An Application of Poincarè's Fundamental Polyhedron Theorem

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Abstract

Given a finitely presented subgroup Γ of $PSL(2, \mathbb{C})$, is Γ discrete? In Section 1, we review the background material necessary for understanding this question in the context of hyperbolic geometry. In Sections 2 and 3, we explain the algorithm proposed in [10] which attempts to answer the question by applying Poincarè's Fundamental Polyhedron Theorem in a manner that is computationally feasible under certain sufficient conditions. In Section 4 we answer a question posed at the end of [10] in the affirmative by showing that one of these sufficient conditions is generically satisfied. Section 5 gives an algorithm for computing the hyperbolic volume of the domain. Section 6 describes our rigorous implementation of the algorithms of the previous sections, with source code available at [11]. Section 7 describes an application of the program to rigorously construct manifolds associated to the exceptional regions defined in [7].

1 Introduction

Recall the quaternionic model of three dimensional hyperbolic space, given by

$$U^3 := \{x + yi + zj \in \mathbb{H} : z > 0\}. \tag{1.1}$$

equipped with Riemannian metric

$$\frac{dx^2 + dy^2 + dz^2}{z}. \tag{1.2}$$

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Then an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2,\mathbb{C})$ acts on U^3 by

$$gq := (aq+b)(cq+d)^{-1}. \tag{1.3}$$

It is straightforward to check that (1.3) defines an action of $SL(2,\mathbb{C})$ by isometries. Clearly $gq = (\lambda g)q$ for any $\lambda \in \mathbb{C}^*$, and in fact

$$PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\pm I \tag{1.4}$$

is the isometry group of U^3. The compact-open topology on $PSL(2,\mathbb{C})$ given by the group action coincides with the topology induced as a quotient of $SL(2,\mathbb{C}) \subset \mathbb{C}^4$ in the subspace topology.

Thus, given $\Gamma < PSL(2,\mathbb{C})$ we consider the quotient topological space U^3/Γ and approach the question of whether Γ is discrete using some tools of hyperbolic geometry, which we now recall.

First suppose that Γ is discrete and U^3/Γ is a compact 3-manifold. Then for any $p \in U^3$ we define the *Dirichlet domain* centered at p by

$$D(p) := \{q \in U^3 : d_H(p,q) < d_H(g(p),q) \ for\ all\ g \neq 1 \in \Gamma\} \tag{1.5}$$

where d_H is the metric on U^3 induced by the Riemannian metric in Equation (1.2). In particular, D is a *fundamental domain* for Γ and its closure $\overline{D}$ is a convex polyhedron with finitely many sides. Each side S corresponds to a group element g_S such that S is contained in the plane

$$P_{g_S} := \{q \in U^3 : d_H(p,q) = d_H(g_S(p),q)\}. \tag{1.6}$$

$D(p)$ has an associated *side-pairing*

$$\Phi := \{g \in \Gamma : g_S \ is\ a\ side\ of\ D(p)\}. \tag{1.7}$$

We also define the closed half-spaces

$$H_g := \{q \in U^3 : d_H(p,q) \leq d_H(g(p),q)\} \tag{1.8}$$

so that

$$\overline{D} = \bigcap_{g\in\Phi} H_g. \tag{1.9}$$

Also recall that for an edge E of a side S, there corresponds a sequence of sides $\{S_i\}$, called an *edge cycle* defined as follows: $S_1 = S$, S_2 is the side of P adjacent S_1 such that $g_{S_1}(S_1' \cap S_2) = E$. For $i > 2$, S_{i+1} is the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.

The Dirichlet domain satisfies several important conditions:

(1) for each side S, there is a side S' such that $g_S(S') = S$;

(2) $g_{S'} = g_S^{-1}$ (side-pairing relation);

(3) there is a least integer k such that $g_{S_1}...g_{S_k} = 1$ (cycle relation);

(4) $\sum_{i=1}^{k} \theta(S_i', S_i+1) = 2\pi$, where θ is the dihedral angle between two sides;

(5) no point of $\overline{D(p)}$ is fixed by a non-trivial element of Γ.

Poincarè's Fundamental Polyhedron Theorem provides a converse to these observations:

Theorem 1.1 (Poincarè). *Let Φ be a side-pairing for a convex polyhedron D satisfying conditions (1)-(4) above. Then the group Γ generated by Φ is discrete, P is a fundamental polyhedron for Γ, and the side-pairing and cycle relations define Γ up to isomorphism. If (5) holds, then U^3/Γ is a compact hyperbolic 3-manifold.*

We apply Poincarè's theorem by constructing the domain corresponding to a finite subset $\{g_i\} \subset G$ and then checking the hypotheses of the theorem. However, without the algebraic coordinates of the polyhedron we cannot check these directly. The following section addresses a sufficient conditions that makes this feasible.

2 Applying Poincarè's Theorem

Let D be a polyhedron with side-pairing Φ. We now show that under the additional assumption that every edge cycle has length three, it is possible to check the hypotheses (1)-(4) of Poincarès fundamental polyhedron theorem using only the solution of the word problem in the group. Note that if the group is in fact discrete, the solution to the word problem can be found by means of an *automatic structure* as described in [6].

Using the solution to the word problem, it is easy to check that for each $g_S \in \Phi$ there is a $g_{S'}$ such that $g_S' = g_S^{-1}$. To verify that $g_S(S') = S$ it suffices to check that for each edge E of S there is a corresponding edge E' of S' such that $g_S(E') = E$.

Let S_E be the unique side not equal to S that has E as an edge. Then $E \subset P_{g_S} \cap P_{g_{S_E}}$. Suppose there is an edge E' of S' such that

$$P_{g_S} \cap P_{g_{S_E}} = P_{g_S} \cap g_S(P_{g_{S_{E'}}}). \tag{2.1}$$

Then since by hypothesis edge cycles have length three, we must have

$$g_{S_E} = g_S \circ g_{S_{E'}} \tag{2.2}$$

which can be checked using the solution of the word problem. But if this holds for all edges on all of the sides of the domain, then since the domain is completely defined by the intersections of planes P_g, conditions (1) and (2) hold.

Condition (3) holds by hypothesis, and can be confirmed using the solution of the word problem.

Each dihedral angle is less than π, so the sum of the dihedral angles along the edge cycle is less than 3π, but the sum must be a multiple of 2π. Hence it is exactly 2π, i.e. condition (4) holds.

To see that no point in $\overline{D}$ is fixed by a nontrivial element of Γ, it suffices to compare the vertices of $\overline{D}$ with the finite number of images of $\overline{D}$ which share vertices with $\overline{D}$, which can be done rigorously using interval arithmetic which is explained in the next section.

3 Constructing Polyhedra

In order to use of the result of the previous section, we need a method of constructing polyhedra. More precisely, given a finite set of words $\{g_i\}$, we wish to construct the domain $\cap_i H_{g_i}$ and its associated side-pairing Φ. Recall the projective disk model of hyperbolic space.

We work in this model because the images of geodesics coincide with those in the Euclidean metric, thus simplifying computations. We begin with the unit cube in R^3, which sits outside of D^3, and proceed by successively intersecting the cube with the planes P_{g_i}. For this to be rigorous, the numerical calculations are performed using interval arithmetic.

That is, suppose we wish to represent a real number x. Instead of using a single floating point approximation, we use an interval $[a, b]$, defined by a pair of floating point values, such that $x \in [a, b]$. When we perform any numerical operations (addition, multiplication, etc.) on x, we actually perform the operations on the floating point values a and b and bound the error on either end of the interval, and enlarging it to guarantee that the number we want is contained in the interval.

And so we build two models of the domain. The first is an approximate construction which uses vertex coordinates. It is stored as a collection of sides, each of which is a collection of edges, each of which is a pair of vertices whose components are intervals as above. This model is used in order to measure distances and calculate intersections of planes. The second model is the side-pairing Φ, consisting of words in the group corresponding to the hyperplane containing each side, and for each $g \in \Phi$, a subset $\Phi_g \subset \Phi$ of words corresponding to sides which share an edge with g. It is this model which is checked against the hypotheses of Poincarè's theorem.

How can we guarantee that Φ actually defines a convex polyhedron? By construction, the only obstruction to our collection of planes defining a polyhedron is if a number of planes which are supposed to share a vertex do not actually intersect in a single point. However, this can only happen if there are at least four planes involved. This shows that under the additional condition that every vertex of the Dirichlet polyhedron is shared by exactly three sides, the polyhedron is able to be rigorously constructed by the above method.

4 Genericity of Hypotheses

In Section 2 we showed that under the condition that every edge cycle of $D(p)$ has length three, the hypotheses of Poincarè's theorem are checked using only the solution to the word problem. Throughout this section we assume that Γ is a discrete subgroup of $PSL(2, \mathbb{C})$ such that U^3/Γ is a compact hyperbolic 3-manifold. In this case, we prove that this condition is satisfied by a generic choice of basepoint. A similar result for two dimensions appears as Theorem 9.4.5(2) in [2], and our proof is modeled on the one given there.

We define a quaternionic cross ratio of four points in $\mathbb{Q} \cup \{\infty\}$ by

$$C(q_1, q_2, q_3, q_4) := (q_1 - q_3)(q_1 - q_4)^{-1}(q_2 - q_4)(q_2 - q_3)^{-1}. \tag{4.1}$$

By studying the geometry of quaternionic Möbius transformations, we have the following result:

Theorem 4.1. *Four pair-wise distinct points $q_1, q_2, q_3, q_4 \in U^3$ lie on a hyperbolic circle if and only if $C(q_1, q_2, q_3, q_4)$ is real.*

Proof. See Theorem 4.9 of [3]. □

Given f, g, h in $PSL(2,\mathbb{C})$, the following modified cross-ratio will be important to the proof of our main theorem:

$$\begin{aligned} C_{f,g,h}(q) :&= C(q, fq, gq, hq) \\ &= (q-gq)(q-hq)^{-1}(fq-hq)(fq-gq)^{-1}. \end{aligned} \tag{4.2}$$

Recall that an element g of $PSL(2,C)$ is hyperbolic if it fixes exactly two points, both of which are on the boundary of U^3. Let $A(g)$ be the *axis* of g, that is the image of the unique geodesic connecting the two fixed points of g. Up to conjugation, a hyperbolic element acts by translating translation along its axis.

Lemma 4.2. *For* Γ *as above, every element of* Γ *is hyperbolic.*

Proof. H^3/Γ is compact, so by Theorem 6.6.6 of [12], Γ contains no parabolic elements. Since Γ acts freely, it cannot contain an elliptic element. □

Lemma 4.3. *For any* f *and* g *in* Γ *as above, their fixed points are either disjoint or coincide.*

Proof. By Lemma 4.2 f and g are hyperbolic, so they each fix two points on the boundary of U^3.

Suppose for contradiction that they have exactly one fixed point q in common. Then the axes $A(f)$ and $A(g)$ get arbitrarily close as they approach q. Let $\pi : U^3 \to U^3/\langle f, g\rangle$ be the quotient map which we know to be a covering map and a local isometry. Then $\pi(A(f))$ and $\pi(A(g))$ are disjoint closed loops. Let $v \in A(g)$ such that $\pi(v) \notin \pi(A(f))$. Then as the loops are closed, the distance from $\pi(v)$ to any point of $\pi(A(f))$ is bounded from below by $\epsilon > 0$. The orbit of v by $\langle g\rangle$ gets arbitrarily close to q and so it gets arbitrarily close to $A(f)$. But since π is a local isometry, this implies that $\pi(v)$ gets arbitrarily close to $\pi(A(f))$, contradicting the ϵ bound. We conclude that the fixed points are either disjoint or coincide. □

We define the *stabilizer* of a point q in $\overline{U^3}$ to be the subgroup of Γ

$$\Gamma_q := \{g \in \Gamma : gq = q\}. \tag{4.3}$$

Lemma 4.4. *For* Γ *as above and any* q *in* $\overline{U^3}$, Γ_q *is cyclic.*

Proof. By Lemmas 4.2 and 4.3, we know that Γ_q consists of hyperbolic elements, all of which fix q and one other point, say v. Hence Γ_q fixes an axis A. Applying an isometry taking A to the j-axis, we may assume that each element of Γ_q is of the form $p \mapsto tp$ for some $t \in \mathbb{R}^*$. Clearly the stabilizer of the j-axis in $PSL(2, \mathbb{C})$ is isomorphic to the multiplicative group $\mathbb{R}_+$, which is isomorphic to the additive group $\mathbb{R}$. Any discrete subgroup of $\mathbb{R}$ is cyclic and generated by the least positive element of the group. Γ_q is discrete as it is a subgroup of Γ, and so it is cyclic. $\square$

Lemma 4.5. *Suppose that for p in U^3, the Dirichlet domain centered at p has four distinct edges E, $f^{-1}E$, $g^{-1}E$, $h^{-1}E$ in a cycle. Then $C_{f,g,h}$ is non-constant and $C_{f,g,h}(p)$ is real.*

Proof. By the construction of $D(p)$, for all points q in E,

$$d_H(p, q) = d_H(fp, q) = d_H(gp, q) = d_H(hp, q). \tag{4.4}$$

That is, for every q in E is the center of a sphere containing p, fp, gp, hp. Since the intersection of distinct spheres is either empty or a hyperbolic circle, this implies that the points lie on a hyperbolic circle with centered at some u. Hence by Theorem 4.1, $C_{f,g,h}$ is real.

Suppose that $C_{f,g,h}$ is constant, say λ. By Lemma 4.2 every element of Γ is hyperbolic. Let z be a point of H^n. Then z is not fixed by any element of Γ, in particular it is not fixed by g, f, $f^{-1}h$, $g^{-1}h$, so none of the factors of (4.2) vanish at z. Thus $C_{f,g,h}(z) = \lambda \neq 0, \infty$. As z tends to a fixed point v of g, the first factor of (4.2) tends to zero, hence so do the second or third factors, and so f or h also fixes v. Without loss of generality, assume v is a fixed point of f. By Lemma 4.4, Γ_v is cyclic so its subgroup $< f, g >$ is also cyclic, generated, say, by k. But w, fw and gw lie on a hyperbolic circle, which, as k is hyperbolic, is impossible. Therefore $C_{f,g,h}$ is non-constant. $\square$

We need one more lemma before we can prove the main result of this section.

Lemma 4.6. *Let $f : U^3 \to U^3 \cup \{\infty\}$ be a non-constant rational function. Then $E := \{q \in U^3 : f(q)$ is real$\}$ has volume zero.*

Proof. Let $\{q_i\}$ be the finite set of zeroes of the denominator of f. Every point of $U^3 \backslash \{z_i\}$ has a neighborhood N on which f is a diffeomorphism and so $E \cap N$ has measure zero. Since $U^3 \backslash \{z_i\}$ is covered by countably many such N, we conclude that E has measure zero. $\square$

Theorem 4.7. *Let Γ be a discrete subgroup of $PSL(2,\mathbb{C})$ such that U^3/Γ is a compact hyperbolic 3-manifold. Then for almost all choices of p in U^3, every edge cycle of the Dirichlet domain centered at p has length three.*

Proof. For p as in the statement of the theorem, by Lemma 4.5 there are group elements f,g,h such that $C_{f,g,h}$ is non-constant and $C_f, g, h(p)$ is real. Hence p is in

$$E_{f,g,h} := \{q \in U^3 : f(q) \textit{ is real}\} \tag{4.5}$$

which by Lemma 4.6 has measure zero. Since there are at most countably many group elements, the union of all such regions has measure zero. ☐

5 Volume

Given a convex polyhedron P, we wish to compute its hyperbolic volume, $VolP$. It is easy to express P as a finite union of disjoint tetrahedra T_i. Then

$$VolP = \sum_i VolT_i \tag{5.1}$$

Thus it suffices to compute the volume of a tetrahedron.

Let T be a tetrahedron with vertices v_1, v_2, v_3, v_4 in D^3. We say that T is in general position if $v_1 = (0,0,0)$ and v_2 also lies on the z-axis. Recall that an ideal tetrahedron in D^3 is one with an ideal vertex, that is a vertex on the boundary of D^3. We associate two ideal tetrahedra to T: IT_1 with vertices $v_1, v_3, v_4, (0,0,1)$ and IT_2 with vertices $v2, v3, v4, (0,0,1)$. Then we have

$$VolT = VolIT_1 - VolIT_2 \tag{5.2}$$

Let IT be a tetrahedron with a single ideal vertex and dihedral angles A, B, C, A', B', and C' such that A, B, and C share exactly one vertex; A and A' share no vertex, and similarly for B and B', and C' and C'. Then by [5], the volume of IT is given by

$$VolIT = \frac{1}{2}[L(\tfrac{A-B-C+\pi}{2}) + L(\tfrac{-A+B-C+\pi}{2}) + L(\tfrac{-A-B+C+\pi}{2}) - L(\tfrac{A+B+C+\pi}{2}) + L(\tfrac{A-B'-C'+\pi}{2}) + L(\tfrac{-A+B'-C'+\pi}{2}) + L(\tfrac{-A-B'+C'+\pi}{2}) - L(\tfrac{A+B'+C'+\pi}{2}) + L(\tfrac{A'-B-C'+\pi}{2}) + L(\tfrac{-A'+B-C'+\pi}{2}) + L(\tfrac{-A'-B+C'+\pi}{2}) - L(\tfrac{A'+B+C'+\pi}{2}) + L(\tfrac{A'-B'-C+\pi}{2}) + L(\tfrac{-A'+B'-C+\pi}{2}) + L(\tfrac{-A'-B'+C+\pi}{2}) - L(\tfrac{A'+B'+C+\pi}{2})] + L(\tfrac{A+A'+B+B'}{2}) + L(\tfrac{A+A'+C+C'}{2}) + L(\tfrac{B+B'+C+C'}{2}) \tag{5.3}$$

where L is Lobachevsky function given by

$$2iL(\theta) = \psi(e^{2i\theta}) - \psi(1) + \pi\theta - \theta^2 \tag{5.4}$$

where $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\theta^n}{n^2}$.

To compute the dihedral angles, we find the side normals in D^3 and compute the angle between them by

$$\theta = \arccos \frac{< u, v >_x}{< u, u >_x \cdot < v, v >_x} \tag{5.5}$$

where the inner product $<, >_x$ at a point x of D^3 is given by

$$< e_i, e_j >_x = \begin{cases} \dfrac{1 - |x|^2 + x_j^2}{(1 - |x|^2)^2} & \text{if } i = j \\ \dfrac{x_i x_j}{(1 - |x|^2)^2} & \text{if } i \neq j \end{cases} \tag{5.6}$$

Thus we compute the volume of an ideal tetrahedron, and hence of a tetrahedron in general position by (5.2). To compute the volume of an arbitrary tetrahedron, we isometrically transform it to general position as follows. We work in the upper-half space model where we represent isometries by elements of $PSL(2, C)$ as in (1.3). First we transform v_1 to the point $(0, 0, 1)$ in U^3 by the element

$$\frac{1}{\sqrt{z}} \begin{pmatrix} 1 & -x - yi \\ 0 & z \end{pmatrix}. \tag{5.7}$$

Then we rotate so that v_2 is in $x - z$ plane, and by inversion with respect to the semi-unit circle, v_2 can be transformed to z-axis and v_1 is fixed:

$$\begin{pmatrix} \frac{(x_2 - y_2 I)}{k} & \frac{(1-n+\sqrt{4k^2+(1-n)^2})}{2k} \\ (x_2 - y_2 I) \cdot \frac{(1-n+\sqrt{4k^2+(1-n)^2})}{2k^2} & 1 \end{pmatrix} \tag{5.8}$$

. where $k = \sqrt{x_2^2 + y_2^2}$ and $n = x_2^2 + y_2^2 + z_2^2$ and we must divide the matrix representation by its determinant so that it is an element of $PSL(2, \mathbb{C})$.

Finally, we return to D^3 where we now have a tetrahedron in general position, whose volume we may compute as before.

6 Computer Program

We implemented the algorithms described above using Sage [14]. The package MPFI [13] enables us to use interval arithmetic for constructing the domains and computing volume. The package kbmag [8] uses the Knuth-Bendix algorithm to determine the automatic structure of the group, thus solving the word problem and enabling us to check the conditions described in Section 2. The source code and documentation are available at [11].

Given the group presentation and matrix entries for the generators, we begin by computing the automatic structure. In practice, this computation has varied in duration from a couple of seconds to several minutes. The structure may be saved for future computations so as not to repeat this process. Next we build the polyhedron centered at a specified point defined by the subset of G containing words up to some specified length on the specified generators. Then we check it against our conditions using the automatic structure. If it passes, the group is discrete and we are done. If not, we proceed by considering the subset of Γ of words of length 2 on the words which made up the previous polyhedron. Again, we check the conditions. And in this manner we continue. In practice, using an initial word length of five allowed the program to succeed within a couple of iterations of this process. If the program is successful, it will print out the alternate group presentation given by Poincarè's theorem.

If the program is unsuccessful, that is it does not terminate, then we have one of the following:

- Γ is not discrete,
- Γ is discrete but the group elements corresponding to sides of the fundamental polyhedron centered at x were not reached.
- Γ is discrete but the the Dirichlet domain with specified center does not satisfy the necessary additional conditions.

7 Application

In [7], Gabai, Meyerhoff and Thurston show that if a closed, irreducible 3-manifold M is homotopy equivalent to a closed hyperbolic 3-manifold N, then in fact M and N are homeomorphic. The proof isolates seven exceptional families containing two-generator subgroups with two quasi-relators, say $\Gamma_i =< f, w | r_1(X_i), r_2(X_i) >$, somewhere inside the region. They conjecture that each family X_i contains a unique manifold M_i with $\pi_1(M_i) = \Gamma_i$. Arithmetic methods in [9] have been successful in proving this, however they do not extend to construct the manifold in the case of X_3. Using the algebraic expressions for the group elements found in [4], we are able to run our program on these groups. In every case, the program succeeds. The hyperbolic volume associated to each domain agrees with [9].

There are several directions in which to extend this work. The most appealing of these is to extend our methods for determining whether the group Γ is discrete to two additional cases: the case where the Dirichlet domain is non-compact (i.e. it has ideal vertices) and the case where U^3/Γ is an orbifold. There are several more practical enhancements to the computer program that would also be beneficial. For example, one could implement the Lobachevsky function in (5.4) using interval arithmetic and hence, by Mostow rigidity, we would have a means of rigorously distinguishing groups (for which the program is successful) up to isomorphism. Also it would be useful to allow the user to input only the group presentation and then solve for the matrix entries of the generators.

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