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Probability and Statistics

Random Experiments and Sample Spaces :-

Experiments involving Randomness

Coin toss, dice roll, etc.

$\Omega \rightarrow$ Set of all possible outcomes of a random experiment. It could be finite or infinite.

$\Omega_c = \{Head, Tail\}$, $\Omega_d = \{1,2,3,4,5,6\}$

2-coins $\rightarrow$ $\Omega_{2c} = \Omega_c \times \Omega_c = \{H,T\} \times \{H,T\}$

Outcomes and Events

(omega) $\rightarrow$ $\omega \in \Omega$ is called a sample point or a possible outcome

A subset $A \subseteq \Omega$ is called an event.

Events in the coin toss experiment $C_1 = \{T\}$ $[\subset \Omega_c]$

Events in the die roll $D_1 = \{6\}$, $D_2 = \{1,3,5\}$ $[\subseteq \Omega_d]$

Events are any subset of Ω, even null sets, but $\mathbb{P}(\phi) = 0$

Probability of an event $A = \mathbb{P}(A)$.

It may or may not be possible to measure / assign $\mathbb{P}$ for every subset A.

$\mathbb{P}_c(T) = 0.5$

Probability measure $\mathbb{P}$ is a set function. It acts on sets and measures the probability of such sets

Set Theory 101:-

A^c = complement of A

ϕ = denotes empty set. [belongs to every set]

$A \cup B$ = A union B

$A \cap B$ = A intersection B

~~A△B~~

$A \setminus B$ = A minus B $= A \cap B^c$

~~Symmetric~~

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

ME = Mutually Exclusive

~~HW → Identity Complement~~,

$|A|$ = no. of elements in A = cardinality of A.

Inclusion - Exclusion Principle = $|A \cup B| = |A| + |B| - |A \cap B|$

Countable Sets and Uncountable Sets

Monotone Seq → $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ [Increasing seq]

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ [Decreasing]

$I_n = [0, 1 - \frac{1}{n}]$

$D_n = [0, \frac{1}{n}]$

Cartesian product :- $A \times B = \{(a,b) : a \in A, b \in B\}$

Powerset [$\mathcal{P}(A)$] :- Set of all possible subsets of A.

$|\mathcal{P}(A)| = 2^{|A|}$ [Only for discrete elements]

$\mathcal{P}([0,1]) = \{(a,b) : a \leq b,\ a, b \in [0,1]\}$

↳ range, not just 2 elements

Functions:-

Fns are maps from elements in the domain D to the range R

$f: D \to R$

$f: \mathbb{R} \to \mathbb{R} \Rightarrow f(x) = x$

Set fns are those fns ~~that~~ who act on sets; D is a collect of sets.

$\mathbb{P}$ is a set fn.

Axioms of probability:-

1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$
2) For a set $A \subseteq \Omega$, $\mathbb{P}(A) \in [0,1]$
3) For a disjoint collection of events $A_1, A_2, \dots, A_j$ where $A_i \subseteq \Omega$, then:

$$\mathbb{P}\left(\bigcup_{i=1}^{j} A_i\right) = \sum_{i=1}^{j} \mathbb{P}(A_j) \quad [j \text{ can be } \infty] \rightarrow \text{but must be countable}$$

where $A_i \cap A_j \;\forall i,j = \phi$

In general, the domain of $\mathbb{P}$ is $\mathcal{P}(\Omega)$.

~~Probability of impossible event = 0~~.

Counter example for $\mathcal{P}(\Omega)$ being satisfactory

1) Pick a number randomly (uniformly) of # from the real line.
2) $\Omega = \mathbb{R}$, hence $\mathbb{P}(\mathbb{R}) = 1$,
3) Domain $= \mathcal{P}(\mathbb{R})$
4) $\mathbb{P} : \mathcal{P}(\mathbb{R}) \rightarrow [0,1]$
5) $\mathbb{P}$ has the property that sets of equal length have equal probability
6) We know that $\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1)$ where $[n, n+1) \in \mathcal{P}(\mathbb{R})$
[Countable union since n is an integer]
7) $\mathbb{P}(\mathbb{R}) = 1 = \sum \mathbb{P}([n, n+1)) = 0 / \infty$ depending
8) This is a contradiction! [Happens when picking segments of unit length]

Powerset is thus a bad choice

Not all set fns can be calibrated to measure all possible subset of the sample space.

Our choice of domain must have certain features.
Let domain = $\mathcal{F}$
then $\phi, \Omega \in \mathcal{F}$; if $B \in \mathcal{F} \rightarrow B^c \in \mathcal{F}$ etc;

$\mathcal{F}$
↑
Bags that meet these criteria are called sigma-algebras

Sigma algebra / Event space [$\mathcal{F}$] associated with a set Ω is a collection of subsets of Ω that satisfy
- $\phi \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$
- $A_1, A_2, \dots, A_n \in \mathcal{F} \rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

→ The σ-algebra is closed under formation of complements and countable unions and countable intersections (by De Morgans)

→ When Ω is countable and finite, then $\mathcal{P}(\Omega)$ is a σ-algebra

If Ω is countable & finite, we will consider $\mathcal{P}(\Omega)$ as the domain.

Probability Space = $\{\Omega, \mathcal{F}, \mathbb{P}\}$

Probability measure :- A probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ is a set fn $\mathbb{P} : \mathcal{F} \rightarrow [0,1]$ s.t.
1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$
2) For a disjoint collection of events $A_1, A_2 \dots$ from $\mathcal{F}$, we have

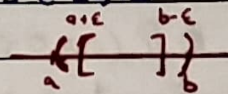
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$\mathbb{P}$ space for $U[0,1]$
$\Omega = [0,1]$
If we generate a σ-algebra, but we get a borel-sigma algebra $\mathcal{B}[0,1]$

Borel σ-algebra $[\mathcal{B}[0,1]]$

Defined when $\Omega = [0,1]$, as the σ-algebra generated by closed sets of the form $[a,b]$ where $a \le b$ & $a,b \in [0,1]$

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \tfrac{1}{n},\ b - \tfrac{1}{n}\right]$$



$$(a,b] = \bigcup_{n=1}^{\infty} \left(a,\ b + \tfrac{1}{n}\right]$$

$[\mathcal{B}[0,1]]$

Borel σ-algebra ↑: is the σ algebra generated by sets of the form $[a,b]$ or (a,b) or $(a,b]$ or even $[a,b)$ where $a \le b$ & $a,b \in [0,1]$

If $\Omega = \mathbb{R}$, the $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by open sets of the form (a,b), where $a<b$ and $a,b \in \mathbb{R}$.

How to define $\mathcal{B}(\mathbb{R}^2)$

Consequences of the Probability Axioms:-

i) $P(A^c) = 1 - P(A)$

$$P(A \cup A^c) = P(A) + P(A^c) = 1.$$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

~~$P(A\cup B)$~~ ~~Prove~~ Prove it

iii) If $A \subseteq B$, prove that $P(A) \le P(B)$ [$A \subseteq B$ can be interpreted as $A \to B$]

If $A \subseteq B \to B = A + \lambda$.

iv) $P\left(\bigcup_{i=1}^{\infty} B_i\right) \le \sum_{i=1}^{\infty} P(B_i)$ [Boole's ineq.]

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Diff b/w Impossible event vs Zero probability event

ϕ ← ↓ finite sized set

In $U[0,1]$, $\mathbb{P}(\omega = 0.5) = 0$
Every experiment outcome of this experiment is a zero probability event.
This implies that zero outcome events could occur.

ϕ on the other hand, can never occur, and are hence impossible events.

$\mathbb{P}(\omega \in [0, 0.25) \cap [0.75, 1]) = 0$

Limits and Continuity:-

Limits:-

Let $a_1, a_2, \dots, a_n$ be a sequence with limit L.
Then $\forall \varepsilon$, $\exists N_\varepsilon$ s.t $\forall n > N_\varepsilon$, $|a_n - L| \leq \varepsilon$

For a fn $f(x)$; $\lim\limits_{x \to c} f(x) = f(c)$

The limit exists only if $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x)$.

All points δ close to c are ε close to $f(c)$.

Continuity:-

LHL = RHL = $f(x)$. If the fn is cont at all pts, it is said to be continuous. As $x \to c$, $f(x) \to f(c)$

But what about $\mathbb{P}$?

For a continuous set fn S as $A_n \to A$, we have $S(A_n) \to S(A)$

Sequence of sets :-

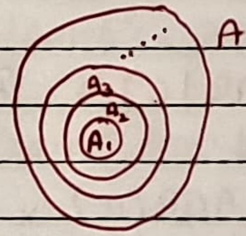
Given $(\Omega, \mathcal{F})$, if $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say that the given seq of sets A_n are increasing to A $(A_n \uparrow A)$

Similarly when $A_1 \supset A_2 \supset \cdots$ is a decreasing seq of events and $\bigcap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$

Alt notation :- increasing seq of sets $A_n \Rightarrow \lim_{n \to \infty} A_n$ for $\bigcup_{n=1}^{\infty} A_n$

decreasing seq of sets $A_n \Rightarrow \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$

Lemma :- For a sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$



Proof :-

$$A = \bigcup_{i=1}^{\infty} A_i$$

(i) Consider an increasing seq:

define $F_n = A_n - A_{n-1}$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} F_n = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

Hence Proved.

(ii) Decreasing seq. [HW]

Conditional Probability :-

Ex:- Given the die outcome is odd, what is $\mathbb{P}(1)$? $[1/3]$

Given $\bar{\omega} \in [0, 0.5]$, what is $\mathbb{P}(\bar{\omega} \in [0, 0.25])$

Defn:- The conditional probability of event A is defined as

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \quad \text{when } \mathbb{P}(A) > 0$$

Theorem:- $\mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \times \mathbb{P}(B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \times \mathbb{P}(A)$$

Bayes' Rule:- $$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \times \mathbb{P}(B)}{\mathbb{P}(A)}$$

(i) What is $\mathbb{P}(A|B \cap C)$

(ii) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B)$

(iii) $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C) \times \mathbb{P}(B|C)$

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(\cancel{C})} \times \frac{\mathbb{P}(\cancel{C})}{\mathbb{P}(B \cap C)} \times \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$$

$$\mathbb{P}(A|B \cap C) \times \mathbb{P}(B|C)$$

(iii) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \times \mathbb{P}(B|A) \times \mathbb{P}(C|A \cap B)$

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)} \times \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \times \mathbb{P}(A)$$

iv) $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n | A_1 \cap \cdots \cap A_n)$

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Q) Draw 4 cards without replacement. What is the $\mathbb{P}(9\text{club}, 8\text{dia}, K\text{sp}, K\text{cl})$

$$\mathbb{P}(9_c) \times \mathbb{P}(8d \mid 9c) \times \mathbb{P}(K_{sp} \mid 9_c \cap 8d) \times \mathbb{P}(K_{cl} \mid 9_c \cap 8d \cap K_{sp})$$

$$= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} = \frac{48!}{52!}$$

When every outcome is equally likely in a finite sample space Ω, $\mathbb{P}(B \mid A) = \frac{|A \cap B|}{|A|}$

Law of Total Probability:-

$A = (A \cap B) \cup (A \cap B^c)$, $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

Same as $\mathbb{P}(A) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\,\mathbb{P}(B^c)$

Let $B_1, B_2, \dots, B_n$ be the partition of the sample space Ω. Then for any event A we have

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

Q) 3 bags with m marbles total. Bag i has R_i red and B_i blue for $i \in [1, 3]$. Find $\mathbb{P}(\text{marble} = \text{red from rand bag})$

$$\frac{1}{3} \times \sum_{i=1}^{3} \frac{R_i}{R_i + B_i} = \frac{1}{3} \times \sum_{i=1}^{3} \mathbb{P}(\text{Red} \mid B_i) \times \mathbb{P}(B_i)$$

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Independence:-

An event A is independent (w.r.t an event B) if
$\mathbb{P}(A|B) = \mathbb{P}(A)$

Ex:- Simultaneously tossing a coin and rolling a dies

$\Omega_1 = \{H, T\}$ $\Omega_2 = \{1,2,3,4,5,6\}$
$\Omega = \Omega_1 \times \Omega_2$
$\mathcal{F} =$ Powerset (Ω)

$\mathbb{P}(\{H, 6\}) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

$\mathbb{P}(\{T, odd\}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \rightleftharpoons$ Independence

If A and B are independent

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$; $\mathbb{P}(B^c) = 1 - \mathbb{P}(B)$
$\mathbb{P}(A^c) \times \mathbb{P}(B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) = 1 - \mathbb{P}(A \cup B)$
Where to now? $= \mathbb{P}(A \cup B)^c$
$= \mathbb{P}(A^c \cap B^c)$

A^c & B^c are independent

Prove $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n} (1 - \mathbb{P}(A_i))$

Mutual and Pairwise independence

A collection of events $\{A_i, i \in I\}$ are said to be mutually independent if the $\mathbb{P}(\bigcap_{j=J} A_j) = \prod_{j \in J} \mathbb{P}(A_j)$ for any subset $J \subseteq I$

A collection of events $\{A_i, i \in I\}$ are said to be pairwise independent if any pair of events from the collection are independent $\Rightarrow$ all pairs.

Ex:- Pick a random number from $\{1, \dots, 10\}$

$\mathbb{P}(A) = \{\omega \mid \omega < 7; \omega \in \{1, \dots 10\}\} = 3/5$

$\mathbb{P}(B) = \{\omega \mid \omega < 8; \omega \in \{1 \dots 10\}\} = 7/10$

$\mathbb{P}(C) = \{\omega \mid \omega \% 2 = 0; \omega \in \{1, \dots 10\}\} = 1/2$

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \rightarrow$ not independent

$\mathbb{P}(A \cap C) = \mathbb{P}(A) \times \mathbb{P}(C) \rightarrow$ independent

$\mathbb{P}(B \cap C) = \frac{3}{10} \neq \mathbb{P}(B) \times \mathbb{P}(C) \rightarrow$ not independent

Correlation b/w events:-

Two events A & B are +vely correlated iff $\mathbb{P}(A|B) > \mathbb{P}(A)$

Two events A & B are -vely ~~not~~ correlated iff $\mathbb{P}(A|B) < \mathbb{P}(A)$

A & B have same correlation as A^c & B^c (Prove)

A & B have opp correlation as A & B^c (Prove)

Mutual Exclusivity:-

Two events are mutually exclusive if $\mathbb{P}(A \cap B) = 0$.

The occurence of one implies the other cannot occur.

Two events with non zero probabilities cannot be both mutually exclusive and independent.

$P(A \cap B) = 0 = P(A)P(B) \rightarrow$ contradiction.

If A & B are ME

$$\mathbb{P}(A|B^c) = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(B^c)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B^c)}$$

When $A \subseteq B$, A & B are not ME nor I

Zero Probability Events are always independent

Let $\mathbb{P}(E) = 0$

· Then for any set F, we want to show that $\mathbb{P}(E \cap F) = 0$

But $E \cap F$ has two choices

(a) $E \cap F = \phi$

(b) $E \cap F \subseteq E$

In either case $\mathbb{P}(E \cap F) = 0$ or $\mathbb{P}(E \cap F) \leq \mathbb{P}(E) = 0$

$\mathbb{P}(E \cap F) = 0$

Conditional Independence:-

Two 0 events A & B are conditionally independent of C with $\mathbb{P}(C) > 0$ if $\mathbb{P}((A \cap B)/C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

It thus follows that $\mathbb{P}(A|B \cap C) = \mathbb{P}(A|C)$

Ex:- Given a fair coin and a fake coin (2 heads). Pick a random coin and toss twice.

A:- First toss $\rightarrow$ H $= 3/4$

B:- Second toss $\rightarrow$ H $=$ Still $3/4$

~~Even~~ C :- First coin is chosen $= 1/2$

$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B \mid C) + \mathbb{P}(A \cap B \mid C^c)$

$= \underbrace{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}_{\text{Fair coin}} \quad \underbrace{\frac{1}{2}}_{\text{Unfair coin}}$

$= \frac{5}{8} \neq \frac{3}{4} \times \frac{3}{4} \rightarrow$ Not independent

C : coin is chosen and tossed twice; not one coin being chosen and tossed twice

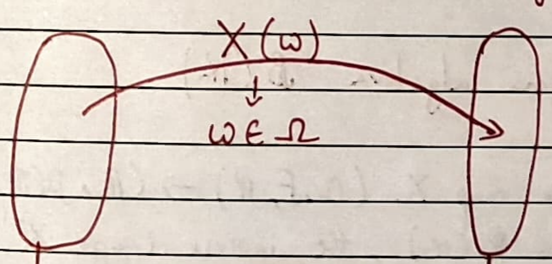
A & B are conditionally independent on C, not C^c

Random Variables

Given a random experiment with associated $(\Omega, \mathcal{F}, \mathbb{P})$, it is sometimes difficult to deal directly with $\omega \in \Omega$
ω is not a number, it is a sequence (vector) of nos.

Random variables are devices that help us make a mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', \mathbb{P}_x)$, as this would help analyze functions of sample pts rather than any single sample pt.

Assume you want to count no. of 6s in 10 rolls



$\Omega =$ set of all 10 vectors $\qquad \Omega' = \{1, \cdots, 10\}$; $\mathcal{F}'$ is the Powerset(Ω')

$\mathbb{P} \qquad \mathbb{P}_x$

A random variable X is a fn $X : \Omega \to \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', \mathbb{P}_X)$ and is $(\mathcal{F}, \mathcal{F}')$-measurable

The map $X : \Omega \to \Omega'$ implies $X(\omega) \in \Omega'$ $\forall$ $\omega \in \Omega$

For the event $B \in \mathcal{F}'$, the preimage $X^{-1}(B)$ is defined as $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$

→ for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$

X^{-1} is a one to many mapping from $\mathcal{F}' \to \mathcal{F}$

$\mathbb{P}_X(\{B\})$ = Probability that X results in $\{B\}$ while ~~being darted~~ acting upon ~~by~~ $\omega \in \Omega$

= ~~Probability of~~ $\mathbb{P}(\omega)$ where $\omega \in \Omega$; $X(\omega) \in B$

$\mathcal{F}$ could have elements that have no images in $\mathcal{F}'$. But $\forall f \in \mathcal{F}'$, f has a pre image in $\mathcal{F}$

$\mathbb{P}_X$ is called induced probability measure

Conventions:-
i) $\Omega' = \mathbb{R}$
ii) $\mathcal{F}'$ = Borel sigma algebra $\mathcal{B}(\mathbb{R})$

A rand var X is a map $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ satisfies

$X^{-1}(B) \in \mathcal{F}$

$\mathbb{P}_X(B) = \mathbb{P}(\omega \in \Omega : X(\omega) \in B) = \mathbb{P}(X^{-1}(B))$

If Ω' is countable, then the random variable is called a discrete random variable. Only here is $\mathcal{F}'$ used as the Powerset

Capital letters X Y Z etc for Rand var, and x, y, z etc are the realizations of said rand vars

Discrete Random Variables :-

Ex:- Roll two dice and calculate the sum

Ω = set of all tuples (x,y) where $x, y \in [1,6]$
Ω' = integers $[2,12]$

$\mathcal{F}$ & $\mathcal{F}'$ are Powersets

$\{X=3\} \in \mathcal{F}' \rightarrow \mathbb{P}_x(3) = \mathbb{P}(\{1,2\},\{2,1\})$

In general for $x \in \Omega'$ $\mathbb{P}_x(x) = \mathbb{P}(\{a,b\} \mid \{a,b\} \in \Omega,\ a+b=x)$

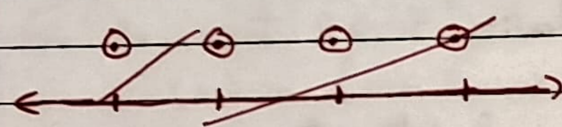
$p_X(x) = \mathbb{P}_X(\{x\})$
→ fn [not a probability measure] → makes x a set and passes as arg to $\mathbb{P}_X$.
→ Probability mass fn.

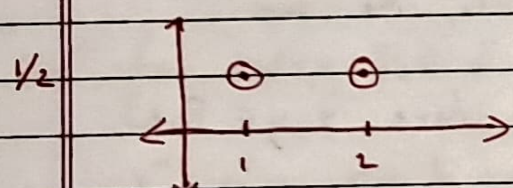
$$p_X(x) = \begin{cases} \frac{x-1}{36} & \forall x \in \text{int}[2,7] \\ \frac{13-x}{36} & \forall x \in \text{int}[8,12] \end{cases}$$

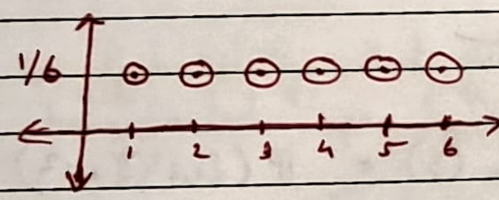
Multiple X_is can have the same Probability Mass fn.
For example, for the exp of rolling a die four times, then divide and conquer using $p_X(x)$ mentioned above
Only if X_is are independent

The fn $p_X(x) := \mathbb{P}_X(\{x\})$ for $x \in \Omega'$ is called as a Probability Mass Function (PMF) of a random variable X.

Q) Plot PMFs for X corresponding to coin tosses or dice rolls.



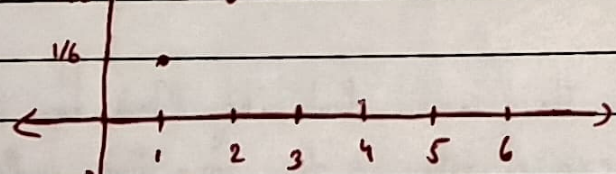




The Cumulative Distribution function (CDF) is defined as

$$F_X(x_1) = \sum_{x \le x_1} p_X(x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x_1\})$$

F_X (Rolling dice)



The X axis is only X not Ω. Here Ω' ~~the mapping X~~ is identical to Ω.

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Expectation and Moments:-

The mean or expectation of a random variable X is denoted by $E(x)$ and is given by $E(x) = \sum_{x \in \Omega'} x \, p_x(x)$

The expected value need not be part of Ω

Depends on how X is defined.

First moment of a rand var.

The n^{th} moment of a random variable X is denoted by $E(X^n)$ and is given by $E(X^n) = \sum_{x \in \Omega'} x^n p_x(x)$

For a fn $g(\cdot)$ of a rand var X, $E(g(X)) = \sum_{x \in \Omega'} g(x) \, p_x(x)$

fns of random variables are random variables with their own probability mass fns.

Consistency of PMF:-

PMF: $p_x(x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$ for $x \in \Omega'$

To prove $\sum_{x \in \Omega'} p_x(x) = 1$

$$\sum_{x \in \Omega'} p_x(x) = \sum_{x \in \Omega'} \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$$

$$= \mathbb{P}\Big(\bigcup_{x \in \Omega'} \{\omega \in \Omega : X(\omega) = x\}\Big)$$

$$= \mathbb{P}(\Omega) = 1$$

Expectance is Linear :-

Let $Y = aX + b$

Then $E(Y) = E(aX+b)$

$$= \sum_{x \in \Omega'} (ax+b)\, p_x(x)$$

$$= a\left(\sum_{x \in \Omega'} x\, p_x(x)\right) + b$$

$$= a\,E(X) + b$$