

that is, in the pure two-particle configuration $|j_k^2(0)\rangle$ describing two nucleons moving in time reversal states of the close shell system $|0\rangle$. If one was able to disentangle the γ and σ dependence of $a^{(1)}$ (as well as that of $a^{(2)}$ and $a^{(NO)}$, see above) from its formfactor dependence, the comparison between the quantities $\sum_n |c^{(n)}|^2$, $\sum_k |X_k^{(n)}|^2$ and $\sum_i |Y_i^{(n)}|^2$ could eventually be phrased in terms of exact sum rules. This not being the case, one has to deal with approximate TNTR sum rules. With this proviso in mind, TNTR sum rules are nonetheless quite useful (see also Sect. 6.2, 3; also end of Sect. 1, 2), summing up, pairing

1.7 Nuclear Field Theory for pedestrians

Nuclear Field Theory (NFT) was tailored after Feynman's graphical version of quantum electrodynamics (QED). It is then natural that in discussing NFT analogies with QED will be recurrent. Pairing is important in atomic nuclei, and Cooper

pairs play a central role both in open and in closed shell nuclei. In the first case through condensation and superfluidity, closely connected with the observation of pairing rotational bands. In the second case, through collective pairing vibrations which change particle number in two⁴⁸. It is then also natural that references to BCS and Cooper condensation and tunneling are widely used in this monograph⁴⁹.

Arguably, as a consequence of special relativity which put an end to the concept of ether, the field-free and matter-free vacuum was rightly considered as *bona fide* empty space. The advent of quantum mechanics changed this situation, the vacuum becoming populated. In quantum mechanics an oscillator, for example, cannot be at rest. The oscillatory nature of the radiation field requires zero point fluctuations (ZPF) of the electromagnetic fields in the vacuum state of lower energy. The occupation of the negative kinetic energy electron states and the subsequent calculation of the cross section for pair creation by photons in the Coulomb field of atomic nuclei, (the nuclear particle-hole excitations), contribute another step in the understanding of the QED vacuum, *let alone the Lamb shift*.

When the fields are expressed in terms of creation and annihilation operators, the fermion and boson fields in its simple form consists on the product of two fermion creation or destruction operator a^\dagger or a and one boson operator Γ^\dagger or Γ : e.g. $a_\nu^\dagger a_\nu \Gamma_\alpha^\dagger$, (see Fig. 1.7.1). That is, bilinear in the fermion fields and linear in the boson fields.

⁴⁸For details concerning pairing rotations and vibrations we refer to Bès and Broglia (1977) and Brink, D. and Broglia (2005) Chs. 4 and 5 respectively, and refs. therein.

⁴⁹Within this context see Broglia, R. A. and Zelevinsky, V. (2013)

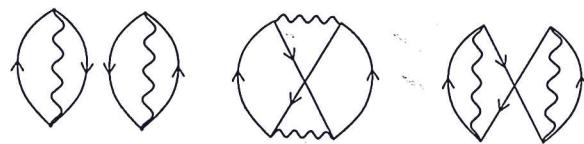


Figure 1.7.1: Oyster diagrams describing the correlation of the nuclear ground state associated with the ZPF of a collective particle-hole excitation is built, and Pauli principle correction processes in which fermions are exchanged.

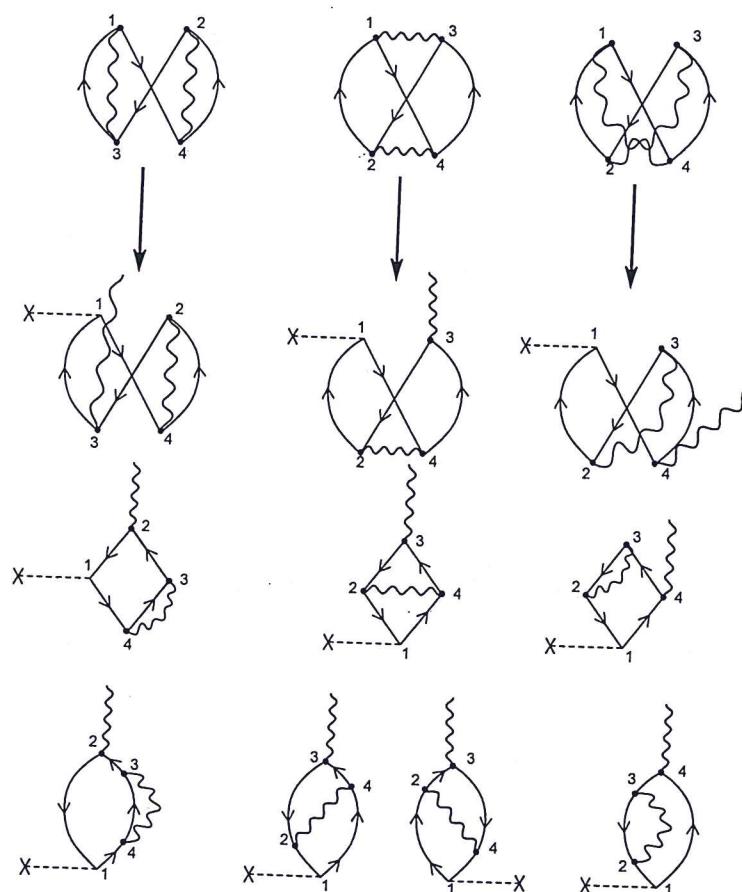


Figure 1.7.2: Some of the possible outcomes resulting from acting with a single-particle field, e.g. that associated with inelastic processes (represented by a horizontal dashed line starting with a \times), on the Pauli corrected ZPF oyster diagrams associated with collective ($p - h$) excitations of the nuclear vacuum (see also Fig. 1.7.1). It is of notice the care with which the contribution of the collective states have to be selected, only ones which can be accurately calculated. Within this context one returns to the question of empirical renormalization mentioned in the text (see Sect. 1.4, cf. also Idini et al. (2015), Broglia et al. (2016), Barranco et al (2017)).

Physical insight to be found at the basis of renormalized (NFT) of structure and reaction (see e.g. Sect. 4.2.2).

1.7. NUCLEAR FIELD THEORY FOR PEDESTRIANS

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A detailed graphical NFT treatment of the vacuum has an important consequence concerning the probing of nuclear structure with reactions. By intervening it with an external field one will excite the modes whose properties can be eventually compared with experiment without further ado.

In other words, if one is in doubt of which are the properly dressed elementary, physical modes of excitation, do not study them, or calculate them and then compare the results with the experimental data. This comes after. One should first find out how to specifically excite the mode in question, by acting with an external field on the ZPF of the vacuum. That is, by carrying out a *gedanken*, NFT-like *experiment* as in Fig. 1.7.2 for *p-h*-vibrations and in Fig. 1.7.3 regarding pairing vibrations. Because the corresponding processes deal with physical states, they translate with ease into a laboratory setup. In keeping with the fact that the vacuum contains all the information (right physical degrees of freedom) of the quantal system under study, forcing virtual processes associated with vacuum ZPF to become real, one is guaranteed to get, in each instance, the real, dressed, physical particle. ~~Of course, once the *gedanken eksperiment* has provided this information, one should use such properly renormalized modes, in all the rest of the calculations, at the risk of neglecting relevant physics.~~

Let us now provide a short introduction of NFT for pedestrians and see how the above considerations become concretely implemented⁵⁰

1.7.1 The concept of elementary modes of excitation⁵¹

The Hamiltonian of a many-body system of noninteracting particles, bosons or fermions, can be written as

$$H = \sum_i H_i, \quad (1.7.1)$$

where the summation is over all the particles of the system and where each H_i depends only on the variables of the i -th particle. The single-particle Schrödinger equation is

$$H_i \psi_k(\mathbf{r}_i) = \epsilon_k \psi_k(\mathbf{r}_i), \quad (1.7.2)$$

where ϵ_k is the single-particle energy eigenvalue and

$$\psi_k(\mathbf{r}_i) \equiv \langle \mathbf{r}_i | a_k^\dagger | 0 \rangle \quad (1.7.3)$$

is the corresponding wave function. The operator a_k^\dagger creates a particle in the state k when acting in the vacuum state $|0\rangle$. The energy levels of the system are given by the equation

$$E_n = \sum_k n_k \epsilon_k, \quad (1.7.4)$$

⁵⁰For details we refer to Bortignon, P. F. et al. (1977) and refs. therein.
⁵¹Bès and Broglia (1977)

(B) - (B)

From p. 8 From bare to renormalized
see next page (51a)

(see version
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1.7. NUCLEAR FIELD THEORY FOR PEDESTRIANS

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is the operator which specifically excites the eigenstates described by $\psi_\tau(\xi)$. Because the excitation energies E_m and observables $|\langle m' | O | m \rangle|^2$ (e.g. two-particle transfer cross-section, electromagnetic-transition probabilities, etc.) are linear combinations of ϵ_τ and A_τ , respectively, the eigenstates with energy ϵ_τ and associated observable A_τ are called the *elementary excitations of the system*.

There lie thus two ideas behind the concept of elementary excitations⁵². First, there is the idea that the total binding energy, although an important quantity by itself, does not have much to do with the behaviour of the physical system. Thus, the state $|0\rangle$ is assumed to exist but to act only as a background whose detailed intrinsic structure one does not need to know to describe the behaviour of the system. What is important is the behaviour of the lower excited states relative to the ground state. The second idea is that the low-lying states often are of a particular simple character, and are amenable to a simple and rigorous mathematical treatment. With the help of experimental probes which couple weakly to the nucleus, i.e. in such a way that the system can be expressed in terms of the properties of the excitation in the absence of probes, it has been possible to identify the following elementary excitations in systems around closed shells.⁵³

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- a) single particle and holes,
- b) shape vibrations,
- c) spin and isospin vibrations and charge exchange modes,
- d) pairing vibrations.

Away from closed shells one has to add to the above modes:

- e) rotations in 3D-space (e.g. quadrupole rotations)
- f) rotations in gauge space (pairing rotations).

Different probes have been utilized in the process of the identification of the different modes. In particular two-neutron transfer reactions induced by tritons and protons have played a central role in unraveling the basic features of the pairing modes.

1.7.2 NFT rules and applications

A field theory can be formulated in which the nuclear elementary modes of excitation play the role of the free fields and in which their mutual interweaving

⁵²This concept was first introduced by Landau (Landau, 1941) to describe the spectrum of HeII. It was subsequently utilized in nuclear physics by Bohr and Mottelson (Bohr and Mottelson (1969); Bohr A. and Mottelson (1975)) to obtain a unified description of the nuclear spectrum.

⁵³The restriction to closed shell nuclei is made to simplify the discussion. The concept of elementary modes of excitation applies equally well to open-shell nuclei.

V. ELEMENTARY MODES OF EXCITATION: EMPIRICAL RENORMALIZATION IN STRUCTURE AND REACTIONS

(B)

(provide)

(giving)

The elementary modes of excitation of a many-body system represent a generalization of the idea of normal modes of vibration. They ~~constitute~~ the building blocks of the excitation spectra, ~~providing~~ insight into the deep nature of the system one is studying, aside from allowing for an economic description of complicated spectra in terms of a gas of, as a rule, weakly interacting bosons and fermions. In the nuclear case they correspond to clothed particles and empirically renormalised vibrations (rotations).

There lie two ideas behind the concept of elementary modes of excitation. First, that one does not need to be able to calculate the total binding energy of a nucleus to accurately describe the low-energy excitation spectrum, in much the same way in which one can calculate the normal modes of a metal rod not knowing how to calculate its total cohesive energy. The second idea is that low-lying states ($\hbar\omega \ll \epsilon_F \ll BE$) are of a particularly simple character, and are amenable to a simple treatment, their interweaving being carried out at profit, in most cases, in perturbation theory¹. Within this context it is necessary to have a microscopic description of the ground state of the system which ensures that it acts as the vacuum state $|\tilde{0}\rangle$ of the elementary modes of excitation. In other words $a_\nu |\tilde{0}\rangle = 0$, $\Gamma_\alpha |\tilde{0}\rangle = 0$, where $a_\nu^\dagger |\tilde{0}\rangle = |\nu\rangle$ and $\Gamma_\alpha^\dagger |\tilde{0}\rangle = |\alpha\rangle$ represent a single-particle and a one-phonon state. This implies, in keeping with the indeterminacy relations $\Delta x \Delta p \geq \hbar/2$, that $|\tilde{0}\rangle = |0\rangle_F |0\rangle_B$ displays quantal zero point fluctuations (ZPF). 1.E

Within the framework of nuclear field theory (NFT) used below, in which single-particle (fermionic, F) and vibrational (bosonic, B) elementary modes of excitation are to be calculated within the framework of HFB and QRPA respectively, $|\tilde{0}\rangle$ must display the associated ZPF (cf. App. D). In particular for (harmonic) vibrational modes $\Delta x \Delta p = \hbar/2$, the associated zero point energy amounting to $\hbar\omega/2$ for each degree of freedom, e.g. $5\hbar\omega/2$ for quadrupole vibrations, $\hbar\omega$ being the energy of the collective vibrational mode under consideration. (renormalization³)

An illustrative example of the above arguments is provided by the low-lying quadrupole vibrational state of ^{120}Sn . Diagonalizing SLy4 in QRPA leads to a value of $B(E2)$ (890 $e^2 \text{ fm}^2$) which is about a factor of 2 smaller than experimentally observed (2030 $e^2 \text{ fm}^2$). Taking into account renormalisation effects in NFT, namely in a conserving approximation (self-energy and vertex corrections, generalised Ward identities), one obtains a value (2150 $e^2 \text{ fm}^2$), which essentially coincides with the experimental findings. One does not know how to accurately calculate the absolute ground state energy E_0 (total binding energy) of e.g. ^{120}Sn , but one can do pretty well to work out the properties of the low-energy mode of this nucleus, also the collective energies $\hbar\omega_L = E_L - E_0$, and thus the associated ZPF and zero point energy E_0 , by renormalizing QRPA solutions to lowest order through self-energy and vertex corrections contributions². Now, if the collective phonons are not the main object of the study, but are to be used to cloth the single-particle states and give rise to the induced pairing interaction, one can make use of phonons which account for the experimental findings (empirical renormalization¹⁷, see also [45, 46]). B to P. 51

It is to be noted that in calculating the $E\lambda$ lifetimes, e.g. the quadrupole lifetime associated with the low-lying quadrupole mode ($T(E2) = 1.22 \times 10^9 \times E_\gamma^5 \times B(E2)$, $E_\gamma = \hbar\omega_{2+}$), the kinematic (E_γ^5) and structure ($B(E2)$) contributions can be treated separately. This is in keeping with the fact that in the case of electromagnetic decay as well as of anelastic processes, the relative motion coordinate is always that of the entrance channel, at variance with particle transfer processes. Consequently, in connections with these processes, structure and reactions are treated separately, a possibility not operative in the case of transfer reactions. Let us extend this discussion to particle transfer process. In particular, to the two-particle pickup reaction $^{120}\text{Sn}(p,t)^{118}\text{Sn}(\text{gs})$. In this case, and to be able to calculate the radial dependence of successive transfer, everything has to be translated in terms of single-particle motion and associated absolute separation energies and radial wave functions in systems with different relative coordinates.

If the k -mass connected with the Perey-Buck energy-dependent term [19] already made the concept of a single mean field potential somewhat illusory (App. E), consider the difficulties one is confronted with in attempting at translating into a single-particle motion description inside a common potential, independent motion of Cooper pairs, composite bosonic particles with binding energies of the order of one tenth of the Fermi energy² ($\approx 2\Delta/\epsilon_F \approx 3 \text{ MeV}/36 \text{ MeV}$) and a correlation length of tens of fm, subject to a strong external field of radius $R_0 \approx 6 \text{ fm}$ and depth $\approx 50 \text{ MeV}$. A way out to this situation is provided by the fact that in superfluid nuclei, one is not very far from an independent particle picture. As a consequence, no major errors are introduced in treating the system accordingly. Also in keeping with the fact that transfer takes place through the single-particle field [14].

Summing up, while one does not know how to calculate the mass of the nucleus, one can accurately calculate $U_j(118)V_j(120)$, as well as the relative value of the clothed single-particle energies. In keeping with the fact that

¹More precisely, and in keeping with the fact that boson degrees of freedom have to decay through linear particle-vibration coupling vertices into their fermionic components to interact with another vibrational mode, the interweaving between the variety of many-body components clothing a single-particle state or a collective vibration will be described at profit in terms of an arrowed matrix which, assuming perturbation theory to be valid, can be transformed, neglecting contributions of the order of g_{pv}^3 or higher, into a co-diagonal matrix, namely a matrix whose non-zero elements are $(i, i-1)$ and $(i, i+1)$, aside from the diagonal ones (i, i) .

²Within this context we note that in ^{120}Sn the two-neutron-separation energy is $S_{2n} = 15.6 \text{ MeV}$, while $S_{1n} = 9.1 \text{ MeV}$, i.e. $(2 \times S_{1n}) - S_{2n} = 2.6 \text{ MeV}$.

>2) Barranco et al (2004)

>3) Idini et al (2015), Broglia et al (2016), Barranco et al (2017)

Appendix D. NFT vacuum polarization

The role zero point fluctuations play in the nuclear ground state, i.e. in the NFT vacuum can be clarified by relating it to the polarisation of the QED vacuum. Let us briefly dwell on the "reality" of such phenomenon, by recalling the fact that to the question of Rabi of whether the polarisation of the QED vacuum could be measured [41] - in particular the change in charge density felt by the electrons of an atom, e.g. the electron of a hydrogen atom, due to virtual creation and annihilation of electron-positron pairs - Lamb gave a quantitative answer, both experimentally and theoretically [42, 43]. The corresponding correction (Lamb shift) implies that the $2s_{1/2}$ level lies higher than the $2p_{1/2}$ level by about 1000 megacycles/s as experimentally observed. 1,4,2(f)

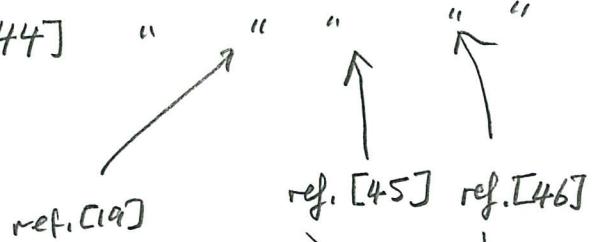
In connection with the discussion of Feynman of vacuum polarisation, where a field produces a pair, the subsequent pair annihilation producing a new field, namely a close loop, he implemented in his space-time trajectories Wheeler's idea of electrons going backwards in time (positrons). Such trajectories would be like an N in time, that is electrons which would back up for a while, and go forward again. Being connected with a minus sign, these processes are associated with Pauli principle in the self-energy of electrons (see Fig. 4(e)). The divergences affecting such calculations could be renormalised by first computing the self-energy diagram in second order and finding the answer which is finite, but contains a cut-off to avoid a logarithmic divergence. Expressing the result in terms of the experimental mass, one can take the limit (cut-off $\rightarrow \infty$) which now exists. Concerning radiative corrections to scattering, in particular that associated with the process in which the potential creates an electron-positron pair which then reannihilates, emitting a quantum which scatters the electron, the renormalisation procedure should be applied to the electric charge, introducing the observed one (Bethe and Pauli, see [44]). ****

****) In the nuclear case, for example Skyrme effective interactions give rise to particle-vibration coupling vertices which, because of the contact character of these interactions may lead to divergent zero point energies, unless a cut-off is introduced. The Gogny force being finite range does not display such problems. Nonetheless, the associated results concerning zero point energies may not be very stable and/or accurate carrying out a complete summation over both collective and non collective contributions. In this case one can eliminate such a problem by going to higher orders in the oyster diagrams (see Fig. 4(f)(a)). The fermion exchange between two of these diagrams (Pauli principle) essentially eliminates all of the non-collective contributions, leading to accurate results.

An economic and quite reliable method to achieve a similar result, is that of using empirical renormalisation. That is, to calculate the lowest order diagrams but introducing, in the intermediate states, the dressed physical (empirical) states ([46, 47]; see also [48]).

1,4,2(a)

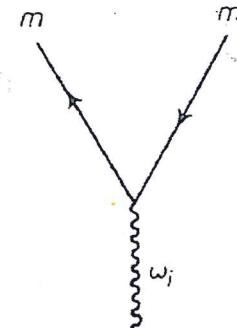
- *) A. Pais & J. Inward bound, ... ref [41] From bare to renormalized...
- **) W. Lamb and R. Rutherford ref. [42], " " " " "
- N. M. Kroll and W. Lamb ref. [43] " " " " "
- **) R. P. Feynman ref [44] " " " " "



Mahaux et al (1985)

(Broglia et al (2016), Barranco et al (2017))

****) Cet alone the fact that the velocity dependent component of these forces weaken the PVC vertices leading to poorly collective low-lying vibrations, and to equally poor clothed valence states. The question emerges of which are the provisos to be taken in the use of effective forces to higher orders of the PVC. Within this context cf. [49], also [45, 46] concerning the implementation of renormalization in both configuration and 3D-spaces within the framework of NFT. In a nutshell, the bare mean field exists but its properties cannot be measured (not any more than the bare electron mass in renormalized quantum electrodynamics), and corresponds to a set of parameters of a Fermi-like function which ensure that the clothed states reproduce all of the experimental findings, both structure and reaction.



✓ Figure 1.7.4: Graphical representation of the amplitude of the collective phonon (wavy line) on a given particle-hole excitation $((m, 1), (m, m - 1))$. This amplitude can be written in terms of the interaction vertex denoted by Λ_i , and the energy denominator $\omega_i - \epsilon'$. The particles (holes) are depicted by upward (downward) going arrowed lines.

Field-theoretical solutions

bare
The free nuclear and fermion fields are the elementary modes of excitation comprising surface vibrations and single particles, respectively. The boson fields are defined through the random-phase approximation, in terms of particle-hole excitations. The basis utilized to describe the nuclear systems is a product of the different free fields. The closed-shell system of the schematic model under consideration corresponds to the lowest ($\sigma = -1$) level filled with Ω particles, while the upper ($\sigma = 1$) level remains empty. The basis particle and hole states are obtained by adding or removing a single particle to/from this closed-shell configuration. The corresponding wave functions and energies, which should include the Hartree-Fock corrections (see Fig. 1.2.1 (a)-(c)) generated by the residual interaction⁵⁶, are

$$\begin{cases} |m, 1\rangle = a_{m,1}^\dagger |0\rangle, & E(m, 1) = \frac{1}{2}(\epsilon + V), \\ |m, -1\rangle = a_{m,-1}^\dagger |0\rangle, & E(m, -1) = \frac{1}{2}(\epsilon + V). \end{cases} \quad (1.7.17)$$

Thus the unperturbed energy for producing a particle-hole excitation with respect to the ground state is

$$\epsilon' = E(m, 1) + E(m, -1) = \epsilon + V. \quad (1.7.18)$$

The contribution V in (1.7.17) is the Hartree-Fock contribution to the particle-hole excitation.

⁵⁶The Hartree-Fock energy associated with the Hamiltonian (1.7.13) can be obtained from the linearization relation $[H, a_{\sigma,m}^\dagger] = E(m, \sigma) a_{\sigma,m}^\dagger$ acting on the Hartree-Fock vacuum, which in this case coincides with the single-particle vacuum defined by $a_{m,-1}^\dagger |0\rangle = a_{m,1}^\dagger |0\rangle = 0$

If we define the creation operator of the normal modes as

$$\beta_\nu^\dagger = \sum_m \lambda_m^\nu a_{m,1}^\dagger a_{m,-1}, \quad (1.7.19)$$

the linearization equation

$$[H, \beta_\nu^\dagger] = \omega_\nu \beta_\nu^\dagger \quad (1.7.20)$$

yields

$$\begin{cases} \omega_1 = \epsilon' - V\Omega, \\ \omega_\nu = \epsilon' \quad (\nu = 2, 3, \dots, \Omega). \end{cases} \quad (1.7.21)$$

Utilizing (1.7.20) and the normalization condition

$$\textcircled{f} \quad \xrightarrow{\quad} \quad [\beta_\nu, \beta_{\nu'}^\dagger] = \delta(\nu, \nu'), \quad (1.7.22)$$

we obtain for the amplitudes associated with the lowest mode

$$\lambda_m^1 = \frac{1}{\sqrt{\Omega}}. \quad (1.7.23)$$

One can also write this amplitude as the ratio between a coupling matrix element and an energy denominator, i.e.

$$\lambda_m^1 = \frac{\Lambda_1}{\omega_1 - \epsilon'} \quad (1.7.24)$$

From (1.7.21), (1.7.23) and (1.7.24) We obtain

$$\Lambda_1 = -V \sqrt{\Omega} \quad (1.7.25)$$

which is the strength with which a particle hole excitation $(m, 1; m, -1)$ couples to the collective phonon (see Fig. 1.7.4). This can also be seen by calculating the matrix element of the interaction Hamiltonian (1.7.15) between the normal modes and the single particle-hole state

$$\Lambda_\nu = \langle n_\nu = 1 | H_{TB} | m, 1; m', -1 \rangle = -V \sqrt{\Omega} \delta(m, m') \delta(\nu, 1). \quad (1.7.26)$$

Note that the particle-vibration coupling strengths associated with the other normal modes lying at an energy ϵ' are equal to zero. As shown in ref. [Bès et al. (1974); Broglia et al. (1976)], The exact solution of (1.7.13) is reproduced by utilizing as the basic degrees of freedom both the vibrations (see (1.7.21)) and the particles (see (1.7.17)) coupled through the interactions (1.7.16) (four-point vertex) and (1.7.26) (particle-vibration coupling). A significant part of the original interaction has already been included in generating the collective mode (1.7.21). This implies that the rules for evaluating the effect of the couplings (1.7.16) and (1.7.26) between fermions and bosons involve a number of restrictions as compared with the usual rules of perturbation theory that are to be utilized in evaluating the effect of the original interaction (1.7.15) acting in a fermion space. They read as follows:

*) Bès et al (1974), Broglia et al (1976)

where, as in (1.7.27), only the energy of the particle-hole excitation is given (see (1.7.18)). One can also displace the zero point of the odd system to $\frac{1}{2} E$, in which case the unperturbed energy of the basis states $|n_i; m, 1\rangle$ is ω_i .

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This is done in this section only in the above 2 legs. and is used throughout in Sect. 1.7.3

- I) In initial and final states, proper diagrams involve collective modes and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. This restriction permits an initial state comprising the configuration $(n_\nu = 1; m)$, but excludes $(m, 1; m : m', 1)$.
- II) The couplings (1.7.16) and (1.7.26) are allowed to act in all orders to generate the different diagrams of perturbation theory; the restriction I) does not apply to internal lines of these diagrams.
- III) The internal lines of diagrams are, however, restricted by the exclusion of diagrams in which a particle-hole pair is created and subsequently annihilated without having participated in subsequent interactions.
- IV) The energies of the uncoupled particle and phonon fields are to be calculated by utilizing the Hartree-Fock approximation (see eq. (1.7.17)) and the RPA (see eq. (1.7.21)), respectively. The contributions of all allowed diagrams are evaluated by the usual rules of perturbation theory.

We note that the external fields acting on the system are allowed to create any state which may generate the different diagrams of perturbation theory. The corresponding matrix elements should be weighted with the amplitude of the component through which the final state is excited. The above rules are also valid for those situations which cannot be treated in perturbation theory and where a full diagonalization is called for. Thus, e.g., when the system displays a spurious state (see Sect. 1.7.3). In what follows we discuss the energy of the $2p - 1h$ -like excitations. We distinguish between two types of states, namely

$$|n_i = 1; m, 1\rangle, \quad \begin{cases} \omega_1 = \epsilon' - V\Omega, & \Lambda_1 = -\sqrt{\Omega}V \\ \omega_i = \epsilon', & \Lambda_i = 0 \end{cases} \quad (i = 1; m = 1, 2, \dots, \Omega), \quad (1.7.27)$$

(i = 2, \dots, \Omega; m = 1, 2, \dots, \Omega),

simplest modes
which can display
spuriousity.

and⁵⁷

$$|m', 1; m, -1; m', 1\rangle, \quad \epsilon' \quad (m, m' = 1, 2, \dots, \Omega), \quad (1.7.28)$$

The physical states are to be written as

$$|qm\rangle = \sum n'_i |n'_i = 1; m, 1\rangle, \quad (1.7.29)$$

as (1.7.28) cannot be basis states according to rule I), but only intermediate states. The quantities ξ_{iqm} are the amplitudes of the physical state in the different components of the product basis of elementary excitations. The model space contains Ω^2 states. While the correct number is $\Omega - 1$. Thus the basis $|n_1 = 1; m, 1\rangle$ contains Ω

⁵⁷Since the states (1.7.28) are restricted to be intermediate states of the perturbation expansion, the configuration $(m, 1; m, -1; m', 1)$ is allowed.

(one for each value of m)

In other words, allowing the quantum number m to be in all possible S_L -states, the

(A) - (A) P/55
handwritten

$S_L (S_L - 1)$
P.F

(A) Rule (I) eliminates most of the double counting of two-particle, one-hole states. The model state contains Ω "proper" states, of the form $|N_i; m, 1\rangle$, in which case the odd particle is in the state $(m, 1)$. That is $|n_1; m, 1\rangle$ ($\omega_1 = \epsilon' - V\Omega$) and $|n_i; m, 1\rangle$ ($\omega_i = \epsilon'$, $i = 2, \dots, \Omega$). However, there are only $\Omega - 1$ two-particle, one-hole states in which the odd particle is in the state $(m, 1)$ (Fig. α). Therefore, a spurious state remains in the spectrum based on elementary modes of excitation.

(55a)

(A)

p. (55)

p. 55

b

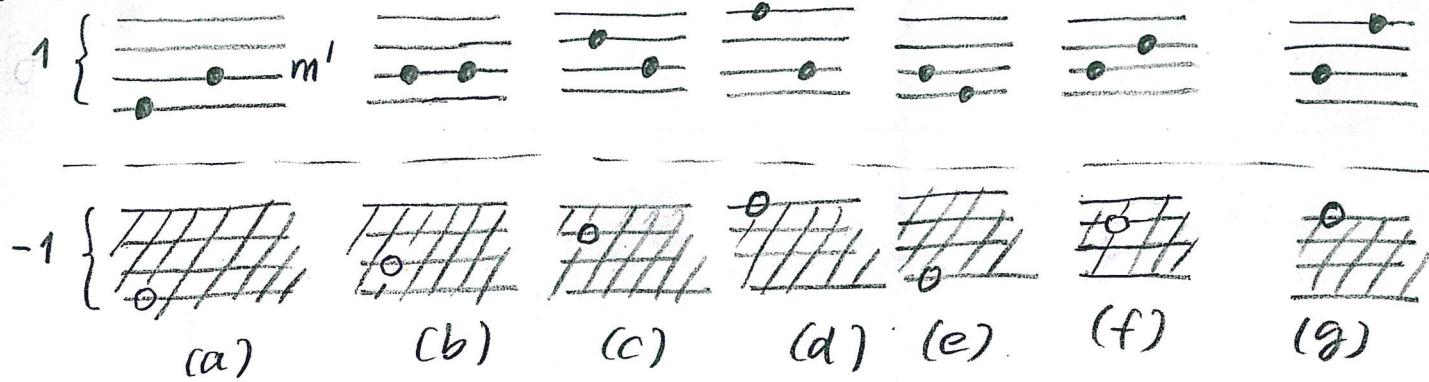


Fig. d
Schematic two-level model

Count of the states $|m, 1; m-1, m', 1\rangle$ in the case in of $j = \frac{3}{2}$ and $\Omega = 2j+1 = 4$. State (b) is not allowed because of Pauli principle. The states ((a), (e)), ((c), (f)) and ((d), (g)) are pairwise identical, in keeping with the indistinguishability of the particles. Thus, the states (a), (c) and (d) (equivalent (e), (f), (g)) exhaust the degrees of freedom of states of type (1, 7, 28). In other words, there are only $\Omega-1=3$ two-particle one-hole states in which the odd particle is in the state $(m'; 1)$.

Magnetic

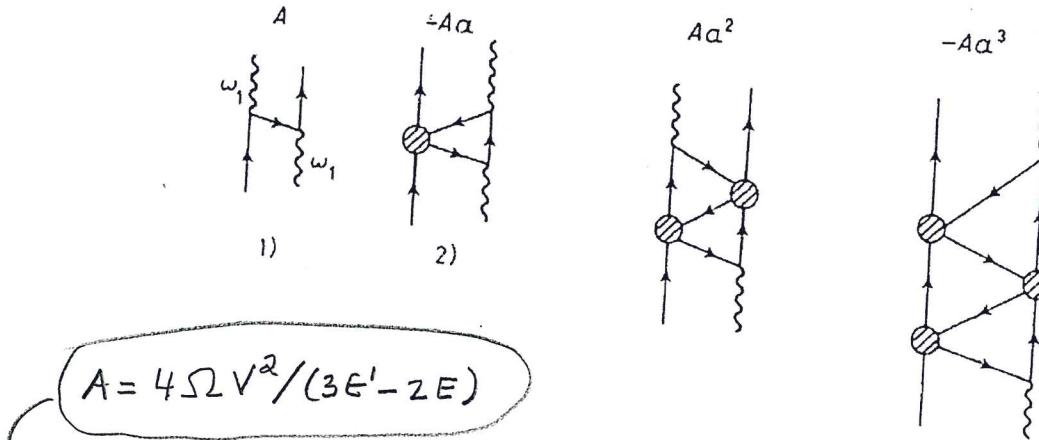


Figure 1.7.5: Contributions to the interaction of a fermion and a collective boson ω_i to order $1/\Omega^4$. The secular equation $E - E^{(0)} = A \sum_n a^n (-1)^n$ is given in terms of the quantities $A = 4\Omega V^2 3 / (3\epsilon' - 2E)$ and $a = 2V / (3\epsilon' - 2E)$.

spurious states. Its origin can be traced back to the violation of the Pauli principle (see also sect. 1.7.3). To obtain the energy of $|qm\rangle$ we have to allow the states $|n_1 = 1; m, 1\rangle$ to interact through the vertices (1.7.16) and (1.7.26) and generate all the different perturbation theory diagrams (see rule II) except those containing bubbles (see rule III)).

The different graphical contributions calculated in the framework of the Brillouin-Wigner perturbation theory are displayed in fig. 1.7.5. There is only one (diagonal) matrix element given by a single summation, which can be carried to all orders in the interaction vertices, and can be written as

$$X_{ii'} = A \sum_n (-1)^n a^n \delta(i, i') = \quad (1.7.30)$$

$$= \frac{A}{1 + a} \delta(i, i') \delta(n, 1) = -K(E) (\sqrt{\Omega} V)^2 \delta(i, i') \delta(i, 1), \quad (1.7.31)$$

where a and A are defined in the caption to the figure and

$$K(E) = \left(\frac{3}{2}\epsilon' - E + V\right)^{-1} \quad (1.7.32)$$

is the effective coupling strength. The associated secular equation

$$|(\omega_i - E)\delta(i, i') + X_{ii'}| = 0 \quad (1.7.33)$$

is equivalent to the dispersion relation

$$\frac{1}{K(E)} = \sum_i \frac{(\sqrt{\Omega} V)^2}{\omega_i - E} \delta(i, 1). \quad (1.7.34)$$

Thus the energies of the system are determined by the equation

$$E = \omega_1 + \frac{\Omega V^2}{\frac{3}{2}\epsilon' - E + V}. \quad (1.7.35)$$

It admits the two solutions

$$E_{qm} = \begin{cases} \frac{3}{2}\epsilon', \\ \frac{1}{2}\epsilon' + \omega_1 + V = \frac{3}{2}\epsilon' - \Omega V + V, \end{cases} \quad (1.7.36)$$

and agree with the exact value⁵⁸.

Because $A = 0$ for $i \neq 1$, there is no summation in (1.7.29) and

$$|qm\rangle = N_{qm}^2 |n_1 = 1; m, 1\rangle, \quad (1.7.37)$$

Where

$$1 = N_{qm}^2 \left(1 - \frac{\partial X_{11}}{\partial E} \right) = N_{qm}^2 \left(1 - \frac{\Omega V^2}{\left(\frac{3}{2}\epsilon' - E + V \right)^2} \right). \quad (1.7.38)$$

For $E_{qm} = \frac{1}{2}\epsilon' + \omega_1 + V$ we obtain

$$N_{qm}^2 = \frac{\Omega}{\Omega - 1}, \quad (1.7.39)$$

while for $E_{qm} = \frac{3}{2}\epsilon'$ the state is nonnormalizable as the quantity in parentheses in (1.7.38) is either negative ($\Omega > 1$) or zero ($\Omega = 1$). The state defined by

$$|q, m\rangle = \sqrt{\frac{\Omega}{\Omega - 1}} |n_1 = 1; m, 1\rangle, \quad (1.7.40)$$

and

$$E_{qm} = \frac{1}{2}\epsilon' + \omega_1 + V = \frac{3}{2}\epsilon' - V(\Omega - 1), \quad (1.7.41)$$

exhausts the inelastic sum rule in agreement with the exact results. Note that (1.7.40) is specifically excited in inelastic processes, as can be seen by direct inspection. The external inelastic field can act in two ways, exciting either a particle-hole pair or a phonon, with amplitudes

$$\langle m, 1; m', -1 | A_1^\dagger | 0 \rangle = \delta(m, m') \quad (1.7.42)$$

and

$$\langle n_i = 1 | A_1^\dagger | 0 \rangle = \sqrt{\Omega} \delta(i, 1) \quad (1.7.43)$$

⁵⁸The exact solutions can be easily obtained by noting that the operators A^\dagger, A and $\frac{1}{2}N_0$ are generators of the SU_2 group (see Bortignon, P. F. et al. (1977)).

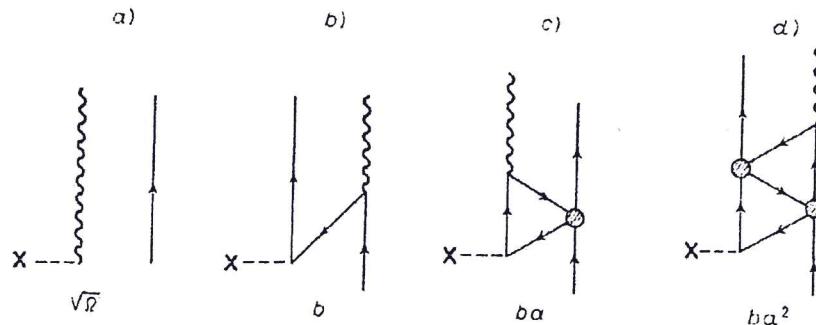


Figure 1.7.6: Graphical representation of the different terms contributing to the matrix element of the transfer operator $\sqrt{\Omega}A^\dagger$ up to order $1/\Omega^3$. Note that the different contributions b), c), etc. have a one-to-one correspondence with the different contributions to E (see fig 1.7.5). $a = -2V/(3\epsilon' - 2E)$, $b = 2\Lambda_1/(3\epsilon' - 2E)$.

respectively. The different graphical contributions to the inelastic-scattering process are displayed in Fig. 1.7.6, and can again be summed to all orders in the interaction vertices giving

$$\langle n_1 = 1; m, 1 | A_1^\dagger | m, 1 \rangle = \sqrt{\Omega} + \frac{\Lambda_1}{\frac{3}{2}\epsilon' - E_{qm} + V}. \quad (1.7.44)$$

For $E_{qm} = \frac{3}{2}\epsilon'$ this quantity is equal to zero. Thus, the corresponding states do not carry any inelastic strength, a feature which is closely related to the fact that they cannot be normalized and that they do not display any correlation energy⁵⁹. On the other hand, the matrix element associated with (1.7.40) is

$$\langle qm | A^\dagger | m, 1 \rangle = \sqrt{\frac{\Omega}{\Omega - 1}} \frac{\Omega - 1}{\sqrt{\Omega}} = \sqrt{\Omega - 1} \quad (1.7.45)$$

which agrees with the exact answer. The results (1.7.41) and (1.7.45) can be traced down to Pauli-principle corrections. In fact, the state $|n_i = 1; m, 1\rangle$ has a nonvanishing matrix element, implying a single particle-vibration coupling vertex, with the state $|m, 1; m, -1; m, 1\rangle$. This component, which is spurious, is removed by the different graphs displayed in Fig. 1.7.5 and 1.7.6. The presence of the odd particle ($m, 1$) blocks the particle-hole excitation ($m, 1; m, -1$) which was present in the uncoupled system. Thus the system increases its energy by a quantity V . The reduction of the inelastic amplitude from $\sqrt{\Omega}$ to $\sqrt{\Omega - 1}$ also indicates that there is one less particle-hole excitation responding to the external probe.

Figs.

⁵⁹Note that, even if $N(E_{qm} = \epsilon_m) \rightarrow \infty$, the matrix elements associated with the different transitions tend to zero more rapidly and the final result converges and is equal to zero as expected.

, i.e. one for each value of m

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1.7.3 Spurious states

While the model space product of elementary modes of excitation discussed in the last section contains Ω^2 states, only $\Omega(\Omega - 1)$ are physically possible, the number of spurious states being Ω . On the other hand, the agreement between the exact and the nuclear-field-theoretical results shows that the effects of those spurious states are eliminated from all the matrix elements associated with physical observables.

In what follows we show that, in fact, the spurious states are isolated in an explicit way in the nuclear field theory⁶⁰. Their energy coincides with the initial unperturbed energy, while all physical operators have zero off-diagonal matrix elements between any physical state and a spurious state, in particular the unit operator, which measures the overlap of the two types of states. For this purpose we use again a schematic model consisting in a number, Ω , of single-particle levels in which particles interact by means of a "monopole" force,

$$H = H_{sp} + H_{int}, \quad (1.7.46)$$

where

$$H_{sp} = \frac{1}{2} \sum_{m=1}^{\Omega} \epsilon_m (a_{m,1}^\dagger a_{m,1} - a_{m,-1}^\dagger a_{m,-1}), \quad (1.7.47)$$

and

$$H_{int} = -VA^\dagger A, \quad (1.7.48)$$

with

$$A^\dagger = \sum_{m=1}^{\Omega} a_{m,1}^\dagger a_{m,1}. \quad (1.7.49)$$

The energy of the i -th phonon is determined by the RPA dispersion relation (see rule IV))

$$\sum_{m=1}^{\Omega} \frac{1}{\epsilon_m - \omega_i} = \frac{1}{V}. \quad (1.7.50)$$

The eigenfunction corresponding to the different modes is

$$|n_i = 1\rangle = \sum_m \frac{\Lambda_i}{\epsilon_m - \omega_i} a_{m,1}^\dagger a_{m,-1} |0\rangle. \quad (1.7.51)$$

The particle-vibration coupling constant is given by

$$\Lambda_i = -\langle n_i = 1 | H_{int} | m, 1; m', -1 \rangle = \left[\sum_m \frac{1}{(\epsilon_m - \omega_i)^2} \right]^{-\frac{1}{2}} \delta(n, n'), \quad (1.7.52)$$

⁶⁰Broglia et al. (1976)

where $|n_i = 1\rangle$ denotes a state containing one phonon, while $|m, 1; m, -1\rangle$ is the eigenstate associated with particle-hole excitation. The other interaction to be included (rule II) is the four-point vertex which has the value

$$\langle m, 1; m', -1 | H_{int} | m'', 1; m', -1 \rangle = -V\delta(m, m')\delta(m'', m''). \quad (1.7.53)$$

The single-particle energies to be used in calculating the different graphs are $\frac{1}{2}\epsilon_m$, as the Hartree-Fock contribution (see rule IV)) of H_{int} is zero.

Similarly to H_{int} the “inelastic operator” has two different matrix elements, namely

$$\langle n_i = 1 | a_{m', 1}^\dagger a_{m', -1} | 0 \rangle = \frac{\Lambda_i}{\epsilon_{m'} - \omega_i} \quad (1.7.54)$$

and

$$\langle m', 1; m'', -1 | a_{m, 1}^\dagger a_{m, -1} | 0 \rangle = \delta(m, m')\delta(m', m''). \quad (1.7.55)$$

(see Fig. α)

In what follows we discuss again the system comprising an odd particle, in the orbit $(m, 1)$, in addition to a single phonon excitation of the vacuum. According to rule I initial and final states may involve both collective mode and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. The exclusion of the states $|m, 1; m', 1; m', -1\rangle$ eliminates most of the double counting of two-particle, one-hole states. The Ω “proper” states of the form $|n_i = 1; m, 1\rangle$ are allowed. However, there are only $\Omega - 1$ (two-particle, one-hole) states in which the odd particle is in the state $(m, 1)$. Therefore, a spurious state remains in the spectrum of the elementary modes of excitation.

If we displace the zero-point energy of the odd system to $\frac{1}{2}\epsilon_m$ the unperturbed energy of the basis state $|n_i = 1, m, 1\rangle$ is ω_i .

The lower-order corrections to this energy which do not contain bubbles are drawn in fig. 1.7.7 (I). Iterating these processes to infinite order we obtain the secular equation

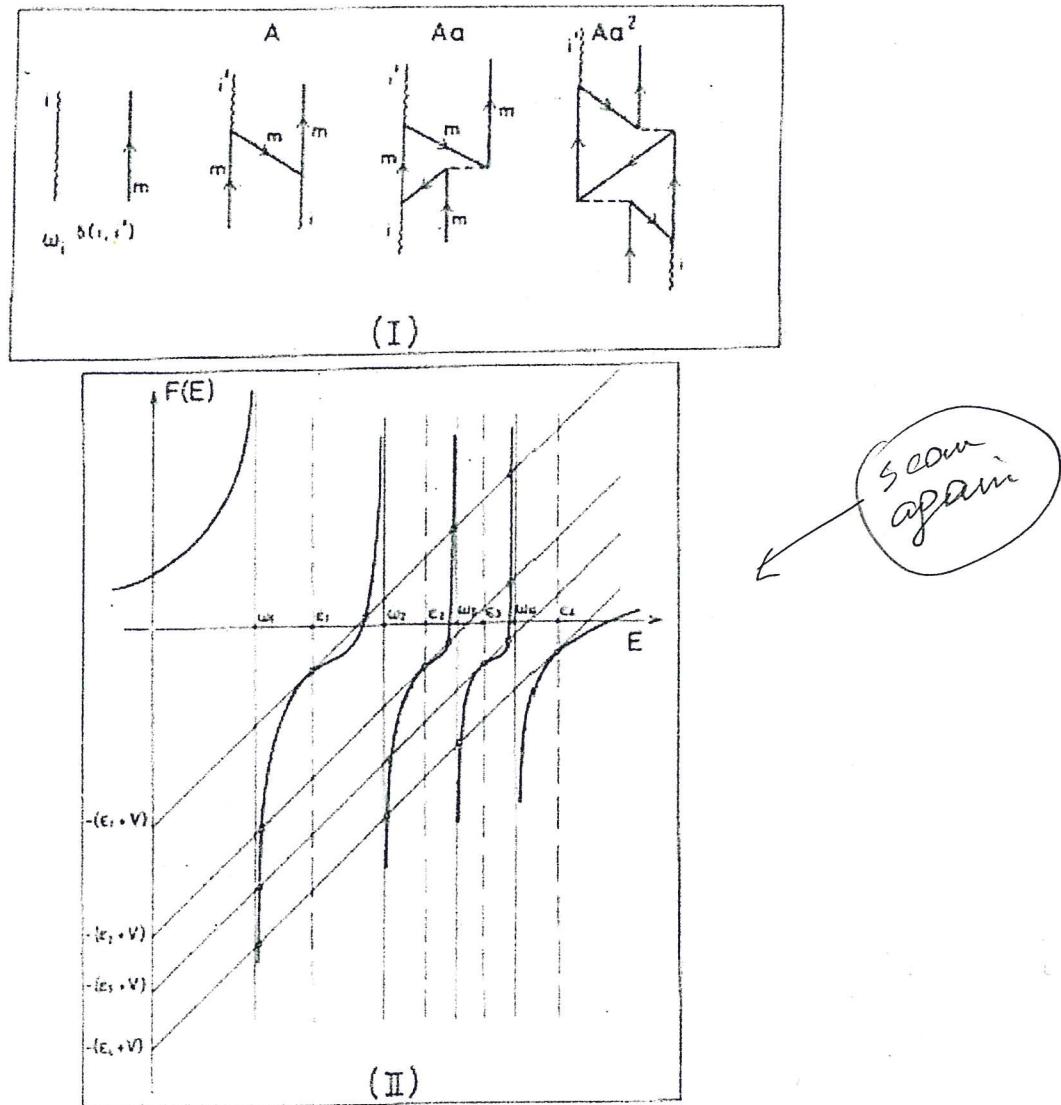
$$|(\omega_i - E)\delta(i, i') + X_{ii'}(E)| = 0, \quad (1.7.56)$$

where

$$X_{ii'} = -\frac{\Lambda_i \Lambda_{i'}}{E - \epsilon_m - V} \quad (1.7.57)$$

The different contributions calculated in the framework of the Brillouin-Wigner perturbation theory are energy dependent, and take into account renormalization effects of the states not explicitly included in the calculations. The dispersion relation fixing the energies E_m of the physical states (see App. 1.C)

$$E - \epsilon_m - V = \sum_{i=1}^{\Omega} \frac{\Lambda_i^2}{\omega_i - E} = F(E) \quad (1.7.58)$$



✓ Figure 1.7.7: Lower order contributions to the energy matrix element between the basis states $|n_i = 1; m, 1\rangle$. The dashed line stands for the model bare interaction (see eq. 1.7.53). The quantity $X_{ii'}(E) = A \sum_n a^n = -\Lambda_i \Lambda'_i / (E - \epsilon_m - V)$, where $A = -\Lambda_i \Lambda'_i / (\Omega - \epsilon_m)$ and $a = V / (\Omega - \epsilon_m)$, is the matrix element iterated to all orders in $1/\Omega$. The secular equation of the problem is $|\langle \omega_i \delta(i, i') \rangle + X_{ii'}| = 0$, and is equivalent to the dispersion relation (1.7.58). II) Graphical solution of the dispersion relation (1.7.58), for the case $\Omega = 4$. The function $F(E) = \sum_i \Lambda_i^2 / (\omega_i - E)$ is displayed as a continuous thick line, while the parallel lines $E - \epsilon_m - V$ have been drawn as thin continuous lines intersecting the ordinates axis at $-(\epsilon_m + V)$. The intersections between the two functions give the eigenvalues of the secular equation. For each value of ϵ_m there are $\Omega + 1$ roots, the root at $E = \epsilon_m$ being double.

There is one equation for each single-particle level because the monopole force cannot change the m-state of the odd particle. The relation (1.7.58) can be solved graphically as shown in Fig. 1.7.7 (II). The energy $E = \epsilon_m$ is always a root of (1.7.58), in fact a double root since

$$\left[\frac{dF(E)}{dE} \right]_{E=\epsilon_m} = \sum_i \frac{\Lambda_i^2}{(\omega_i - \epsilon_m)^2} = 1 \quad (1.7.59)$$

and the line $E - \epsilon_m - V$ is at 45° . The remaining intersections of this line and the function $F(E)$ give rise to $\Omega - 1$ additional roots denoted by (qm) , whose energy E_{qm} agrees with the physical eigenvalues obtained from the exact solution of the model. The eigenvectors associated with the physical states (qm) are

$$|qm\rangle_F = \sum_i \xi_{iqm} |i; m, 1\rangle, \quad (1.7.60)$$

where

$$\xi_{iqm} = -N_{qm} \frac{\Lambda_i}{\omega_i - E_{qm}} = \langle i; m, 1 | qm \rangle_F. \quad (1.7.61)$$

The normalization condition which determines N_{qm} is

$$\begin{aligned} {}_F \langle qm | qm \rangle_F &= 1 = \sum_{i,i'} \left(\delta(i, i') - \frac{\partial X_{ii'}}{\partial E} \right) \xi_{iqm}^* \xi_{i'qm} = \\ &= N_{qm}^2 \left[\sum_i \frac{\Lambda_i^2}{(\omega_i - E_{qm})^2} - \frac{1}{(E_{qm} - \epsilon_m - V)^2} \sum_{i,i'} \frac{\Lambda_i^2 \Lambda_{i'}^2}{(\omega_i - E_{qm})(\omega_{i'} - E_{qm})} \right] = \\ &= N_{qm}^2 \left[\sum_i \frac{\Lambda_i^2}{(\omega_i - E_{qm})^2} - 1 \right], \end{aligned} \quad \text{and defined in (1.7.57).} \quad (1.7.62)$$

where the dispersion relation (1.7.58) has been utilized, and where $X_{ii'}$ is the matrix element appearing in (1.7.56). For $E_{qm} = \epsilon_m$ the factor multiplying N_{qm}^2 is zero (see eq. (1.7.59)). Thus, there are only $\Omega - 1$ states which can be normalized when solving the Hamiltonian (1.7.46) in the framework of the nuclear field theory. The full spuriousity of the elementary-mode product basis is concentrated in a single state.⁽⁶¹⁾

The subscript F has been utilized in (1.7.60) to indicate that we are dealing with the nuclear-field solution of the Hamiltonian (1.7.46) (for simplicity it will not be used in the following). Note that these eigenvectors are expressed in terms of only the allowed initial or final states (see rule I))

$$|i; m, 1\rangle \equiv a_{m,1}^\dagger |i\rangle, \quad (1.7.63)$$

⁽⁶¹⁾ Note that the mathematical relation $N^2 f(E) = 1, N^2$, being the norm of the state with energy E , implies that such state is spurious if $f(E) = 0$ or $f(E) < 0$ (see eq. (1.7.38) and subsequent discussion).

which are assumed to form an orthonormal basis, in particular in deriving the relation (1.7.62). This is equivalent to the basic assumption of the nuclear field theory of the independence of the different modes of excitation, i.e., in the present case,

$$[\Gamma_i, a_{m,1}^\dagger] = 0. \quad (1.7.64)$$

Rules I)–IV) discussed in the last section give the proper mathematical framework to this ansatz, which has played a basic role in developing a unified theory of nuclear structure. The above discussion can be illuminated by utilizing a conventional treatment of the residual interaction. Expanding the states $|n_i = 1; m, 1\rangle$ in terms of particle and hole states, we can write, with the help of (1.7.51),

$$a_{m,1}^\dagger |n_i = 1\rangle = a_{m,1}^\dagger \sum_{m' \neq m} \frac{\Lambda_i}{\epsilon' - \omega_i} a_{m',1}^\dagger |0\rangle \quad (1.7.65)$$

The overlap between the states $|n_i = 1; m, 1\rangle$ is thus given by

$$\begin{aligned} Z(i, i') &= \langle i' | a_{m,i} a_{m,1}^\dagger | i \rangle = \leftarrow \rightarrow \langle i' | a_{m,1} a_{m,1}^\dagger | i \rangle \\ &= \sum_{m' \neq m} \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_{m'} - \omega_i)(\epsilon_{m'} - \omega_{i'})} = \delta(i, i') - \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_m - \omega_i)(\epsilon_m - \omega_{i'})}, \end{aligned} \quad (1.7.66)$$

where the orthogonality relation

$$\sum_{m'} \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_{m'} - \omega_i)(\epsilon_{m'} - \omega_{i'})} = \delta(i, i') \quad (1.7.67)$$

of the RPA solutions in the even system has been utilized. Because of the nonorthogonality of the basis, the eigenvalues of the system are determined by the relation*)

$$|Z(E)(H - E)| = 0. \quad (1.7.68)$$

This is fulfilled for

$$|H - E| = 0, \quad (1.7.69)$$

which yields the $\Omega - 1$ physical roots, as well as for

$$|Z(E)| = 0. \quad (1.7.70)$$

This solution corresponds to the spurious root $E_{qm} = \epsilon_m$. In fact,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sum_i \xi_{iqm}(E_{qm} = \epsilon_m + \delta) Z_{ii'} &= \lim_{\delta \rightarrow 0} N_{qm}(E_{qm} = \epsilon_m + \delta) \\ &\times \sum_i \frac{\Lambda_i}{\omega_i - (\epsilon_m + \delta)} \sum_{m' \neq m} \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_{m'} - \omega_i)(\epsilon_{m'} - \omega_{i'})} = 0, \end{aligned} \quad (1.7.71)$$

**) Within the context of renormalization, one first calculates for a finite value of δ and once found the finite result takes the limit.

*) See also Broglia et al (1976).

since

$$\sum_{m \neq m'} \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_{m'} - \omega_i)(\epsilon_{m'} - \omega_{i'})} = \delta(m, m'). \quad (1.7.72)$$

(c) case

Note that this solution in terms of the overlap Z gives the exact answer in the present case, because of the simplicity of the model. In a general case which includes ground-state correlations this may not be true any longer.

Before discussing the consequences of the above discussion in connection with reaction matrix elements (one-particle transfer amplitudes), let us return to (1.7.62).

The physical amplitudes ξ_{iqm} are connected to $\tilde{\xi}_{iqm}$ by the relation

dealing with

$$\xi_{iqm} = \frac{\tilde{\xi}_{iqm}}{\sqrt{N_{qm}}}. \quad (1.7.73)$$

Thus,

$$N_{qm} = \sum_{i,i'} \left(\delta(i, i') - \frac{\partial X_{ii'}}{\partial E} \right) \tilde{\xi}_{qm}^* \tilde{\xi}_{qm} = \sum_{i,i'} \tilde{M}_{ii'}^{mm} \tilde{\xi}_{qm}^* \tilde{\xi}_{qm}, \quad (1.7.74)$$

In usual perturbation theory

$$\frac{\partial X_{ii'}}{\partial E} \tilde{\xi}_{qm}^* \tilde{\xi}_{qm} < 0, \quad (1.7.75)$$

and N_{qm} is always > 1 . In the present case, however, because the matrix elements of the effective Hamiltonian have to be calculated excluding the contributions containing bubbles, the quantity

$$\sum_{ii'} \frac{\partial X_{ii'}}{\partial E} \tilde{\xi}_{qm}^* \tilde{\xi}_{qm}, \quad (1.7.76)$$

can be either positive, or negative. From the above discussion it can be concluded that N_{qm} can vanish for certain states, eliminating the redundant degrees of freedom. Examples are discussed in Sect. 1.7.4 (see also 1.C.2). ↩

We now calculate the one-particle stripping process leading to the odd system. This calculation illustrates the explicit concentration of the whole spuriousity into a single state which has zero correlation energy⁶² and zero amplitude for the different physical processes exciting the $\Omega - 1$ physical states.

One has first to calculate the amplitude for the transition to a basis component ($n = l; m, 1$) including only those graphs in which all intermediate states are excluded from appearing as initial or final states. reflects the fact that the diagonalization procedure has included all interaction effects that link these allowed states. The final amplitude for the transition to the state (qm) is obtained by summing

⁶²This is because the spurious state has zero phase space to correlate.

* Within this context see Bes et al (1976 b)
in particular App. B, footnote p. 25)

lower case
B