Geometric Modeling

Assignment 2: Implicit Surface Reconstruction

Acknowledgements: Olga Diamanti, Julian Panetta

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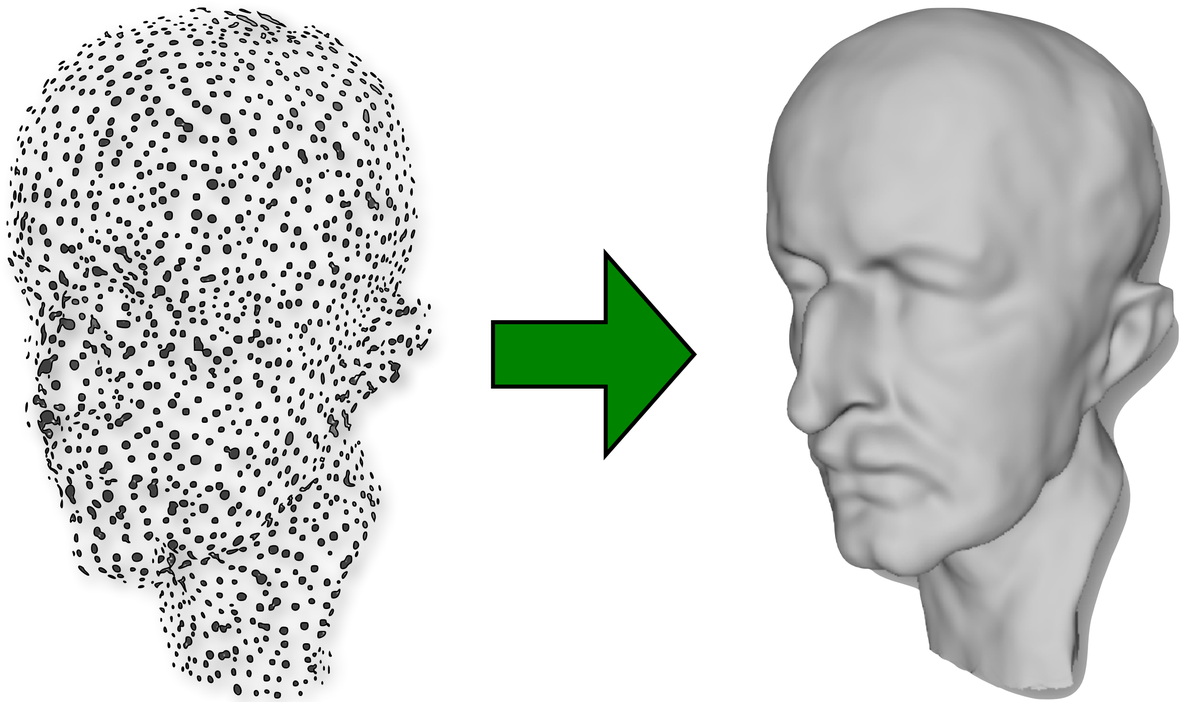
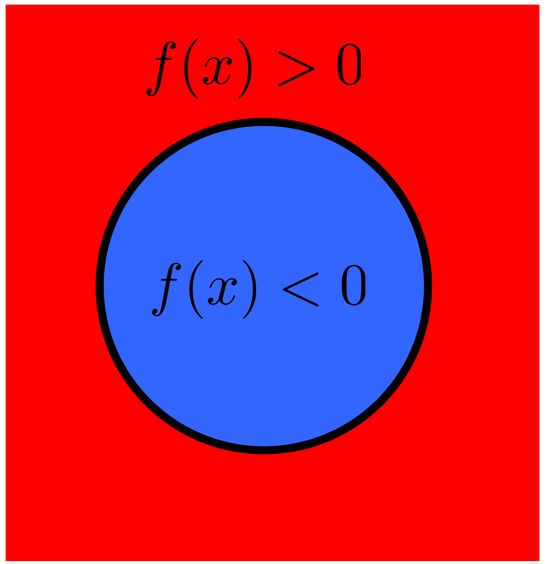
HW 2: Surface Reconstruction

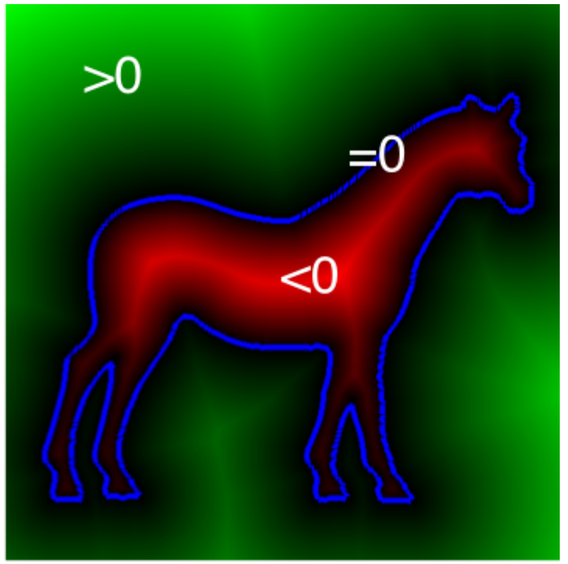


- **Input:** point cloud with normals
- **Output:** smooth surface mesh passing near each point
- **How?** Find out tomorrow from Daniele's lecture! (Or keep listening…)

Implicit Surface Reconstruction

- Remember: **surface representation matters!**
- Implicit representation bypasses many headaches an explicit approach would encounter.
- Guarantees by construction:
  - 2-Manifold
  - No holes (watertight)
- Robust to noisy point clouds

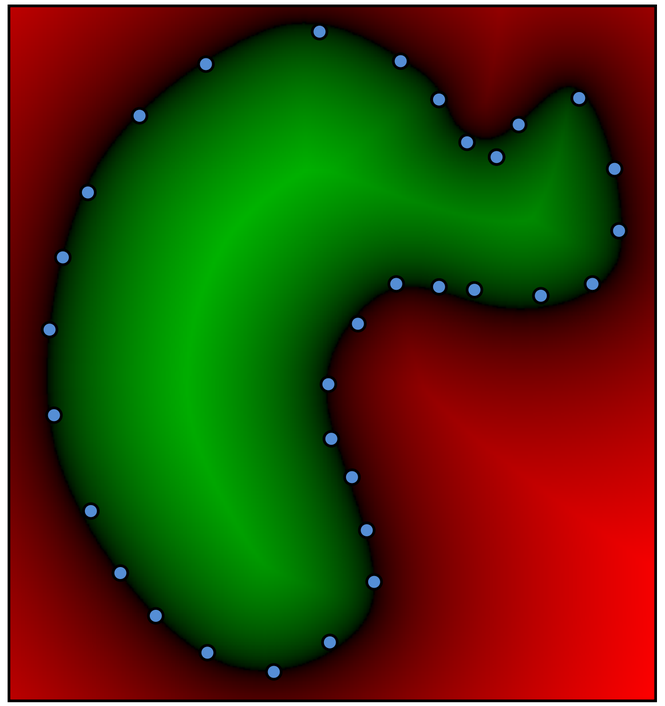




Simplifies the Problem

Surface interpolation

Scalar field interpolation

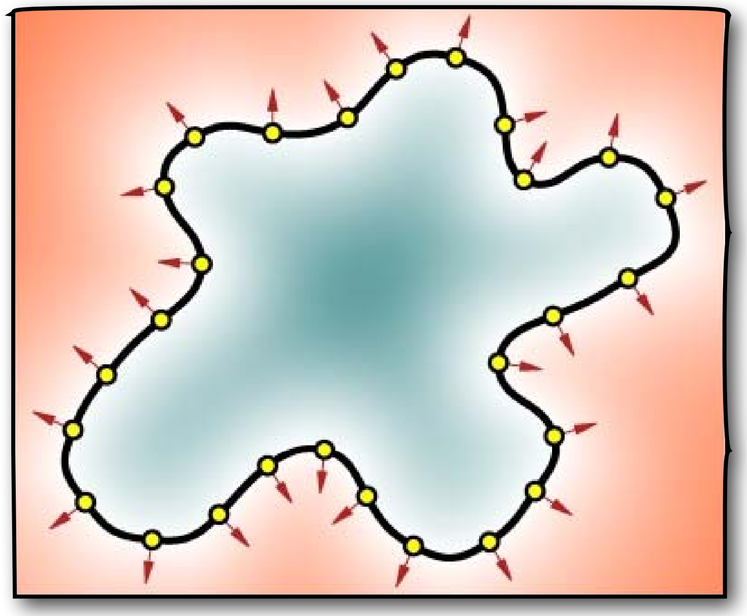


Constructing the Scalar Field

- Interpolate information from the input cloud:
  - **Points** tell us where the zero level set should go:

$$f(\mathbf{p}_i) = 0$$

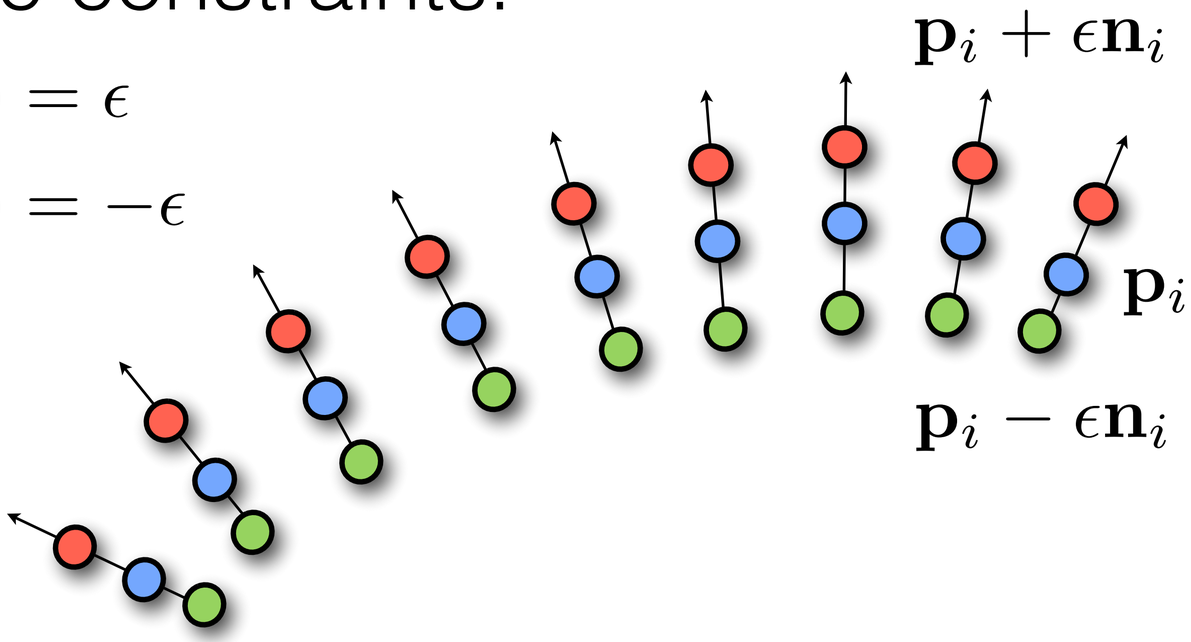
  - **Normals** define (locally) inside/outside

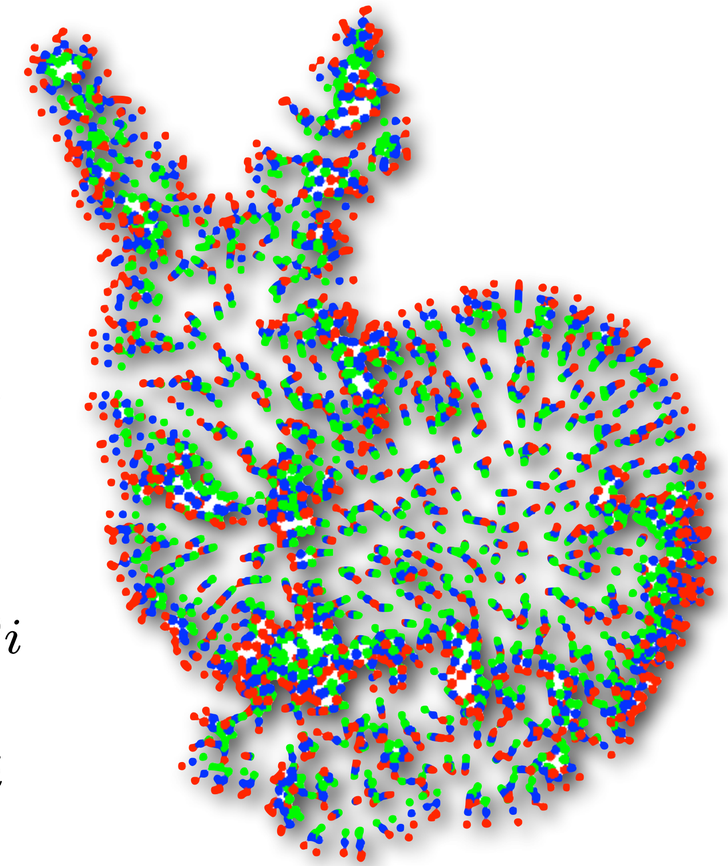


Step 1: Build the Constraint Set

- Point constraints $f(\mathbf{p}_i) = 0$ are insufficient (trivial solution f = 0)
- Incorporate normal info with additional off-surface constraints:

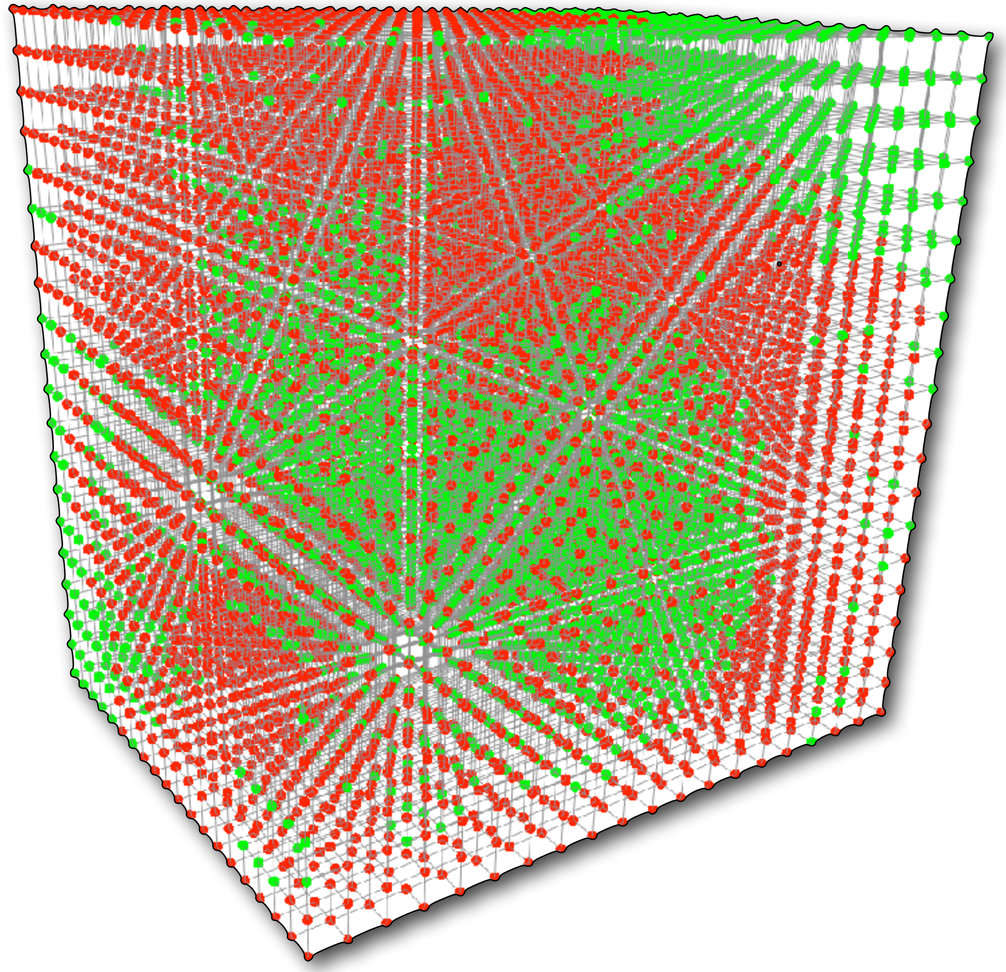
$$f(\mathbf{p}_i + \epsilon\mathbf{n}_i) = \epsilon$$
$$f(\mathbf{p}_i - \epsilon\mathbf{n}_i) = -\epsilon$$





Step 2: Construct Interpolant

- Construct regular grid
- Compute nodal scalar field satisfying constraints (approximately).
- Method: **MLS (Moving Least Squares)**



Interpolation Problem

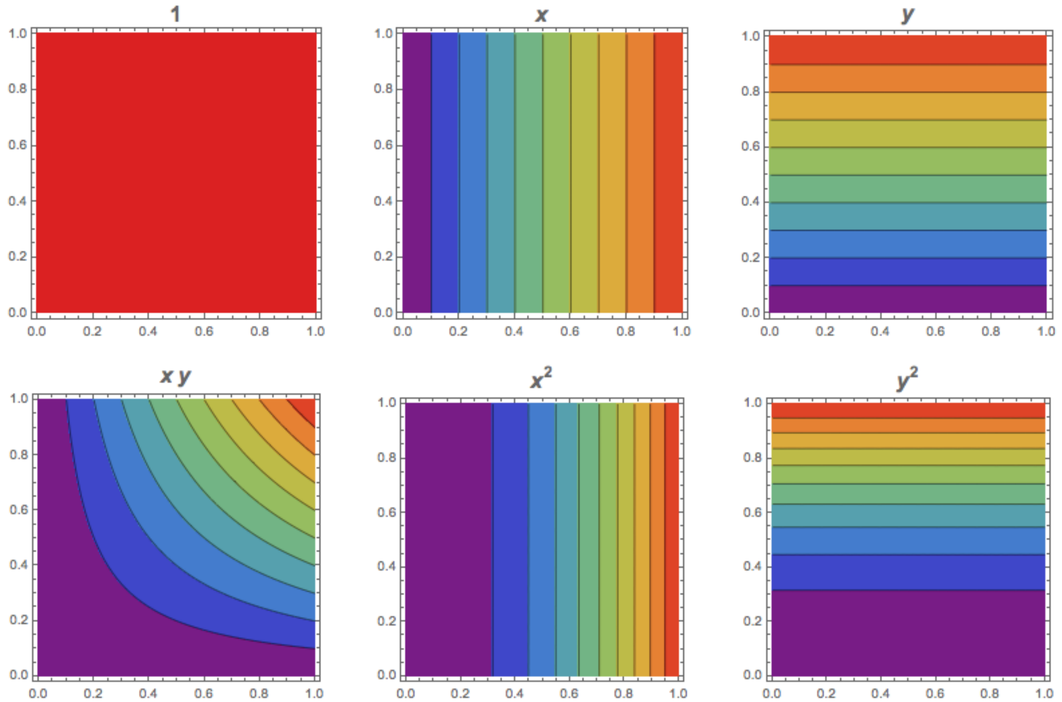
- List of 3N constraint locations, $\mathbf{c}_i$ (e.g. $\mathbf{p}_0$, $\mathbf{p}_0+\varepsilon\mathbf{n}_0$, …)
- List of 3N values, d_i
- Together, they describe 3N constraints of the form $f(\mathbf{c}_i) = d_i$
- **Goal:** find the "best" f in the span of chosen basis functions b(**x**):

$$f(\mathbf{x}) = \sum_j b_j(\mathbf{x}) a_j$$

(By tuning weights a_j to best approximate constraints)

Basis Functions

- For this assignment, we'll use **polynomial basis functions**:



(but in 3D)

Constraints in the Basis

- We can express our constraints in this basis:

$$f(\mathbf{c}_i) = \sum_j b_j(\mathbf{c}_i) a_j = d_i$$

In matrix form:

$$B\mathbf{a} = \mathbf{d}$$

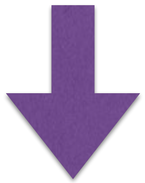
- Where matrix $B_{ij} := b_j(c_i)$
  (columns hold basis function's value at every constraint location).

$$B = \begin{bmatrix} 1 & x_1 & y_1 & \cdots \\ 1 & x_2 & y_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Overconstrained Linear System

- We'll have many more constraints than basis functions…
- Least-squares solution?

$$\min_f \sum_i (f(\mathbf{c}_i) - d_i)^2$$

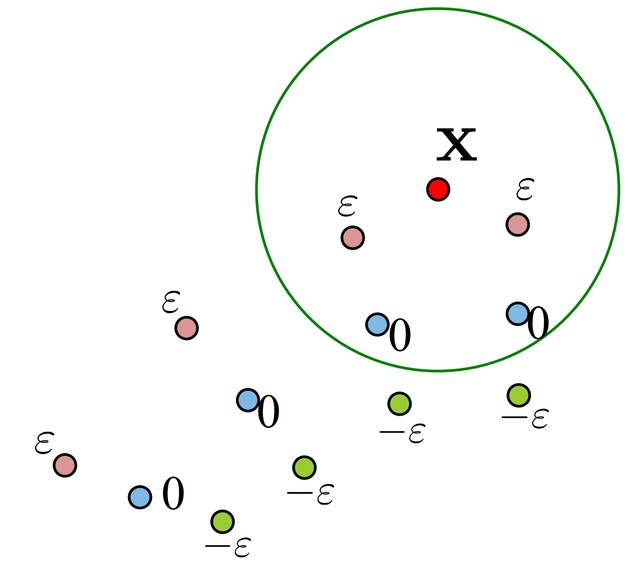


$$\min_{\mathbf{a}} \|B\mathbf{a} - \mathbf{d}\|^2$$

- What's bad about this?

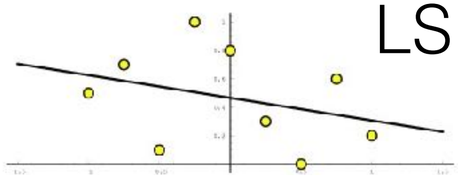
Problems with Least-squares

- **Global problem:** large matrices (even if basis functions are local)
- **Need many, high-degree basis functions**
  - Evaluating interpolant becomes expensive
- Better idea:
  - Construct **low degree**, **local interpolants** and stitch them together

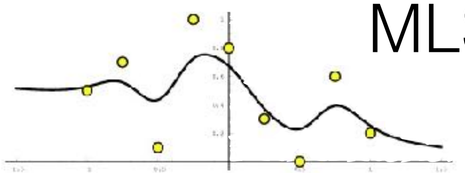


Moving Least Squares (MLS)

LS



MLS



- MLS builds distinct local interpolant **around every eval pt!**
- But final stitched function is still guaranteed smooth.
- Idea: weight the constraints based on distance to eval pt **x:**

$$f_{\mathbf{x}} := \underset{f}{\operatorname{argmin}} \sum_i w\left(\|\mathbf{x} - \mathbf{c_i}\|\right)\left(f(\mathbf{c}_i) - d_i\right)^2$$

- **Constraints with zero weight disappear!**
  (Choose weight function so few kept ==> **small linear system**)

MLS in Matrix Form

$$\min_f \sum_i w\left(\|\mathbf{x}-\mathbf{c}_i\|\right)\left(f(\mathbf{c}_i)-d_i\right)^2$$

$$\min_a \|B\mathbf{a}-\mathbf{d}\|^2_{W(\mathbf{x})}$$

Note: some papers call this $W(\mathbf{x})^2$

$$\|B\mathbf{a}-\mathbf{d}\|^2_{W(\mathbf{x})} := \left(B\mathbf{a}-\mathbf{d}\right)^T W(\mathbf{x})\left(B\mathbf{a}-\mathbf{d}\right)$$

$$W(\mathbf{x}) = \begin{bmatrix} w(\|\mathbf{x}-\mathbf{c}_1\|) & & \\ & \ddots & \\ & & w(\|\mathbf{x}-\mathbf{c}_{3N}\|) \end{bmatrix}$$

MLS Coefficients, Closed Form

- MLS objective function is quadratic in coefficients **a**; find optimum by differentiating and solving a linear system:

$$\begin{aligned} 0 =& \nabla_{\mathbf{a}} \left(\left(B\mathbf{a} - \mathbf{d} \right)^T W(\mathbf{x}) \left(B\mathbf{a} - \mathbf{d} \right) \right) \\ =& \quad 2B^T W(\mathbf{x}) B\mathbf{a} - 2B^T W(\mathbf{x})\mathbf{d} \end{aligned}$$

- Thus the coefficients **for point x** are given by solving the system:

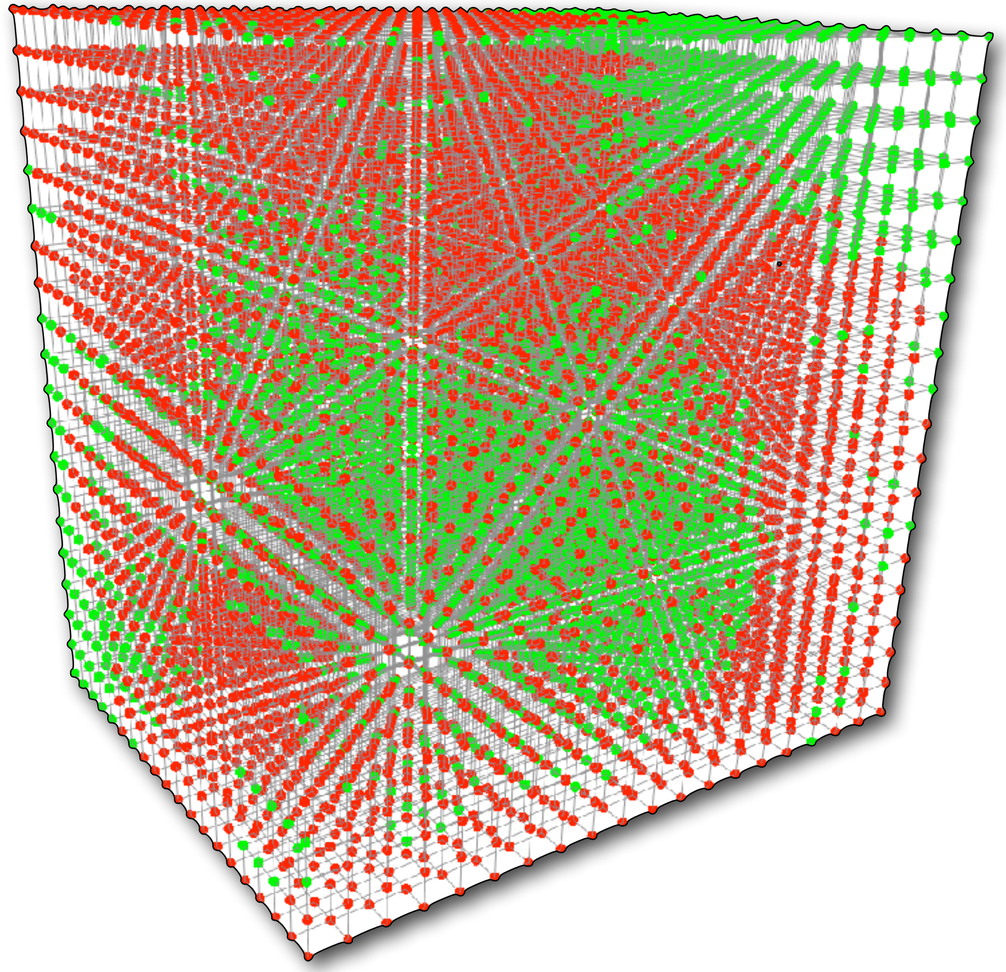
$$\left(B^T W(\mathbf{x}) B \right) \mathbf{a}(\mathbf{x}) = B^T W(\mathbf{x}) \mathbf{d}$$

for **a(x)**.

Step 2: Construct Interpolant

- Finally, fill in the grid!
- Evaluate local MLS interpolant at each grid point **x**.

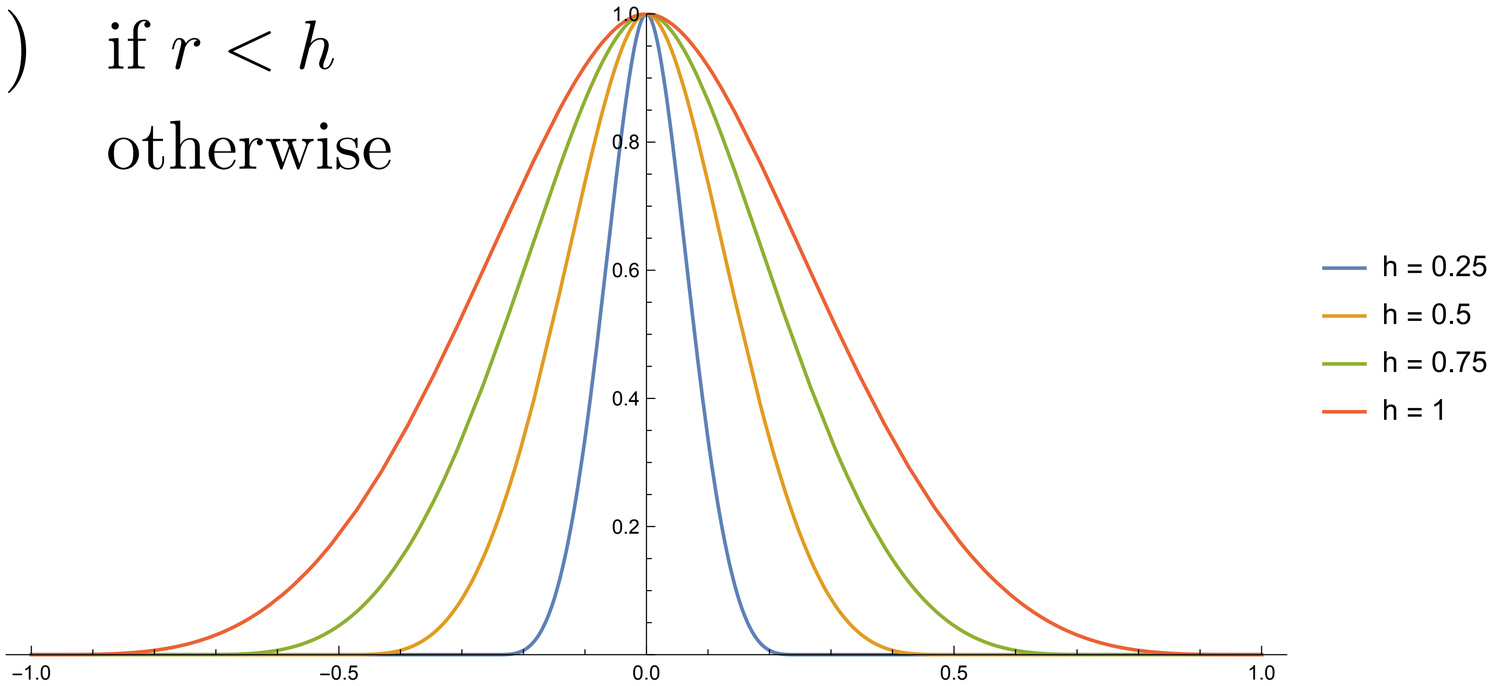
$$f_{\mathbf{x}}(\mathbf{x}) = \sum_j b_j(\mathbf{x}) a_j(\mathbf{x})$$



Wendland Weights

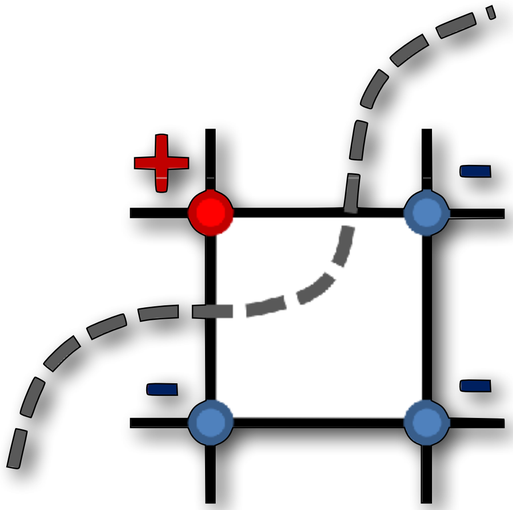
- You'll use **Wendland weights** for w in this assignment
- Vanish at dist "h" from eval pt (most constraints disappear)

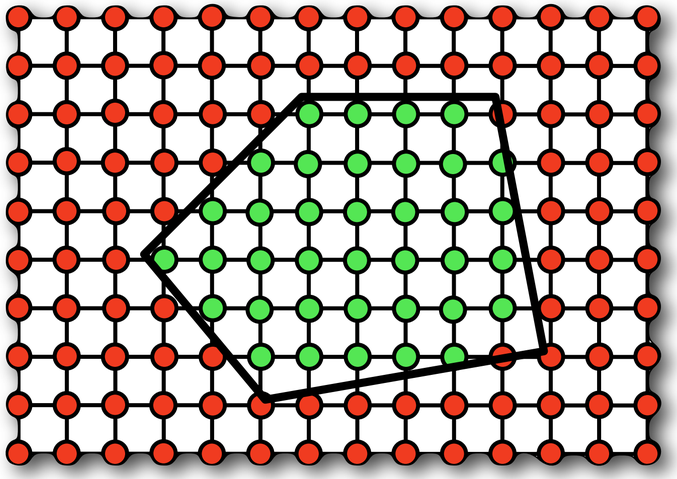
$$w(r) := \begin{cases} \left(1 - \frac{r}{h}\right)^4 \left(4\frac{r}{h} + 1\right) & \text{if } r < h \\ 0 & \text{otherwise} \end{cases}$$

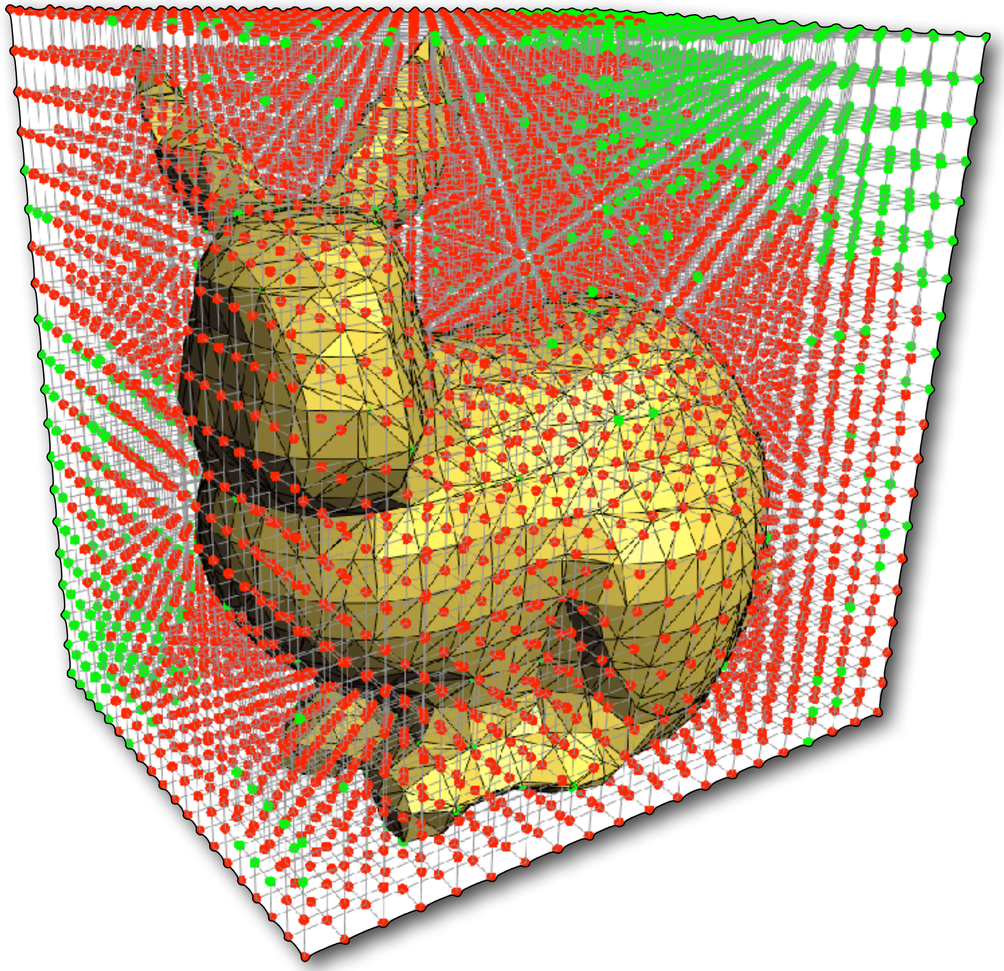


Step 3: Extract Zero Level Set

- Use the **marching cubes** algorithm to extract the grid function's zero isosurface
- Just call igl::copyleft::marching_cubes

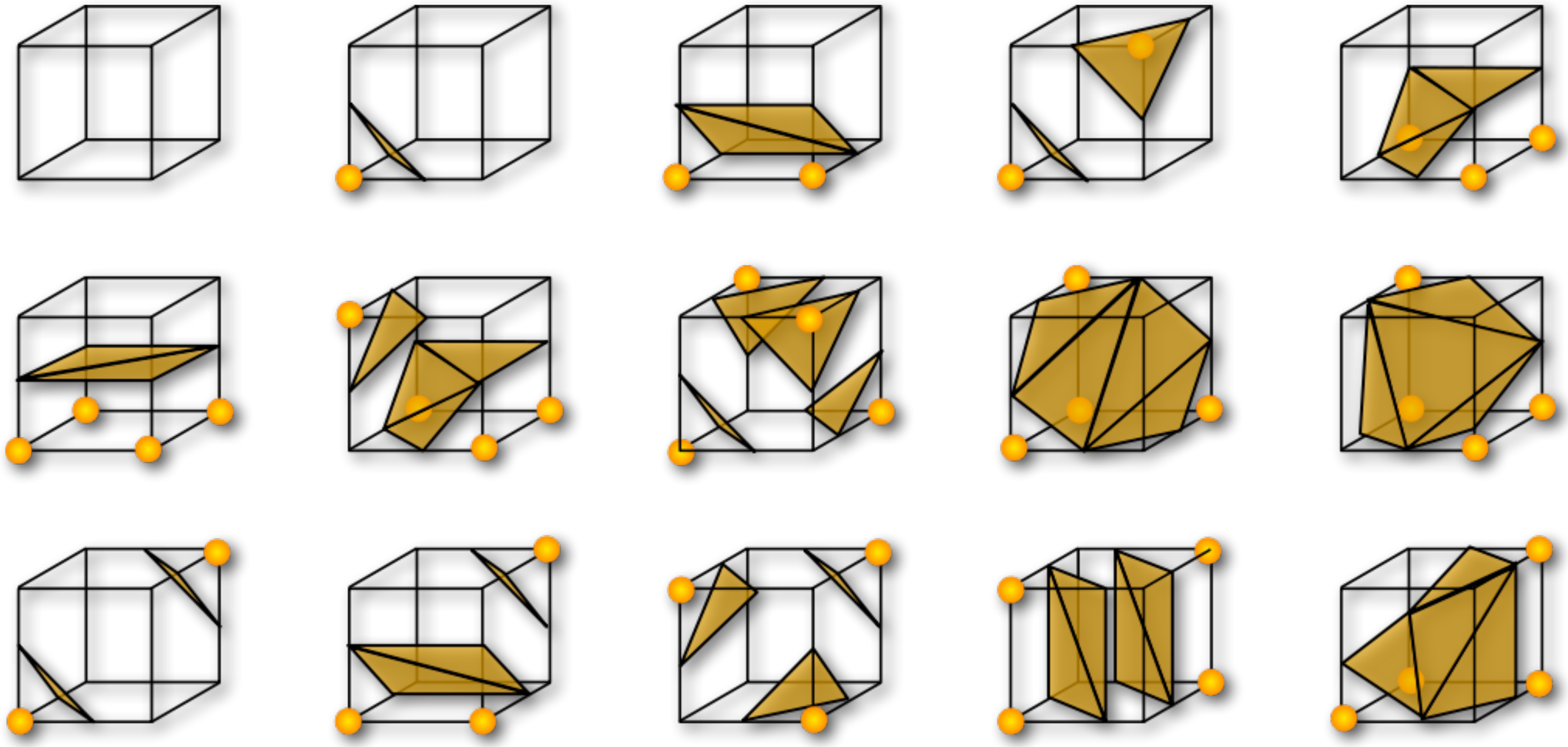




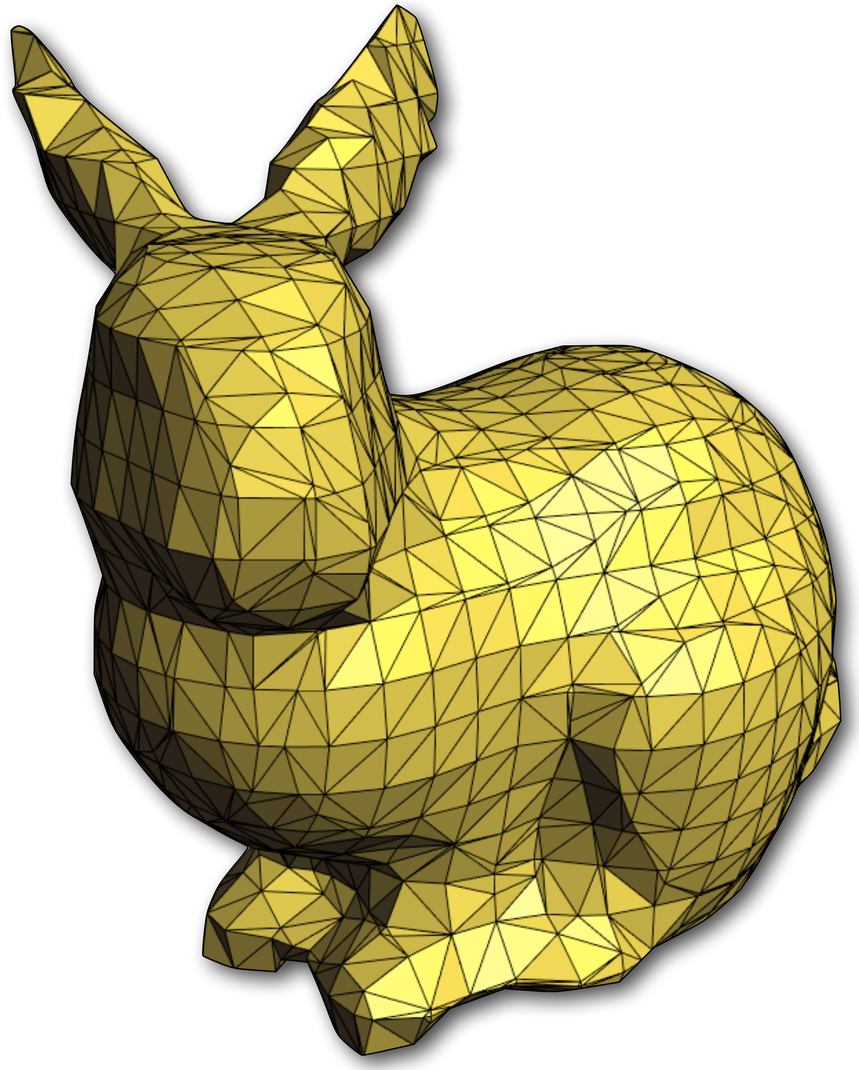


Marching Cubes: General Idea

- Look up triangles to create in each grid cell based on corner values:

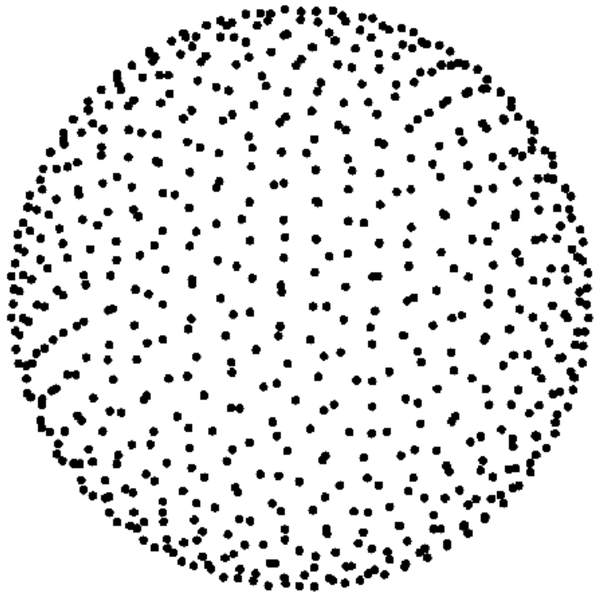


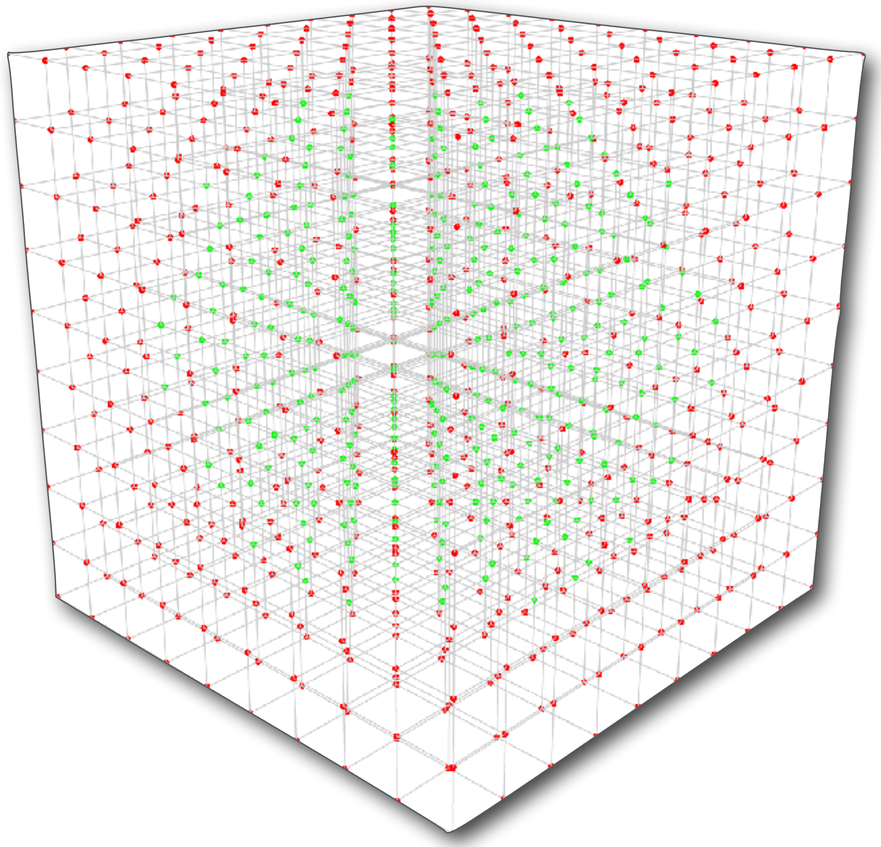
Final Result from Marching Cubes

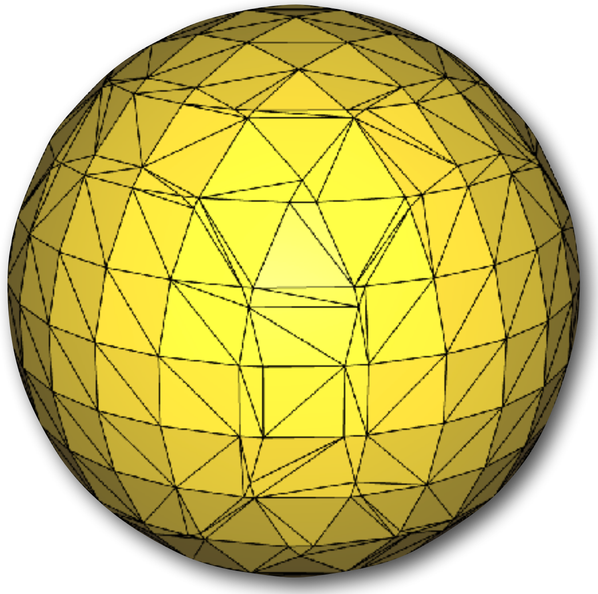


Provided Code

- Implements pipeline but uses analytic signed distance fn for sphere in place of MLS

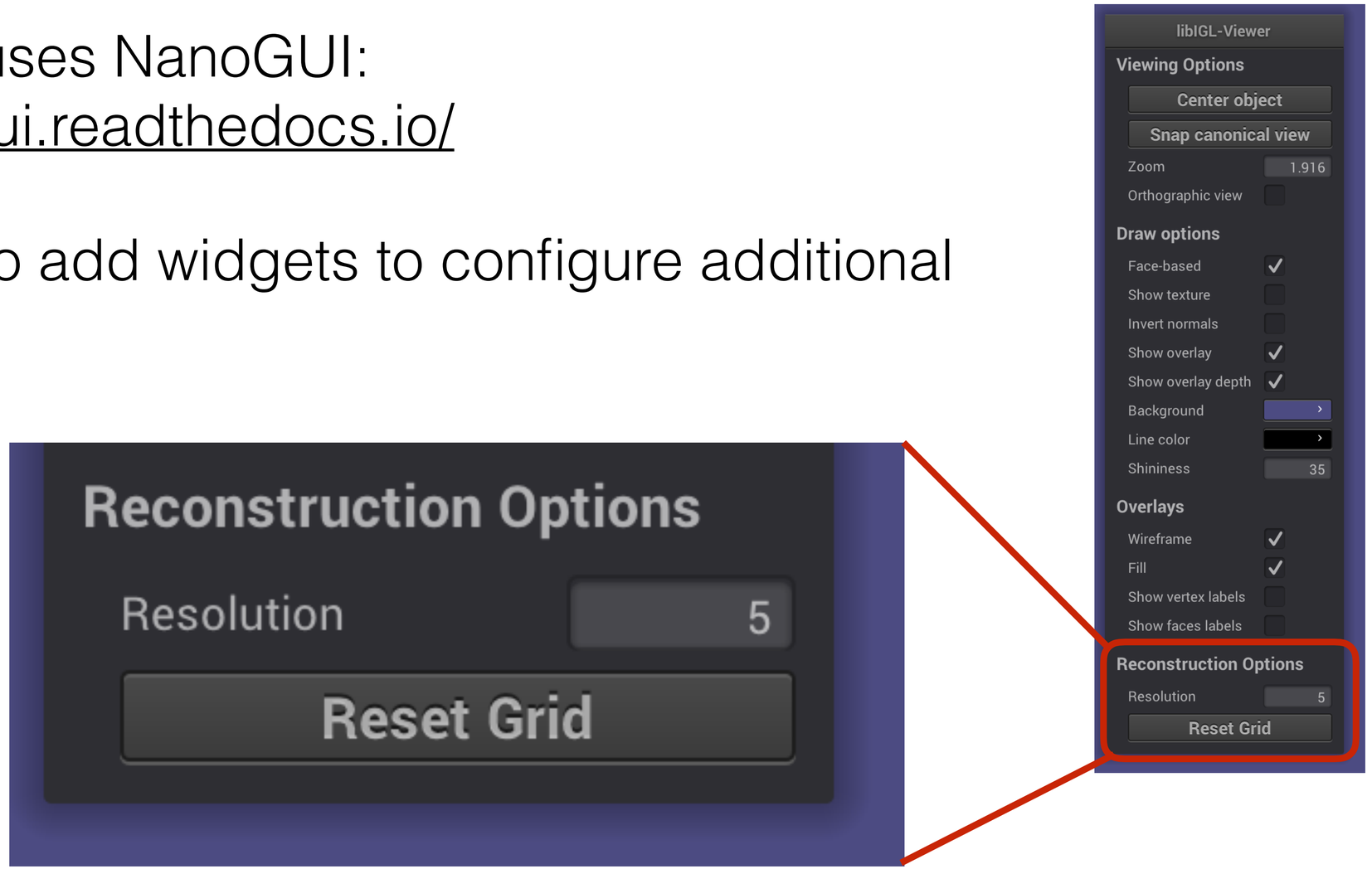






NanoGUI

- IGL Viewer uses NanoGUI: http://nanogui.readthedocs.io/
- You'll need to add widgets to configure additional variables.



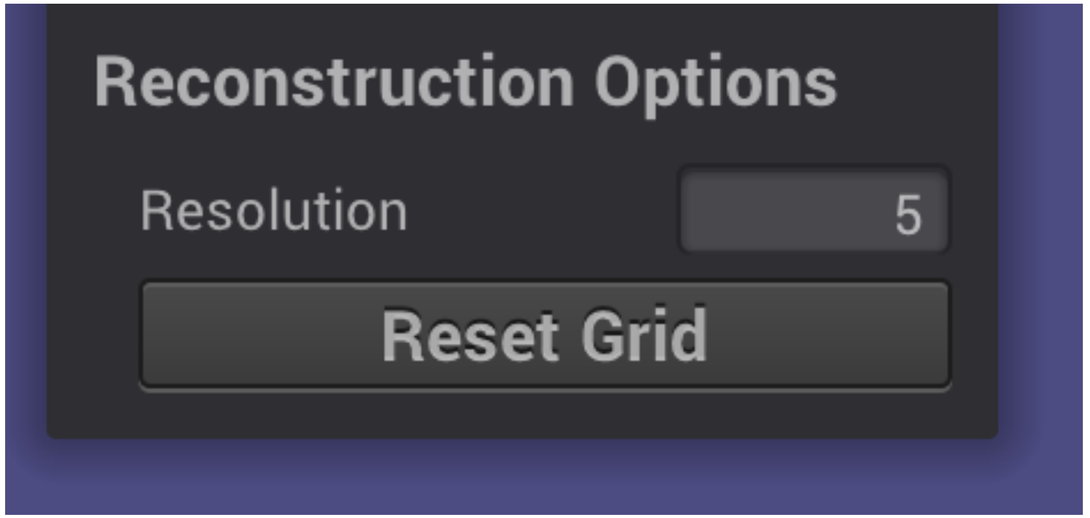
NanoGUI: Adding Settings

- Thankfully, this is really easy:

```
viewer.callback_init = [&](Viewer &v) {
    // Add widgets to the sidebar.
    v.ngui->addGroup   ("Reconstruction Options");
    v.ngui->addVariable("Resolution", resolution);
    v.ngui->addButton  ("Reset Grid", [&](){
        // Recreate the grid
        createGrid();
        // Switch view to show the grid
        callback_key_down(v, '3', 0);
    });

    // Add more parameters to tweak here...

    v.screen->performLayout();
    return false;
};
```



- (C++ lambda expressions)

Provided Example: Implicit Sphere

- Step 1: Compute an axis-aligned bounding box

```
/******** createGrid() ********/
// Grid bounds: axis-aligned bounding box
Eigen::RowVector3d bb_min, bb_max;
bb_min = P.colwise().minCoeff();
bb_max = P.colwise().maxCoeff();

// Bounding box dimensions
Eigen::RowVector3d dim = bb_max - bb_min;
```

Provided Example: Implicit Sphere

- Step 2: construct a grid over the bounding box

```
/******** createGrid() ********/
// Grid spacing
const double dx = dim[0] / (double)(resolution - 1);
const double dy = dim[1] / (double)(resolution - 1);
const double dz = dim[2] / (double)(resolution - 1);
// 3D positions of the grid points -- see slides or marching_cubes.h for ordering
grid_points.resize(resolution * resolution * resolution, 3);
// Create each gridpoint
for (unsigned int x = 0; x < resolution; ++x) {
    for (unsigned int y = 0; y < resolution; ++y) {
        for (unsigned int z = 0; z < resolution; ++z) {
            // Linear index of the point at (x,y,z)
            int index = x + resolution * (y + resolution * z);
            // 3D point at (x,y,z)
            grid_points.row(index) = bb_min + Eigen::RowVector3d(x * dx, y * dy, z * dz);
        }
    }
}
```

Provided Example: Implicit Sphere

- Step 3: Fill grid with the values of the implicit function

```
/******** evaluateImplicitFunc() ********/
// Scalar values of the grid points (the implicit function values)
grid_values.resize(resolution * resolution * resolution);

// Evaluate sphere's signed distance function at each gridpoint.
for (unsigned int x = 0; x < resolution; ++x) {
    for (unsigned int y = 0; y < resolution; ++y) {
        for (unsigned int z = 0; z < resolution; ++z) {
            // Linear index of the point at (x,y,z)
            int index = x + resolution * (y + resolution * z);
            // Value at (x,y,z) = implicit function for the sphere
            grid_values[index] = (grid_points.row(index) - center).norm() - radius;
        }
    }
}
```

$$f(\mathbf{x}) = \|\mathbf{x} - \mathbf{c}\| - r$$

Provided Example: Implicit Sphere

- Step 4: run marching cubes

```
igl::copyleft::marching_cubes(grid_values, grid_points, resolution, resolution, resolution, V, F);
```

input: implicit function values at grid points

Provided Example: Implicit Sphere

- Step 4: run marching cubes

```
igl::copyleft::marching_cubes(grid_values, grid_points, resolution, resolution, resolution, V, F);
```

input: grid point positions

Provided Example: Implicit Sphere

- Step 4: run marching cubes

```
igl::copyleft::marching_cubes(grid_values, grid_points, resolution, resolution, resolution, V, F);
```

input: grid size (x, y, z)

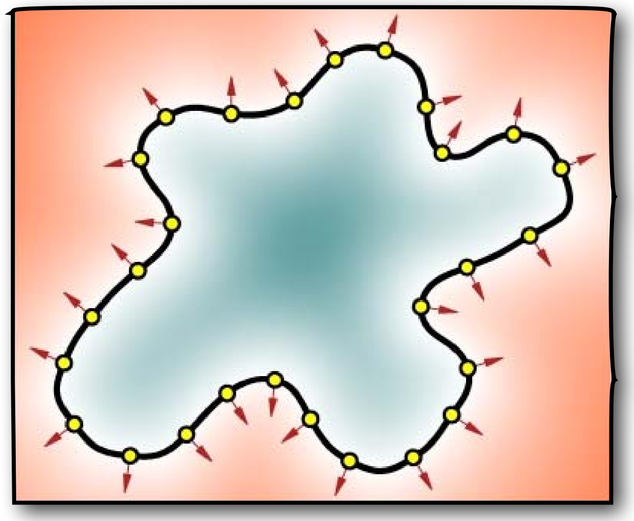
Provided Example: Implicit Sphere

- Step 4: run marching cubes

```
igl::copyleft::marching_cubes(grid_values, grid_points, resolution, resolution, resolution, V, F)
```

output: vertices and faces

Bonus: Better Normal Constraints



- Our method implemented only point constraints
- Normals “constrained” using inward- and outward-offset value constraints
    - Leads to undesirable surface oscillation
- Solution: use the normal to define a linear function at each sample point; interpolate these **functions** with MLS.

- Chen Shen, James F. O'Brien, and Jonathan R. Shewchuk. "**Interpolating and Approximating Implicit Surfaces from Polygon Soup**". In *Proceedings of ACM SIGGRAPH 2004*, pages 896–904. ACM Press, August 2004. **(Section 3.3)**

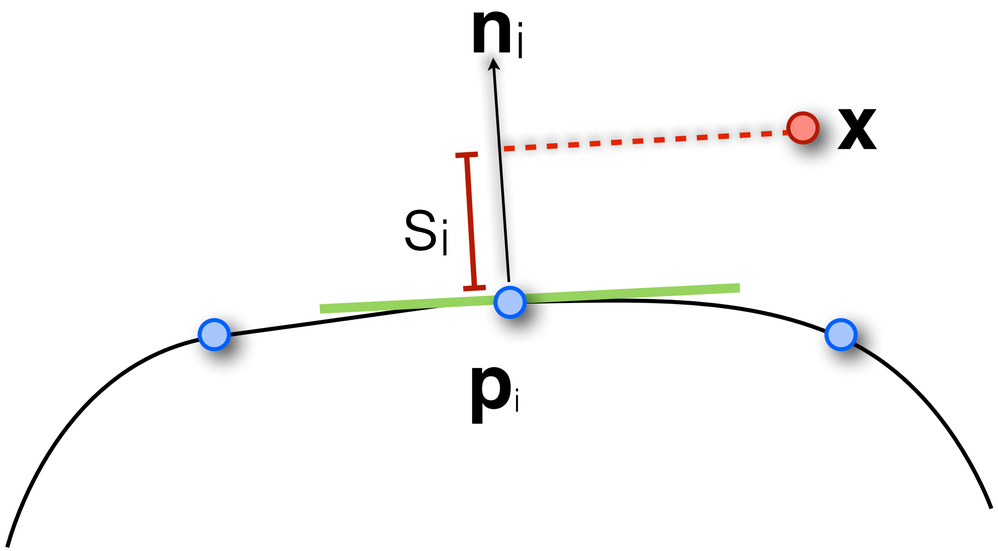
Bonus: Better Normal Constraints

- Recall, we computed our interpolant by solving:

$$\min_a \|B\mathbf{a} - \mathbf{d}\|^2_{W(\mathbf{x})}$$

  with constraint value d_i for the 3N constraint locations.

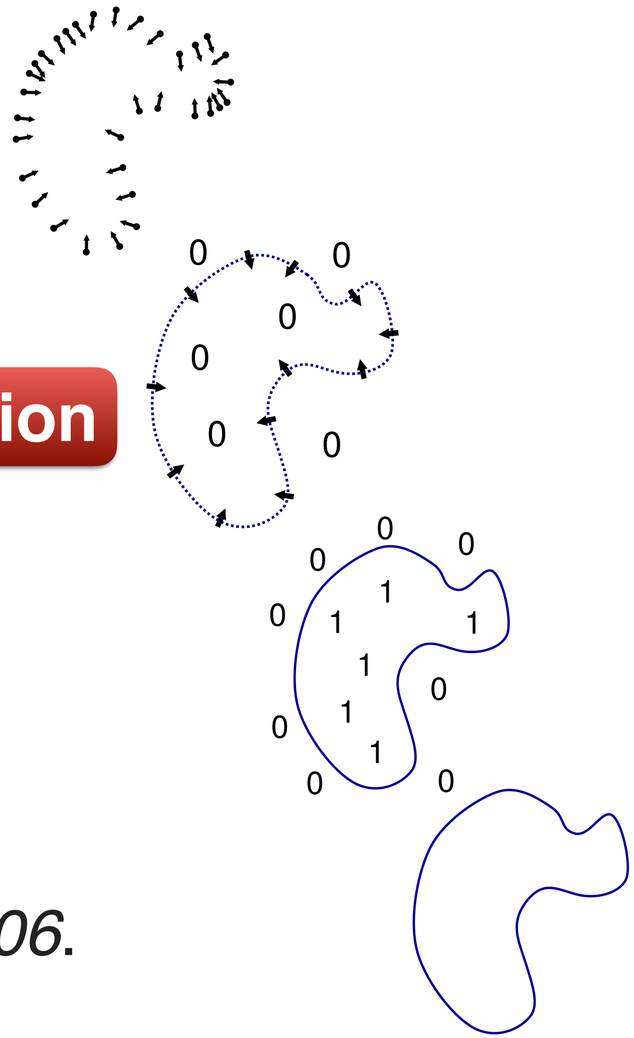
- **New scheme:** use just one constraint per sample pt
- Replace d_i with: $s_i(\mathbf{x}) = (\mathbf{x} - \mathbf{p}_i) \cdot \mathbf{n}_i$
- s_i is the linear function computing signed distance to $\mathbf{p}_i$'s tangent plane
- Note: $\nabla_{\mathbf{x}} s_i = \mathbf{n}_i$



Bonus: Poisson Reconstruction

- Explicitly a fit scalar function's gradient to the normals.
  - Smooth out sampled normals to create a global vector field $\vec{V}$
  - Find scalar function χ whose gradient best approximates this vector field: $min_\chi \|\nabla\chi - \vec{V}\|$
- Advantages:
  - No spurious sheets far from the surface!
  - Robust to noise
- Michael Kazhdan,Matthew Bolitho, Hugues Hoppe. "**Poisson Surface Reconstruction.**" In *Eurographics Symposium on Geometry Processing, 2006*.

Approximate indicator function



Questions?