

Real analysis

Final (Fall 2024)

Duration: 2 hours
Maximum marks: 100

Question 1: (6 marks) Give two examples of sets that are connected but not path connected (No justification required, just examples).

Question 2: (10 marks) Let $\{f_n\} \subset C(\mathbb{R})$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of functions from \mathbb{R} to \mathbb{R} .

1. What does it mean to say that the sequence $\{f_n\}$ converges pointwise on A to a function $f: A \rightarrow \mathbb{R}$?
2. What does it mean to say that the sequence $\{f_n\}$ converges uniformly on A to a function $f: A \rightarrow \mathbb{R}$?

Question 3: (12 marks)

Let $A, B \subset \mathbb{R}$. State True or False with justification:

1. A is compact and B is closed implies $A \cap B$ is compact. (Note here that compact means closed and bounded)
2. A is open and B is closed implies $A \cap B$ is closed.
3. Let X and Y be metric spaces, and $f: X \rightarrow Y$ a continuous function. If $B \subset X$ is bounded, then $f(B)$ is bounded.
4. Let $\{f_n\}$ be a sequence of continuous functions that converge pointwise to a function $f: X \rightarrow \mathbb{R}$. If f is continuous, then the functions must converge uniformly.

Question 4: (10 marks)

Find the pointwise limit of the sequence $f_n(z) = \frac{1}{n} \sin \frac{1}{z}$ on \mathbb{R} . Is this convergence uniform?

Question 5: (10 marks)

Let $f: A \rightarrow \mathbb{R}$ be continuous on A . If $K \subset A$ is compact, show that $f(K)$ is also compact.

Question 6: (12 marks)

Consider the set $S = \{0\} \cup (2, 3)$. Classify each of the points $\{0\}, \{1\}, \{2\}, \{3\}, \{1.5\}$.

1. Boundary
2. Interior
3. Accumulation
4. Adherent

Question 7: (10 marks)

Given a metric space X and $D \subset X$. Then prove that D is dense in X if and only if every point of X is an adherent point of D .

Question 8: (15 marks)

Let z^* be an accumulation point of a set S . Prove that every neighbourhood of z^* contains infinitely many points of S .

Question 9: (15 marks)

Show that the union of two connected sets is connected if their intersection is nonempty.