XOR Simplex

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1 Basic Idea

We construct the following matrix for the XOR constraints:

$$\left(\begin{array}{ccc|ccc|c} a_{1,1}x_1 & \cdots & a_{1,n}x_n & s_1 & & \bar{0} & b_1 \\ \vdots & \ddots & \vdots & & \ddots & & \vdots \\ a_{m,1}x_1 & \cdots & a_{m,n}x_n & \bar{0} & & s_m & b_m \end{array}\right)$$

The matrix has the trivial solution assigning 0 to all variables $x_{i,j}$ and b_i to s_i. We are interested in solutions where

- all s_i are zero, and
- the value of x_j corresponds to the truth value of Boolean variable v_j.

It should be possible to update values of variables as described in the paper *"Integrating Simplex with* DPLL(T)*"*. It should hopefully be easier because as far as I can see, the only operations necessary to implement pivoting is exchanging columns and adding rows.

Looking at the examples below, it seems like we can decide if a set of XOR constraints is satsifibale with at most $m - 1$ pivots.

2 UNSAT Example

We consider the following example, where variables x, y, and z are zero and variables s_1, s_2, and s_3 are one. A solution requires the slack variables s_1, s_2, and s_3 to be zero:

$$\begin{array}{ccc|ccc|c} x & y & 0 & s_1 & 0 & 0 & 1 \\ 0 & y & z & 0 & s_2 & 0 & 1 \\ x & 0 & z & 0 & 0 & s_3 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

2.1 Pivot s_1 and x

Swap column 1 and 4:

$$\begin{array}{ccc|ccc|c} s_1 & y & 0 & x & 0 & 0 & 1 \\ 0 & y & z & 0 & s_2 & 0 & 1 \\ 0 & 0 & z & x & 0 & s_3 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

Add row 1 to row 3:

$$\begin{array}{ccc|ccc|c} s_1 & y & 0 & x & 0 & 0 & 1 \\ 0 & y & z & 0 & s_2 & 0 & 1 \\ s_1 & y & z & 0 & 0 & s_3 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

2.2 Pivot s_2 and y

Swap column 2 and 5:

$$\begin{array}{ccc|ccc|c} s_1 & 0 & 0 & x & y & 0 & 1 \\ 0 & s_2 & z & 0 & y & 0 & 1 \\ s_1 & 0 & z & 0 & y & s_3 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

Add row 2 to rows 1 and 3:

$$\begin{array}{ccc|ccc|c} s_1 & s_2 & z & x & 0 & 0 & 0 \\ 0 & s_2 & z & 0 & y & 0 & 1 \\ s_1 & s_2 & 0 & 0 & 0 & s_3 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & \end{array}$$

We have show that the equations are unsatisfiable because there is no way to pivot s_3! We also see that we derived the constraint $s_1 \oplus s_2 \oplus s_3 = 1$. Clearly, this constraint cannot be satisfied by setting the slack variables to zero.

3 SAT Example

We consider the following example, where variables x, y, and z are zero and variables s_1 and s_2 are one and s_3 is zero. A solution requires the slack variables s_1, s_2, and s_3 to be zero:

$$\begin{array}{ccc|ccc|c} x & y & 0 & s_1 & 0 & 0 & 1 \\ 0 & y & z & 0 & s_2 & 0 & 1 \\ x & 0 & z & 0 & 0 & s_3 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

3.1 Pivot s_1 and x

Swap column 1 and 4:

$$\begin{array}{ccc|ccc|c}
s_1 & y & 0 & x & 0 & 0 & 1 \\
0 & y & z & 0 & s_2 & 0 & 1 \\
0 & 0 & z & x & 0 & s_3 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 &
\end{array}$$

Add row 1 to row 3:

$$\begin{array}{ccc|ccc|c}
s_1 & y & 0 & x & 0 & 0 & 1 \\
0 & y & z & 0 & s_2 & 0 & 1 \\
s_1 & y & z & 0 & 0 & s_3 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 &
\end{array}$$

3.2 Pivot s_2 and y

Swap column 2 and 5:

$$\begin{array}{ccc|ccc|c}
s_1 & 0 & 0 & x & y & 0 & 1 \\
0 & s_2 & z & 0 & y & 0 & 1 \\
s_1 & 0 & z & 0 & y & s_3 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 &
\end{array}$$

Add row 2 to rows 1 and 3:

$$\begin{array}{ccc|ccc|c}
s_1 & s_2 & z & x & 0 & 0 & 0 \\
0 & s_2 & z & 0 & y & 0 & 1 \\
s_1 & s_2 & 0 & 0 & 0 & s_3 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

We have show that the equations are satisfiable with x and z set to zero and y set to one.