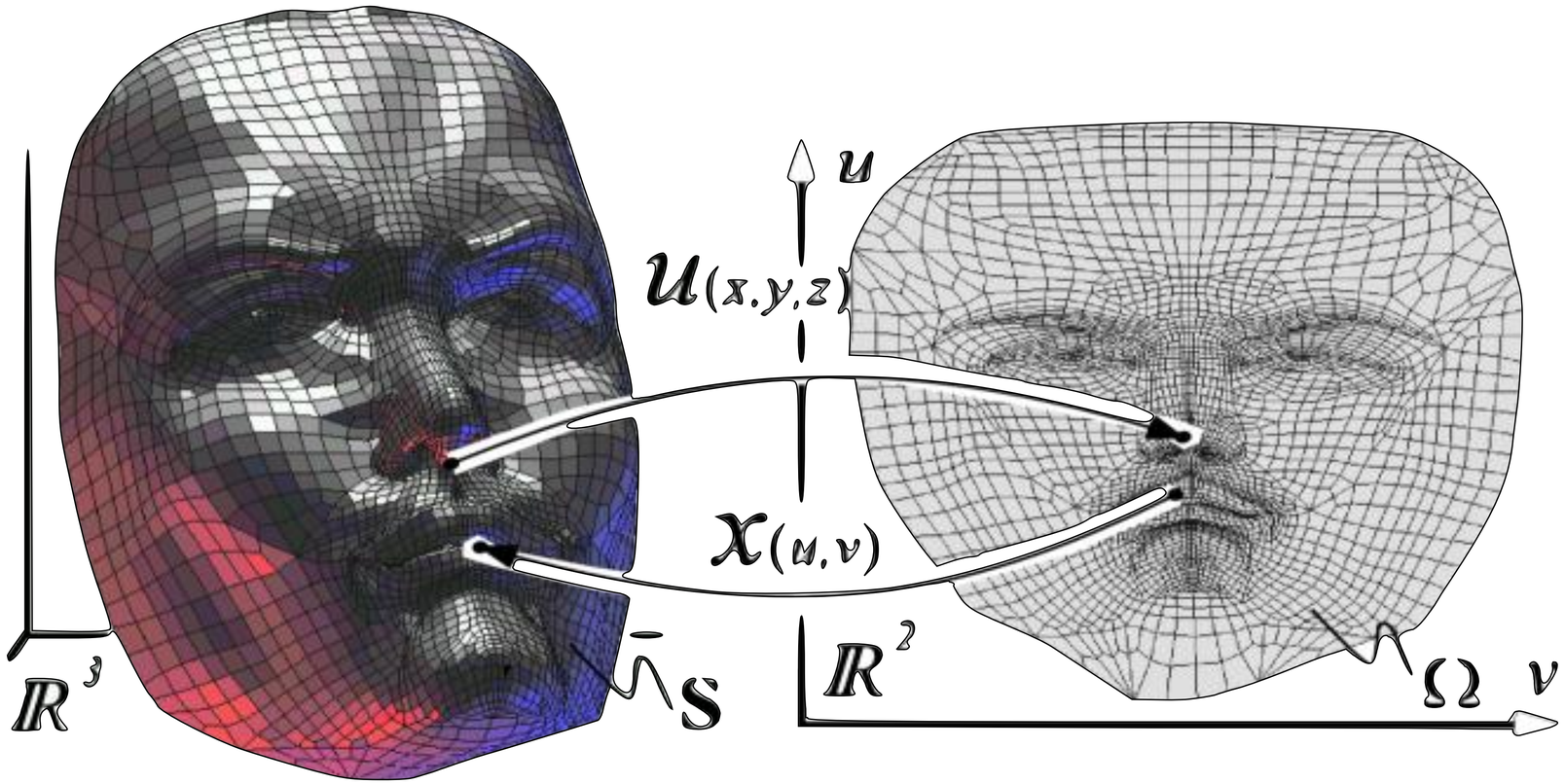
Geometric Modeling

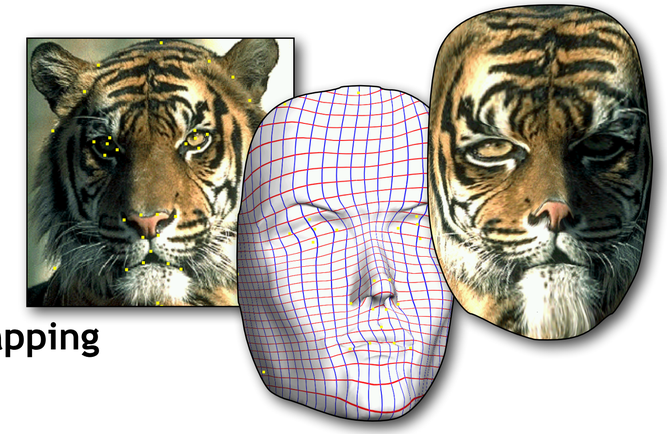
Assignment 4: Mesh Parametrization

Acknowledgements: Olga Diamanti, Julian Panetta

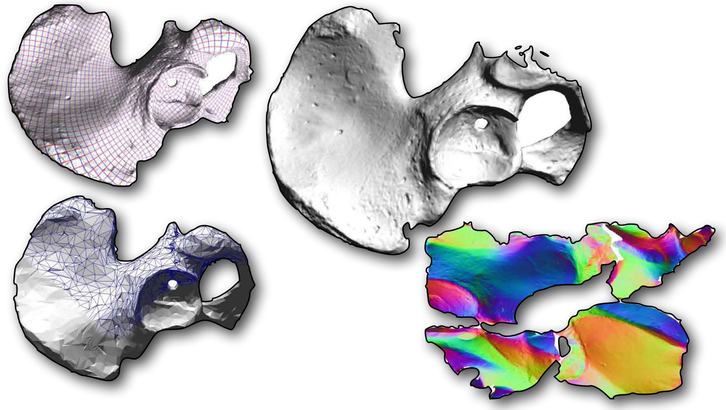
Mesh Parametrization



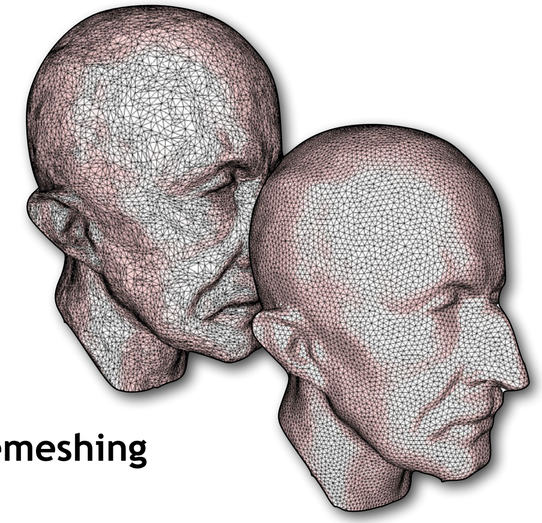
Parametrization Applications



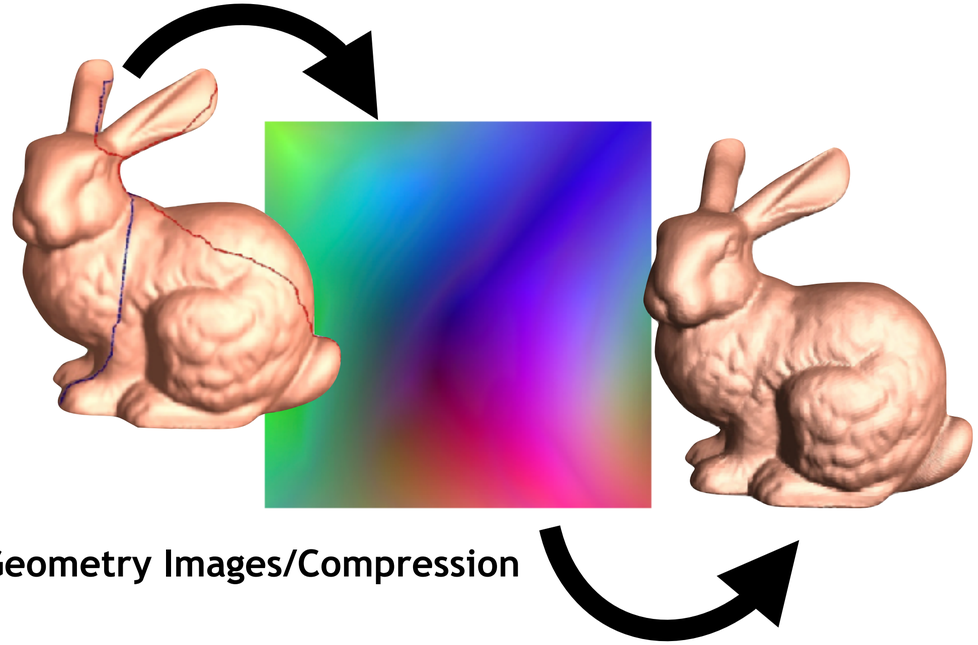
Texture Mapping



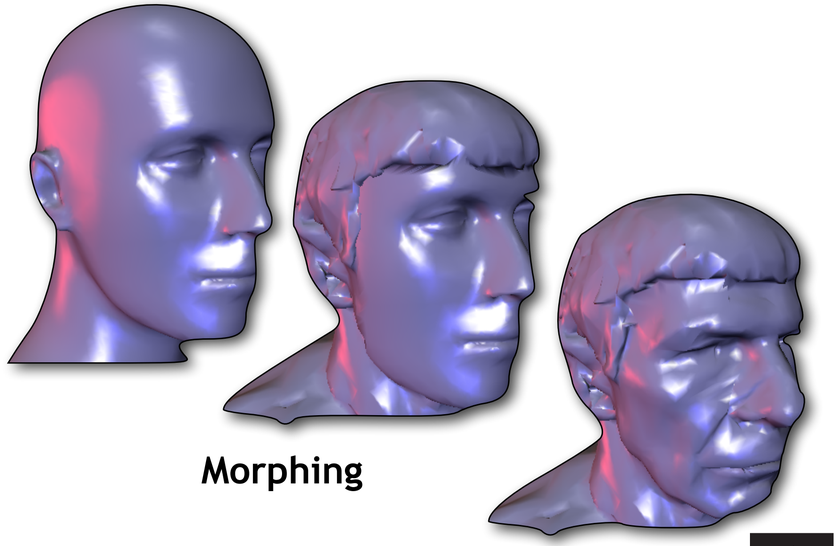
Normal/Bump Mapping



Remeshing



Geometry Images/Compression

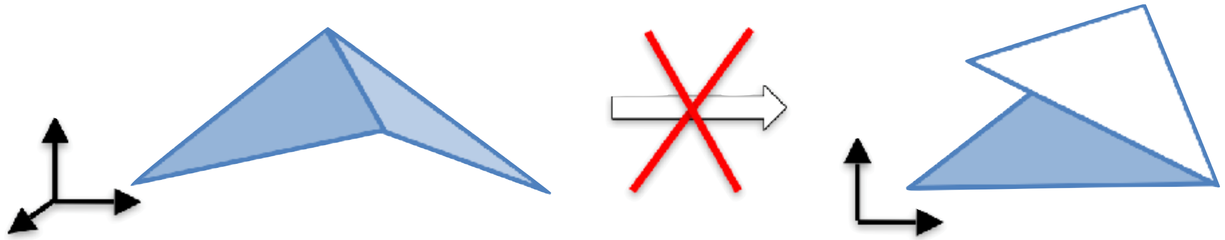


Morphing

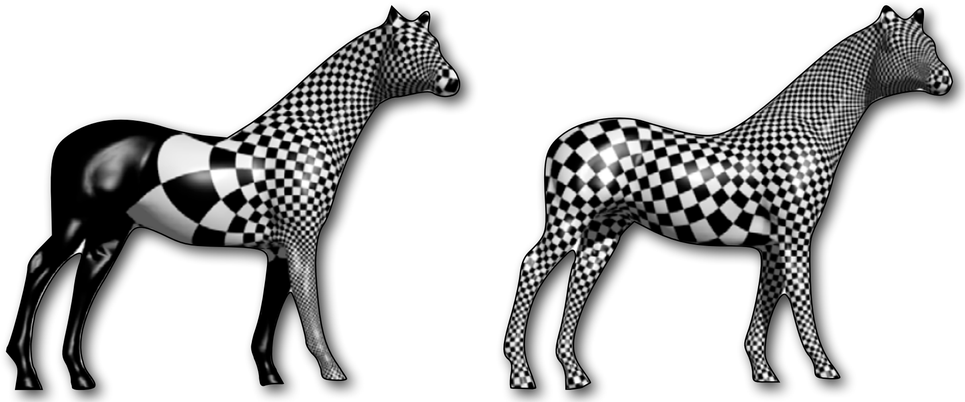
- Detail Transfer
- Mesh Completion
- Editing
- Surface Fitting
- ...

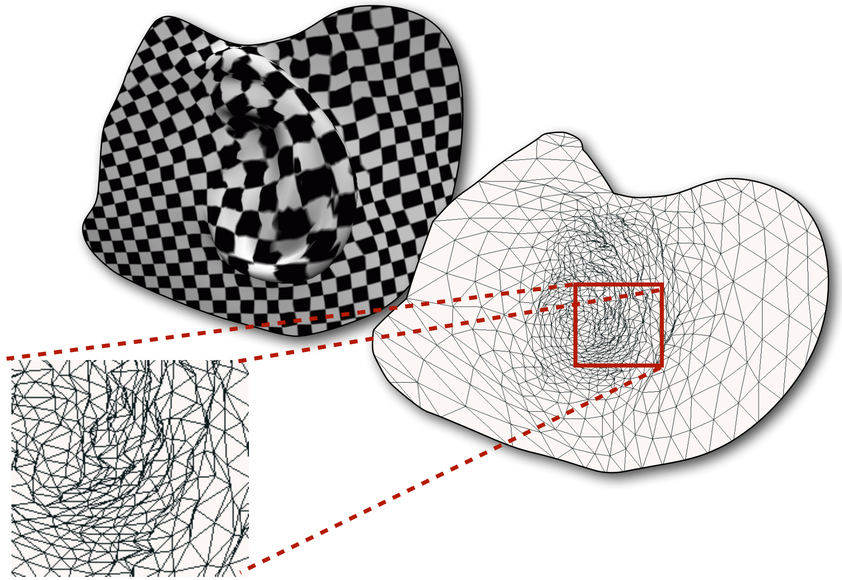
Desirable Properties

- Bijective (invertible): avoid flips/overlaps



- Minimal distortion
  - –Area
  - –Angle





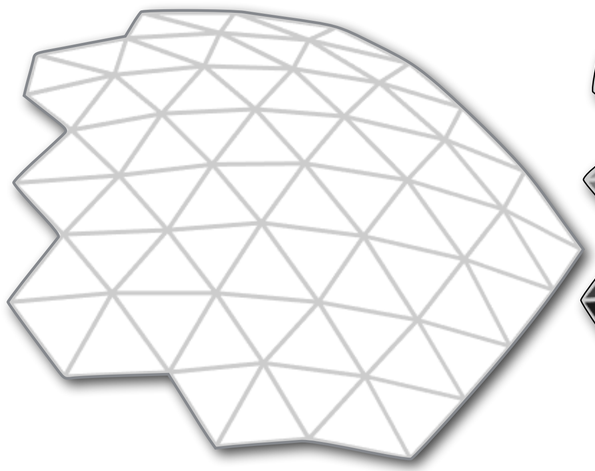
- Efficiently computable

Parametrization Approaches

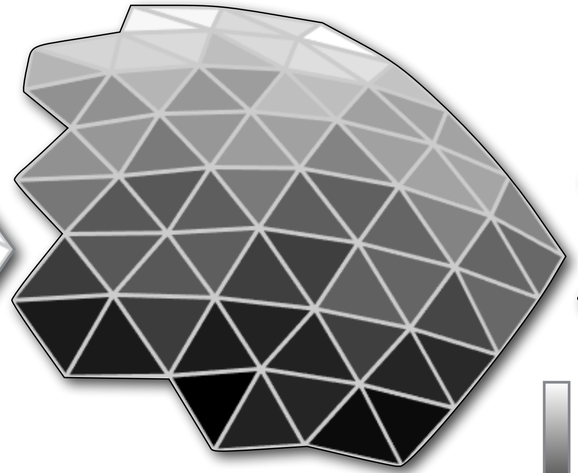
- This assignment will consider 3 approaches
  - Least-Squares Conformal (LSCM)
  - Harmonic
  - Field-Guided
- LSCM, Harmonic implemented in libigl

Formulation: Design Scalar Fields

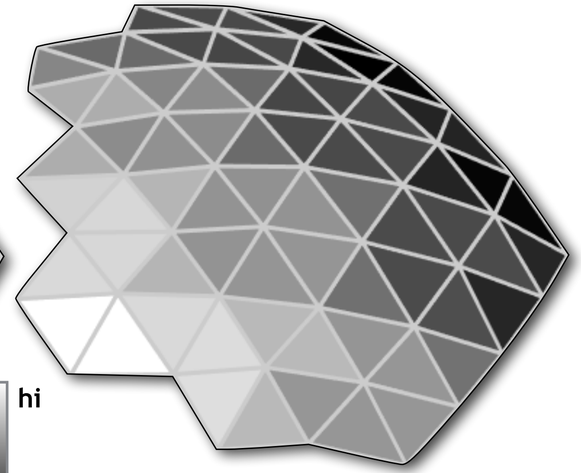
- Compute two scalar fields (u, v) giving the plane coordinates of each mesh point



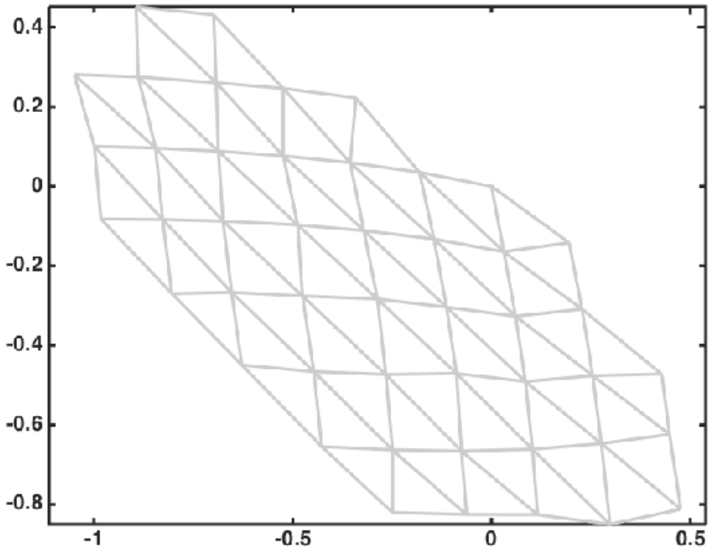
mesh



U-function



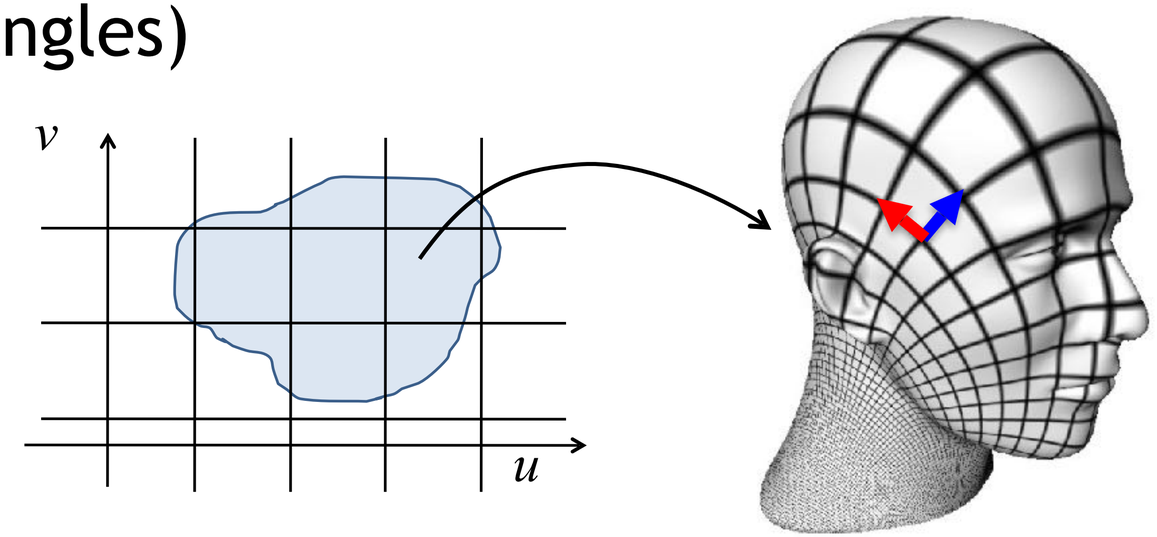
V-function



flattened mesh (UV-domain)

Least-Squares Conformal

- **Conformal** maps have **zero angle distortion** (preserve 90° angles)



- Least-Squares Conformal: minimize angle distortion

$$\min_{u,v} \int_M \|\nabla u^\perp - \nabla v\|^2 \, \mathrm{d}V$$

- Encourage perpendicular, equally scaled coordinate vectors

Least-Squares Conformal

$$\min_{u,v} \frac{1}{2} \int_M \|\nabla u^\perp - \nabla v\|^2 \, \mathrm{d}V$$

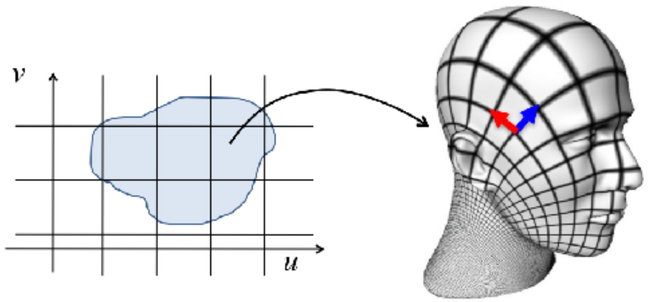
$$= \min_{u,v} \frac{1}{2} \int_M (\hat{n} \times \nabla u - \nabla v) \cdot (\hat{n} \times \nabla u - \nabla v) \, \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2} \int_M \|\hat{n} \times \nabla u\|^2 - 2(\hat{n} \times \nabla u) \cdot \nabla v + \|\nabla v\|^2 \, \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2} \int_M \|\nabla u\|^2 + \|\nabla v\|^2 - 2(\hat{n} \times \nabla u) \cdot \nabla v \, \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2} \int_M \|\nabla u\|^2 + \|\nabla v\|^2 - 2 \det \begin{pmatrix} \nabla u^T \\ \nabla v^T \\ \hat{n}^T \end{pmatrix} \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2} \int_M \|\nabla u\|^2 \, \mathrm{d}V + \frac{1}{2} \int_M \|\nabla v\|^2 \, \mathrm{d}V - A(u,v)$$



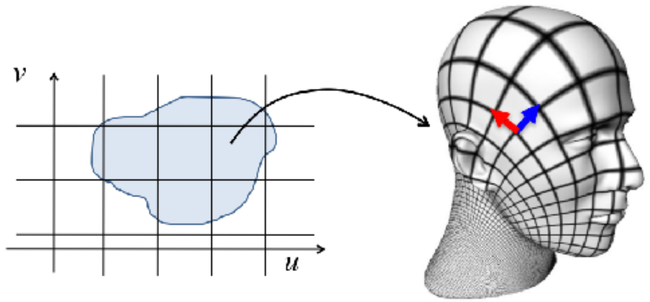
Jacobian Determinant
(Area scaling at each pt)

Area in (u,v) plane!

LSCM ==> Harmonic

- We showed LSCM decomposes into:

$$\frac{1}{2}\int_M \|\nabla u^\perp - \nabla v\|^2 \,\mathrm{d}V = \frac{1}{2}\int_M \|\nabla u\|^2 \,\mathrm{d}V + \frac{1}{2}\int_M \|\nabla v\|^2 \,\mathrm{d}V - A(u,v)$$



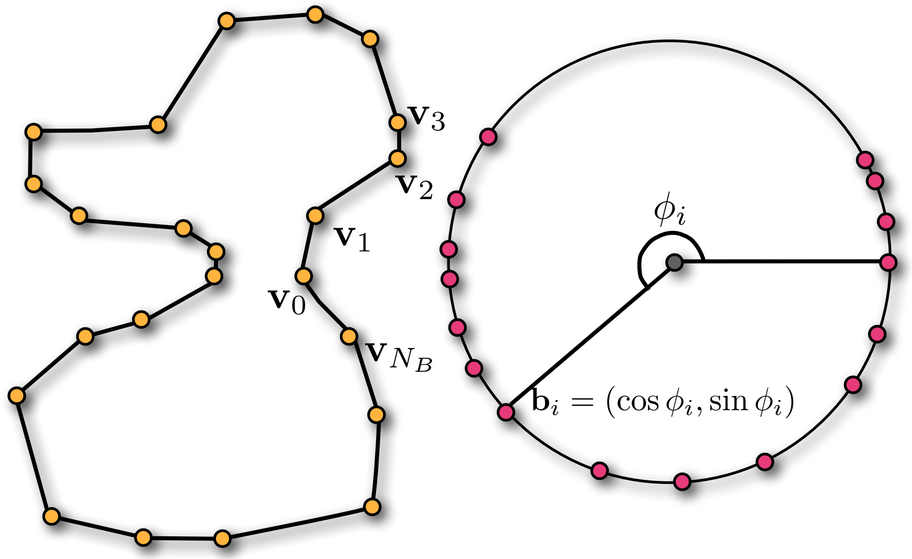
Smoothness of u and v
("Dirichlet Energy")

Area in plane

- If we **fix the flattened boundary** (area), LSCM is equivalent to designing **smooth scalar fields**

Harmonic Parametrization

- Map mesh boundary to a convex polygon in plane

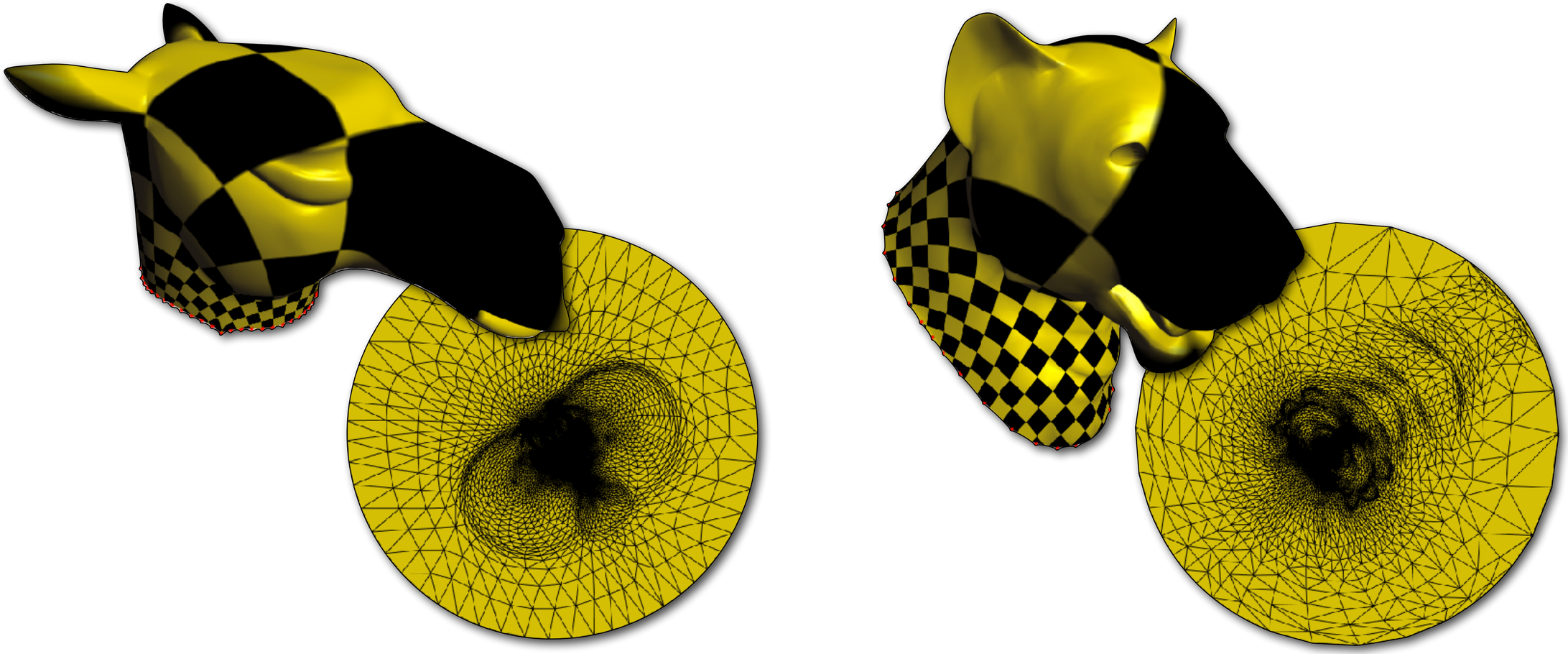


- Solve for smoothly interpolating (u, v):

$$\min_{u=u_b \text{ on } \partial M} \frac{1}{2}\int_M \|\nabla u\|^2 \,\mathrm{d}V \quad \Longrightarrow \quad \begin{cases} \Delta u = 0 & \text{in } M \\ u = u_b & \text{on } \partial M \end{cases}$$

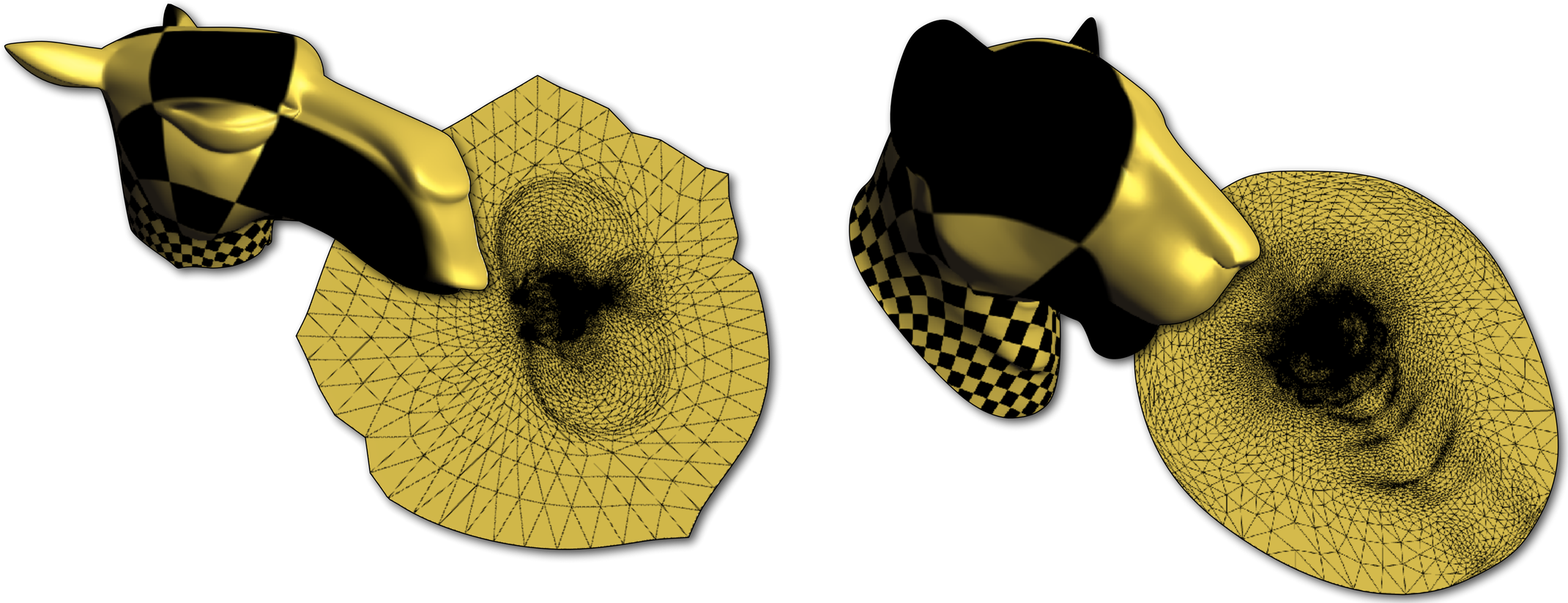
Just solve Laplace equation with fixed boundary values!

Harmonic Parametrization



Already implemented in libigl!
See tutorial 501.

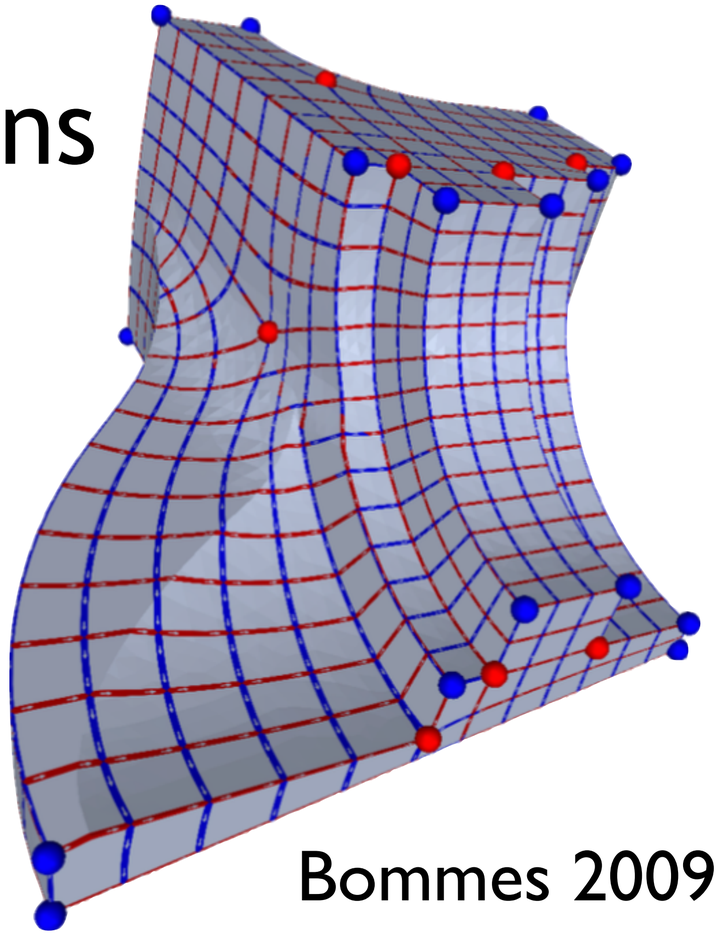
LSCM Parametrization



Already implemented in libigl!
See tutorial 502.

Field-Guided Parametrization

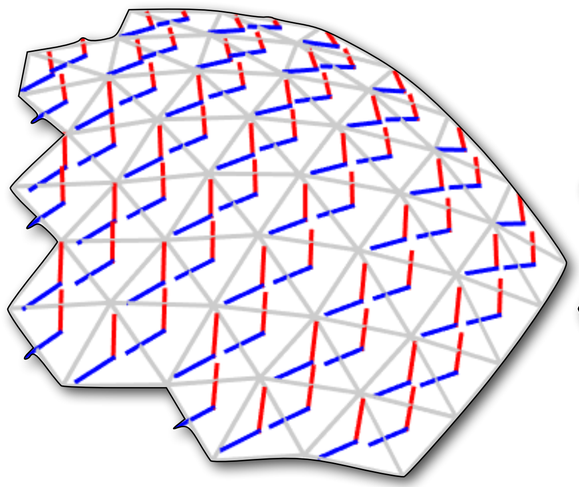
- Often we want coordinate directions to **align** with mesh features.
- We can do this by **specifying vector fields** and **fitting (u, v) gradients** (coordinate directions) to them.
- Orthonormal vectors ==> minimize **angle and area distortion**



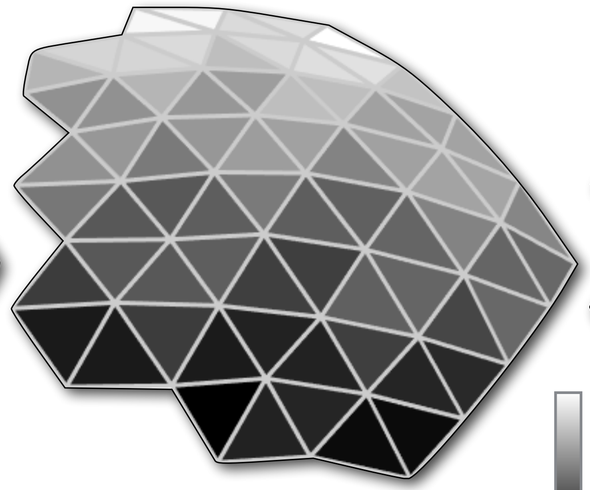
Bommes 2009

Field-Guided Parametrization

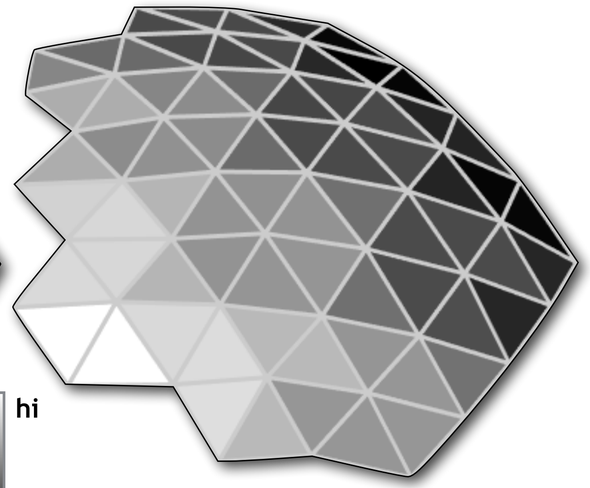
- We'll consider piecewise linear (per-vertex) scalar fields, so gradients and vector fields are constant per triangle:



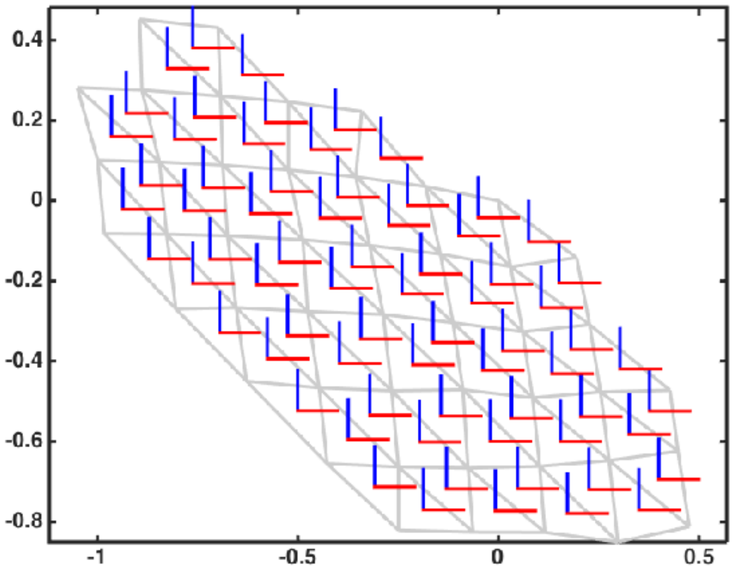
mesh



U-function



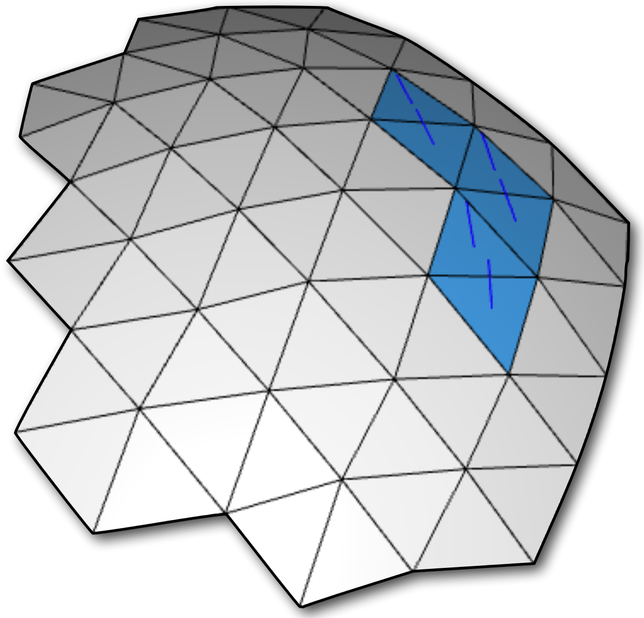
V-function



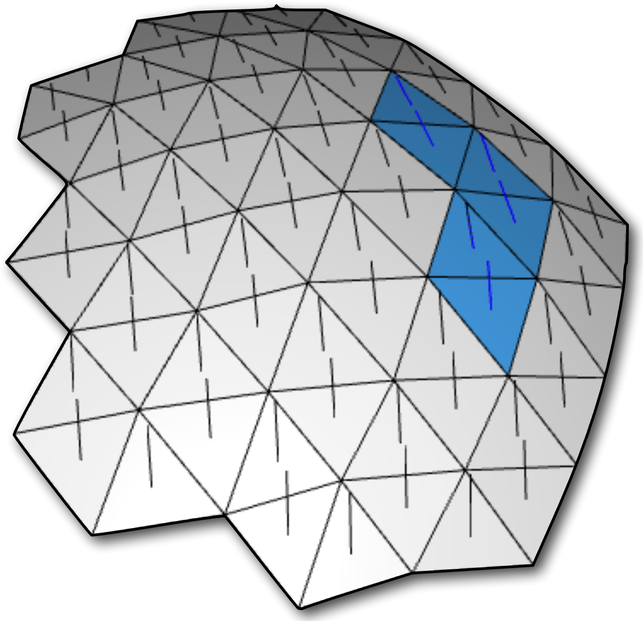
flattened mesh (UV-domain)

Vector-Guided Scalar Fields

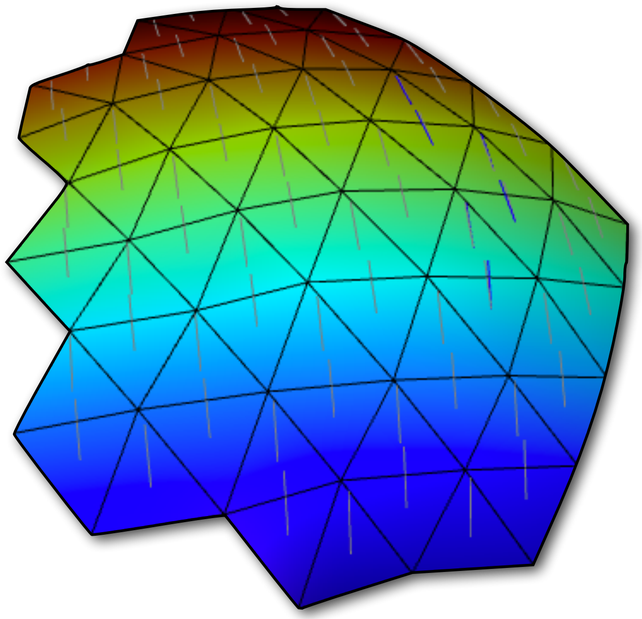
- Main HW4 task: design smooth vector fields and compute a scalar field aligned to them.



vector constraints

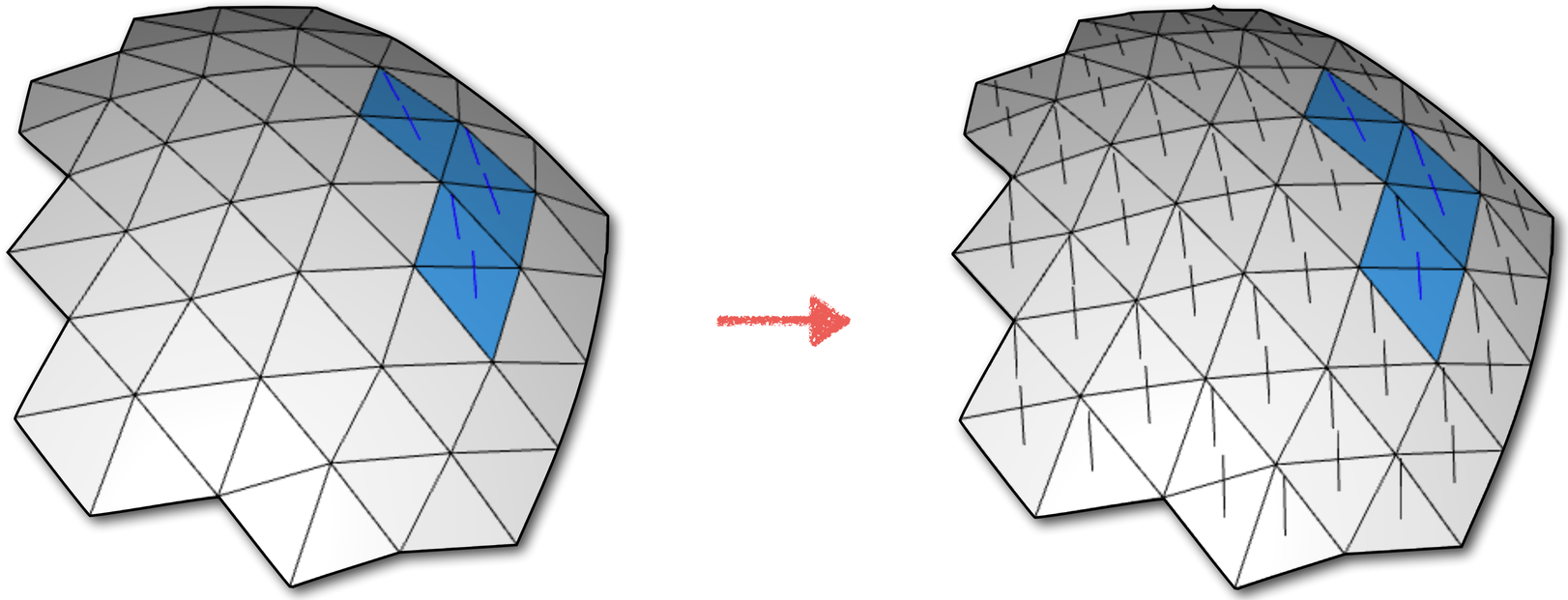


smooth interpolated field



reconstructed scalar function and gradient

Step 1: Vector Field Design

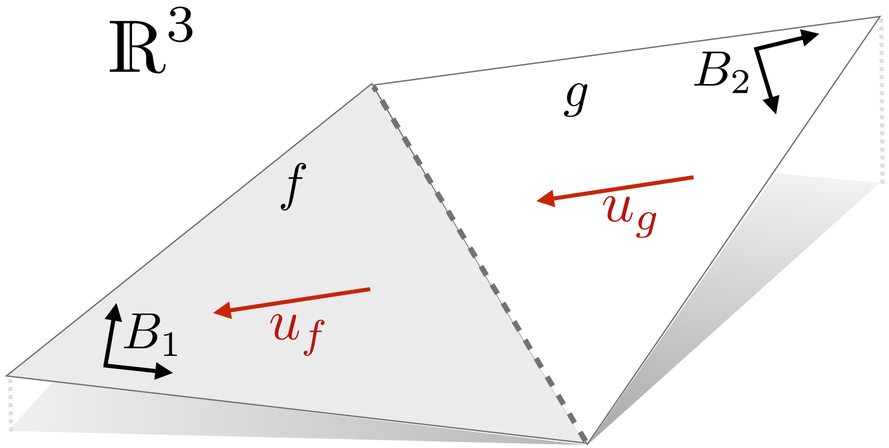


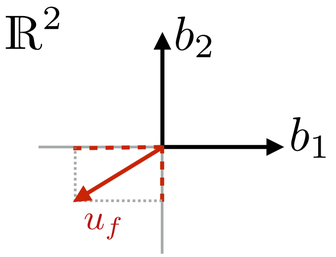
Paint constraint vectors on some faces

Smoothly interpolated field

Vector Field Representation

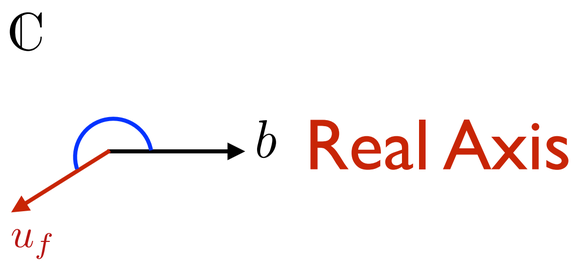
- Tangent vector per triangle: represent in an arbitrary local basis for each triangle:





$u_f = (u_f^x, u_f^y) \in \mathbb{R}^2$

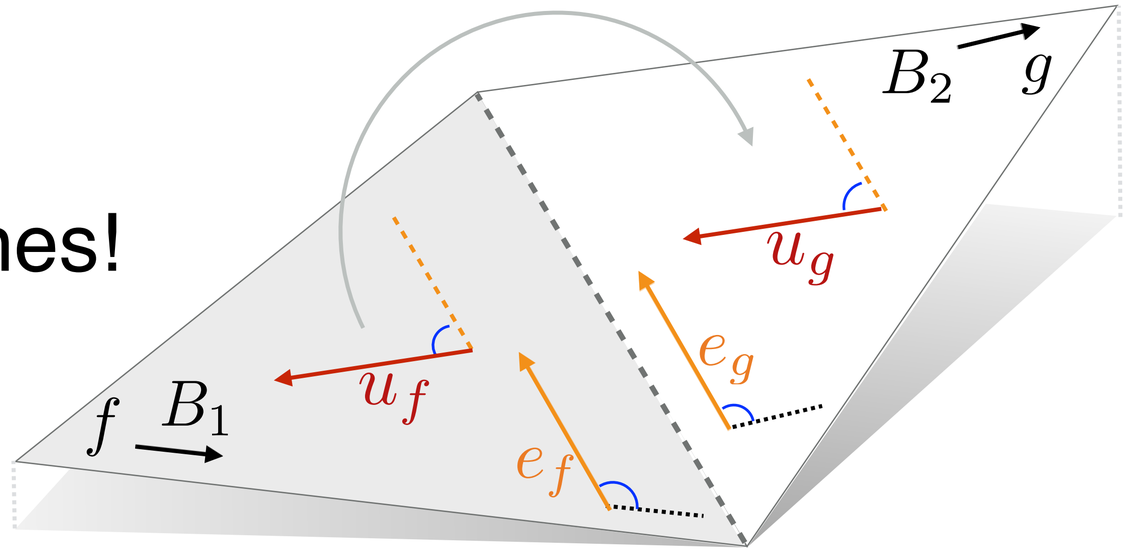
- (Optional) notation simplification: Vector = complex number (Tangent plane = complex plane)



$u_f = u_f^x + iu_f^y \in \mathbb{C}$

Vector Field Smoothness

- Try to keep vectors constant
- Vectors lie in different tangent planes! Must transport to compare.
- Idea: unfold triangle pair along shared edge, compare vectors
- Equivalent: represent vectors in orthonormal basis determined by shared edge.
  (Rotate so shared edge is real axis)



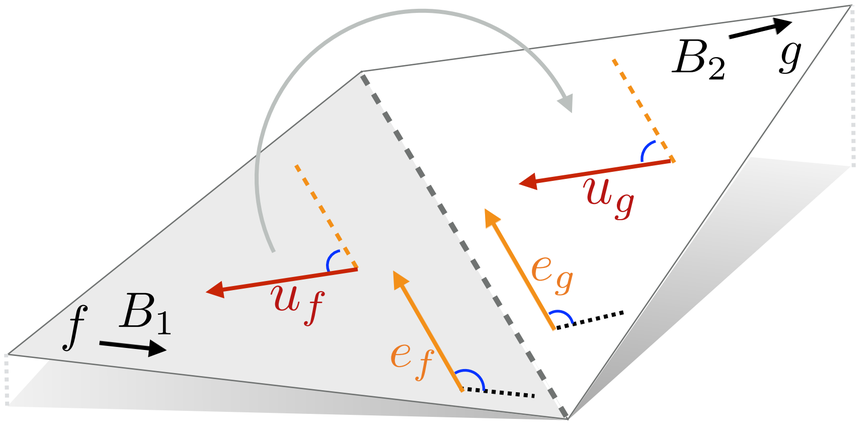
Vectors in common basis:

$$\tilde{u}_f = u_f \bar{e}_f$$
$$\tilde{u}_g = u_g \bar{e}_g$$
$$|\tilde{u}_f - \tilde{u}_g|^2$$

Since for unit complex numbers:

$$e_f^{-1} = \overline{e_f}$$

Vector Field Smoothness



- Minimize non-smoothness across all edges:

$$\sum_{\text{edge } (f,g)} |u_f \overline{e_f} - u_g \overline{e_g}|^2 = \sum_{\text{edge } (f,g)} \begin{bmatrix} \overline{u_f} & \overline{u_g} \end{bmatrix} Q_{fg} \begin{bmatrix} u_f \\ u_g \end{bmatrix}$$

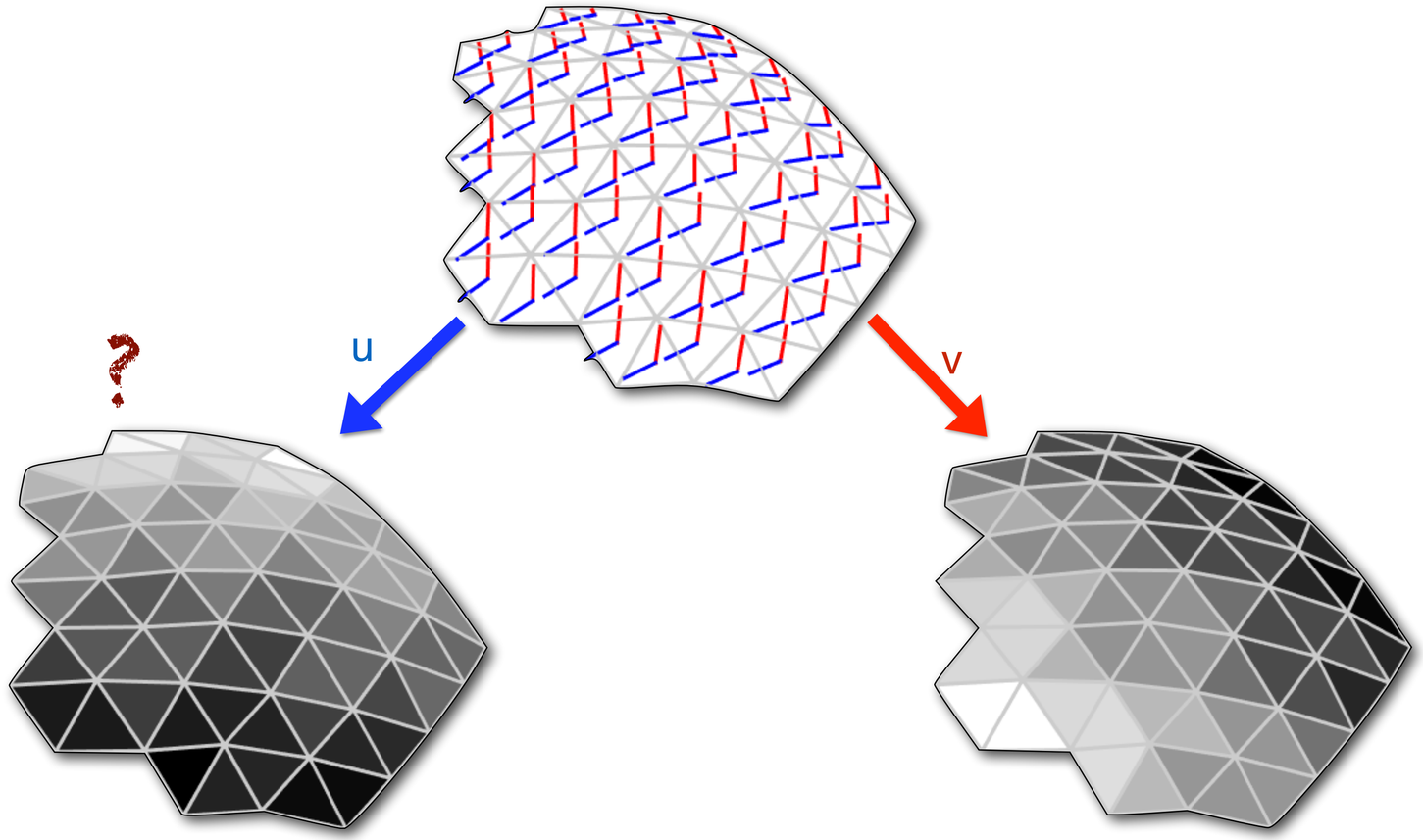
- Can be re-written as one large complex quadratic form:

$$\min_{\mathbf{u}|_{cf}=\vec{c}} \mathbf{u}^* Q \mathbf{u} \quad \Longleftrightarrow \quad \frac{\partial}{\partial \overline{\mathbf{u}}} \mathbf{u}^* Q \mathbf{u} = Q\mathbf{u} = 0$$

Complex equiv to gradient

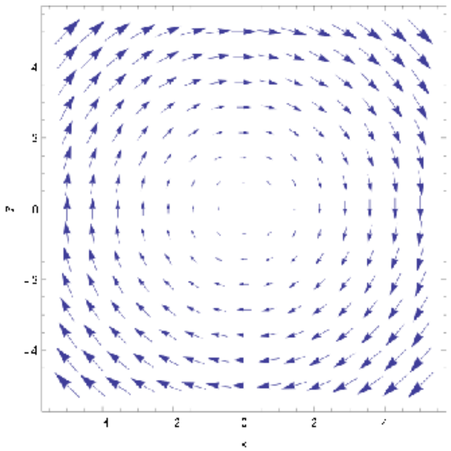
Step 2: Scalar Field Design

- Find a scalar field whose gradients match the vector field:

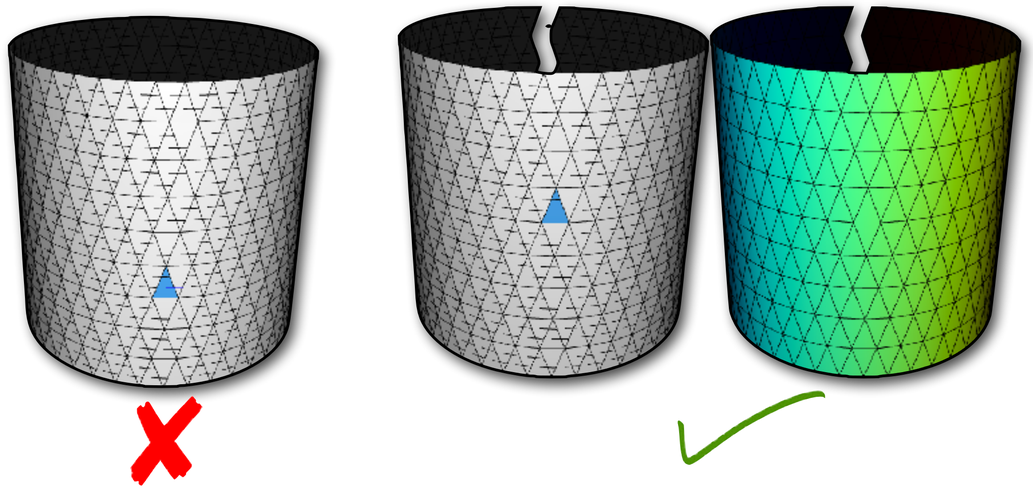


Integrating the Vector Field

- Why not just assign zero to some vertex and “integrate” the vector field outward to get all other vertex values?
- Problem: most vector fields are not actually integrable (i.e. not the gradient of a scalar field)
- This can happen for local or global reasons:



Local: nonzero curl



Global: non-simply-connected domains

Integrating the Vector Field

- **The fix:** integrate in the “least-squares sense”

$$\min \sum_{\text{face } t} w_t \|\mathbf{g}_t - \mathbf{u}_t\|^2 \qquad \mathbf{g}_t = \nabla f|_t$$

- This turns out to be equivalent to the Poisson equation:

$$\nabla \cdot \nabla f = \nabla \cdot u \quad \text{in } M$$
$$n \cdot \nabla f = n \cdot u \quad \text{on } \partial M$$

- Can also be written as a quadratic form:

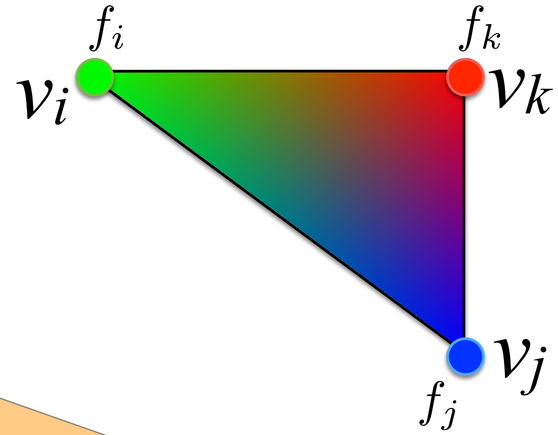
$$\frac{1}{2}\mathbf{f}^T K \mathbf{f} + \mathbf{f}^T \mathbf{b} + c$$

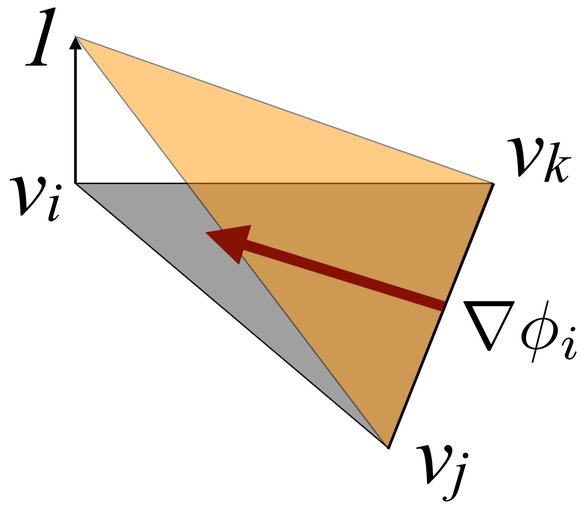
- Only determined up to constant: fix one vertex to zero!

Expressing as Quadratic Form

$$\min \sum_{\text{face } t} w_t \|\mathbf{g}_t - \mathbf{u}_t\|^2 \qquad \mathbf{g}_t = \nabla f|_t$$

- Recall, piecewise linear f can be written as a sum of “hat” basis functions:





$$f(x) = f_i\phi_i(x) + f_j\phi_j(x) + f_k\phi_k(x)$$

$$\nabla f(x) = f_i\nabla\phi_i(x) + f_j\nabla\phi_j(x) + f_k\nabla\phi_k(x)$$

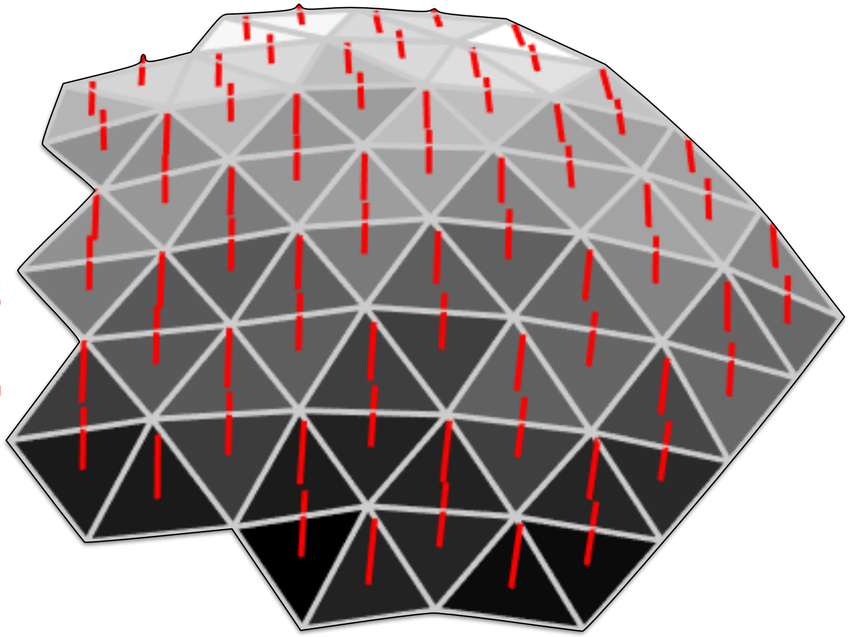
The “Gradient Matrix”

- So the gradient is just a linear combination of vertex scalar field values

$$\begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{t_1} \\ \vdots \\ \frac{\partial f}{\partial x}\Big|_{t_{\#F}} \\ \frac{\partial f}{\partial y}\Big|_{t_1} \\ \vdots \\ \frac{\partial f}{\partial z}\Big|_{t_{\#F}} \end{bmatrix} = G \begin{bmatrix} f|_{v_1} \\ \vdots \\ f|_{v_{\#V}} \end{bmatrix}$$

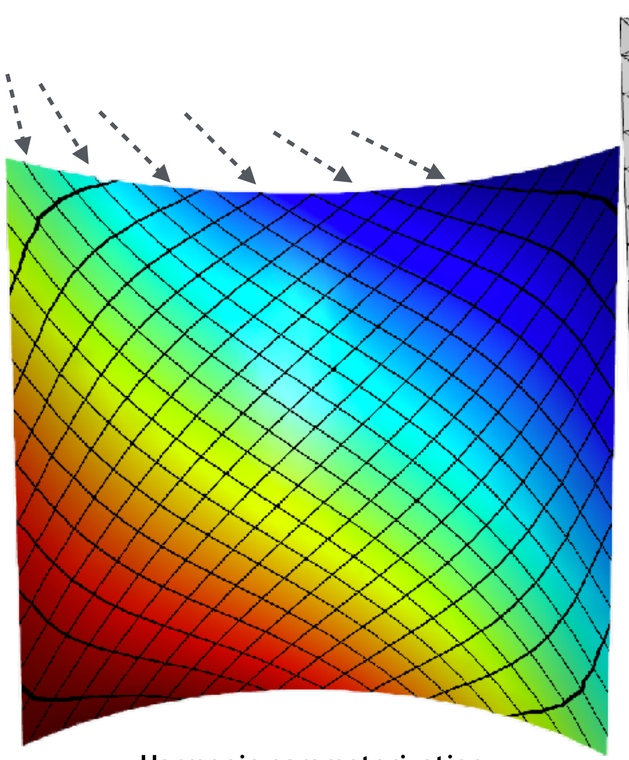
$3\#F \times 1$ $\quad$ G $\quad$ $3\#F \times \#V$ $\quad$ $\#V \times 1$

Maps per-vertex scalars to per-triangle gradient vectors

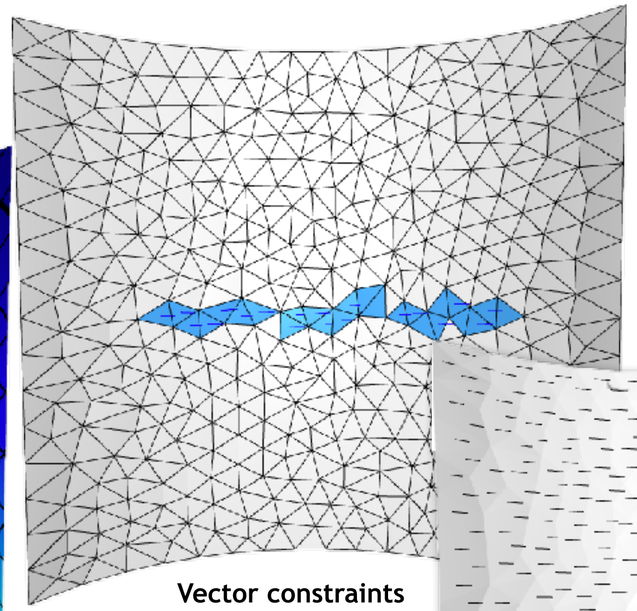


```
void igl::grad(const Eigen::MatrixXd &V,
               const Eigen::MatrixXi &F,
               Eigen::SparseMatrix<double> &G);
```

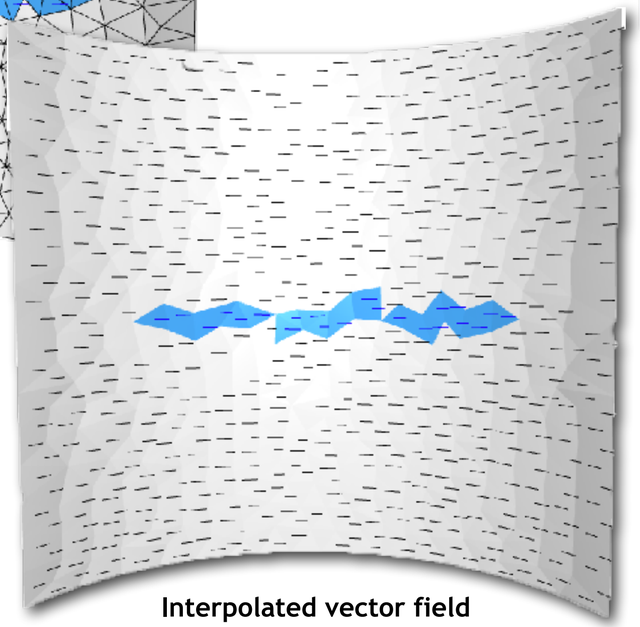
Application: Edit Parameterizations



Harmonic parameterization
(initial u function)

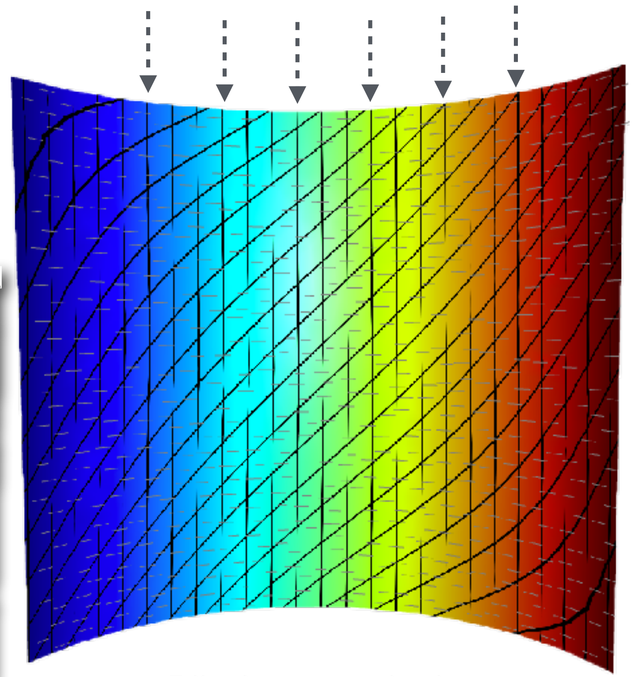


Vector constraints
for u's gradient



Interpolated vector field

(approximate u gradient)



Edited parameterization
(new u function)

Equality Constraints

- Both steps involve equality constraints
  - Painted vectors on constrained faces
  - Scalar field value on single vertex fixed to zero
- Two approaches:
  - Lagrange multipliers (standard approach)
  - Variable elimination (much more efficient here)

Variable Elimination via Row/Col Removal

- I'll derive this since it's less discussed than Lagrange multipliers
- Most efficient way to fix variables to constants:

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c \quad \text{s.t.} \quad \mathbf{x}_c = \mathbf{d}$$

$$\left(\mathbf{x} = \begin{bmatrix} \mathbf{x}_f \\ \mathbf{x}_c \end{bmatrix}\right)$$

Free Variables

Constrained Variables

- For derivation, assume vars ordered like this; Trick works for any ordering

Row/Col Removal

$$\frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$$

$$= \frac{1}{2}\begin{bmatrix}\mathbf{x}_f^T & \mathbf{x}_c^T\end{bmatrix}\begin{bmatrix}A_{ff} & A_{fc}\\ A_{cf} & A_{cc}\end{bmatrix}\begin{bmatrix}\mathbf{x}_f\\ \mathbf{x}_c\end{bmatrix} - \begin{bmatrix}\mathbf{b}_f^T & \mathbf{b}_c^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_f\\ \mathbf{x}_c\end{bmatrix} + c$$

$$= \frac{1}{2}\left(\mathbf{x}_f^T A_{ff}\mathbf{x}_f + 2\mathbf{x}_f^T A_{fc}\mathbf{x}_c + \mathbf{x}_c^T A_{cc}\mathbf{x}_c\right) - \mathbf{b}_f^T\mathbf{x}_f - \mathbf{b}_c^T\mathbf{x}_c + c$$

const

const

$$\min_{\mathbf{x}_f} \quad \Longrightarrow \quad 0 = A_{ff}\mathbf{x}_f + A_{fc}\mathbf{x}_c - \mathbf{b}_f$$

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$$

$$\tilde{A} = A_{ff} \quad \tilde{\mathbf{x}} = \mathbf{x}_f \quad \tilde{\mathbf{b}} = \mathbf{b}_f - A_{fc}\mathbf{x}_c$$

Rows/cols corresponding to fixed variables removed!

Columns corresponding to fixed vars contribute to RHS

Eigen Complex Sparse Matrices

```
Eigen::SparseMatrix<std::complex<double> > Q;


Eigen::SparseLU< Eigen::SparseMatrix<std::complex<double> > > solver;
```

Questions?