Probability and Statistics

Random Experiments and Sample Spaces:-

Experiments involving Randomness

Coin toss, dice roll, etc.

Ω → Set of all possible outcomes of a random experiment. It could be finite or infinite.

$\Omega_c = \{Head, Tail\}$, $\Omega_d = \{1,2,3,4,5,6\}$

2-coins → $\Omega_{2c} = \Omega_c \times \Omega_c = \{H,T\} \times \{H,T\}$

Outcomes and Events

(omega) → $\omega \in \Omega$ is called a sample point or a possible outcome

A subset $A \subseteq \Omega$ is called an event.

Events in the coin toss experiment $C_1 = \{T\}$ $[\subset \Omega_c]$

Events in the die roll $D_1 = \{6\}$, $D_2 = \{1,3,5\}$ $[\subseteq \Omega_d]$

Events are any subset of Ω, even null sets, but $\mathbb{P}(\phi) = 0$

Probability of an event $A = \mathbb{P}(A)$.

It may or may not be possible to measure/assign $\mathbb{P}$ for every subset A.

$\mathbb{P}_c(T) = 0.5$

Probability measure $\mathbb{P}$ is a set function. It acts on sets and measures the probability of such sets

Set Theory 101:-

A^c = complement of A

ϕ = denotes empty set. [belongs to every set]

$A \cup B$ = A union B

$A \cap B$ = A intersection B

~~$A \Delta B$~~

$A \setminus B$ = A minus B $= A \cap B^c$

~~Symmetric~~

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

ME = Mutually Exclusive

~~HW → Identity Complement~~

$|A|$ = no. of elements in A = cardinality of A.

Inclusion - Exclusion Principle = $|A \cup B| = |A| + |B| - |A \cap B|$

Countable Sets and Uncountable Sets

Monotone Seq → $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ [Increasing seq]

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ [Decreasing]

$I_n = [0, 1 - \frac{1}{n}]$

$D_n = [0, \frac{1}{n}]$

Cartesian product :- $A \times B = \{(a,b) : a \in A, b \in B\}$

Powerset [$\mathcal{P}(A)$] :- Set of all possible subsets of A.

$|\mathcal{P}(A)| = 2^{|A|}$ [Only for discrete elements]

$\mathcal{P}([0,1]) = \{(a,b) : a \leq b, a, b \in [0,1]\}$

↳ range, not just 2 elements

Functions:-

Fns are maps from elements in the domain D to the range R

$f: D \to R$

$f: \mathbb{R} \to \mathbb{R} \Rightarrow f(x) = x$

Set fns are those fns ~~do~~ who act on sets; D is a collect of sets.

$\mathbb{P}$ is a set fn.

Axioms of probability:-

1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$
2) For a set $A \subseteq \Omega$, $\mathbb{P}(A) \in [0,1]$
3) For a disjoint collection of events $A_1, A_2, \ldots, A_j$ where $A_i \subseteq \Omega$, then:

$$\mathbb{P}\left(\bigcup_{i=1}^{j} A_i\right) = \sum_{i=1}^{j} \mathbb{P}(A_j) \quad [j \text{ can be } \infty] \to \text{but must be countable}$$

where $A_i \cap A_j \ \forall i,j = \phi$

In general, the domain of $\mathbb{P}$ is $\mathcal{P}(\Omega)$.

~~Probability of impossible event = 0~~.

Counter example for $\mathcal{P}(\Omega)$ being satisfactory

1) Pick a number randomly (uniformly) of # from the real line.
2) $\Omega = \mathbb{R}$, hence $\mathbb{P}(\mathbb{R}) = 1$,
3) Domain $= \mathcal{P}(\mathbb{R})$
4) $\mathbb{P} : \mathcal{P}(\mathbb{R}) \to [0,1]$
5) $\mathbb{P}$ has the property that sets of equal length have equal probability
6) We know that $\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1)$ where $[n, n+1) \in \mathcal{P}(\mathbb{R})$
[Countable union since n is an integer]
7) $\mathbb{P}(\mathbb{R}) = 1 = \sum \mathbb{P}([n, n+1)) = 0 / \infty$ depending
8) This is a contradiction! [Happens when picking segments of unit length]

Powerset is thus a bad choice

Not all set fns can be calibrated to measure all possible subset of the sample space.

Our choice of domain must have certain features.
Let domain = $\mathcal{F}$
then $\phi, \Omega \in \mathcal{F}$; if $B \in \mathcal{F} \rightarrow B^c \in \mathcal{F}$ etc;

$\mathcal{F}$
$\uparrow$
Bags that meet these criteria are called sigma-algebras

Sigma algebra / Event space $[\mathcal{F}]$ associated with a set Ω is a collection of subsets of Ω that satisfy
- $\rightarrow \phi \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
- $\rightarrow A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$
- $\rightarrow A_1, A_2, \ldots, A_n \in \mathcal{F} \rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

$\rightarrow$ The σ-algebra is closed under formation of complements and countable unions and countable intersections (by De Morgans)

$\rightarrow$ When Ω is countable and finite, then $\mathcal{P}(\Omega)$ is a σ-algebra

If Ω is countable & finite, we will consider $\mathcal{P}(\Omega)$ as the domain.

Probability Space = $\{\Omega, \mathcal{F}, \mathbb{P}\}$

Probability measure :- A probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ is a set fn $\mathbb{P} : \mathcal{F} \rightarrow [0,1]$ s.t.
1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$
2) For a disjoint collection of events $A_1, A_2 \ldots$ from $\mathcal{F}$, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$\mathbb{P}$ space for $U[0,1]$

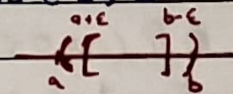
$\Omega = [0,1]$

If we generate a σ-algebra, but we get a borel-sigma algebra $\mathcal{B}[0,1]$

Borel σ-algebra $[\mathcal{B}[0,1]]$

Defined when $\Omega = [0,1]$, as the σ-algebra generated by closed sets of the form $[a,b]$ where $a \le b$ & $a, b \in [0,1]$

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n},\ b - \frac{1}{n}\right]$$



$$(a,b] = \bigcup_{n=1}^{\infty} \left(a,\ b + \frac{1}{n}\right]$$

$[\mathcal{B}[0,1]]$

Borel σ-algebra ↑: is the σ algebra generated by sets of the form $[a,b]$ or (a,b) or $(a,b]$ or even $[a,b)$ where $a \le b$ & $a, b \in [0,1]$

If $\Omega = \mathbb{R}$, the $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by open sets of the form (a,b), where $a<b$ and $a, b \in \mathbb{R}$.

How to define $\mathcal{B}(\mathbb{R}^2)$

Consequences of the Probability Axioms:-

i) $P(A^c) = 1 - P(A)$

$$P(A \cup A^c) = P(A) + P(A^c) = 1.$$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

~~P(A∪B)~~ ~~Prove~~ Prove it

iii) If $A \subseteq B$, prove that $P(A) \le P(B)$ [$A \subseteq B$ can be interpreted as $A \to B$]

If $A \subseteq B \to B = A + \lambda$.

iv) $P\left(\bigcup_{i=1}^{\infty} B_i\right) \le \sum_{i=1}^{\infty} P(B_i)$ [Boole's ineq.]

Diff b/w Impossible event vs Zero probability event

ϕ (impossible event) — finite sized set (zero probability event)

In $U[0,1]$, $\mathbb{P}(\omega = 0.5) = 0$
Every experiment outcome of this experiment is a zero probability event.
This implies that zero outcome events could occur.

ϕ on the other hand, can never occur, and are hence impossible events.

$$\mathbb{P}(\omega \in [0, 0.25) \cap [0.75, 1]) = 0$$

Limits and Continuity:-

Limits:-

Let $a_1, a_2, \dots, a_n$ be a sequence with limit L.
Then $\forall \varepsilon$, $\exists N_\varepsilon$ s.t $\forall n > N_\varepsilon$, $|a_n - L| \leq \varepsilon$

For a fn $f(x)$; $\lim_{x \to c} f(x) = f(c)$

The limit exists only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$.

All points δ close to c are ε close to $f(c)$.

Continuity:-

$LHL = RHL = f(x)$. If the fn is cont at all pts, it is said to be continuous. As $x \to c$, $f(x) \to f(c)$

But what about $\mathbb{P}$?

For a continuous set fn S as $A_n \to A$, we have $S(A_n) \to S(A)$

Sequence of sets :-

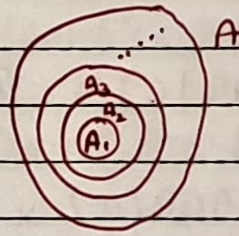
Given $(\Omega, \mathcal{F})$, if $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say that the given seq of sets A_n are increasing to A $(A_n \uparrow A)$

Similarly when $A_1 \supset A_2 \supset \cdots$ is a decreasing seq of events and $\bigcap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$

Alt notation :- increasing seq of sets $A_n \Rightarrow \lim_{n\to\infty} A_n$ for $\bigcup_{n=1}^{\infty} A_n$

decreasing seq of sets $A_n \Rightarrow \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$

Lemma :- For a sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$



Proof :-

$$A = \bigcup_{i=1}^{\infty} A_i$$

(i) Consider an increasing seq:

define $F_n = A_n - A_{n-1}$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} F_n = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

Hence Proved.

(ii) Decreasing seq. [HW]

Conditional Probability :-

Ex:- Given the die outcome is odd, what is $\mathbb{P}(1)$? [1/3]

Given $\bar{\omega} \in [0, 0.5]$, what is $\mathbb{P}(\bar{\omega} \in [0, 0.25])$

Defn:- The conditional probability of event A is defined as

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \quad \text{when } \mathbb{P}(A) > 0$$

Theorem:- $\mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \times \mathbb{P}(B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \times \mathbb{P}(A)$$

Bayes Rule:- $$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \times \mathbb{P}(B)}{\mathbb{P}(A)}$$

(i) What is $\mathbb{P}(A | B \cap C)$

(ii) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B)$

(iii) $\mathbb{P}(A \cap B | C) = \mathbb{P}(A | B \cap C) \times \mathbb{P}(B|C)$

$$\downarrow$$

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(\cancel{C})} \times \frac{\mathbb{P}(\cancel{C})}{\mathbb{P}(B \cap C)} \times \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$$

$$\downarrow$$

$$\mathbb{P}(A | B \cap C) \times \mathbb{P}(B|C)$$

(iii) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \times \mathbb{P}(B|A) \times \mathbb{P}(C | A \cap B)$

$$\downarrow$$

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)} \times \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \times \mathbb{P}(A)$$

iv) $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n | A_1 \cap \cdots \cap A_{n-1})$

classmate

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Q) Draw 4 cards without replacement. What is the $\mathbb{P}$(9 club, 8 dia, K sp, K cl)

$$\mathbb{P}(9_c) \times \mathbb{P}(8d \mid 9c) \times \mathbb{P}(K_{sp} \mid 9_c \cap 8d) \times \mathbb{P}(K_{cl} \mid 9_c \cap 8d \cap K_{sp})$$

$$= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} = \frac{48!}{52!}$$

When every outcome is equally likely in a finite sample space Ω, $\mathbb{P}(B \mid A) = \frac{|A \cap B|}{|A|}$

Law of Total Probability:-

$A = (A \cap B) \cup (A \cap B^c)$, $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

Same as $\mathbb{P}(A) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\,\mathbb{P}(B^c)$

Let $B_1, B_2, \ldots, B_n$ be the partition of the sample space Ω. Then for any event A we have

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

Q) 3 bags with m marbles total. Bag i has R_i red and B_i blue for $i \in [1,3]$. Find $\mathbb{P}$(marble = red from rand bag)

$$\frac{1}{3} \times \sum_{i=1}^{3} \frac{R_i}{R_i + B_i} = \frac{1}{3} \times \sum_{i=1}^{3} \mathbb{P}(\text{Red} \mid B_i) \times \mathbb{P}(B_i)$$