Chapter 7

Two-nucleon transfer

Ch. 7 React

(A) - (A) Ch. 7 p. 1 below

7.1 Summary of 2nd order DWBA

~~0.1 Details of the Calculation~~ (the theory of second order DWBA two-nucleon transfer with the subject discussed in 7.2.10

Let us illustrate the calculation with ~~a~~ $A+t \rightarrow B(\equiv A+2)+p$ reaction, in which $A+2$ and A are even nuclei in their 0^+ ground state. The extension of the following expressions to the transfer of pairs coupled to arbitrary angular momentum is ~~straightforward~~. The wavefunction of the nucleus $A+2$ ~~is~~ can be written as

$$\Psi_{A+2}(\xi_A, \mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \psi_A(\xi_A) \sum_{l_i, j_i} [\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^0_0, \tag{1}$$

where

$$\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \sum_{nm} a_{nm} \left[\varphi^{A+2}_{n, l_i, j_i}(\mathbf{r}_{A1}, \sigma_1)\varphi^{A+2}_{m, l_i, j_i}(\mathbf{r}_{A2}, \sigma_2)\right]^0_0, \tag{2}$$

while ~~and~~ the wavefunctions $\varphi^{A+2}_{n, l_i, j_i}(\mathbf{r})$ are eigenfunctions of a Woods-Saxon potential

$$U(r) = -\frac{V_0}{1+\exp\left[\frac{r-R_0}{a}\right]}, \qquad R_0 = r_0 A^{1/3}, \tag{3}$$

of ~~The~~ depth V_0 ~~is~~ adjusted to reproduce the experimental single–particles energies. The ~~spatial part of~~ radial dependence of the wavefunction of the two neutrons in the ~~tritium~~ triton is written as $\phi_t(\mathbf{r}_{p1}, \mathbf{r}_{p2}) = \rho(r_{p1})\rho(r_{p2})\rho(r_{12})$, where r_{p1}, r_{p2}, r_{12} are the distances between neutron 1 and the proton, neutron 2 and the proton and between neutrons 1 and 2 respectively, ~~and~~ while $\rho(r)$ is the hard core potential ($r_{core} = 0.45$ fm) wavefunction ~~with hard core at r = 0.45 fm, as~~ depicted in Fig. 7.1.

The two-nucleon transfer differential cross section is written as,

$$\frac{d\sigma}{d\Omega} = \frac{\mu_i \mu_f}{(4\pi\hbar^2)^2} \frac{k_f}{k_i} \left|T^{(1)} + T^{(2)}_{succ} - T^{(2)}_{NO}\right|^2, \tag{4}$$

definition of all quantities

where ~~the three amplitudes contributing to the transfer are~~ (see also [1]),

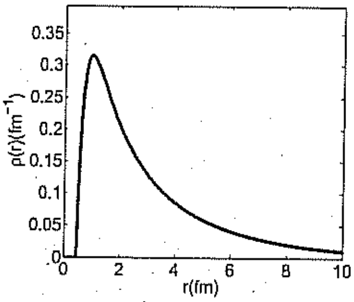
$$T^{(1)} = 2 \sum_{l_i, j_i} \sum_{\sigma_1 \sigma_2} \int d\mathbf{r}_{tA} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^{0*}_0 \chi^{(-)*}_{pB}(\mathbf{r}_{pB})$$
$$\times v(\mathbf{r}_{p1})\phi_t(\mathbf{r}_{p1}, \mathbf{r}_{p2})\chi^{(+)}_{tA}(\mathbf{r}_{tA}), \tag{5a}$$

$$T^{(2)}_{succ} = 2 \sum_{l_i, j_i} \sum_{l_f, j_f, m_f} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma'_1 \sigma'_2}} \int d\mathbf{r}_{dF} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^{0*}_0 \chi^{(-)*}_{pB}(\mathbf{r}_{pB}) v(\mathbf{r}_{p1})$$
$$\times \phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f, j_f, m_f}(\mathbf{r}_{A2}) \int d\mathbf{r}'_{dF} d\mathbf{r}'_{p1} d\mathbf{r}'_{A2} G(\mathbf{r}_{dF}, \mathbf{r}'_{dF})$$
$$\times \phi_d(\mathbf{r}'_{p1})^* \varphi^{A+1*}_{l_f, j_f, m_f}(\mathbf{r}'_{A2}) \frac{2\mu_{dF}}{\hbar^2} v(\mathbf{r}'_{p2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}), \tag{5b}$$

Sabama BenBaymam

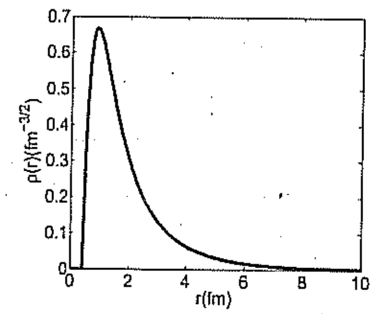
$$T^{(2)}_{NO} = 2 \sum_{l_i, j_i} \sum_{l_f, j_f, m_f} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma'_1 \sigma'_2}} \int d\mathbf{r}_{dF} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^{0*}_0 \chi^{(-)*}_{pB}(\mathbf{r}_{pB}) v(\mathbf{r}_{p1})$$
$$\times \phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f, j_f, m_f}(\mathbf{r}_{A2}) \int d\mathbf{r}'_{p1} d\mathbf{r}'_{A2} d\mathbf{r}'_{dF}$$
$$\times \phi_d(\mathbf{r}'_{p1})^* \varphi^{A+1*}_{l_f, j_f, m_f}(\mathbf{r}'_{A2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}). \tag{5c}$$

The quantities μ_i, μ_f (k_i, k_f) are the reduced masses (relative linear momenta) in both entrance (initial, i) and ~~exit~~ exit (final, f) channels, respectively.



7.1

Figure ~~1~~: Tritium wavefunction



7.2

Figure ~~2~~: Deuteron wavefunction

Although the are a number of ways to exactly treat such states, discretization processes may be sufficiently accurate. They can be implemented by, for example

the above

being

lie

7.2

sufficiently

In ~~these~~ expressions, $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ are the wavefunctions describing the intermediate states of the nucleus $F(\equiv A+1)$, generated as solutions of a Woods–Saxon potential, and $\phi_d(\mathbf{r}_{p2})$ ~~is~~ the wavefunction ~~of the~~ deuteron bound ~~state~~ (see ~~Fig. 2~~). Note that some or all of the $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ may ~~be~~ in the continuum ~~for unbound or loosely bound~~, ~~and some discretization procedure is required in order to deal with these states. In this case, they are generated by~~ embedding the Woods–Saxon potential in a spherical box of large ~~enough~~ radius. In actual calculations, ~~we got convergence with less than~~ 20 continuum states ~~in a~~ and a box of 30 fm radius ~~box. As for the wavefunction of the neutrons in the tritium, it is~~ generated with the $p-n$ Tang–Herndon interaction

(case in which the nucleus F is loosely bound or unbound,

(3)

single-particle states described by the wavefunctions

involving the halo nucleus ^{11}Li, and where $|F\rangle = |^{10}$Li$\rangle$, one achieved convergence making use of about

$$v(r) = -v_0 \exp\left(-k(r-r_c)\right) \quad r > r_c \tag{6}$$

$$v(r) = \infty \quad r < r_c, \tag{7}$$

Concerning the component

50 as

where $k = 2.5\ \mathrm{fm}^{-1}$ and $r_c = 0.45$ fm, ~~and~~ the depth v_0 ~~is~~ adjusted to reproduce the experimental separation energies. The positive–energy wavefunctions $\chi^{(+)}_{tA}(\mathbf{r}_{tA})$ and $\chi^{(-)}_{pB}(\mathbf{r}_{pB})$ are the ingoing distorted wave in the initial channel and the outgoing distorted wave in the final channel respectively. They are continuum solutions of the Schrödinger equation associated with the corresponding optical potentials. being

of the the triton wavefunction describing the relative motion of the dineutron, it was

The transition potential responsible for the transfer of the pair is, in the *post* representation,

$$V_\beta = v_{pB} - U_\beta, \tag{8}$$

where v_{pB} is the interaction between the proton and nucleus B, and U_β is the optical potential in the final channel. We make the assumption that v_{pB} can be decomposed into a term containing the interaction between A and p and the potential describing the interaction between the proton and each of the transferred nucleons, namely

$$v_{pB} = v_{pA} + v_{p1} + v_{p2}, \tag{9}$$

where v_{p1} and v_{p2} is the hard–core potential (6). The transition potential is

$$V_\beta = v_{pA} + v_{p1} + v_{p2} - U_\beta. \tag{10}$$

Assuming that $\langle\beta|v_{pA}|\alpha\rangle \approx \langle\beta|U_\beta|\alpha\rangle$ (i.e, assuming that the matrix element of the core–core interaction between the initial and final states is very similar to the matrix element of the real part of the optical potential), one obtains the final expression of the transfer potential in the *post* representation, namely,

$$V_\beta \simeq v_{p1} + v_{p2} = v(\mathbf{r}_{p1}) + v(\mathbf{r}_{p2}). \tag{11}$$

???

We make the further approximation of using the same interaction potential in all (e.g. initial, intermediate and final) the channels.

and if not?

The extension to a heavy–ion reaction $A + a(\equiv b+2) \longrightarrow B(\equiv A+2)$ ~~+ b~~ imply no essential modifications in the formalism. The deuteron and triton ~~states~~ wavefunctions appearing in ~~(5a,5b,5c)~~ Eqs. 5(a), 5(b) and 5(c) ~~must~~ can to be substituted with the corresponding wavefunctions $\Psi_{b+2}(\xi_b, \mathbf{r}_{b1}, \sigma_1, \mathbf{r}_{b2}, \sigma_2)$, constructed in a similar way ~~as in~~ to those introduced in Eqs. (1) and (2). The interaction potential used in ~~(5a,5b,5c)~~ Eqs 5(a), 5(b) and 5(c) will now be the Woods–Saxon used to define the initial (final) state in the post (prior) representation, instead of the proton–neutron interaction (6).

The Green's function $G(\mathbf{r}_{dF}, \mathbf{r}'_{dF})$ appearing in 5(b) propagates the intermediate channel d, F. ~~and can~~ It can be expanded in partial waves as

$$G(\mathbf{r}_{dF}, \mathbf{r}'_{dF}) = i \sum_l \sqrt{2l+1}\, \frac{f_l(k_{dF}, r_<)\, g_l(k_{dF}, r_>)}{k_{dF}\, r_{dF}\, r'_{dF}} \left[Y^l(\hat{r}_{dF}) Y^l(\hat{r}'_{dF}) \right]^0_0 . \tag{12}$$

?

γ for a

The $f_l(k_{dF}, r)$ and ~~$g_l(k_{dF}, r)$~~ are the regular and the irregular solutions of a Schrödinger equation ~~with a~~ suitable optical potential and an energy equal to the kinetic energy ~~in~~ the intermediate state. In most cases of interest, the result is hardly altered if we use the same energy of ~~the~~ relative motion ~~between nuclei~~ for all the intermediate states. This representative energy is calculated when both intermediate nuclei are in their corresponding ground states. ~~However,~~ the validity of this approximation can break down in some particular cases. If, for example, some relevant intermediate state become off shell, its contribution is significantly quenched. An interesting situation can arise when this happens to all possible intermediate states, so they can only be virtually populated.

of

It is of notice that

Bibliography

[1] B. F. Bayman and J. Chen. One-step and two-step contributions to two-nucleon transfer reactions. *Phys. Rev. C*, 26:1509, 1982.

to the first page

A

Chapter 7

Two–particle transfer

Cooper pairs are the building blocks of pairing correlations in many-body fermionic systems. In particular in atomic nuclei. As a consequence, nuclear superfluidity can be specifically probed through Cooper pair tunneling.

In the simultaneous transfer of two nucleons, one nucleon goes over from target to projectile, or viceversa, under the influence of the nuclear interaction responsible of the existence of a mean field potential, while the other follows suit by profiting of: 1) pairing correlations (simultaneous transfer); 2) the fact that the single-particle wavefunctions describing the motion of Cooper pair partners in both target and projectile are solutions of different single-particle potentials (non-orthogonality transfer). In the limit of independent particle motion, in which all of the nucleon-nucleon interaction is used up in generating a mean field, both contributions to the transfer process (simultaneous and non-orthogonality) cancel out exactly (cf. [redacted]). app A Sect. 7.13

e

In keeping with the fact that nuclear Cooper pairs are weakly bound, this cancellation is, in actual nuclei, quite strong. Consequently, successive transfer, a process in which the mean field acts twice is, as a rule, the main mechanism at the basis of Cooper pair transfer. Because of the same reason (weak binding), the correlation length of ~~Cooper pairs~~ is larger than nuclear dimensions, a fact which allows the two members of a Cooper pair, to move between target and projectile, essentially as a whole, also in the case of successive transfer.

a Three appendixes are provided. One in which the cancellations existing between the different contributions to the two–nucleon transfer spectroscopic amplitudes (successive, simultaneous and non–orthogonality) are discussed in detail within the framework of the semi–classical approximation. (App. A, now Sect. 7.13) Another one in which simple estimates of the relative importance of successive and of simultaneous transfer are worked out. numerical (App. B Sect. 7.6 Finally, a derivation of first order DWBA simultaneous transfer is worked out ~~within a~~ formalism tailored to focus the attention on the nuclear structure correlations aspects of the process leading to effective two–nucleon transfer form factors. (App. C Sect. 7.5 7.1

a The present Chapter is structured in the following way. In section 7.1 we present a summary of two–nucleon transfer reaction theory. These are all the elements needed to calculate the absolute two–nucleon transfer differential cross sections in second order DWBA, and thus to compare theory with experiment. ~~In this way~~, after reading this section one can go directly to the ~~next~~ Chapter 8 containing ~~which contains~~ examples of the applications of this formalism.

For the more theoretically oriented reader we provide in section 7.2 a detailed derivation of the equations presented in section 7.1 and which are implemented in the

We shall also concentrate new mechs of Cooper pair binding

Bardeen-Pines Fröhlich

COOP, KNOCK...

software s/ ~~COOPS~~ used in the applications (cf. also app. A Ch. 8)

A

@-@ from p. 1

Gregory has to write

Detailed derivation of 2nd order DWBA

7.2.1 Simultaneous transfer: distorted waves

~~7.1 simultaneous transfer~~

~~7.1.1 distorted waves~~

For a (t, p) reaction, the triton is represented by an incoming total distorted wave. ~~We make the assumption that the two transferred neutrons are in the $S = 0$ (singlet) state and that the triton has orbital angular momentum $L = 0$, so the spin~~ is entirely due to the spin of the proton. We will explicitly treat it, as, ~~unlike in [?],~~ we will consider a spin–orbit term in the optical potential acting between the triton and the ~~heavy ion~~ target. ~~We~~ In what follows we will use the notation of [?]. ~~After [??]~~ we can write the triton distorted wave as following,

of ref

$$\psi^{(+)}_{m_t}(\mathbf{R}, \mathbf{k}_i, \sigma_p) = \sum_{l_t} \exp\left(i\sigma^t_{l_t}\right) g_{l_t j_t} Y^{l_t}_0(\hat{\mathbf{R}}) \frac{\sqrt{4\pi(2l_t+1)}}{k_i R} \chi_{m_t}(\sigma_p), \tag{7.1}$$

where ~~we have used~~ use was made of $Y^{l_t}_0(\hat{\mathbf{k}}_i) = i^{l_t}\sqrt{\frac{2l_t+1}{4\pi}}\delta_{m_t,0}$, ~~as~~ in keeping with the fact that $\mathbf{k}_i$ is oriented along the z–axis. Note the phase difference with eq. (7) of [?], due to the use of time-reversal rather than Condon–Shortley phase convention. ~~If we write~~ Making use of the relation

$$Y^{l_t}_0(\hat{\mathbf{R}})\chi_{m_t}(\sigma_p) = \sum_{j_t} \langle l_t\ 0\ 1/2\ m_t | j_t\ m_t\rangle \left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}, \tag{7.2}$$

we have

$$\begin{aligned}\psi^{(+)}_{m_t}(\mathbf{R}, \mathbf{k}_i, \sigma_p) = &\sum_{l_t, j_t} \exp\left(i\sigma^t_{l_t}\right) \frac{\sqrt{4\pi(2l_t+1)}}{k_i R} g_{l_t j_t}(R) \\ &\times \langle l_t\ 0\ 1/2\ m_t | j_t\ m_t\rangle \left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}.\end{aligned} \tag{7.3}$$

Following ( )

We now turn our attention to the outgoing proton distorted wave, which, ~~after (??), is~~ we can write it as

$$\psi^{(-)}_{m_p}(\zeta, \mathbf{k}_f, \sigma_p) = \sum_{l_p j_p} \frac{4\pi}{k_f \zeta} i^{l_p} \exp\left(-i\sigma^p_{l_p}\right) f^*_{l_p j_p}(\zeta) \sum_m Y^{l_p}_m(\hat{\zeta}) Y^{l_p *}_m(\hat{\mathbf{k}}_f)\chi_{m_p}(\sigma_p). \tag{7.4}$$

Making use of the relation ~~Now,~~

$$\begin{aligned}\sum_m Y^{l_p}_m(\hat{\zeta}) Y^{l_p *}_m(\hat{\mathbf{k}}_f)\chi_{m_p}(\sigma_p) &= \sum_{m, j_p} Y^{l_p *}_m(\hat{\mathbf{k}}_f)\langle l_p\ m\ 1/2\ m_p | j_p\ m + m_p\rangle \\ &\times \left[Y^{l_p}(\hat{\zeta})\chi_{m_p}(\sigma_p)\right]^{j_p}_{m+m_p} \\ &= \sum_{m, j_p} Y^{l_p *}_{m-m_p}(\hat{\mathbf{k}}_f)\langle l_p\ m - m_p\ 1/2\ m_p | j_p\ m\rangle \left[Y^{l_p}(\hat{\zeta})\chi_{m_p}(\sigma_p)\right]^{j_p}_{m},\end{aligned} \tag{7.5}$$

one obtains ~~and, finally,~~

$$\begin{aligned}\psi^{(-)}_{m_p}(\zeta, \mathbf{k}_f, \sigma_p) = \frac{4\pi}{k_f \zeta} &\sum_{l_p j_p m} i^{l_p} \exp\left(-i\sigma^p_{l_p}\right) f^*_{l_p j_p}(\zeta) Y^{l_p *}_{m-m_p}(\hat{\mathbf{k}}_f) \\ &\times \langle l_p\ m - m_p\ 1/2\ m_p | j_p\ m\rangle \left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p}_{m}.\end{aligned} \tag{7.6}$$

App. 7A

We make the assumption that the two transferred neutrons are in an $S = L = 0$ state, and that the relative motion of the proton with respect to the dineutron is also $l = 0$. Consequently, the total spin of the triton

7.~~1~~2.2 matrix element for the transition amplitude (I)

We now turn our attention to the evaluation of

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= \frac{(4\pi)^{3/2}}{k_ik_f}\sum_{l_pl_tj_pj_tm}((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sqrt{2l_t+1}\\
&\times\langle l_p\ m-m_p\ 1/2\ m_p|j_p\ m\rangle\langle l_t\ 0\ 1/2\ m_t|j_t\ m_t\rangle\, i^{-l_p}\exp[i(\sigma^p_{l_p}+\sigma^t_{l_t})]\\
&\times 2Y^{l_p}_{m-m_p}(\hat{\mathbf{k}}_f)\sum_{\sigma_1\sigma_2\sigma_p}\int\frac{d\zeta d\mathbf{r}d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]^{0*}_0\\
&\times f_{l_pj_p}(\zeta)g_{l_tj_t}(R)\left[\chi(\sigma_1)\chi(\sigma_2)\right]^{0*}_0\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m V(r_{1p})\\
&\times\theta^0_0(\mathbf{r},\mathbf{s})\left[\chi(\sigma_1)\chi(\sigma_2)\right]^0_0\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t},
\end{aligned}
\tag{7.7}
$$

where

$$
\begin{aligned}
\mathbf{r} &= \mathbf{r}_2-\mathbf{r}_1\,,\\
\mathbf{s} &= \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)-\mathbf{r}_p\,,\\
\eta &= \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\,,\\
\zeta &= \mathbf{r}_p-\frac{\mathbf{r}_1+\mathbf{r}_2}{A+2}.
\end{aligned}
\tag{7.8}
$$

The sum over σ_1,σ_2 in (7.7) is ~~readily~~ found to be ~~1~~ equal to (por la referencia mas tarde after). We will now simplify the term $\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}$, ~~first~~ noting that, ~~after~~

$$
\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m = (-1)^{1/2-\sigma_p+j_p-m}\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}_{-m}.
\tag{7.9}
$$

and that ~~On the other hand~~,

$$
\begin{aligned}
\left[Y^{l_p}(\hat{\zeta})\ \chi(-\sigma_p)\right]^{j_p}_{-m}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t} &= \sum_{JM}\langle j_p\ -m\ j_t\ m_t|J\ M\rangle\\
&\times\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^J_M
\end{aligned}
\tag{7.10}
$$

The only terms which do not vanish after the integration is performed is the one in which ~~In order to survive the integration, the~~ angular and spin functions ~~must~~ are couple to $L=0,S=0,J=0$, ~~so the only term that remains is~~ Thus

$$
\begin{aligned}
\cdots\langle j_p\ -m\ j_t\ m_t|0\ 0\rangle&\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^0_0\delta_{l_pl_t}\delta_{j_pj_t}\delta_{mm_t}\\
&= \frac{(-1)^{j_p+m_t}}{\sqrt{2j_p+1}}\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^0_0\delta_{l_pl_t}\delta_{j_pj_t}\delta_{mm_t}.
\end{aligned}
\tag{7.11}
$$

Coupling ~~We couple~~ separately the spin and spatial functions, one obtains

$$
\begin{aligned}
&\left\{\left[Y^l(\hat{\zeta})\chi(-\sigma_p)\right]^j\left[Y^l(\hat{\mathbf{R}})\chi(\sigma_p)\right]^j\right\}^0_0\\
&\qquad= ((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\left[\chi(-\sigma_p)\chi(\sigma_p)\right]^0_0\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]^0_0.
\end{aligned}
\tag{7.12}
$$

We substitute (7.9),(7.30),(7.31) in (7.7) to obtain

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle = &-\frac{(4\pi)^{3/2}}{k_ik_f}\sum_{lj}((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sqrt{\frac{2l+1}{2j+1}}\\
&\times\langle l\ m_t-m_p\ 1/2\ m_p|j\ m_t\rangle\langle l\ 0\ 1/2\ m_t|j\ m_t\rangle\, i^{-l}\exp[i(\sigma_l^p+\sigma_l^t)]\\
&\times 2Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int\frac{d\zeta d\mathbf{r} d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*}\\
&\times f_{lj}(\zeta)g_{lj}(R)\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0 V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})\\
&\times((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sum_{\sigma_p}(-1)^{1/2-\sigma_p}\left[\chi(-\sigma_p)\chi(\sigma_p)\right]_0^0.
\end{aligned}
\tag{7.13}
$$

The last sum over σ_p ~~is~~ leads to

$$
\begin{aligned}
\sum_{\sigma_p}(-1)^{1/2-\sigma_p}\left[\chi(-\sigma_p)\chi(\sigma_p)\right]_0^0 &= \sum_{\sigma_p m}(-1)^{1/2-\sigma_p}\langle 1/2\ m\ 1/2\ -m|0\ 0\rangle\\
&\quad\times\chi_m(-\sigma_p)\chi_{-m}(\sigma_p)\\
&= \frac{1}{\sqrt{2}}\sum_{\sigma_p m}(-1)^{1/2-\sigma_p}(-1)^{1/2-m}\delta_{m,-\sigma_p}\delta_{-m,\sigma_p} = -\sqrt{2}.
\end{aligned}
\tag{7.14}
$$

The 9j symbols can be evaluated to find

$$
\begin{aligned}
((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 &= \sqrt{\frac{2k+1}{2(2\lambda+1)}},\\
((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 &= \sqrt{\frac{2j+1}{2(2l+1)}}.
\end{aligned}
\tag{7.15}
$$

Consequently

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle = &\frac{(4\pi)^{3/2}}{k_ik_f}\sum_{lj}\sqrt{\frac{2k+1}{2\lambda+1}}\\
&\times\langle l\ m_t-m_p\ 1/2\ m_p|j\ m_t\rangle\langle l\ 0\ 1/2\ m_t|j\ m_t\rangle\, i^{-l}\exp[i(\sigma_l^p+\sigma_l^t)]\\
&\times\sqrt{2}Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int\frac{d\zeta d\mathbf{r} d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*}\\
&\times f_{lj}(\zeta)g_{lj}(R)\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0 V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s}).
\end{aligned}
\tag{7.16}
$$

~~We now check~~ The ~~possible~~ values of the Clebsh–Gordan coefficients ~~, finding, for~~ are, for $j = l-1/2$,

$$
\langle l\ m_t-m_p\ 1/2\ m_p|l-1/2\ m_t\rangle\langle l\ 0\ 1/2\ m_t|l-1/2\ m_t\rangle
= \begin{cases} \dfrac{l}{2l+1} & \text{if } m_t = m_p\\[2mm] -\dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\tag{7.17}
$$

and, for $j = l + 1/2$:

$$\langle l\ m_t - m_p\ 1/2\ m_p | l + 1/2\ m_t \rangle \langle l\ 0\ 1/2\ m_t | l + 1/2\ m_t \rangle = \begin{cases} \dfrac{l+1}{2l+1} & \text{if } m_t = m_p \\ \dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p\ . \end{cases} \tag{7.18}$$

One thus can write, ~~Substituting, we get~~

$$\begin{aligned} \langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{3/2}}{k_i k_f} \sum_l \frac{1}{(2l+1)} \sqrt{\frac{(2k+1)}{(2\lambda+1)}} \exp[i(\sigma_l^p + \sigma_l^t)] i^{-l} \\ &\times \sqrt{2} Y^l_{m_t - m_p}(\hat{\mathbf{k}}_f) \int \frac{d\zeta d\mathbf{r} d\eta}{\zeta R} u_{\lambda k}(r_1) u_{\lambda k}(r_2) \left[Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2) \right]_0^{0*} \\ &\times V(r_{1p}) \theta_0^0(\mathbf{r}, \mathbf{s}) \left[Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}}) \right]_0^0 \\ &\times \Big[\big(f_{ll+1/2}(\zeta) g_{ll+1/2}(R)(l+1) + f_{ll-1/2}(\zeta) g_{ll-1/2}(R) l \big) \delta_{m_p, m_t} \\ &+ \big(f_{ll+1/2}(\zeta) g_{ll+1/2}(R) \sqrt{l(l+1)} - f_{ll-1/2}(\zeta) g_{ll-1/2}(R) \sqrt{l(l+1)} \big) \delta_{m_p, -m_t} \Big] . \end{aligned} \tag{7.19}$$

We can further simplify this expression using

$$\begin{aligned} \left[Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2) \right]_0^{0*} = \left[Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2) \right]_0^0 &= \sum_m \langle \lambda\ m\ \lambda\ -m | 0\ 0 \rangle Y^\lambda_m(\hat{\mathbf{r}}_1) Y^\lambda_{-m}(\hat{\mathbf{r}}_2) \\ &= \sum_m (-1)^{\lambda - m} \langle \lambda\ m\ \lambda\ -m | 0\ 0 \rangle Y^\lambda_m(\hat{\mathbf{r}}_1) Y^{\lambda *}_m(\hat{\mathbf{r}}_2) \\ &= \frac{1}{\sqrt{2\lambda+1}} \sum_m Y^\lambda_m(\hat{\mathbf{r}}_1) Y^{\lambda *}_m(\hat{\mathbf{r}}_2) \\ &= \frac{\sqrt{(2\lambda+1)}}{4\pi} P_\lambda(\cos\theta_{12}). \end{aligned} \tag{7.20}$$

Note that when using Condon–Shortley phases this last expression ~~would~~ is to be multiplied by $(-1)^\lambda$, ~~Now~~ and that

$$\begin{aligned} \left[Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}}) \right]_0^0 &= \sum_m \langle l\ m\ l\ -m | 0\ 0 \rangle Y^l_m(\hat{\zeta}) Y^l_{-m}(\hat{\mathbf{R}}) \\ &= \frac{1}{\sqrt{(2l+1)}} \sum_m (-1)^{l+m} Y^l_m(\hat{\zeta}) Y^l_{-m}(\hat{\mathbf{R}}). \end{aligned} \tag{7.21}$$

Because ~~We can see that~~ the integral of the above expression is independent of m, ~~so we~~ one can ~~drop~~ eliminate the m-sum and multiply by $2l+1$ the $m = 0$ term, ~~leaving~~ leading to

$$\begin{aligned} \left[Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}}) \right]_0^0 &\Rightarrow (-1)^l \sqrt{(2l+1)}\, Y^l_0(\hat{\zeta})_0 Y^l(\hat{\mathbf{R}}) \\ &= \sqrt{(2l+1)} Y^l_0(\hat{\zeta}) Y^{l*}_0(\hat{\mathbf{R}}). \end{aligned} \tag{7.22}$$

We now change the integration variables from $(\zeta, \mathbf{r}, \eta)$ to $(\mathbf{R}, \alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p})$, The quantity

$$\left| \frac{\partial(\mathbf{r}, \eta, \zeta)}{\partial(\mathbf{R}, \alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p})} \right| = r_{12} r_{1p} r_{2p} \sin\beta\ , \tag{7.23}$$

being the Jacobian of the transformation. Finally,

$$
\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= \frac{\sqrt{8}\pi}{k_i k_f}\sum_l \sqrt{\frac{2k+1}{2l+1}}\exp[i(\sigma_l^p+\sigma_l^t)]i^{-l} \\
&\times Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int d\mathbf{R}\,Y_0^{l*}(\hat{\mathbf{R}})\int \frac{d\alpha\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta}{\zeta R}Y_0^l(\hat\zeta) \\
&\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})P_\lambda(\cos\theta_{12})r_{12}r_{1p}r_{2p} \\
&\times\Big[\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)(l+1)+f_{ll-1/2}(\zeta)g_{ll-1/2}(R)l\big)\delta_{m_p,m_t} \\
&+\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\sqrt{l(l+1)}-f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\sqrt{l(l+1)}\big)\delta_{m_p,-m_t}\Big].
\end{aligned}
\qquad (7.24)
$$

spherical harmonics

under rotations

~~We note~~ It is noted that the ~~inner~~ second integral is a function of **R** ~~alone~~ solely, ~~and that it transforms~~ transforming as $Y_0^l(\hat{\mathbf{R}})$ ~~under rotations~~, ~~because all the~~ in keeping with the fact that the full dependence on the orientation of **R** is contained in the ~~term~~ $Y_0^l(\hat\zeta)$. The ~~inner~~ Second integral can thus be cast into the form

$$
A(R)Y_0^l(\hat{\mathbf{R}}) = \int d\alpha\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta \times F(\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p},R_x,R_y,R_z). \qquad (7.25)
$$

To evaluate $A(R)$, we ~~put~~ set **R** ~~on~~ along the z–axis

$$
A(R) = 2\pi i^{-l}\sqrt{\frac{4\pi}{2l+1}}\int d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta \times F(\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p},0,0,R), \qquad (7.26)
$$

where a factor 2π ~~has been included as the~~ results from the integration ~~of the integral~~ over α, ~~since~~ the integrand ~~clearly does~~ not depending on α. ~~We substitute~~ Substituting (7.25) and (7.26) in (7.24), and after integrating over the angular variables of **R**, we obtain

$$
\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= 2\frac{(2\pi)^{3/2}}{k_i k_f}\sum_l \sqrt{\frac{2k+1}{2l+1}}\exp[i(\sigma_l^p+\sigma_l^t)]i^{-l} \\
&\times Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int dR\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,R\sin\beta\,r_{12}r_{1p}r_{2p} \\
&\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta) \\
&\times\Big[\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)(l+1)+f_{ll-1/2}(\zeta)g_{ll-1/2}(R)l\big)\delta_{m_p,m_t} \\
&+\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\sqrt{l(l+1)}-f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\sqrt{l(l+1)}\big)\delta_{m_p,-m_t}\Big]/\zeta,
\end{aligned}
\qquad (7.27)
$$

where ~~we have~~ used, was made of the relation

$$
Y_0^l(\hat\zeta) = i^l\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta_\zeta). \qquad (7.28)
$$

The final expression of the differential cross section involves a sum over the spin orientations:

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{k_f}{k_i}\frac{\mu_i\mu_f}{(2\pi\hbar^2)^2}\frac{1}{2}\sum_{m_t m_p}|\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle|^2. \qquad (7.29)
$$

When $m_p = 1/2, m_t = 1/2$ or $m_p = -1/2, m_t = -1/2$, the terms proportional to δ_{m_p,m_t} ~~will include~~ including the factor

$$
|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,m_t}| = |Y_0^l(\hat{\mathbf{k}}_f)| = \left|i^l\sqrt{\frac{2l+1}{4\pi}}P_l^0(\cos\theta)\right|, \qquad (7.30)
$$

~~when~~ in the case in which $m_p = -1/2, m_t = 1/2$

$$|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y^l_1(\hat{\mathbf{k}}_f)| = \left| i^l \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}} P^1_l(\cos\theta) \right|, \tag{7.31}$$

and when $m_p = 1/2, m_t = -1/2$

$$|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y^l_{-1}(\hat{\mathbf{k}}_f)| = |Y^l_1(\hat{\mathbf{k}}_f)| = \left| i^l \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}} P^1_l(\cos\theta) \right|, \tag{7.32}$$

~~It is easily checked that, after~~ Taking the squared modulus of (7.27), the sum over m_t and m_p yields a factor 2 multiplying each one of the 2 different terms of the sum ($m_t = m_p$ and $m_t = -m_p$). This is equivalent to multiply each amplitude by $\sqrt{2}$, so the final constant that multiply the amplitudes is

$$\frac{8\pi^{3/2}}{k_i k_f}. \tag{7.33}$$

Now, for the ~~tritium we can take~~ triton wavefunction we use

$$\theta^0_0(\mathbf{r},\mathbf{s}) = \rho(r_{1p})\rho(r_{2p})\rho(r_{12}), \tag{7.34}$$

$\rho(r)$ being a Tang–Herndon wave function as done in [?]. We the obtain

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{1}{2E_i^{3/2}E_f^{1/2}}\sqrt{\frac{\mu_f}{\mu_i}}\left(|I^{(0)}_{\lambda k}(\theta)|^2 + |I^{(1)}_{\lambda k}(\theta)|^2\right), \tag{7.35}$$

~~with~~ where

$$\begin{aligned} I^{(0)}_{\lambda k}(\theta) = &\sum_l P^0_l(\cos\theta)\sqrt{2k+1}\exp[i(\sigma^p_l + \sigma^t_l)] \\ &\times \int dR\, d\beta\, d\gamma\, dr_{12}\, dr_{1p}\, dr_{2p}\, R\sin\beta\, \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ &\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ &\times \left(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\,(l+1) + f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\,l\right)/\zeta, \end{aligned} \tag{7.36}$$

and

$$\begin{aligned} I^{(1)}_{\lambda k}(\theta) = &\sum_l P^1_l(\cos\theta)\sqrt{2k+1}\exp[i(\sigma^p_l + \sigma^t_l)] \\ &\times \int dR\, d\beta\, d\gamma\, dr_{12}\, dr_{1p}\, dr_{2p}\, R\sin\beta\, \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ &\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ &\times \left(f_{ll+1/2}(\zeta)g_{ll+1/2}(R) - f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\right)/\zeta. \end{aligned} \tag{7.37}$$

Note the absence of the $(-1)^\lambda$ factor with respect to what ~~can be~~ is found in [?], is due to the use of time–reversed phases instead of Condon–Shortley phasing. This is compensated in the total result ~~with the same~~ by a similar difference in the expression of the spectroscopic ~~factors~~ amplitudes. This ensures that, in either case, the contribution of all the single particle transitions tend to have the same phase for superfluid nuclei, adding coherently to enhance the transfer cross section.

If we are dealing with a heavy ion reaction, $\theta_0^0(\mathbf{r},\mathbf{s})$ will be the spatial part of the wavefunction

$$\begin{aligned}\Psi(\mathbf{r}_{b1},\mathbf{r}_{b2},\sigma_1,\sigma_2) &= \left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]_0^0 \\ &= \theta_0^0(\mathbf{r},\mathbf{s})\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^0,\end{aligned} \tag{7.38}$$

where $\mathbf{r}_{b1},\mathbf{r}_{b2}$ are the positions of the two neutrons with respect to the b core. It can be shown to be

$$\theta_0^0(\mathbf{r},\mathbf{s}) = \frac{u_{l_i j_i}(r_{b1})u_{l_i j_i}(r_{b2})}{4\pi}\sqrt{\frac{2j_i+1}{2}}P_{l_i}(\cos\theta_i), \tag{7.39}$$

where θ_i is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. ~~If we neglect~~ Neglecting the spin–orbit term in the optical potential, as is usually done for heavy ion reactions, ~~we obtain~~ one obtains,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{\mu_f\mu_i}{16\pi^2\hbar^4k_i^3k_f}|T_l^{j_i,j_f}(\theta)|^2, \tag{7.40}$$

~~with~~ where

$$\begin{aligned}T_l^{j_i,j_f}(\theta) = &\sum_l(2l+1)P_l(\cos\theta)\sqrt{(2j_i+1)(2j_f+1)}\exp[i(\sigma_l^p+\sigma_l^t)] \\ &\times\int dR\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,R\sin\beta\,u_{l_i j_i}(r_{b1})u_{l_i j_i}(r_{b2}) \\ &\times u_{l_f j_f}(r_{A1})u_{l_f j_f}(r_{A2})V(r_{b1})P_\Lambda(\cos\theta_{12})P_l(\cos\theta_\zeta) \\ &\times r_{12}r_{b1}r_{b2}P_{l_i}(\cos\theta_i)\frac{f_l(\zeta)g_l(R)}{\zeta},\end{aligned} \tag{7.41}$$

~~where~~ $\mathbf{r}_{A1},\mathbf{r}_{A2}$ ~~are~~ being the ~~positions of the two~~ coordinate of the two transferred neutrons with respect to the A core.

Heavy ion Reactions

~~heavy–ion version~~

The distorted waves for a reaction taking place between spinless nuclei are

$$\psi^{(+)}(\mathbf{r}_{Aa},\mathbf{k}_{Aa}) = \sum_l \exp\left(i\sigma_l^i\right)g_lY_0^l(\hat{\mathbf{r}}_{aA})\frac{\sqrt{4\pi(2l+1)}}{k_{aA}r_{aA}}, \tag{7.42}$$

and

$$\psi^{(-)}(\mathbf{r}_{bB},\mathbf{k}_{bB}) = \frac{4\pi}{k_{bB}r_{bB}}\sum_{\tilde{l}} i^{\tilde{l}}\exp\left(-i\sigma_{\tilde{l}}^f\right)f_{\tilde{l}}^*(r_{bB})\sum_m Y_m^{\tilde{l}*}(\hat{\mathbf{k}}_{bB})Y_m^{\tilde{l}}(\hat{\mathbf{r}}_{bB}). \tag{7.43}$$