Corrected 30/11/2013

Chapter 7

Two–particle transfer

Cooper pairs are the building blocks of pairing correlations in many-body fermionic systems. In particular in atomic nuclei. As a consequence, nuclear superfluidity can be specifically probed through Cooper pair tunneling. In the simultaneous transfer of two nucleons, one nucleon goes over from target to projectile, or viceversa, under the influence of the nuclear interaction responsible of the existence of a mean field potential, while the other follows suit by profiting of: 1) pairing correlations (simultaneous transfer); 2) the fact that the single-particle wavefunctions describing the motion of Cooper pair partners in both target and projectile are solutions of different single-particle potentials (non-orthogonality transfer). In the limit of independent particle motion, in which all of the nucleon-nucleon interaction is used up in generating a mean field, both contributions to the transfer process (simultaneous and non-orthogonality) cancel out exactly (cf. App. 7.C)

In keeping with the fact that nuclear Cooper pairs are weakly bound ($E_{corr} \ll \epsilon_F$), this cancellation is, in actual nuclei, quite strong. Consequently, successive transfer, a process in which the nuclear interaction acts twice is, as a rule, the main mechanism at the basis of Cooper pair transfer. Because of the same reason (weak binding), the correlation length of Cooper pairs is larger than nuclear dimensions ($\xi = \hbar v_F/E_{corr} \gg R$), a fact which allows the two members of a Cooper pair to move between target and projectile, essentially as a whole, also in the case of successive transfer. In other words, because of its (intrinsic, virtual extension) Cooper pair transfer display equivalent pairing correlations both in simultaneous as in successive transfer.[1]

[1]In order for a nucleon to display independent particle motion, all other nucleons must act coherently so as to leave the way free making feel their pullings and pushings only when the nucleon in question tries to leave the self–bound system,thus acting as a reflecting surface which inverts the momentum of the particle. It is then natural to consider the nuclear mean field the most striking and fundamental collective feature in all nuclear phenomena. A close second is provided by the BCS mean field, resulting from the condensation of strongly overlapping Cooper pairs (i.e. $\langle BCS|\sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar\nu}|BCS\rangle = \alpha_0 \neq 0$) and leading to independent quasiparticle motion. It is a rather unfortunate perversity of popular terminology that regards these collective fields (HF and HFB) as well as successive transfer, as in some sense an antithesis to the nuclear collective modes (Mottelson (1962)) and to simultaneous transfer respectively. Within this context it is of notice that the

The present Chapter is structured in the following way. In section 7.1 we present a summary of two–nucleon transfer reaction theory. It provides the elements needed to calculate the absolute two–nucleon transfer differential cross sections in second order DWBA, and thus to compare theory with experiment. Within this context one can, after reading this section, move directly to Chapter 8 containing examples of applications of this formalism. For the more theoretically oriented reader we provide in section 7.2 and in Appendices 7.K and 7.L, a detailed derivation of the equations presented in section 7.1. These equations are implemented and made operative in the softwares used in the applications (cf. App. 8.~~A~~). D

In the present Chapter a number of Appendices are provided. In particular one (App. 7.B) in which the derivation of first order DWBA simultaneous transfer is worked out within a formalism tailored to focus the attention on the nuclear structure correlations aspects of the process leading to effective two–nucleon transfer form factors. Another one (App. 7.C) in which the variety of contributions to two–nucleon transfer amplitudes (successive, simultaneous and non–orthogonality) are discussed in detail within the framework of the semi–classical approximation, and other two (App. 7.D and App. 7.E) in which simple numerical estimates of the relative importance of successive and of simultaneous transfer are worked out. Appendices 7.G, 7.H and 7.I provide ~~elements to be~~ used in the order of magnitude estimates mentioned above, while Appendix 7.A deals with nuclear structure processes renormalizing the properties of single particle and collective states and their relation with two–nucleon transfer processes. Within this context, App. 7.J provides an example of coherent states. Appendix 7.F ~~provides~~ simple estimates of the relative importance of final state interactions, while Appendices 7.M and 7.N contain relations used in the derivation of two–nucleon transfer spectroscopic amplitudes. Finally Appendix 7.O provides a glimpse of material which was instrumental to render quantitative studies of two–nucleon transfer, studies which can now be carried out in terms of absolute cross sections and not relative ones as done previously (cf. e.g. Broglia et al. (1973), Potel et al. (2013) and refs. therein).

contain

original

7.1 Summary of second order DWBA

Let us illustrate the theory of second order DWBA two–nucleon transfer reactions with the $A + t \rightarrow B(\equiv A+2) + p$ reaction, in which $A+2$ and A are even nuclei in their 0^+ ground state. The extension of the expressions to the transfer of pairs coupled to arbitrary angular momentum is discussed in subsection 7.2.10.

The wavefunction of the nucleus $A+2$ can be written as

$$\Psi_{A+2}(\xi_A, \mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \psi_A(\xi_A) \sum_{l_i,j_i} [\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^0_0, \qquad (7.1.1)$$

transfer connecting

two–nucleon differential cross section associated with the ~~transition between~~ the ground state of superfluid nuclei is proportional to α_0^2 and not to Δ^2. In fact, Cooper pairs partners remain correlated even over regions in which $G = 0$.

Nota para Gregory: los detalles de como ajustas V_0 y que potencial $\vec{l}\cdot\vec{s}$ usas (tipo Bohr + Mottelson, Vol I, para ^{11}Li, otro para Sn, etc) lo escribes en App. 8.D software description COOP, etc)

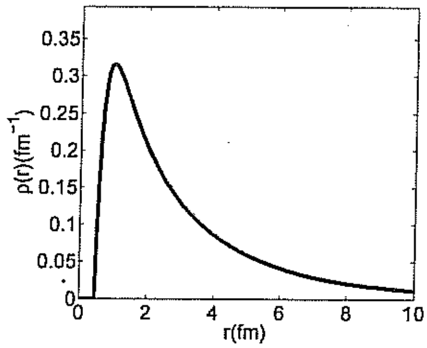

Figure 7.1.1: Radial function $\rho(r)$ (hard core 0.45 fm) entering the tritium wavefunction (cf Tang and Herndon (1965)).

cf.

where

$$[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^0_0 = \sum_{nm} a_{nm}\left[\varphi^{A+2}_{n,l_i,j_i}(\mathbf{r}_{A1},\sigma_1)\varphi^{A+2}_{m,l_i,j_i}(\mathbf{r}_{A2},\sigma_2)\right]^0_0, \qquad (7.1.2)$$

while the wavefunctions $\varphi^{A+2}_{n,l_i,j_i}(\mathbf{r})$ are eigenfunctions of a Saxon–Woods potential

$$U(r) = -\frac{V_0}{1+\exp\left[\frac{r-R_0}{a}\right]}, \qquad R_0 = r_0 A^{1/3}, \qquad (7.1.3)$$

of depth V_0 adjusted to reproduce the experimental single–particles energies, together with a standard spin–orbit potential. The radial dependence of the wavefunction of the two neutrons in the triton is written as $\phi_t(\mathbf{r}_{p1},\mathbf{r}_{p2}) = \rho(r_{p1})\rho(r_{p2})\rho(r_{12})$, where r_{p1}, r_{p2}, r_{12} are the distances between neutron 1 and the proton, neutron 2 and the proton and between neutrons 1 and 2 respectively, while $\rho(r)$ is the hard core (r_{core} = 0.45 fm) potential wavefunction depicted in Fig 7.1.1.

The two–nucleon transfer differential cross section is written as

$$\frac{d\sigma}{d\Omega} = \frac{\mu_i\mu_f}{(4\pi\hbar^2)^2}\frac{k_f}{k_i}\left|T^{(1)} + \underbrace{T^{(2)}_{succ} - T^{(2)}_{NO}}\right|^2, \qquad (7.1.4)$$

if one adds indices or variables like e.g. Eq (7.2.40) write $T^{(1)}(j_i, j_f, l; \theta)$

$$
\begin{tikzcd}
T^{j_i j_f}_{\ell} \arrow[r] \arrow[d] & T^{1\,step}_{2N}(\theta) \\
\boxed{T^{(1)}(j_i, j_f, \ell; \theta)} &
\end{tikzcd}
$$

$$T^{j_i j_f}_{\ell}(\theta)$$

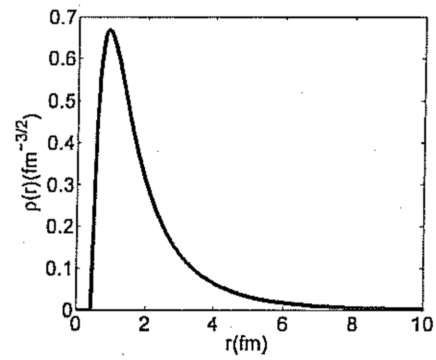

Figure 7.1.2: Radial wavefunction $\rho_d(r)$ (hard core 0.45 fm) entering the deuteron wavefunction (cf,Tang and Herndon (1965)).

cf.

(2) second order in # of channels

Gregory: no seria conveniente de tener un set the coordenadas usado en el calculo de los diferentes amplitudes, p. ej. los que he dibujado yo en las notas de two-nucleon transfer in a nutshell

v=1

NO

where (see e.g. Bayman and Chen (1982) and App. 7.O),

$$T^{(1)} = 2\sum_{l_i,j_i}\sum_{\sigma_1\sigma_2}\int d\mathbf{r}_{tA}d\mathbf{r}_{p1}d\mathbf{r}_{A2}[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^{0*}_0\chi^{(-)*}_{pB}(\mathbf{r}_{pB})$$
$$\times v(\mathbf{r}_{p1})\phi_t(\mathbf{r}_{p1},\mathbf{r}_{p2})\chi^{(+)}_{tA}(\mathbf{r}_{tA}), \quad (7.1.5a)$$

Notación

$$T^{(2)}_{succ} = 2\sum_{l_i,j_i}\sum_{l_f,j_f,m_f}\sum_{\substack{\sigma_1\sigma_2\\ \sigma'_1\sigma'_2}}\int d\mathbf{r}_{dF}d\mathbf{r}_{p1}d\mathbf{r}_{A2}[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^{0*}_0\chi^{(-)*}_{pB}(\mathbf{r}_{pB})v(\mathbf{r}_{p1})$$
$$\times \phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A2})\int d\mathbf{r}'_{dF}d\mathbf{r}'_{p1}d\mathbf{r}'_{A2}G(\mathbf{r}_{dF},\mathbf{r}'_{dF})$$
$$\times \phi_d(\mathbf{r}'_{p1})^*\varphi^{A+1*}_{l_f,j_f,m_f}(\mathbf{r}'_{A2})\frac{2\mu_{dF}}{\hbar^2}v(\mathbf{r}'_{p2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}), \quad (7.1.5b)$$

$$T^{(2)}_{NO} = 2\sum_{l_i,j_i}\sum_{l_f,j_f,m_f}\sum_{\substack{\sigma_1\sigma_2\\ \sigma'_1\sigma'_2}}\int d\mathbf{r}_{dF}d\mathbf{r}_{p1}d\mathbf{r}_{A2}[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^{0*}_0\chi^{(-)*}_{pB}(\mathbf{r}_{pB})v(\mathbf{r}_{p1})$$
$$\times \phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A2})\int d\mathbf{r}'_{p1}d\mathbf{r}'_{A2}d\mathbf{r}'_{dF}$$
$$\times \phi_d(\mathbf{r}'_{p1})^*\varphi^{A+1*}_{l_f,j_f,m_f}(\mathbf{r}'_{A2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}). \quad (7.1.5c)$$

The quantities $\mu_i, \mu_f (k_i, k_f)$ are the reduced masses (relative linear momenta) in both entrance (initial, i) and exit (final, f) channels, respectively. In the above expressions, $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ are the wavefunctions describing the intermediate states of the nucleus $F(\equiv (A+1))$, generated as solutions of a Woods–Saxon potential, $\phi_d(\mathbf{r}_{p2})$ being the the deuteron bound wavefunction (see Fig. 7.1.2). Note that some or all of the single–particle states described by the wavefunctions $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ may lie in the continuum (case in which the nucleus F is loosely bound or unbound). Although there are a number of ways to exactly treat such states, discretization processes may be sufficiently accurate. They can be implemented by, for example, embedding the Woods–Saxon potential in a spherical box of sufficiently large radius. In actual calculations involving the halo nucleus ^{11}Li, and where $|F\rangle = |^{10}\text{Li}\rangle$, one achieved convergence making use of approximately 20 continuum states and a box of 30 fm of radius. Concerning the components of the triton wavefunction describing the relative motion of the dineutron, it was generated with the $p-n$ interaction (Tang and Herndon, 1965)

$$v(r) = -v_0 \exp(-k(r-r_c)) \quad r > r_c \quad (7.1.6)$$
$$v(r) = \infty \quad r < r_c, \quad (7.1.7)$$

where $k = 2.5$ fm^{-1} and $r_c = 0.45$ fm, the depth v_0 being adjusted to reproduce the experimental separation energies. The positive–energy wavefunctions $\chi^{(+)}_{tA}(\mathbf{r}_{tA})$

gregory

Fig. 7.C.2 is not call; why?

and $\chi^{(-)}_{pB}(\mathbf{r}_{pB})$ are the ingoing distorted wave in the initial channel and the outgoing distorted wave in the final channel respectively. They are continuum solutions of the Schrödinger equation associated with the corresponding optical potentials.

The transition potential responsible for the transfer of the pair is, in the *post* representation (cf. Fig. 7.C.1),

$$V_\beta = v_{pB} - U_\beta, \tag{7.1.8}$$

where v_{pB} is the interaction between the proton and nucleus B, and U_β is the optical potential in the final channel. We make the assumption that v_{pB} can be decomposed into a term containing the interaction between A and p and the potential describing the interaction between the proton and each of the transferred nucleons, namely

$$v_{pB} = v_{pA} + v_{p1} + v_{p2}, \tag{7.1.9}$$

where v_{p1} and v_{p2} is the hard–core potential (7.1.6). The transition potential is

$$V_\beta = v_{pA} + v_{p1} + v_{p2} - U_\beta. \tag{7.1.10}$$

Assuming that $\langle\beta|v_{pA}|\alpha\rangle \approx \langle\beta|U_\beta|\alpha\rangle$ (i.e, assuming that the matrix element of the core–core interaction between the initial and final states is very similar to the matrix element of the real part of the optical potential), one obtains the final expression of the transfer potential in the *post* representation, namely,

$$V_\beta \simeq v_{p1} + v_{p2} = v(\mathbf{r}_{p1}) + v(\mathbf{r}_{p2}). \tag{7.1.11}$$

We make the further approximation of using the same interaction potential in all the (i.e. initial, intermediate and final) channels.

The extension to a heavy–ion reaction $A + a(\equiv b+2) \longrightarrow B(\equiv A+2) + b$ imply no essential modifications in the formalism. The deuteron and triton wavefunctions appearing in Eqs. (7.1.5a), (7.1.5b) and (7.1.5c) are to be substituted with the corresponding wavefunctions $\Psi_{b+2}(\xi_b, \mathbf{r}_{b1}, \sigma_1, \mathbf{r}_{b2}, \sigma_2)$, constructed in a similar way as those appearing in (7.1.1 and 7.1.2). The interaction potential used in Eqs. (7.1.5a), (7.1.5b) and (7.1.5c) will now be the Saxon–Woods used to define the initial (final) state in the post (prior) representation, instead of the proton–neutron interaction (7.1.6).

The Green's function $G(\mathbf{r}_{dF}, \mathbf{r}'_{dF})$ appearing in (7.1.5b) propagates the intermediate channel d, F. It can be expanded in partial waves as,

$$G(\mathbf{r}_{dF}, \mathbf{r}'_{dF}) = i\sum_l \sqrt{2l+1}\frac{f_l(k_{dF}, r_<)g_l(k_{dF}, r_>)}{k_{dF}r_{dF}r'_{dF}}\left[Y^l(\hat{r}_{dF})Y^l(\hat{r}'_{dF})\right]^0_0. \tag{7.1.12}$$

The $f_l(k_{dF}, r)$ and $g_l(k_{dF}, r)$ are the regular and the irregular solutions of a Schrödinger equation for a suitable optical potential and an energy equal to the kinetic energy of the intermediate state. In most cases of interest, the result is hardly altered if we use the same energy of relative motion for all the intermediate states. This

Heavy–ion Reactions

In dealing with a heavy ion reaction, $\theta_0^0(\mathbf{r},\mathbf{s})$ are be the spatial part of the wavefunction

$$\begin{aligned}\Psi(\mathbf{r}_{b1},\mathbf{r}_{b2},\sigma_1,\sigma_2) &= \left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]_0^0 \\ &= \theta_0^0(\mathbf{r},\mathbf{s})\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^0,\end{aligned} \qquad (7.2.38)$$

where $\mathbf{r}_{b1},\mathbf{r}_{b2}$ are the positions of the two neutrons with respect to the b core. It can be shown to be

$$\theta_0^0(\mathbf{r},\mathbf{s}) = \frac{u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})}{4\pi}\sqrt{\frac{2j_i+1}{2}}P_{l_i}(\cos\theta_i), \qquad (7.2.39)$$

old version

where θ_i is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. Neglecting the spin–orbit term in the optical potential, as is usually done for heavy ion reactions, one obtains

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{\mu_f\mu_i}{16\pi^2\hbar^4k_i^3k_f}|T_{2N}^{1step}(\theta)|^2, \qquad (7.2.40)$$

$T_l^{j_ij_f}$

where

check what is better

$$\begin{aligned}T_{2N}^{1step}(\theta) &= \sum_l (2l+1)P_l(\cos\theta)\sqrt{(2j_i+1)(2j_f+1)}\exp[i(\sigma_l^p+\sigma_l^t)] \\ &\times\int dR\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,R\sin\beta\,u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\ &\times u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})V(r_{b1})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta) \\ &\times r_{12}r_{b1}r_{b2}P_{l_i}(\cos\theta_i)\frac{f_l(\zeta)g_l(R)}{\zeta},\end{aligned} \qquad (7.2.41)$$

simult.

obtained by using Eq. (7.2.39) in Eq. (7.2.7) instead of (7.2.34), $\mathbf{r}_{A1},\mathbf{r}_{A2}$ being the coordinates of the two transferred neutrons with respect to the A core.

For control, in what follows we work out the same transition amplitude but starting from the distorted waves for a reaction taking place between spinless nuclei, namely

$$\psi^{(+)}(\mathbf{r}_{Aa},\mathbf{k}_{Aa}) = \sum_l \exp\left(i\sigma_l^i\right)g_lY_0^l(\hat{\mathbf{r}}_{aA})\frac{\sqrt{4\pi(2l+1)}}{k_{aA}r_{aA}}, \qquad (7.2.42)$$

and

$$\psi^{(-)}(\mathbf{r}_{bB},\mathbf{k}_{bB}) = \frac{4\pi}{k_{bB}r_{bB}}\sum_{\tilde{l}} i^{\tilde{l}}\exp\left(-i\sigma_{\tilde{l}}^f\right)f_{\tilde{l}}^*(r_{bB})\sum_m Y_m^{\tilde{l}*}(\hat{\mathbf{k}}_{bB})Y_m^{\tilde{l}}(\hat{\mathbf{r}}_{bB}). \qquad (7.2.43)$$

multiplicative factor $2\pi\sqrt{\frac{4\pi}{2l+1}}$), the above expression becomes

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{2\pi}{k_{aA}k_{bB}}\sum_l\sqrt{(2j_f+1)(2j_i+1)}\\ &\times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]Y_0^l(\hat{\mathbf{k}}_{bB})\\ &\times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2}\\ &\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})\\ &\times f_l(r_{bB})g_l(r_{aA})Y_0^{l*}(\hat{\mathbf{r}}_{bB})V(r_{1p})/r_{bB},\end{aligned}\tag{7.2.47}$$

which eventually can be recasted, through the use of Legendre polynomials, in the expression,

$$\begin{aligned}T_{2N}^{1step} = \langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{1}{2k_{aA}k_{bB}}\sum_l\sqrt{(2j_f+1)(2j_i+1)}\\ &\times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]P_l(\cos\theta)(2l+1)\\ &\times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2}\\ &\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})V(r_{1p})\\ &\times u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})f_l(r_{bB})g_l(r_{aA})P_l(\cos\theta_{if})/r_{bB},\end{aligned}\tag{7.2.48}$$

expression which gives the same results as (7.2.41)

7.2.3 Coordinates for the calculation of simultaneous transfer

In what follows we explicit the coordinates used in the calculation of the above equations. Making use of the notation of Bayman (1971), we find the expression of the variables appearing in the integral as functions of the integration variables $r_{1p}, r_{2p}, r_{12}, R, \beta, \gamma$ (remember that $\mathbf{R} = R\,\hat{\mathbf{z}}$, see last section). $\mathbf{R}$ being the center of mass coordinate. Thus, one can write

$$\mathbf{R} = \frac{1}{3}\left(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_p\right) = \frac{1}{3}\left(\mathbf{R}+\mathbf{d}_1+\mathbf{R}+\mathbf{d}_2+\mathbf{R}+\mathbf{d}_p\right),\tag{7.2.49}$$

so

$$\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_p = 0.\tag{7.2.50}$$

Together with

$$\mathbf{d}_1+\mathbf{r}_{12} = \mathbf{d}_2 \qquad \mathbf{d}_2+\mathbf{r}_{2p} = \mathbf{d}_p,\tag{7.2.51}$$

we find

$$\mathbf{d}_1 = \frac{1}{3}\left(2\mathbf{r}_{12}+\mathbf{r}_{2p}\right),\tag{7.2.52}$$

(Gregory, insert the corresponding Fig. of coordinates)

The rest of the formulae are identical to the (t, p) ones. We list them for convenience,

$$\mathbf{r}_{A1} = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \tag{7.2.75}$$

$$\mathbf{r}_{A2} = \begin{bmatrix} d_1 \sin(\beta) + r_{12} \cos(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \sin(\beta) \cos(\alpha) \\ r_{12} \sin(\gamma) \sin(\alpha) \\ R + d_1 \cos(\beta) - r_{12} \sin(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \cos(\alpha) \cos(\beta) \end{bmatrix}. \tag{7.2.76}$$

We we also find

$$\mathbf{r}_{b1} = \frac{1}{m_b}(\mathbf{r}_{A2} + (m_b + 1)\mathbf{r}_{A1} - m_a \mathbf{R}), \tag{7.2.77}$$

and

$$\mathbf{r}_{b2} = \frac{1}{m_b}(\mathbf{r}_{A1} + (m_b + 1)\mathbf{r}_{A2} - m_a \mathbf{R}). \tag{7.2.78}$$

One can readily obtain

$$\cos\theta_{12} = \frac{r_{A1}^2 + r_{A2}^2 - r_{12}^2}{2 r_{A1} r_{A2}}, \tag{7.2.79}$$

and

$$\cos\theta_i = \frac{r_{b1}^2 + r_{b2}^2 - r_{12}^2}{2 r_{b1} r_{b2}}. \tag{7.2.80}$$

7.2.4 Matrix element for the transition amplitude (alternative derivation)

In what follows we work an alternative derivation of T_{2N}^{1step}, more closely related to heavy ion reactions. Following Bayman and Chen (1982) it can be written as

$$\begin{aligned} T_{2NT}^{1step} =& 2\frac{(4\pi)^{3/2}}{k_{Aa}k_{Bb}} \sum_{l_p j_p m l_t j_p} i^{-l_p} \exp[i(\sigma_{l_p}^p + \sigma_{l_t}^t)] \sqrt{2l_t + 1} \\ & \times \langle l_p\, m - m_p\; 1/2\, m_p | j_p\, m \rangle \langle l_t\, 0\; 1/2\, m_t | j_t\, m_t \rangle Y_{m-m_p}^{l_p}(\hat{\mathbf{k}}_{Bb}) \\ & \times \sum_{\sigma_1 \sigma_2 \sigma_p} \int d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A1}, \sigma_1)\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)\right]_0^{0*} \\ & \times v(r_{b1}) \left[\psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\psi^{j_i}(\mathbf{r}_{b2}, \sigma_2)\right]_0^0 \frac{g_{l_t j_t}(r_{Aa}) f_{l_p j_p}(r_{Bb})}{r_{Aa} r_{Bb}} \\ & \times \left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]_{m_t}^{j_t} \left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]_m^{j_p *}. \end{aligned} \tag{7.2.81}$$

$T^{(1)}$

2

and

$$\begin{aligned}
&\left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^0_0 \\
&= ((l_i\tfrac{1}{2})_{j_i}(l_i\tfrac{1}{2})_{j_i}|(l_il_i)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 u_{l_i}(r_{b1})u_{l_i}(r_{b2}) \\
&\times \left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^0_0 [\chi(\sigma_1)\chi(\sigma_2)]^0_0 \\
&= \sqrt{\frac{2j_i+1}{2(2l_i+1)}}u_{l_i}(r_{b1})u_{l_i}(r_{b2}) \\
&\times \left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^0_0 [\chi(\sigma_1)\chi(\sigma_2)]^0_0 \\
&= \sqrt{\frac{2j_i+1}{2}}\frac{u_{l_i}(r_{b1})u_{l_i}(r_{b2})}{4\pi}P_{l_i}(\cos\omega_b)\,[\chi(\sigma_1)\chi(\sigma_2)]^0_0,
\end{aligned} \tag{7.2.86}$$

where ω_A is the angle between $\mathbf{r}_{A1}$ and $\mathbf{r}_{A2}$, and ω_b is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. Consequently

$T^{(1)}$

$$\begin{aligned}
T^{1step}_{2NT} &= (4\pi)^{-3/2}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l i^{-l}\frac{\exp[i(\sigma^p_l+\sigma^t_l)]}{\sqrt{2l+1}}Y^l_{m_t-m_p}(\hat{\mathbf{k}}_{Bb}) \\
&\times \int \frac{d\mathbf{r}_{Cc}d\mathbf{r}_{b1}d\mathbf{r}_{A2}}{r_{Aa}r_{Bb}}P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if}) \\
&\times v(r_{b1})u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2}) \\
&\times \Big[\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})(l+1)+f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})l\Big)\delta_{m_p,m_t} \\
&+\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})\sqrt{l(l+1)}-f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})\sqrt{l(l+1)}\Big)\delta_{m_p,-m_t}\Big],
\end{aligned} \tag{7.2.87}$$

where ω_{if} is the angle between $\mathbf{r}_{Aa}$ and $\mathbf{r}_{Bb}$. For heavy ions, we can consider that the the optical potential does not have a spin–orbit term, and the distorted waves are independent of j. We thus have

$$\begin{aligned}
T^{1step}_{2NT} &= (4\pi)^{-3/2}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l i^{-l}\exp[i(\sigma^p_l+\sigma^t_l)]Y^l_0(\hat{\mathbf{k}}_{Bb})\sqrt{2l+1} \\
&\times \int \frac{d\mathbf{r}_{Cc}d\mathbf{r}_{b1}d\mathbf{r}_{A2}}{r_{Aa}r_{Bb}}P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if}) \\
&\times v(r_{b1})u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2})f_l(r_{Bb})g_l(r_{Aa}).
\end{aligned} \tag{7.2.88}$$

Changing variables one obtains,

$$\begin{aligned}
T^{1step}_{2NT} &= (4\pi)^{-1}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l \exp[i(\sigma^p_l+\sigma^t_l)]P_l(\cos\theta)(2l+1) \\
&\times \int dr_{1A}\,dr_{2A}\,dr_{Aa}\,d(\cos\beta)\,d(\cos\omega_A)\,d\gamma\,r^2_{1A}r^2_{2A}r^2_{Aa} \\
&\times P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if})v(r_{b1}) \\
&\times u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2})f_l(r_{Bb})g_l(r_{Aa}).
\end{aligned} \tag{7.2.89}$$

?

coma

The expression of the remaining quantities appearing in the integral are now straightforward.

$$
\begin{aligned}
r_{b1} &= m_b^{-1} \left|(m_b+1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b+2)\mathbf{r}_{Aa}\right| \\
&= m_b^{-1}\Big((m_b+2)^2 r_{Aa}^2 + (m_b+1)^2 r_{A1}^2 + r_{A2}^2 \\
&\quad - 2(m_b+2)(m_b+1)\mathbf{r}_{Aa}\,\mathbf{r}_{A1} - 2(m_b+2)\mathbf{r}_{Aa}\,\mathbf{r}_{A2} + 2(m_b+1)\mathbf{r}_{A1}\mathbf{r}_{A2}\Big)^{1/2},
\end{aligned}
\tag{7.2.101}
$$

$$
\begin{aligned}
r_{b2} &= m_b^{-1} \left|(m_b+1)\mathbf{r}_{A2} + \mathbf{r}_{A1} - (m_b+2)\mathbf{r}_{Aa}\right| \\
&= m_b^{-1}\Big((m_b+2)^2 r_{Aa}^2 + (m_b+1)^2 r_{A2}^2 + r_{A1}^2 \\
&\quad - 2(m_b+2)(m_b+1)\mathbf{r}_{Aa}\,\mathbf{r}_{A2} - 2(m_b+2)\mathbf{r}_{Aa}\,\mathbf{r}_{A1} + 2(m_b+1)\mathbf{r}_{A2}\mathbf{r}_{A1}\Big)^{1/2},
\end{aligned}
\tag{7.2.102}
$$

$$
\begin{aligned}
r_{Bb} &= \left|\frac{m_b+2}{m_b}\mathbf{r}_{Aa} - \frac{m_A+m_b+2}{(m_A+2)m_b}(\mathbf{r}_{A1}+\mathbf{r}_{A2})\right| \\
&= \left[\left(\frac{m_b+2}{m_b}\right)^2 r_{Aa}^2 + \left(\frac{m_A+m_b+2}{(m_A+2)m_b}\right)^2 (r_{A1}^2 + r_{A2}^2 + 2\mathbf{r}_{A1}\mathbf{r}_{A2}) \right. \\
&\quad \left. - 2\frac{(m_b+2)(m_A+m_b+2)}{(m_A+2)m_b^2}\mathbf{r}_{Aa}(\mathbf{r}_{A1}+\mathbf{r}_{A2})\right]^{1/2},
\end{aligned}
\tag{7.2.103}
$$

$$
\begin{aligned}
r_{Cc} &= \left|\frac{m_b+2}{m_b+1}\mathbf{r}_{Aa} - \frac{m_b+2+m_A}{(m_b+1)(m_A+1)}\mathbf{r}_{A2}\right| \\
&= \left[\left(\frac{m_a}{(m_a-1)}\right)^2 r_{Aa}^2 + \left(\frac{m_A+m_a}{(m_A+1)(m_a-1)}\right)^2 r_{A2}^2 \right. \\
&\quad \left. - 2\frac{m_A m_a + m_a^2}{(m_A+1)(m_a-1)^2}\mathbf{r}_{Aa}\mathbf{r}_{A2}\right]^{1/2},
\end{aligned}
\tag{7.2.104}
$$

$$
\cos\omega_b = \frac{\mathbf{r}_{b1}\mathbf{r}_{b2}}{r_{b1}r_{b2}},
\tag{7.2.105}
$$

$$
\cos\omega_{if} = \frac{\mathbf{r}_{Aa}\mathbf{r}_{Bb}}{r_{Aa}r_{Bb}},
\tag{7.2.106}
$$

with

$$
\mathbf{r}_{Aa}\mathbf{r}_{A1} = r_{Aa}r_{A1}\cos\alpha,
\tag{7.2.107}
$$

$$
\mathbf{r}_{Aa}\mathbf{r}_{A2} = r_{Aa}r_{A2}(\sin\alpha\cos\gamma\sin\omega_A + \cos\alpha\cos\omega_A),
\tag{7.2.108}
$$

$$
\mathbf{r}_{A1}\mathbf{r}_{A2} = r_{A1}r_{A2}\cos\omega_A.
\tag{7.2.109}
$$

In the previous version it read T^{vv}_{2NT}

7.2.6 Successive transfer

The successive two–neutron transfer amplitudes can be written as (Bayman and Chen (1982)):

$T^{(2)}_{succ}$

$$
\begin{aligned}
T^{2step}_{2NT} = \frac{4\mu_{Cc}}{\hbar^2} \sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'\\ KM}} \int & d^3r_{Cc}d^3r_{b1}d^3r_{A2}d^3r'_{Cc}d^3r'_{b1}d^3r'_{A2}\chi^{(-)*}(\mathbf{k}_{Bb},\mathbf{r}_{Bb}) \\
& \times \left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]_0^{0*} v(r_{b1}) \left[\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]_M^K \\
& \times G(\mathbf{r}_{Cc},\mathbf{r}'_{Cc}) \left[\psi^{j_f}(\mathbf{r}'_{A2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]_M^{K*} v(r'_{c2}) \\
& \times \left[\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\right]_0^0 \chi^{(+)}(\mathbf{r}'_{Aa}).
\end{aligned}
\tag{7.2.110}
$$

?

It is of notice that the time–reversal phase convention is used throughout. Expanding the Green function and the distorted waves in a basis of angular momentum eigenstate one can write,

$$\chi^{(-)*}(\mathbf{k}_{Bb},\mathbf{r}_{Bb}) = \sum_{\bar{l}} \frac{4\pi}{k_{Bb}r_{Bb}} i^{-\bar{l}} e^{i\sigma_f^{\bar{l}}} F_{\bar{l}} \sum_m Y^{\bar{l}}_m(\hat{r}_{Bb}) Y^{\bar{l}*}_m(\hat{k}_{Bb}), \tag{7.2.111}$$

the sum over m being

$$\sum_m (-1)^{\bar{l}-m} Y^{\bar{l}}_m(\hat{r}_{Bb}) Y^{\bar{l}}_{-m}(\hat{k}_{Bb}) = \sqrt{2\bar{l}+1}\left[Y^{\bar{l}}(\hat{r}_{Bb})Y^{\bar{l}}(\hat{k}_{Bb})\right]_0^0, \tag{7.2.112}$$

where we have used (7.K.2) and (7.K.18), so

$$\chi^{(-)*}(\mathbf{k}_{Bb},\mathbf{r}_{Bb}) = \sum_{\bar{l}} \sqrt{2\bar{l}+1}\frac{4\pi}{k_{Bb}r_{Bb}} i^{-\bar{l}} e^{i\sigma_f^{\bar{l}}} F_{\bar{l}}(r_{Bb}) \left[Y^{\bar{l}}(\hat{r}_{Bb})Y^{\bar{l}}(\hat{k}_{Bb})\right]_0^0. \tag{7.2.113}$$

Similarly,

$$\chi^{(+)}(\mathbf{r}'_{Aa}) = \sum_l i^l \sqrt{2l+1}\frac{4\pi}{k_{Aa}r'_{Aa}} e^{i\sigma_i^l} F_l(r'_{Aa}) \left[Y^l(\hat{r}'_{Aa})Y^l(\hat{k}_{Aa})\right]_0^0 \tag{7.2.114}$$

where we have taken into account the choice $\hat{k}_{Aa} \equiv \hat{z}$. The Green function can be written as

$$G(\mathbf{r}_{Cc},\mathbf{r}'_{Cc}) = i\sum_{l_c} \sqrt{2l_c+1}\frac{f_{l_c}(k_{Cc},r_<)P_{l_c}(k_{Cc},r_>)}{k_{Cc}r_{Cc}r'_{Cc}} \left[Y^{l_c}(\hat{r}_{Cc})Y^{l_c}(\hat{r}'_{Cc})\right]_0^0. \tag{7.2.115}$$

in the previous version it read T^{VV}_{2NT}

Finally

$T^{(2)}_{succ}$

$$
\begin{aligned}
T^{2step}_{2NT} = & \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa}k_{Bb}k_{Cc}} \sum_{l,l_c,\tilde{l}} e^{i(\sigma_i^l+\sigma_f^{\tilde{l}})} i^{l-\tilde{l}} \sqrt{(2l+1)(2l_c+1)(2\tilde{l}+1)} \\
& \times \sum_{\substack{\sigma_1\sigma_2 \\ \sigma_1'\sigma_2'}} \int d^3r_{Cc} d^3r_{b1} d^3r_{A2} d^3r'_{Cc} d^3r'_{b1} d^3r'_{A2} v(r_{b1}) v(r'_{c2}) \left[Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb})\right]_0^0 \\
& \times \left[Y^{l}(\hat{r}'_{Aa}) Y^{l}(\hat{k}'_{Aa})\right]_0^0 \left[Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc})\right]_0^0 \frac{F_{\tilde{l}}(r_{Bb})}{r_{Bb}} \frac{F_l(r'_{Aa})}{r'_{Aa}} \\
& \times \frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r_{Cc} r'_{Cc}} \left[\psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)\right]_0^{0*} \\
& \times \left[\psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2)\right]_0^0 \sum_{KM} \left[\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]_M^K \\
& \times \left[\psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1)\right]_M^{K*}.
\end{aligned}
\tag{7.2.116}
$$

?

Let us now perform the integration over $\mathbf{r}_{A2}$,

$$
\begin{aligned}
& \sum_{\sigma_1,\sigma_2} \int d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)\right]_0^{0*} \left[\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]_M^K \\
& = \sum_{\sigma_1,\sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \int d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2)\right]_0^0 \left[\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]_M^K \\
& = -\sum_{\sigma_1,\sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \int d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2) \psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1)\right]_0^0 \left[\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]_M^K \\
& = -((j_f j_f)_0 (j_f j_i)_K | (j_f j_f)_0 (j_f j_i)_K)_K \sum_{\sigma_1,\sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \\
& \times \int d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)\right]_0^0 \left[\psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]_M^K \\
& = \frac{1}{2j_f+1} \sqrt{2j_f+1} ((l_f \tfrac{1}{2})_{j_f} (l_i \tfrac{1}{2})_{j_i} | (l_f l_i)_K (\tfrac{1}{2}\tfrac{1}{2})_0)_K \\
& \times u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \left[Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1})\right]_M^K \sum_{\sigma_1} (-1)^{1/2-\sigma_1} \left[\chi^{1/2}(-\sigma_1) \chi^{1/2}(\sigma_1)\right]_0^0 \\
& = -\sqrt{\frac{2}{2j_f+1}} ((l_f \tfrac{1}{2})_{j_f} (l_i \tfrac{1}{2})_{j_i} | (l_f l_i)_K (\tfrac{1}{2}\tfrac{1}{2})_0)_K \left[Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1})\right]_M^K u_{l_f}(r_{A1}) u_{l_i}(r_{b1}),
\end{aligned}
\tag{7.2.117}
$$

where we have evaluated the $9j$–symbol

$$
((j_f j_f)_0 (j_f j_i)_K | (j_f j_f)_0 (j_f j_i)_K)_K = \frac{1}{2j_f+1}, \tag{7.2.118}
$$

as well as (7.K.19). We proceed in a similar way to evaluate the integral over $\mathbf{r}'_{b1}$,

$$
\begin{aligned}
&\sum_{\sigma'_1,\sigma'_2}\int d\mathbf{r}'_{b1}\left[\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\right]^0_0\left[\psi^{j_f}(\mathbf{r}'_{A2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]^{K*}_M \\
&\quad= -(-1)^{K-M}\sum_{\sigma'_1,\sigma'_2}\int d\mathbf{r}'_{b1}\left[\psi^{j_f}(\mathbf{r}'_{A2},-\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},-\sigma'_1)\right]^K_{-M} \\
&\quad\times\left[\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]^0_0(-1)^{1/2-\sigma'_1+1/2-\sigma'_2} \\
&\quad= -(-1)^{K-M}((j_f j_i)_K(j_i j_i)_0|(j_f j_i)_K(j_i j_i)_0)_K(-\sqrt{2j_i+1}) \\
&\quad\times((l_f \tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_f l_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K(-\sqrt{2})u_{l_f}(r'_{A2})u_{l_i}(r'_{b2})\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]^K_{-M} \\
&\quad= -\sqrt{\frac{2}{2j_i+1}}((l_f \tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_f l_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]^{K*}_M u_{l_f}(r'_{A2})u_{l_i}(r'_{b2}).
\end{aligned}
\tag{7.2.119}
$$

Setting the different elements together one obtains

$T^{(2)}_{2ucc}$

$$
\begin{aligned}
T^{2step}_{2NT} &= \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa}k_{Bb}k_{Cc}}\frac{2}{\sqrt{(2j_i+1)(2j_f+1)}}\sum_{K,M}((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_f l_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)^2_K \\
&\times\sum_{l_c,l,\bar{l}} e^{i(\sigma^l_i+\sigma^{\bar{l}}_f)}\sqrt{(2l_c+1)(2l+1)(2\bar{l}+1)}\, i^{l-\bar{l}} \\
&\times\int d^3r_{Cc}d^3r_{b1}d^3r'_{Cc}d^3r'_{A2}v(r_{b1})v(r'_{c2})u_{l_f}(r_{A1})u_{l_i}(r_{b1})u_{l_f}(r'_{A2})u_{l_i}(r'_{b2}) \\
&\times\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]^{K*}_M\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^K_M\frac{F_l(r'_{Aa})F_{\bar{l}}(r'_{Bb})f_{l_c}(k_{Cc},r_<)P_{l_c}(k_{Cc},r_>)}{r'_{Aa}r_{Bb}r_{Cc}r'_{Cc}} \\
&\times\left[Y^{\bar{l}}(\hat{r}_{Bb})Y^{\bar{l}}(\hat{k}_{Bb})\right]^0_0\left[Y^{l}(\hat{r}'_{Aa})Y^{l}(\hat{k}_{Aa})\right]^0_0\left[Y^{l_c}(\hat{r}_{Cc})Y^{l_c}(\hat{r}'_{Cc})\right]^0_0.
\end{aligned}
\tag{7.2.120}
$$

?

We now proceed to write this expression in a more compact way. For this purpose one writes

$$
\begin{aligned}
&\left[Y^{\bar{l}}(\hat{r}_{Bb})Y^{\bar{l}}(\hat{k}_{Bb})\right]^0_0\left[Y^{l}(\hat{r}'_{Aa})Y^{l}(\hat{k}_{Aa})\right]^0_0 = \\
&\qquad((l\,l)_0(\bar{l}\,\bar{l})_0|(l\,\bar{l})_0(l\,\bar{l})_0)_0\left[Y^{\bar{l}}(\hat{r}_{Bb})Y^{l}(\hat{r}'_{Aa})\right]^0_0\left[Y^{\bar{l}}(\hat{k}_{Bb})Y^{l}(\hat{k}_{Aa})\right]^0_0 \\
&\qquad= \frac{\delta_{\bar{l}l}}{2l+1}\left[Y^{l}(\hat{r}_{Bb})Y^{l}(\hat{r}'_{Aa})\right]^0_0\left[Y^{l}(\hat{k}_{Bb})Y^{l}(\hat{k}_{Aa})\right]^0_0.
\end{aligned}
\tag{7.2.121}
$$

Taking into account the relations

$$
\left[Y^{l}(\hat{k}_{Bb})Y^{l}(\hat{k}_{Aa})\right]^0_0 = \frac{(-1)^l}{\sqrt{4\pi}}Y^l_0(\hat{k}_{Bb})i^l,
\tag{7.2.122}
$$