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Appendix 7.J

~~A~~ spherical harmonics and angular momenta

With Condon–Shortley phases

$$Y^l_m(\hat{z}) = \delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}, \quad Y^{l*}_m(\hat{r}) = (-1)^m Y^l_{-m}(\hat{r}).$$

7.J.1 ~~(220)~~

Time–reversed phases consist in multiplying Condon–Shortley phases with a factor i^l, so

$$Y^l_m(\hat{z}) = \delta_{m,0}\, i^l\sqrt{\frac{2l+1}{4\pi}}, \quad Y^{l*}_m(\hat{r}) = (-1)^{l-m} Y^l_{-m}(\hat{r}).$$

7.J.2 ~~(221)~~

With this phase convention, the relation with the associated Legendre polynomials includes an extra i^l factor with respect to the Condon–Shortley phase,

$$Y^l_m(\theta,\phi) = i^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(\cos\theta)e^{im\phi}.$$

7.J.3 ~~(222)~~

J ~~A~~.1 addition theorem

The addition theorem for the spherical harmonics states that

$$P_l(\cos\theta_{12}) = \frac{4\pi}{2l+1}\sum_m Y^l_m(\mathbf{r}_1)Y^{l*}_m(\mathbf{r}_2),$$

4 ~~(223)~~

where θ_{12} is the angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. This result is independent of the phase convention. With *time–reversed phases*,

$$P_l(\cos\theta_{12}) = \frac{4\pi}{\sqrt{2l+1}}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]^0_0.$$

5 ~~(224)~~

With *Condon–Shortley phases*,

$$P_l(\cos\theta_{12}) = (-1)^l\frac{4\pi}{\sqrt{2l+1}}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]^0_0.$$

6 ~~(225)~~

7.J.2 ~~A~~.2 expansion of the delta function

The Dirac delta function can be expanded in multipoles, yielding

$$\begin{aligned}\delta(\mathbf{r}_2-\mathbf{r}_1) &= \sum_l \delta(r_1-r_2)\frac{2l+1}{4\pi r_1^2}P_l(\cos\theta_{12})\\ &= \sum_l \delta(r_1-r_2)\frac{1}{r_1^2}\sum_m Y^l_m(\mathbf{r}_1)Y^{l*}_m(\mathbf{r}_2).\end{aligned}$$

7 ~~(226)~~

This result is independent of the phase convention. With *time–reversed phases*, 8

$$\delta(\mathbf{r}_2-\mathbf{r}_1) = \sum_l \delta(r_1-r_2)\frac{\sqrt{2l+1}}{r_1^2}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]^0_0. \qquad (227)$$

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~~A.3~~ 7.J.3 coupling and complex conjugation

If $\Psi^{I_1*}_{M_1} = (-1)^{I_1-M_1}\Psi^{I_1}_{-M_1}$ and $\Phi^{I_2*}_{M_2} = (-1)^{I_2-M_2}\Phi^{I_2}_{-M_2}$, as it happens to be the case for spherical harmonics with time-reversed phases, then

$$\begin{aligned}
\left[\Psi^{I_1}\Phi^{I_2}\right]^{I*}_{M} &= \sum_{\substack{M_1 M_2\\(M_1+M_2=M)}} \langle I_1\ I_2\ M_1\ M_2|IM\rangle \Psi^{I_1*}_{M_1}\Phi^{I_2*}_{M_2} \\
&= \sum_{\substack{M_1 M_2\\(M_1+M_2=M)}} (-1)^{I-M_1-M_2}\langle I_1\ I_2\ -M_1\ -M_2|I-M\rangle \Psi^{I_1}_{-M_1}\Phi^{I_2}_{-M_2} \\
&= (-1)^{I-M} \sum_{\substack{M_1 M_2\\(M_1+M_2=M)}} \langle I_1\ I_2\ -M_1\ -M_2|I-M\rangle \Psi^{I_1}_{-M_1}\Phi^{I_2}_{-M_2} \\
&= (-1)^{I-M}\left[\Psi^{I_1}\Phi^{I_2}\right]^{I}_{-M},
\end{aligned} \qquad 9\ \cancel{(228)}$$

where we have used ~~(242)~~ (7.J.23).

Let us care now about the spinor functions $\chi^{1/2}_m(\sigma)$, which have the form

$$\chi^{1/2}(\sigma = 1/2) = \begin{bmatrix}1\\0\end{bmatrix} \quad \chi^{1/2}(\sigma = -1/2) = \begin{bmatrix}0\\1\end{bmatrix}, \qquad 10\ \cancel{(229)}$$

or

$$\chi^{1/2}_m(\sigma) = \delta_{m,\sigma}. \qquad 11\ \cancel{(230)}$$

Thus, $\chi^{1/2*}_m(\sigma) = \chi^{1/2}_m(\sigma) = \delta_{m,\sigma}$, but we can also write

$$\chi^{1/2*}_m(\sigma) = (-1)^{1/2-m+1/2-\sigma}\chi^{1/2}_{-m}(-\sigma). \qquad 11\ \cancel{(231)}$$

This trick enable us to write

$$\left[Y^l(\hat{r})\chi^{1/2}(\sigma)\right]^{J*}_M = (-1)^{1/2-\sigma+J-M}\left[Y^l(\hat{r})\chi^{1/2}(-\sigma)\right]^J_{-M}, \qquad \#13\ \cancel{(232)}$$

which can be derived in a similar way as ~~(228)~~. (7.J.9)

A.4 angular momenta coupling

Relation between Clebsh–Gordan and $3j$ coefficients:

$$\langle j_1\ j_2\ m_1\ m_2|JM\rangle = (-1)^{j_1-j_2+M}\sqrt{2J+1}\begin{pmatrix} j_1 & j_2 & J\\ m_1 & m_2 & -M\end{pmatrix}. \qquad 14\ \cancel{(233)}$$

Relation between Wigner and $9j$ coefficients:

$$((j_1j_2)_{j_{12}}(j_3j_4)_{j_{34}}|(j_1j_3)_{j_{13}}(j_2j_4)_{j_{24}}))_j = \sqrt{(2j_{12}+1)(2j_{13}+1)(2j_{24}+1)(2j_{34}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & j_4 & j_{34}\\ j_{13} & j_{24} & j\end{Bmatrix}. \qquad 15\ \cancel{(234)}$$

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7.J.5

7.J.~~A~~.5 integrals

Let us now prove

$$\int d\Omega \left[Y^l(\hat{r})Y^l(\hat{r})\right]^I_M = \delta_{M,0}\delta_{I,0}\sqrt{2l+1}. \qquad 16 \; (\cancel{235})$$

$$\begin{aligned}
\int d\Omega \left[Y^l(\hat{r})Y^l(\hat{r})\right]^I_M &= \sum_{\substack{m_1,m_2 \\ (m_1+m_2=M)}} \langle l\, l\, m_1\; m_2|IM\rangle \int d\Omega Y^l_{m_1}(\hat{r})Y^l_{m_2}(\hat{r}) \\
&= \sum_{\substack{m_1,m_2 \\ (m_1+m_2=M)}} (-1)^{l+m_1}\langle l\, l\, -m_1\; m_2|IM\rangle \int d\Omega Y^{l*}_{m_1}(\hat{r})Y^l_{m_2}(\hat{r}) \\
&= \delta_{M,0}\sum_m (-1)^{l+m}\langle l\, l\, -m\; m|I0\rangle \\
&= \delta_{M,0}\sqrt{2l+1}\sum_m \langle l\, l\, -m\; m|I0\rangle\langle l\, l\, -m\; m|00\rangle \\
&= \delta_{M,0}\delta_{I,0}\sqrt{2l+1},
\end{aligned} \qquad 17 \; (\cancel{236})$$

where we have used

$$\langle l\, l\, -m\; m|0\; 0\rangle = \frac{(-1)^{l+m}}{\sqrt{2l+1}} \qquad 18 \; (\cancel{237})$$

Let us now prove

$$\sum_\sigma \int d\Omega (-1)^{1/2-\sigma}\left[\Psi^j(\hat{r},-\sigma)\Psi^j(\hat{r},\sigma)\right]^I_M = -\delta_{M,0}\delta_{I,0}\sqrt{2j+1}. \qquad 19 \; (\cancel{238})$$

$$\begin{aligned}
&\sum_\sigma \int d\Omega (-1)^{1/2-\sigma}\left[\Psi^j(\hat{r},-\sigma)\Psi^j(\hat{r},\sigma)\right]^I_M \\
&= \sum_{\substack{m_1,m_2 \\ (m_1+m_2=M)}} \langle j\, j\, m_1\; m_2|IM\rangle \sum_\sigma \int d\Omega \Psi^j_{m_1}(\hat{r},-\sigma)\Psi^j_{m_2}(\hat{r},\sigma) \\
&= \sum_{\substack{m_1,m_2 \\ (m_1+m_2=M)}} \langle j\, j\, m_1\; m_2|IM\rangle \sum_\sigma (-1)^{j+m_1} \int d\Omega \Psi^{j*}_{-m_1}(\hat{r},\sigma)\Psi^j_{m_2}(\hat{r},\sigma) \\
&= \sum_{\substack{m_1,m_2 \\ (m_1+m_2=M)}} \langle j\, j\, m_1\; m_2|IM\rangle (-1)^{j+m_1}\delta_{-m_1,m_2} \\
&= \delta_{M,0}\sum_m (-1)^{j+m}\langle j\, j\, m\; -m|I0\rangle \\
&= -\delta_{M,0}\sqrt{2j+1}\sum_m (-1)^{j+m}\langle j\, j\, m\; -m|I0\rangle\langle j\, j\, m\; -m|00\rangle \\
&= -\delta_{M,0}\delta_{I,0}\sqrt{2j+1}.
\end{aligned} \qquad 20 \; (\cancel{239})$$

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7.J.6 ~~A.6~~ symmetry properties

Note also another useful property

$$\left[\Psi^{I_1}\Psi^{I_2}\right]^I_M = (-1)^{I_1+I_2-I}\left[\Psi^{I_2}\Psi^{I_1}\right]^I_M, \qquad 21\ \cancel{(240)}$$

by virtue of the symmetry property of the Clebsh-Gordan coefficients

$$\langle I_1\ I_2\ m_1\ m_2|IM\rangle = (-1)^{I_1+I_2-I}\langle I_2\ I_1\ m_2\ m_1|IM\rangle. \qquad 22\ \cancel{(241)}$$

Here's another symmetry property of the Clebsh-Gordan coefficients

$$\langle I_1\ I_2\ m_1\ m_2|IM\rangle = (-1)^{I_1+I_2-I}\langle I_1\ I_2\ -m_2\ -m_1|I-M\rangle. \qquad 23\ \cancel{(242)}$$

Another one, which can be derived from the simpler properties of $3j$ symbols

$$\langle I_1\ I_2\ m_1\ m_2|IM\rangle = (-1)^{I_1-m_1}\sqrt{\frac{2I+1}{2I_2+1}}\langle I_1\ I\ m_1\ -M|I_2m_2\rangle. \qquad 24\ \cancel{(243)}$$

Let us use this last property to calculate sums of the type

$$\sum_{m_1,m_3}|\langle I_1\ I_2\ m_1\ m_2|I_3m_3\rangle|^2. \qquad 25\ \cancel{(244)}$$

Using (7.J.24) ~~(243)~~, we have

$$\sum_{m_1,m_3}|\langle I_1\ I_2\ m_1\ m_2|I_3m_3\rangle|^2 =$$

$$\frac{2I_3+1}{2I_2+1}\sum_{m_1,m_3}|\langle I_1\ I_3\ m_1\ -m_3|I_2m_2\rangle|^2 = \frac{2I_3+1}{2I_2+1}, \qquad 26\ \cancel{(245)}$$

since

$$\sum_{m_1,m_3}|\langle I_1\ I_3\ m_1\ -m_3|I_2m_2\rangle|^2 = \sum_{m_1,m_3}|\langle I_1\ I_3\ m_1\ m_3|I_2m_2\rangle|^2 = 1. \qquad 27\ \cancel{(246)}$$

appendix 7.K. ~~B~~ distorted waves

Let us have a closer look at the partial wave expansion of the distorted waves

$$\chi^{(+)}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^l e^{i\sigma^l} F_l \sum_m Y^l_m(\hat{r}) Y^{l*}_m(\hat{k}). \qquad 7.K.1\ \cancel{(247)}$$

Of notice the very important fact that *this definition is independent of the phase convention*, since the l–dependent phase is multiplied by its complex conjugate.

$$\chi^{(-)}(\mathbf{k},\mathbf{r}) = \chi^{(+)*}(-\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^{-l} e^{-i\sigma^l} F_l^* \sum_m Y^{l*}_m(\hat{r}) Y^l_m(-\hat{k}). \qquad 7.K.2\ \cancel{(248)}$$

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As $Y^l_m(-\hat{k}) = (-1)^l Y^l_m(\hat{k})$, we have

$$\chi^{(-)}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^l e^{-i\sigma^l} F_l^* \sum_m Y^{l*}_m(\hat{r}) Y^l_m(\hat{k}), \qquad 3 \quad \cancel{(249)}$$

which is also independent of the phase convention. With time–reversed phase convention

$$\chi^{(+)}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^l \sqrt{2l+1} e^{i\sigma^l} F_l \left[Y^l(\hat{r}) Y^l(\hat{k}) \right]^0_0, \qquad 4 \quad \cancel{(250)}$$

while with Condon–Shortley phase convention we get an extra $(-1)^l$ factor:

$$\chi^{(+)}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^{-l} \sqrt{2l+1} e^{i\sigma^l} F_l \left[Y^l(\hat{r}) Y^l(\hat{k}) \right]^0_0. \qquad 5 \quad \cancel{(251)}$$

The partial–wave expansion of the Green function $G(\mathbf{r},\mathbf{r}')$ is

$$G(\mathbf{r},\mathbf{r}') = i \sum_l \frac{f_l(k,r_<) P_l(k,r_>)}{krr'} \sum_m Y^l_m(\hat{r}) Y^{l*}_m(\hat{r}'), \qquad 6 \quad \cancel{(252)}$$

where $f_l(k,r_<)$ and $P_l(k,r_>)$ are the regular and the irregular solutions of the homogeneous problem respectively. With *time–reversed* phase convention

$$G(\mathbf{r},\mathbf{r}') = i \sum_l \sqrt{2l+1} \frac{f_l(k,r_<) P_l(k,r_>)}{krr'} \left[Y^l(\hat{r}) Y^l(\hat{r}') \right]^0_0. \qquad 7 \quad \cancel{(253)}$$

7.L ~~C~~ hole states and time reversal

Let us consider the state $|(jm)^{-1}\rangle$ obtained by removing a ψ_{jm} single–particle state from a $J = 0$ closed shell $|0\rangle$. The antisymmetrized product state

$$\sum_m \mathcal{A}\{\psi_{jm} |(jm)^{-1}\rangle\} \propto |0\rangle \qquad 7.L.1 \quad \cancel{(254)}$$

is clearly proportional to $|0\rangle$. This gives us the transformation rules of $|(jm)^{-1}\rangle$ under rotations, which must be such that, when multiplied by a j, m spherical tensor and summed over m, yields a $j = 0$ tensor. It can be seen that these properties imply that $|(jm)^{-1}\rangle$ transforms like $(-1)^{j-m} T_{j-m}$, T_{j-m} being a spherical tensor. It also follows that the hole state $|(j\bar{m})^{-1}\rangle$ transforms like a j, m spherical tensor if $\psi_{j\bar{m}}$ is defined as the $\mathcal{R}$–conjugate to ψ_{jm} by the relation

$$\psi_{j\bar{m}} \equiv (-1)^{j+m} \psi_{j-m}. \qquad 7.L.2 \quad \cancel{(255)}$$

In other words, with the latter definition a *hole state* transforms under rotations with the right phase. We will now show that $\mathcal{R}$–conjugation is equivalent to a rotation of spin and spatial coordinates through an angle $-\pi$ about the y–axis:

$$e^{i\pi J_y} \psi_{jm} = (-1)^{j+m} \psi_{j-m} \equiv \psi_{j\bar{m}}. \qquad 3 \quad \cancel{(256)}$$

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Let us begin by calculating $e^{i\pi L_y} Y_l^m$. The rotation matrix about the y–axis is

$$R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

4 ~~(257)~~

so for $R_y(-\pi)$ we get

$$R_y(-\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

5 ~~(258)~~

When applied to the generic direction $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, we obtain $(-\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), -\cos(\theta))$, which corresponds to making the substitutions

$$\theta \to \pi - \theta, \quad \phi \to \pi - \phi.$$

6 ~~(259)~~

When we substitute these angular transformations in the spherical harmonic $Y_l^m(\theta, \phi)$, we obtain the rotated $Y_l^m(\theta, \phi)$:

$$e^{i\pi L_y} Y_l^m = (-1)^{l+m} Y_l^{-m}.$$

7 ~~(260)~~

Let us now turn our attention to the spin coordinates rotation $e^{i\pi s_y}\chi_m$. The rotation matrix in spin space is

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

8 ~~(261)~~

which, for $\theta = -\pi$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

9 ~~(262)~~

Applying it to the spinors, we find the rule

$$e^{i\pi s_y}\chi_m = (-1)^{1/2+m}\chi_{-m},$$

10 ~~(263)~~

so

$$\begin{aligned} e^{i\pi J_y}\psi_{jm} &= \sum_{m_l m_s} \langle l\ m_l\ 1/2\ m_s | j\ m\rangle\ e^{i\pi L_y} Y_l^{m_l}\ e^{i\pi s_y}\chi_{m_s} \\ &= \sum_{m_l m_s} (-1)^{1/2+m_s+l+m_l} \langle l\ m_l\ 1/2\ m_s | j\ m\rangle\ Y_l^{-m_l}\ \chi_{-m_s} \\ &= \sum_{m_l m_s} (-1)^{1+m-j+2l} \langle l\ -m_l\ 1/2\ -m_s | j\ -m\rangle\ Y_l^{-m_l}\ \chi_{-m_s} \\ &= (-1)^{m+j}\psi_{j-m} \equiv \psi_{j\bar{m}}, \end{aligned}$$

11 ~~(264)~~

where we have used $(-1)^{1+m-j+2l} = -(-1)^{m-j} = (-1)^{m+j}$, as j, m are always half–integers and l is always an integer.

We now turn our attention to the time reversal operation, which amounts to the transformations

$$\mathbf{r} \to \mathbf{r}, \quad \mathbf{p} \to -\mathbf{p}.$$

~~(265)~~ 12

58

This is enough to define the operator of time reversal of a spinless particle (see Messiah). In the position representation, in which $\mathbf{r}$ is real and $\mathbf{p}$ pure imaginary, this (unitary antilinear) operator is the complex conjugation operator.

As angular momentum $\mathbf{l} = \mathbf{r}\times\mathbf{p}$ changes sign under time reversal, so does spin: 13

7.L.12
$$\mathbf{s} \to -\mathbf{s}, \qquad \cancel{(266)}$$

which, along with ~~(265)~~, completes the set of rules that define the time reversal operation on a particle with spin. In the representation of eigenstates of $\mathbf{s}^2$ and s_z, complex conjugation alone changes only the sign of s_y, so an additional rotation of $-\pi$ around the y–axis is necessary to change the sign of s_x, s_z and implement the transformation ~~(266)~~). If we call K the time–reversal operator, we have 14

7.L.13
$$K\psi_{jm} = e^{i\pi s_y}\psi^*_{jm}. \qquad \cancel{(267)}$$

This is completely general and independent of the phase convention. It only depends on the fact that we have used the $\mathbf{r}$ representation for the spatial wave function and the representation of the eigenstates of $\mathbf{s}^2$ and s_z for the spin part. *If we use time–reversal phases for the spherical harmonics* (see ~~(221)~~)), 15

7.J.2
$$Y^{l*}_m = (-1)^{l+m}Y^l_{-m} = e^{i\pi L_y}Y^l_m. \qquad \cancel{(268)}$$

So we can write 16

$$K\psi_{jm} = e^{i\pi J_y}\psi_{jm} = \psi_{j\bar{m}}. \qquad \cancel{(269)}$$

Note again that this last expression is valid only if we use time–reversal phases for the spherical harmonics. Only in this case time–reversal coincides with $\mathcal{R}$–conjugation and hole states.

In BCS theory, the quasiparticles are defined in terms of linear combinations of particles and holes. With time–reversal phases, holes are equivalent to time–reversed states, and we get the definitions

7.L.17
$$\begin{aligned}
\alpha^\dagger_\nu &= u_\nu a^\dagger_\nu - v_\nu a_{\bar{\nu}} & a^\dagger_\nu &= u_\nu \alpha^\dagger_\nu + v_\nu \alpha_{\bar{\nu}} \\
\alpha^\dagger_{\bar{\nu}} &= u_\nu a^\dagger_{\bar{\nu}} + v_\nu a_\nu & a^\dagger_{\bar{\nu}} &= u_\nu \alpha^\dagger_{\bar{\nu}} - v_\nu \alpha_\nu \\
\alpha_\nu &= u_\nu a_\nu - v_\nu a^\dagger_{\bar{\nu}} & a_\nu &= u_\nu \alpha_\nu + v_\nu \alpha^\dagger_{\bar{\nu}} \\
\alpha_{\bar{\nu}} &= u_\nu a_{\bar{\nu}} + v_\nu a^\dagger_\nu & a_{\bar{\nu}} &= u_\nu \alpha_{\bar{\nu}} - v_\nu \alpha^\dagger_\nu
\end{aligned} \qquad \cancel{(270)}$$

7.M

amplitude

~~D~~ spectroscopic ~~factors in the~~ BCS approximation

Two-nucleon

The creation operator of a pair of fermions coupled to J, M can be expressed in second quantization as

7.M.1
$$P^\dagger(j_1, j_2, JM) = N\sum_m \langle j_1\, m\, j_2\, M-m | J\, M\rangle\, a^\dagger_{j_1 m} a^\dagger_{j_2 M-m}, \qquad \cancel{(271)}$$

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7.M.1

where N is a normalization constant. To determine it, we write the wave function resulting from the action of ~~(271)~~ on the vacuum

$$\Psi = P^{\dagger}(j_1, j_2, JM)|0\rangle = \frac{N}{\sqrt{2}} \sum_m \langle j_1\; m\; j_2\; M-m|J\; M\rangle$$
$$\times \left(\phi_{j_1 m}(\mathbf{r}_1)\phi_{j_2 M-m}(\mathbf{r}_2) - \phi_{j_2 M-m}(\mathbf{r}_1)\phi_{j_1,m}(\mathbf{r}_2)\right).$$

2 ~~(272)~~

The norm is

$$|\Psi|^2 = \frac{N^2}{2} \sum_{mm'} \langle j_1\; m\; j_2\; M-m|J\; M\rangle\langle j_1\; m'\; j_2\; M-m'|J\; M\rangle$$
$$\times \left(\phi_{j_1 m}(\mathbf{r}_1)\phi_{j_2 M-m}(\mathbf{r}_2) - \phi_{j_2 M-m}(\mathbf{r}_1)\phi_{j_1,m}(\mathbf{r}_2)\right)$$
$$\times \left(\phi_{j_1 m'}(\mathbf{r}_1)\phi_{j_2 M-m'}(\mathbf{r}_2) - \phi_{j_2 M-m'}(\mathbf{r}_1)\phi_{j_1,m'}(\mathbf{r}_2)\right).$$

3 ~~(273)~~

Integrating we get

$$1 = \frac{N^2}{2} \sum_{mm'} \langle j_1\; m\; j_2\; M-m|J\; M\rangle\langle j_1\; m'\; j_2\; M-m'|J\; M\rangle$$
$$\times \left(2\delta_{m,m'} - 2\delta_{j_1,j_2}\delta_{m,M-m'}\right)$$
$$= N^2\Bigg(\sum_m \langle j_1\; m\; j_2\; M-m|J\; M\rangle^2$$
$$-\delta_{j_1,j_2}\sum_m \langle j_1\; m\; j_2\; M-m|J\; M\rangle\langle j_1\; M-m\; j_2\; m|J\; M\rangle\Bigg)$$
$$= N^2\left(1-\delta_{j_1,j_2}(-1)^{2j-J}\right),$$

4 ~~(274)~~

(7.5.22)

where we have used the closure condition for Clebsh–Gordan coefficients and ~~(241)~~, and δ_{j_1,j_2} must be interpreted as a δ function regarding all the quantum numbers but the magnetic one. We see that two fermions with identical quantum numbers (but the magnetic one) *cannot couple to J odd*. If J is even, the normalization constant is

$$N = \frac{1}{\sqrt{1+\delta_{j_1,j_2}}}.$$

5 ~~(275)~~

To sum up,

$$P^{\dagger}(j_1, j_2, JM) = \frac{1}{\sqrt{1+\delta_{j_1,j_2}}} \sum_m \langle j_1\; m\; j_2\; M-m|J\; M\rangle\, a^{\dagger}_{j_1 m} a^{\dagger}_{j_2 M-m}.$$

6 ~~(276)~~

(cf. also Broglia, Hansen and Riedel, 1973)

The spectroscopic amplitude for finding in a $A+2, J_f, M_f$ nucleus a couple of nucleons with quantum numbers j_1, j_2 coupled to J on top of a A, J_i nucleus is

$$B(J, j_1, j_2) = \sum_{M,M_i} \langle J_i\; M_i\; JM|J_f\; M_f\rangle\langle\Psi_{J_f M_f}|P^{\dagger}(j_1, j_2, JM)|\Psi_{J_i M_i}\rangle.$$

7 ~~(277)~~

60

7,L,17

This is completely general. It depends on the structure model only through the way the $A+2$ and A nuclei are treated. We now want to turn our attention to the expression of $B(J, j_1, j_2)$ in the BCS approximation when both the $A+2$ and the A are 0^+, zero–quasiparticle ground states. In order to do this, we write (276) in terms of quasiparticle operators using ($\cancel{270}$)[1]:

$$\begin{aligned}
P^\dagger(j_1, j_2, JM) = & \frac{1}{\sqrt{1+\delta_{j_1,j_2}}} \sum_{m_1,m_2} \langle j_1\ m_1\ j_2\ m_2 | J\ M\rangle \Big(U_{j_1} U_{j_2} \alpha^\dagger_{j_1 m_1} \alpha^\dagger_{j_2 m_2} \\
& + (-1)^{j_1+j_2-M} V_{j_1} V_{j_2} \alpha_{j_1 -m_1} \alpha_{j_2 -m_2} \\
& + (-1)^{j_2-m_2} U_{j_1} V_{j_2} \alpha^\dagger_{j_1 m_1} \alpha_{j_2 -m_2} \\
& - (-1)^{j_1-m_1} V_{j_1} U_{j_2} \alpha^\dagger_{j_2 m_2} \alpha_{j_1 -m_1} \\
& + (-1)^{j_1-m_1} V_{j_1} U_{j_2} \delta_{j_1 j_2} \delta_{-m_1 m_2} \Big).
\end{aligned} \tag{8 \; \cancel{278}}$$

(7,M,7)

If both nuclei are in zero–quasiparticle states, the only term that survives is the last one in the above expression, and ($\cancel{277}$) becomes

$$\begin{aligned}
B(0, j, j) = & \frac{1}{\sqrt{2}} \sum_m \langle j\ m\ j\ -m | 0\ 0\rangle (-1)^{j-m} V_j U_j \\
= & \frac{1}{\sqrt{2}} \sum_m \frac{(-1)^{j-m}}{\sqrt{(2j+1)}} (-1)^{j-m} V_j U_j \\
= & \frac{1}{\sqrt{2}} \sum_m \frac{1}{\sqrt{(2j+1)}} V_j U_j .
\end{aligned} \tag{9 \; \cancel{279}}$$

After doing the sum, we finally find

$$B(0, j, j) = \sqrt{j+1/2}\ V_j U_j . \tag{16 \; \cancel{280}}$$

Note that in this final expression V_j refers to the A nucleus, while U_j is related to the $A+2$ nucleus. In practice, it does not make a big difference to calculate both for the same nucleus.

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[1] In what follows, we use the phase convention $\alpha_{j\bar{m}=(-1)^{j-m}\alpha_{j-m}}$ instead of $\alpha_{j\bar{m}=(-1)^{j+m}\alpha_{j-m}}$, consistent with ($\cancel{255}$). I ~~don't know why, but it seems to be common practice... Had we stick to the definition ($\cancel{255}$), the amplitude $B(0, j, j)$ calculated below would have a minus sign, which would not have any physical consequence.~~

~~even for rather small energy losses. The relation of this open question to the observed "macroscopic" transfer of mass observed in, for example, the Ca + U reaction is a central topic in the field of heavy ion reaction.~~

~~The results of the above model, which account for the main features experimentally observed, can be summarized as follows. At an early stage of the collision when the two surfaces get into contact, energy and angular momentum is absorbed at a fast rate by the damped giant resonances. Low-lying modes with small restoring forces are important towards the final stages of the collision where they give rise to large deformations keeping the nuclear surfaces into contact. This "neck" formation is responsible for the experimentally observed fact that the two nuclei often emerge with relative kinetic energy which is below the Coulomb barrier of the corresponding spherical nuclei. The exchange of nucleons between the nuclei removes energy and angular momentum from relative motion throughout the collision.~~

A ~~The~~ central feature of heavy ion collisions seems to be the importance of the coherent response of the different degrees of freedom. Thus in the description of the excitation of the surface modes it is not enough to know the population of the vibrational states but also the relative phases which determine the shape of the nuclei as a function of time ~~(cf. fig. 14)~~ 15. ~~Although it is difficult to document such a result in a more intuitive way it is possible to obtain a more accurate mathematical description. Thus solving~~ e.g. the problem quantum mechanically ~~we obtain~~ leads to the following ~~total~~ wavefunction (cf. Broglia and Winther, 2004 and refs. therein, cf. Glauber 1969) ~~accompanied~~

sect. II.4 and App. II.B of

~~In a heavy ion collision in which the two nuclei interact through the Coulomb~~

Example of coherent states which display vibrational modes

Let us assume a ~~[illegible]~~ collision in which the ions interact only through the Coulomb field,. Solving

App. 7. N

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Cf. ~~Sect 4 and App. B~~ P. 93 Ch II HAR Broglia Winther

$$|\psi(t)\rangle = e^{-i\frac{H_o t}{\hbar}}|\phi(t)\rangle$$
$$= \sum_{\{n_\mu\}}\left(\prod_\mu e^{-\frac{|I_\mu(t)|^2}{2}}\frac{(I_\mu(t))^{n_\mu}}{\sqrt{n_\mu!}}\right)|\{n_\mu\}\rangle,$$

where

$$I_\mu(t) = \frac{1}{\hbar}\int^t f_\mu^*(t')e^{i\omega t'}dt',$$

and where H_o is the Hamiltonian describing the intrinsic degrees of freedom of each nuclei. The wavefunction $\phi(t)$ is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\phi}{\partial t} = \tilde{V}\phi$$

where $\tilde{V} = \exp(i H_o t/h)\, V \exp(-i H_o t/h)$, V being the external field.

The integral I_μ is related to the average number of phonons by

$$\langle n_\mu\rangle = |I_\mu(t)|^2.$$

~~the corresponding values for the reaction Xe + Pb at the instant of maximum deformation are quoted in table 1.~~

The state $|\psi(t)\rangle$ is known in quantum mechanics as a coherent state (Glauber 1969). Its name stems from the fact that the associated uncertainty relations in momemtum and coordinate associated with it fulfills the absolute minimum consistent with quantum mechanics, that is,

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$$\Delta x_\varphi \, \Delta \pi_\varphi = \frac{\hbar}{2}$$

Note that this value is normally associated with the ground state. In general states described by a wavefunction of the type $\exp\{\frac{i\hat{O}}{\hbar}\}\phi(t)$ exhaust the energy weighted sum rule of the associated operator[15] which in the present case is the Hamiltonian.

Heavy ion collisions seem thus specific to study the nuclear spectroscopy of the coherent nuclear state. Note that we have left behind the field of experiments where the system that is probed can be described as if the probe was not present.

The coherent state which pictorially looks so simple, being almost a classical state, arises from the excitation and delicate phase relation of many collective and non-collective states of the individual nuclei. Thus, the full response function is tested in these reactions in a totally novel way. Note that collective vibrations as those e.g. example surface modes, ~~discussed in connection with fig. 3~~ e.g. ~~[illegible]~~ are also coherent states and arise from the correlated efforts of many particle-hole excitations.

It is interesting to speculate whether the coherent excitation of the gas of phonons will lead to new super-collectivities displaying different condensation or phases as a function of the continuous excitation energy.

~~The behavior of the total energy absorbed by the coherent state in the reaction Kr + Pb as a function of angle (or linear momentum) shown in fig. 15 14 is suggestively reminiscent of the behavior of the coherent state excited in liquid helium (cf. fig. 1).~~