København 11/03/2018 (1)

App. 3.E Coherent state (PRC **87**, 054321 (2013))

The BCS ground state can be written as,

$$|BCS(\phi)\rangle_{\mathcal{K}} = \prod_{\nu>0}(U_\nu + V_\nu a^\dagger_\nu a^\dagger_{\bar\nu})|0\rangle = \prod_{\nu>0} U_\nu\left(1+\frac{V_\nu}{U_\nu}a^\dagger_\nu a^\dagger_{\bar\nu}\right)|0\rangle$$
$$=\left(\prod_{\nu>0}U_\nu\right)\left(\prod_{\nu>0}(1+c_\nu P^\dagger_\nu)|0\rangle\right); \quad c_\nu = \frac{V_\nu}{U_\nu}; \; P^\dagger_\nu = a^\dagger_\nu a^\dagger_{\bar\nu} \qquad (3.E.1)$$

where

$$c_\nu = \frac{U_\nu}{V_\nu} \quad \text{and} \quad P^\dagger_\nu = a^\dagger_\nu a^\dagger_{\bar\nu} \qquad (3.E.2)$$

$\nu = 1,2$ (two pairs):

$$\prod_{\nu>0}(1+c_\nu P^\dagger_\nu) = (1+c_1P^\dagger_1)(1+c_2P^\dagger_2) = 1 + c_1P^\dagger_1 + c_2P^\dagger_2 + c_1c_2P^\dagger_1P^\dagger_2$$
$$= 1 + \sum_{\nu>0} c_\nu P^\dagger_\nu + \frac{1}{2!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^2, \qquad (3.E.3)$$

where use has been made of

$$(c_1P^\dagger_1 + c_2P^\dagger_2)^2 = 2c_1c_2P^\dagger_1P^\dagger_2 \quad (3.E.4)$$

and of the fact that

$$(P^\dagger_1)^2 = (P^\dagger_2)^2 = 0, \qquad [P^\dagger_i, P^\dagger_j] = 0 \quad (3.E.5)$$

$\nu = 1,2,3$ (3 pairs)

$$\prod_{\nu>0}(1+c_\nu P^\dagger_\nu) = (1+c_3P^\dagger_3)(1+c_2P^\dagger_2)(1+c_1P^\dagger_1) = (1+c_3P^\dagger_3)(1+c_1P^\dagger_1+c_2P^\dagger_2+c_1c_2P^\dagger_1P^\dagger_2)$$
$$= 1 + (c_1P^\dagger_1 + c_2P^\dagger_2 + c_3P^\dagger_3) + (c_1c_2P^\dagger_1P^\dagger_2 + c_1c_3P^\dagger_1P^\dagger_3 + c_2c_3P^\dagger_2P^\dagger_3) + c_1c_2c_3P^\dagger_1P^\dagger_2P^\dagger_3$$
$$= 1 + \sum_{\nu>0}c_\nu P^\dagger_\nu + \frac{1}{2!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^2 + \frac{1}{3!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^3 \qquad (3.E.6),$$

where use has been made of the relations (3.E.5)

$$[(a+b+c)(a+b+c)] = ab + ac + ba + bc + ca + cb = 2ab + 2ac + 2bc \qquad a^2=b^2=c^2=0$$
$$(a+b+c)[(a+b+c)(a+b+c)] = 2abc + 2bac + 2cab = 6abc. \qquad (3.E.7)$$

Thus,

$$\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^3 = 6c_1c_2c_3P^\dagger_1P^\dagger_2P^\dagger_3 = 3!\,c_1c_2c_3P^\dagger_1P^\dagger_2P_3 \qquad (3.E.8)$$

Making use of

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots \quad ; \quad c_\nu = e^{-2i\phi}c'_\nu \qquad (3.E.9),$$

one can write

$$|BCS(\phi)\rangle_{\mathcal{K}} = \left(\prod_{\nu>0}U_\nu\right)\left\{1 + \frac{1}{1!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right) + \frac{1}{2!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^2 + \frac{1}{3!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^3 + \dots\right\}|0\rangle$$
$$= \left(\prod_{\nu>0}U'_\nu\right)\left(1 + \frac{e^{-2i\phi}}{1!}\left(\sum_{\nu>0}c'_\nu P^\dagger_\nu\right) + \frac{e^{-4i\phi}}{2!}\left(\sum_{\nu>0}c'_\nu P^\dagger\right)^2 + \frac{e^{-6i\phi}}{3!}\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)^3 + \dots\right\}|0\rangle \qquad (3.E.10)$$

where

$$c_\nu = e^{-2i\phi}c'_\nu, \quad c'_\nu = V'_\nu/U'_\nu. \qquad (3.E.11)$$

Thus,

$$|BCS(\phi)\rangle_{\mathcal{K}} = \left(\prod_{\nu>0}U'_\nu\right)\sum_{N\,\text{even}}\frac{e^{-iN\phi}}{(N/2)!}\left(\sum_{\nu>0}c'_\nu P_\nu\right)^{N/2}|0\rangle = \left(\prod_{\nu>0}U_\nu\right)\exp\left(\sum_{\nu>0}c_\nu P^\dagger_\nu\right)|0\rangle \qquad (3.E.12)$$

and

$$|N_0\rangle = \int d\phi\, e^{iN_0\phi}|BCS(\phi)\rangle_{\mathcal{K}} = \left(\prod_{\nu>0}U'_\nu\right)\sum_{N\,\text{even}}\int d\phi\, e^{iN_0\phi}\frac{e^{-iN\phi}}{(N/2)}\left(\sum_{\nu>0}c'_\nu P^\dagger_\nu\right)^{\frac{N}{2}}|0\rangle$$
$$\sim \left(\sum_{\nu>0}c'_\nu P^\dagger_\nu\right)^{N_0/2}|0\rangle \qquad (3.E.13)$$

is the member with N_0 particles of the pairing rotational band, while

$$\Big(\sum_{\nu>0} c_\nu' P_\nu^\dagger\Big)|0\rangle \qquad (3.E.14)$$

is the Cooper pair state. Because $U_\nu' \to 0$ for $\varepsilon_\nu \ll \varepsilon_F$, (3.E.14) is to be interpreted to be valid for values of ε_ν close to ε_F.

Making use of the single j-shell model

$$V' = \sqrt{\frac{N}{2\Omega}}\,, \quad U' = \sqrt{1-\frac{N}{2\Omega}}\,, \qquad (3.E.15)$$

and

$$\frac{V'}{U'} = \sqrt{\frac{N}{2\Omega - N}} \approx U'V' \qquad (3.E.16)$$

for a number of particles considerably smaller than the full degeneracy of the single-particle subspace in which nucleons can correlate, that is for $N \ll 2\Omega$. Consequently

$$|\tilde{0}\rangle = \frac{1}{\sqrt{\mathcal{N}}}\sum_{\nu>0}(\alpha_0')_\nu\, P_\nu^\dagger|0\rangle\,, \qquad (3.E.17).$$

where

$$(\alpha_0')_\nu = \langle BCS|P_\nu'^\dagger|BCS\rangle = U_\nu' V_\nu' \qquad (3.E.18)$$

and

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