$$PAP^{-1} = \left[\begin{array}{c:c} \begin{bmatrix} A_z & \star \\ \star & A_\theta \end{bmatrix} \oplus \begin{bmatrix} A_x & \star \\ \star & A_\delta \end{bmatrix} \oplus \begin{bmatrix} A_y & \star \\ \star & A_\psi \end{bmatrix} & \star \\ \hdashline 0_{8\times32} & -{\color{red}\lambda} I_{8\times8} \end{array}\right]$$

Similar state matrix with simplifications

$$
PAP^{-1} = \left[\begin{array}{cccccc|c}
A_z & 0 & & & & & \\
0 & A_\theta & & & & & \\
 & & A_x & 0 & & & \\
 & & 0 & A_\delta & & & \star \\
 & & & & A_y & 0 & \\
 & & & & 0 & A_\psi & \\
\hline
 & & & 0_{8\times 32} & & & -\lambda I_{8\times 8}
\end{array}\right]
$$

Assumptions

- Identical UAVs: $m_1 = m_2 = M$
- Identical cables: $l_1 = l_2 = l$
- Opposite contact points: $d_1 = -d_2 = d$

Similar state matrix with simplifications
Assumptions
Identical UAVs: $m_1 = m_2 = M$
Identical cables: $l_1 = l_2 = l$
Opposite contact points: $d_1 = -d_2 = d$

- Block triangular matrix, where bigger block is a block diagonal matrix.
- **6 smaller problems to check** (the block diagonal matrices need to be Hurwitz).

$$
PAP^{-1} = \begin{bmatrix} A_z \oplus A_\theta \oplus A_x \oplus A_\delta \oplus A_y \oplus A_\psi & \star \\ 0_{8\times 32} & -\lambda I_{8\times 8} \end{bmatrix}
$$

Lateral-motion

- Bar's linear lateral-motion y + bar's angular lateral-motion ψ
- Linearized motions governed by state matrices

$$A_y = \Gamma_5(p,k)\,|_{p=(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar{\theta}}),k=(k_{p,y},k_{d,y})}$$
$$A_\psi = \Gamma_5(p,k)\,|_{p=(\frac{g}{l}\frac{l^2 m}{J},\frac{g}{l}\frac{m}{2M},k_{\bar{\theta}}),k=(k_{p,y},k_{d,y})}$$

where

$$\Gamma_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_3 p_1 k_p & p_3 p_1 k_d & p_3(k_p + p_1 + p_2) & p_3 k_d + p_1 + p_2 & p_3 \end{bmatrix}$$

- A_y and A_ψ are Hurwitz iff $k_{\bar{\theta}} > \frac{k_{p,y}}{k_{d,y}}$.
- Both motions are those of 5th order integrators

$$y^{(5)} = (A_y)_{5,5}y^{(4)} + \cdots + (A_y)_{5,1}y^{(0)}$$
$$\psi^{(5)} = (A_\psi)_{5,2}\psi^{(2)} + \cdots + (A_\psi)_{5,1}\psi^{(0)}$$

- linear y-motion of bar
- angular y-motion of bar
- View: the 5th order integrator motions are stable if the attitude inner loop is fast with respect to the ratio $\frac{k_{p,y}}{k_{d,y}}$.
- View: the 5th order integrator motions are stable if the ratio $\frac{k_{p,y}}{k_{d,y}}$ is small enough with respect to the attitude inner loop gain.

Lateral-motion

- Bar's linear lateral-motion y + bar's angular lateral-motion ψ
- Linearized motions governed by state matrices

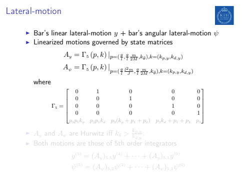
$$A_y = \Gamma_5\left(p,k\right)\big|_{p=(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar\theta}),k=(k_{p,y},k_{d,y})}$$
$$A_\psi = \Gamma_5\left(p,k\right)\big|_{p=(\frac{g}{l}\frac{l^2m}{J},\frac{g}{l}\frac{m}{2M},k_{\bar\theta}),k=(k_{p,y},k_{d,y})}$$

where

$$\Gamma_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_3p_1k_p & p_3p_1k_d & p_3(k_p+p_1+p_2) & p_3k_d+p_1+p_2 & p_3 \end{bmatrix}$$

- A_y and A_ψ are Hurwitz iff $k_{\bar\theta} > \frac{k_{p,y}}{k_{d,y}}$.
- Both motions are those of 5th order integrators

$$y^{(5)} = (A_y)_{5,5}y^{(4)} + \cdots + (A_y)_{5,1}y^{(0)}$$
$$\psi^{(5)} = (A_\psi)_{5,2}\psi^{(2)} + \cdots + (A_\psi)_{5,1}\psi^{(0)}$$



- linear y-motion of bar
- angular y-motion of bar
- View: the 5th order integrator motions are stable if the attitude inner loop is fast with respect to the ratio $\frac{k_{p,y}}{k_{d,y}}$.
- View: the 5th order integrator motions are stable if the ratio $\frac{k_{p,y}}{k_{d,y}}$ is small enough with respect to the attitude inner loop gain.

Lateral-motion

- Bar's linear lateral-motion y + bar's angular lateral-motion ψ
- Linearized motions governed by state matrices

$$A_y = \Gamma_5\left(p,k\right)\big|_{p=\left(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar\theta}\right),k=\left(k_{p,y},k_{d,y}\right)}$$
$$A_\psi = \Gamma_5\left(p,k\right)\big|_{p=\left(\frac{g}{l}\frac{l^2 m}{J},\frac{g}{l}\frac{m}{2M},k_{\bar\theta}\right),k=\left(k_{p,y},k_{d,y}\right)}$$

where

$$\Gamma_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_3 p_1 k_p & p_3 p_1 k_d & p_3(k_p + p_1 + p_2) & p_3 k_d + p_1 + p_2 & p_3 \end{bmatrix}$$

- A_y and A_ψ are Hurwitz iff $k_{\bar\theta} > \frac{k_{p,y}}{k_{d,y}}$.
- Both motions are those of 5th order integrators

$$y^{(5)} = (A_y)_{5,5} y^{(4)} + \cdots + (A_y)_{5,1} y^{(0)}$$
$$\psi^{(5)} = (A_\psi)_{5,2} \psi^{(2)} + \cdots + (A_\psi)_{5,1} \psi^{(0)}$$

- linear y-motion of bar
- angular y-motion of bar
- View: the 5th order integrator motions are stable if the attitude inner loop is fast with respect to the ratio $\frac{k_{p,y}}{k_{d,y}}$.
- View: the 5th order integrator motions are stable if the ratio $\frac{k_{p,y}}{k_{d,y}}$ is small enough with respect to the attitude inner loop gain.

Lateral-motion

- Bar's linear lateral-motion y + bar's angular lateral-motion ψ
- Linearized motions governed by state matrices

$$A_y = \Gamma_5(p,k)\left|_{p=(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar{\theta}}),k=(k_{p,y},k_{d,y})}\right.$$
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where

$$\Gamma_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_3p_1k_p & p_3p_1k_d & p_3(k_p+p_1+p_2) & p_3k_d+p_1+p_2 & p_3 \end{bmatrix}$$

- A_y and A_ψ are Hurwitz iff $k_{\bar{\theta}} > \frac{k_{p,y}}{k_{d,y}}$.
- Both motions are those of 5th order integrators

$$y^{(5)} = (A_y)_{5,5}y^{(4)} + \cdots + (A_y)_{5,1}y^{(0)}$$
$$\psi^{(5)} = (A_\psi)_{5,2}\psi^{(2)} + \cdots + (A_\psi)_{5,1}\psi^{(0)}$$

- linear y-motion of bar
- angular y-motion of bar
- View: the 5th order integrator motions are stable if the attitude inner loop is fast with respect to the ratio $\frac{k_{p,y}}{k_{d,y}}$.
- View: the 5th order integrator motions are stable if the ratio $\frac{k_{p,y}}{k_{d,y}}$ is small enough with respect to the attitude inner loop gain.

Longitudinal-motion

- Bar's longitudinal motion $= x$
- Longitudinal relative motion between UAVs $= \delta$
- Linearized motions governed by state matrices

$$A_x = \Gamma_5\left(p,k\right)|_{p=(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar{\theta}}),k=(k_{p,x},k_{d,x})}$$

$$A_\delta = \Gamma_3\left(p,k\right)|_{p=(\frac{g}{l},\frac{g}{l}\frac{m}{2M},k_{\bar{\theta}}),k=(k_{p,x},k_{d,x})}$$

where

$$\Gamma_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_3 p_1 k_p & p_3 p_1 k_d & p_3(k_p + p_1 + p_2) & p_3 k_d + p_1 + p_2 & p_3 \end{bmatrix} \text{ and } \Gamma_3 = \ldots$$

- A_x and A_δ are Hurwitz iff $k_{\bar{\theta}} > \frac{k_{p,x}}{k_{d,x}}$.
- x-motion is of a 5th$^{\text{order}}$ int., and δ-motion is of a 3rd$^{\text{order}}$ int.

$$x^{(5)} = (A_x)_{5,5}x^{(4)} + \cdots + (A_x)_{5,1}x^{(0)}$$

$$\delta^{(3)} = (A_\delta)_{3,2}\delta^{(2)} + \cdots + (A_\delta)_{3,1}\delta^{(0)}$$

- $x =$ bar longitudinal motion
- $\delta =$ longitudinal relative motion between UAVs
- View: the matrices are Hurwitz if the attitude inner loop is fast with respect to the ratio $\frac{k_{p,x}}{k_{d,x}}$.
- View: the matrices are Hurwitz if the ratio $\frac{k_{p,x}}{k_{d,x}}$ is small enough with respect to the attitude inner loop gain.

Vertical-motion

- Bar's linear vertical-motion z + bar's angular vertical-motion θ
- Linearized motions governed by state matrices A_z and A_θ

$$A_j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_j k_{i,z} & \gamma_j k_{p,z} & \gamma_j k_{d,z} \end{bmatrix}$$

where

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

- A_θ and A_θ are Hurwitz iff

$$k_{i,z} < \min(\gamma_z, \gamma_\theta)\, k_{p,z} k_{d,z}$$

- z- and θ-motions are of a 3rd$^{\text{order}}$ integrator

$$z^{(2)} = (A_z)_{3,3} z^{(1)} + (A_z)_{3,2} z^{(0)} + (A_z)_{3,1} z^{(-1)}$$
$$\theta^{(2)} = (A_\theta)_{3,3} \theta^{(1)} + (A_\theta)_{3,2} \theta^{(0)} + (A_\theta)_{3,1} \theta^{(-1)}$$

- Note that $e_1^T P_z z = \xi_1 + \xi_2$ and that $e_2^T P_z z =: 2e_3^T p =: 2p_z$, i.e., the sum of the integral errors is related to the z-position of the bar.
- Note that $e_1^T P_\theta z = \xi_1 - \xi_2$ and $e_2^T P_\theta z = -2le_3^T n =: -2l\theta$, i.e., difference between the integral errors is related to the z-attitude of the bar.
- Vertical motion stability does not depend on attitude inner loop gain
- When d is really small, hard to guarantee that matrices are Hurwitz (**confirms/validates intuition**)
- When mass is known

$$\gamma_z = 1 \text{ and } \gamma_\theta = \frac{2d^2M + d^2m}{2d^2M + J}$$

- When mass is unknown

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

Vertical-motion

- Bar's linear vertical-motion z + bar's angular vertical-motion θ
- Linearized motions governed by state matrices A_z and A_θ

$$A_j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_j k_{i,z} & \gamma_j k_{p,z} & \gamma_j k_{d,z} \end{bmatrix}$$

where

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

- A_θ and A_θ are Hurwitz iff

$$k_{i,z} < \min(\gamma_z, \gamma_\theta)\, k_{p,z} k_{d,z}$$

- z- and θ-motions are of a 3rd$^{\text{order}}$ integrator

$$z^{(2)} = (A_z)_{3,3} z^{(1)} + (A_z)_{3,2} z^{(0)} + (A_z)_{3,1} z^{(-1)}$$
$$\theta^{(2)} = (A_\theta)_{3,3} \theta^{(1)} + (A_\theta)_{3,2} \theta^{(0)} + (A_\theta)_{3,1} \theta^{(-1)}$$

- Note that $e_1^T P_z z = \xi_1 + \xi_2$ and that $e_2^T P_z z =: 2e_3^T p =: 2p_z$, i.e., the sum of the integral errors is related to the z-position of the bar.
- Note that $e_1^T P_\theta z = \xi_1 - \xi_2$ and $e_2^T P_\theta z = -2l e_3^T n =: -2l\theta$, i.e., difference between the integral errors is related to the z-attitude of the bar.
- Vertical motion stability does not depend on attitude inner loop gain
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$$\gamma_z = 1 \text{ and } \gamma_\theta = \frac{2d^2M + d^2m}{2d^2M + J}$$

- When mass is unknown

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

Vertical-motion

- Bar's linear vertical-motion z + bar's angular vertical-motion θ
- Linearized motions governed by state matrices A_z and A_θ

$$A_j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_j k_{i,z} & \gamma_j k_{p,z} & \gamma_j k_{d,z} \end{bmatrix}$$

where

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

- A_θ and A_θ are Hurwitz iff

$$k_{i,z} < \min\left(\gamma_z, \gamma_\theta\right) k_{p,z} k_{d,z}$$

- z- and θ-motions are of a 3rd$^{\text{order}}$ integrator

$$z^{(2)} = (A_z)_{3,3} z^{(1)} + (A_z)_{3,2} z^{(0)} + (A_z)_{3,1} z^{(-1)}$$

$$\theta^{(2)} = (A_\theta)_{3,3} \theta^{(1)} + (A_\theta)_{3,2} \theta^{(0)} + (A_\theta)_{3,1} \theta^{(-1)}$$

- Note that $e_1^T P_z z = \xi_1 + \xi_2$ and that $e_2^T P_z z =: 2e_3^T p =: 2p_z$, i.e., the sum of the integral errors is related to the z-position of the bar.
- Note that $e_1^T P_\theta z = \xi_1 - \xi_2$ and $e_2^T P_\theta z = -2l e_3^T n =: -2l\theta$, i.e., difference between the integral errors is related to the z-attitude of the bar.
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$$\gamma_z = 1 \text{ and } \gamma_\theta = \frac{2d^2M + d^2m}{2d^2M + J}$$

- When mass is unknown

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2M}{J+2d^2M}$$

Vertical-motion

- Bar's linear vertical-motion z + bar's angular vertical-motion θ
- Linearized motions governed by state matrices A_z and A_θ

$$A_j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_j k_{i,z} & \gamma_j k_{p,z} & \gamma_j k_{d,z} \end{bmatrix}$$

  where

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2 M}{J+2d^2 M}$$

- A_θ and A_θ are Hurwitz iff

$$k_{i,z} < \min\left(\gamma_z, \gamma_\theta\right) k_{p,z} k_{d,z}$$

- z- and θ-motions are of a 3rd$^{\text{order}}$ integrator

$$z^{(2)} = (A_z)_{3,3} z^{(1)} + (A_z)_{3,2} z^{(0)} + (A_z)_{3,1} z^{(-1)}$$
$$\theta^{(2)} = (A_\theta)_{3,3} \theta^{(1)} + (A_\theta)_{3,2} \theta^{(0)} + (A_\theta)_{3,1} \theta^{(-1)}$$

- Note that $e_1^T P_z z = \xi_1 + \xi_2$ and that $e_2^T P_z z =: 2e_3^T p =: 2p_z$, i.e., the sum of the integral errors is related to the z-position of the bar.
- Note that $e_1^T P_\theta z = \xi_1 - \xi_2$ and $e_2^T P_\theta z = -2le_3^T n =: -2l\theta$, i.e., difference between the integral errors is related to the z-attitude of the bar.
- Vertical motion stability does not depend on attitude inner loop gain
- When d is really small, hard to guarantee that matrices are Hurwitz (**confirms/validates intuition**)
- When mass is known

$$\gamma_z = 1 \text{ and } \gamma_\theta = \frac{2d^2 M + d^2 m}{2d^2 M + J}$$

- When mass is unknown

$$\gamma_z = \frac{2M}{2M+m} \text{ and } \gamma_\theta = \frac{2d^2 M}{J+2d^2 M}$$

Main result

Theorem

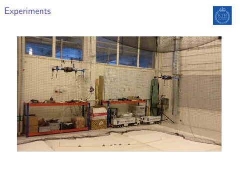
If

- *attitude inner loop is sufficiently fast, i.e.,* $k_{\bar{\theta}} > \min\left(\frac{k_{p,x}}{k_{d,x}}, \frac{k_{p,y}}{k_{d,y}}\right)$
- *integral gain is sufficiently small, i.e.,* $k_{i,z} < \min\left(\gamma_z, \gamma_\theta\right) k_{p,z} k_{d,z}$

then the equilibrium $z^\star$ *is exponentially stable.*

Main result
Theorem
If
- attitude inner loop is sufficiently fast, i.e., $k_{\bar{\theta}} > \min\left(\frac{k_{p,x}}{k_{d,x}}, \frac{k_{p,y}}{k_{d,y}}\right)$
- integral gain is sufficiently small, i.e., $k_{i,z} < \min\left(\gamma_z, \gamma_\theta\right) k_{p,z} k_{d,z}$
then the equilibrium $z^\star$ is exponentially stable.

Experiments



- Two off-the-shelf quadrotors (IRISes+), with $M = 1.442$kg
- Cables of length $l_1 = l_2 = l = 1.4$m
- Bar of length 2m and $m = 0.33$kg
- Contact points away from center-of-mass by $d_1 = -d_2 = d = 1$m
- Gains found in article
- Experiment available at https://youtu.be/ywwPvZuVpF0

Thank you! Questions?

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