check

W

Appendix X: Elastic scattering cross section in DWBA

Let us consider the scattering by a potential $U(r)$. The scattering wavefunction (DWBA) can be written as

$$X^{(+)}(\vec{k},\vec{r}) = \frac{4\pi}{r}\sum_{l,m} i^l f_l(k,r) Y_m^{l*}(\hat{k}) Y_m^l(\hat{r}) \tag{X.1}$$

where $f_l(k,r)$ is solution of the equation

$$\left\{\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \bar{U} - k^2\right\} f_l(k,r) = 0 \tag{X.2}$$

with the boundary condition

$$f_l(k,0) = 0\ . \tag{X.3}$$

Starting at $r = 0$ integrate out to $r \to \infty$, imposing the asymptotic condition

$$\lim_{r\to\infty} f_l(k,r) \to \frac{1}{2ikr}\left[\mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)} - \eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}\right]\ . \tag{X.4}$$

Of notice that

$$\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} = \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}\mathrm{e}^{\ln\eta_l} = \mathrm{e}^{i\left(kr-\frac{l\pi}{2}-i\ln\eta_l\right)}\ . \tag{X.5}$$

In other words, η_l is associated with the phase shift included by the potential $U(r)$.

Replacing in (X.12) one obtains,

$$\begin{aligned}
X^{(+)}(\vec{k},\vec{r}) &\to -4\pi\sum_l i^l \frac{\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} - \mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)}}{2ikr}\sum_m Y_m^{l*}(\hat{k})Y_m^l(\hat{r})\ , \\
&= -4\pi\sum_l i^l \frac{\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} - \mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)} + 2i\sin\left(kr-\frac{l\pi}{2}\right)}{2ikr}\frac{2l+1}{4\pi}P_l(\hat{k},\hat{r})\ , \\
&= -\sum_l i^l(2l+1)P_l(\hat{k},\hat{r})\frac{\sin\left(kr-\frac{l\pi}{2}\right)}{kr}\ , \\
&= -\sum_l i^l \frac{(\eta_l - 1)}{2ikr}\mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}(2l+1)P_l(\hat{k},\hat{r})\ .
\end{aligned} \tag{X.6}$$

Making use of the plane wave expansion

$$\mathrm{e}^{i\vec{k}\cdot\vec{r}} = \sum_l i^l(2l+1)P_l(\cos\theta')\mathcal{I}_l(kr)\ , \tag{X.7}$$

and of the asymptotic expression of the Bessel function

$$\mathcal{I}_l(kr) \to_{r\to\infty} \frac{\sin\left(kr-\frac{l\pi}{2}\right)}{kr}\ , \tag{X.8}$$

one obtains,

$$\lim_{r\to\infty} \mathrm{e}^{i\vec{k}\cdot\vec{r}} = \sum_l i^l(2l+1)P_l(\cos\theta')\frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} . \tag{X.9}$$

Making also use of the relation

$$\mathrm{e}^{-i\frac{\pi l}{2}} = \cos\frac{l\pi}{2} - i\sin\frac{l\pi}{2} = \left\{\begin{array}{ll} -i & l=1 \\ -1 & l=2 \\ +i & l=3 \\ +1 & l=4 \end{array}\right\} = (-1)^l i^l = i^{-l}$$

on can write

$$\lim_{r\to\infty} X^{(+)}(\vec{k},\vec{r}) \to -\mathrm{e}^{i\vec{k}\cdot\vec{r}} - \frac{\mathrm{e}^{ikr}}{r}\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r}) . \tag{X.10}$$

From the relation

$$\frac{\mathrm{d}\sigma_{el}(\theta)}{\mathrm{d}\Omega} = |f_{\alpha\alpha'}(E,\theta)|$$

where

$$\Psi_{\text{elast scatt}} \to \mathrm{e}^{i\vec{k}\cdot\vec{r}} + \frac{\mathrm{e}^{ikr}}{r} f_{\alpha\alpha'}(E,\theta)$$

one obtains

$$\left(\frac{\mathrm{d}\sigma_{el}(\theta)}{\mathrm{d}\Omega}\right)_{DWBA} = \left|\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r})\right|^2 . \tag{X.11}$$

1. Order of magnitude estimate

Elastic cross section ^{16}O+^{208}Pb at $E_{lab} \approx 85$ MeV

$$E_B = \frac{Z_A Z_A e^2}{r_b}\left(1 - \frac{a}{r_B}\right) \qquad \left.\frac{\mathrm{d}U}{\mathrm{d}r}\right|_{r_B} = 0$$

$$r_B = \left[1.07\left(A_a^{1/3} + A_A^{1/3}\right) + 2.72\right] \text{ fm} = [(2.7+6.3)+2.72] \text{ fm} \approx 11.7 \text{ fm}$$

$$a = 0.65 \text{ fm}$$

$$e^2 = 1444 \text{ fm MeV}$$

$$E_B = \frac{8\times 82\times 1.44 \text{ MeV fm}}{11.7 \text{ fm}}\left(1 - \frac{0.65 \text{ fm}}{11.7 \text{ fm}}\right) \approx 80\times 0.9 \approx 72 \text{ MeV}$$

Grazing angular momentum

$$L_g = (r_g)_{fm} \left(\frac{1}{20} \frac{A_a A_A}{A_a + A_A} (E - E_B)_{\text{MeV}} (1 + c) \right)^{1/2} \hbar$$

$$r_g = r_B - \delta = 11.7 \text{ fm} - 0.13 \text{ fm} \approx 11.6 \text{ fm}$$

$$\delta = a \ln \left(1 + \frac{2(E - E_B)}{E_B} \right) = 0.13 \text{ fm}$$

$$c \approx \frac{2a}{r_B} \frac{E - E_B}{E_B} = 0.10 \times 0.11 = 0.01$$

$$L_g = 11.6(0.05 \times 14.9 \times 8.6 \times 1.01)^{1/2} \hbar \approx 30\hbar$$

$$E_{CM} = \frac{M_{\text{target}}}{M_{\text{target}} + M_{\text{project}}} E_{\text{lab}} = \frac{208}{208 + 16} 85 \text{ MeV} = 0.93 \times 85 \text{ MeV} \approx 79 \text{ MeV}$$

$$E = \frac{\hbar^2 k^2}{2\mu}; \qquad k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

$$\mu = \frac{16 \times 208}{224} M_n = 14.9 M_n; \qquad \frac{2\mu}{\hbar^2} \approx 30 \frac{M_c^2}{(\hbar c)^2} = \frac{30}{40 \text{ fm}^2 \text{ MeV}} = 0.75 \text{ fm}^{-2} \text{ MeV}^{-1}$$

$$k = \sqrt{\frac{0.75}{\text{MeV fm}^2} \times 79 \text{ MeV}} \approx 7.7 \text{ fm}^{-1}$$

$$\hbar k (R_a + R_A) \approx 8 \text{ fm}^{-1} \times (3 \text{ fm} + 7 \text{ fm}) \hbar \approx 80\hbar$$

which is completely out of mark (see Fig.).

To get the grazing angular momentum we have to use, instead of E the quantity $E_{CM} - E_B \approx 7$ MeV. Thus

$$k = \sqrt{\frac{2\mu}{\hbar^2}(E_{CM} - E_B)} = \sqrt{\frac{0.75}{\text{fm}^2 \text{ MeV}} \times 7 \text{ MeV}} = 2.3 \text{ fm}^{-1}$$

$$\hbar k (R_a + R_A) \approx 2.3 \text{ fm}^{-1} \times 10.13 \text{ fm} \hbar \approx 23\hbar$$

which is not inconsistent with L_g.

$$L_R \approx \eta_{cl} \cot \frac{\Theta_R}{2}$$

$$\Theta_R = 2 \cot^{-1} \left(\frac{L_R}{\eta_{cl}} \right) = 2 \cot^{-1} \left(\frac{30}{139} \right)$$

$$\cot x = \frac{1}{\tan x}$$

$$\cot \frac{\Theta_R}{2} = \frac{L_R}{\eta_{cl}}$$

$$\frac{1}{\tan\frac{\Theta_R}{2}} = \frac{L_R}{\eta_{cl}}$$

$$\frac{\eta_{cl}}{L_R} = \tan\frac{\Theta_R}{2}$$

$$\Theta_R = 2\arctan\frac{\eta_{cl}}{L_R} = 111^\circ$$

Sommerfeld par. p.38 Broglia and Winther

$$\eta_{cl} = \frac{Z_a Z_A e^2}{\hbar v} = \frac{8 \times 82 \times 1.44 \text{ MeV fm}}{\hbar v} = \frac{945 \text{ MeV fm}}{\hbar v}$$

$$\hbar k = \mu v \; ; \qquad v = \frac{\hbar k}{\mu}$$

To calculate the parameter η_{cl} (how classical the trajectory is) we need to use E_{lab} (asymptotic condition). Thus

$$\hbar v = \frac{\hbar^2 k}{\mu} = \frac{(\hbar c)^2}{Mc^2}\frac{1}{15} \times 8 = \frac{40 \times 8}{15} \text{ fm MeV} \approx 21.3$$

$$\eta_{cl} = \frac{945}{21.3} \approx 44$$

$$\eta_l \approx 0.5 \; ; \qquad \eta_l = 1 - 0.5 = 0.5 \; ; \qquad P_l = \frac{1}{2}$$

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} &= \left|\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r})\right|^2 \\
&\sim \frac{(0.5)^2}{4k^2}\frac{(2L_g+1)^2}{4} \sim \frac{(61)^2}{64 \times (2.5 \text{ fm}^{-1})^2}\frac{1}{\text{sr}} \\
&\sim 9 \text{ fm}^{-2} = 0.09 \times 10^2 \times 10^{-26}\frac{\text{cm}^2}{\text{sr}} \\
&\approx 0.1 \text{ b} = \frac{100 \text{ mb}}{\text{sr}} \qquad (\Theta = \Theta_g)
\end{aligned}$$

$$\frac{\mathrm{d}\sigma(\Theta \approx 110^\circ)}{\mathrm{d}\omega} \sim \frac{100 \text{ mb}}{\text{sr}}$$

appendix δ

~~2.~~ Elastic scattering

The scattered wave (asymptotic region) must be the solution of the *free field equation*

$$H_\alpha \psi_{\rm scatt} = E\psi_{\rm scatt} \ , \tag{X.12}$$

with

$$E = \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha} \ , \tag{X.13}$$

where

$$H_\alpha = T_\alpha \ , \tag{X.14}$$

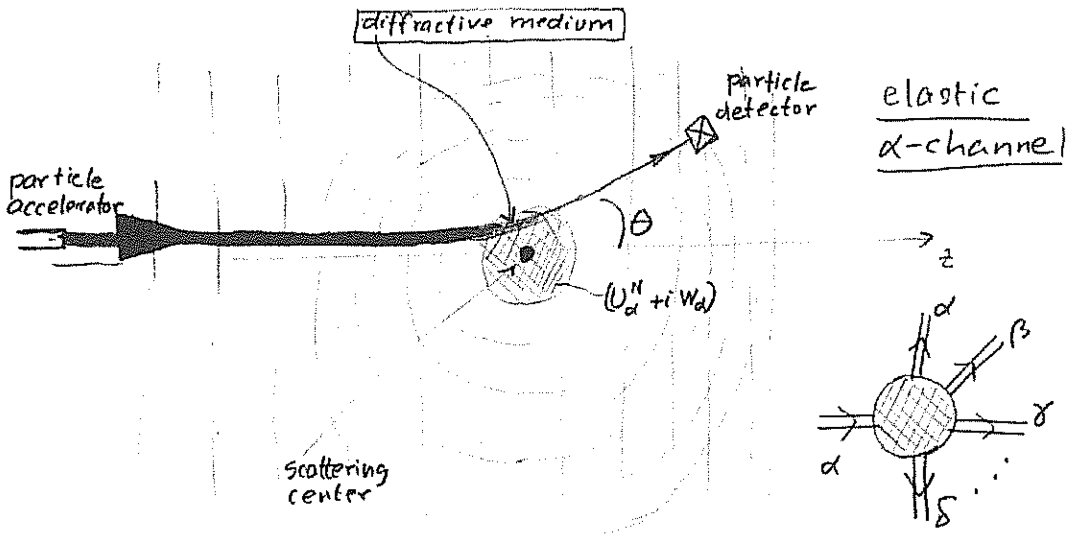
with

$$T_\alpha = \frac{p^2}{2\mu_\alpha} = -\frac{\hbar^2}{2\mu_\alpha}\nabla^2 = -\frac{\hbar^2}{2\mu_\alpha}\left\{\frac{1}{r_\alpha^2}\frac{\partial}{\partial r_\alpha}\left(r_\alpha^2\frac{\partial}{\partial r_\alpha}\right) + \frac{\hat{L}_2}{r\alpha^2}\right\} \ , \tag{X.15}$$

and

$$\mu_\alpha = \frac{M_a M_A}{M_a + M_A} \ , \tag{X.16}$$

is the reduced mass in the entrance channel $\alpha \equiv (a, A)$, while k_α and r_α are the relative momentum and coordinate associated with the elastic process $a + A \to a + A$.



Asymptotically $r \gg R_a + R_A, H_\alpha = T_\alpha$ and the centrifugal barrier term drops out as $1/r^2$, and can be neglected. Consequently, the asymptotic solution is

$$\psi_{\rm scatt} = \frac{e^{ik_\alpha r_\alpha}}{r_\alpha} f_{\alpha\alpha}(E, \theta, \phi)\psi_\alpha(\xi_\alpha) \ . \tag{X.17}$$

Making use of the fact that the incident scattering wavefunction is

$$\psi_{\rm inc} = e^{ik_\alpha z_\alpha} \ , \tag{X.18}$$

one can calculate the differential elastic scattering cross section, as the ratio of the scattered flux going through an asymptotic differential surface perpendicular to the (asymptotic) scattering beam and the incident intensity. One obtains (see appendix Y)

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\omega} = |f_{\alpha\alpha}(E,\theta)|^2 \ , \tag{X.19}$$

making the assumption of the spherical symmetry of the process (independent on ϕ, $\frac{1}{2}\int_0^\pi \mathrm{d}\phi \sin\phi = \frac{1}{2}\int_0^\pi(-\mathrm{d}\cos\theta) = \frac{1}{2}(\cos\theta)_0^\pi = 1$).

The next step consists in the calculation of $f_{\alpha\alpha}(E,\theta,\phi)$. To do this we need to find the scattering wavefunction $\Psi^{(+)}(\vec{k_\alpha},\vec{r_\alpha},\xi_\alpha)$, that is the solution of the full Hamiltonian

$$\begin{aligned} H &= T_a + T_A + H_a + H_A + V_{aA} \\ &= T_{CM} + T_\alpha + H_\alpha + V_\alpha \\ (&= T_{CM} + T_\beta + H_\beta + V_\beta = \ldots) \end{aligned} \tag{X.20}$$

in the center of mass system. In other words, one has to solve the equation

$$\left[-\frac{\hbar^2}{2\mu_\alpha}\nabla^2_{r_\alpha} + H_\alpha(\xi_\alpha) + V_\alpha(\xi_\alpha;\vec{r_\alpha})\right]\Psi^{(+)}_\alpha(\vec{k_\alpha},\vec{r_\alpha}) = E_\alpha\Psi^{(+)}_\alpha(\vec{k_\alpha},\vec{r_\alpha}) \ , \tag{X.21}$$

where

$$\mu_\alpha = \frac{M_a M_A}{M_a + M_A} \ , \tag{X.22}$$

$$\xi_\alpha = \xi_a + \xi_A \ , \tag{X.23}$$

$$\phi_\alpha(\xi_\alpha) = \phi_a(\xi_a)\phi_A(\xi_A) \ , \tag{X.24}$$

and

$$H_\alpha\phi_\alpha(\xi_\alpha) = \varepsilon_\alpha\phi_\alpha(\xi_\alpha) \ . \tag{X.25}$$

Of notice that the relative, intrinsic variables ξ_α refer to the structure information contained int he "exact" scattering wavefunction $\Psi^{(+)}_\alpha(\vec{k_\alpha},\vec{r_\alpha};\xi_\alpha)$. In particular

$$H_\alpha(\xi_\alpha)\Psi^{(+)}_\alpha = \varepsilon_\alpha\Psi^{(+)}_\alpha \ , \tag{X.26}$$

where $H_\alpha(\xi_\alpha) = H_a(\xi_a) + H_A(\xi_A)$ is the total intrinsic Hamiltonian describing the structure of nuclei a and A by themselves, that is, far away from the range in which the interaction

$$V_\alpha(\vec{r_\alpha};\xi_\alpha) = \sum_{\substack{i\in a, j\in A\\ i<j}} V_\alpha(|\vec{r_i} - \vec{r_j}|) , \tag{X.27}$$

is effective.

As seen from the above relations, the structure information associated with the intrinsic degrees of freedom related to the variables ξ_α are melted together the scattering degrees of freedom associated with the variables $(\vec{k_\alpha}, \vec{r_\alpha})$.

Introducing the mean field potential

$$U_\alpha(r_\alpha) = \int \mathrm{d}\xi_\alpha |\psi_\alpha(\xi_\alpha)|^2 V_\alpha(\vec{r},\xi_\alpha) , \tag{X.28}$$

one can write

$$\left(-\frac{\hbar^2}{2\mu_\alpha}\nabla_r^2 + U_\alpha - \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha}\right)\Psi^{(+)} = -V'_\alpha \Psi^{(+)} , \tag{X.29}$$

in keeping with the fact that the difference $E_\alpha - \varepsilon_\alpha$ is equal to the kinetic energy of relative motion in channel α, that is,

$$E_\alpha - \varepsilon_\alpha = \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha} . \tag{X.30}$$

Multiplying the above Schrödinger equation from the left with the intrinsic wavefunction $\psi_\alpha(\xi_\alpha)$, integrating over ξ_α and introducing the definitions

$$V'_\alpha = V_\alpha - U_\alpha , \tag{X.31}$$

$$\bar{U_\alpha} = \frac{2\mu_\alpha}{\hbar^2} U_\alpha(r_\alpha) , \tag{X.32}$$

and

$$\varphi_\alpha(\vec{k_\alpha}, \vec{r_\alpha}) = \int \mathrm{d}\xi_\alpha, \psi^*(\xi_\alpha)\Psi^{(+)} = \langle \psi_\alpha, \Psi_\alpha^{(+)} \rangle , \tag{X.33}$$

one can write

$$\left(-\nabla_{r_\alpha}^2 + \bar{U}(r_\alpha) - k_\beta^2\right)\varphi_\alpha^{(+)}(\vec{k_\alpha}, \vec{r_\alpha}) = -\frac{2\mu_\alpha}{\hbar^2}\langle \psi_\alpha(\xi_\alpha), V'_\alpha \psi_\alpha^{(+)} \rangle . \tag{X.34}$$

The asymptotic form of $\varphi_\alpha^{(+)}(\vec{k_\alpha}, \vec{r_\alpha})$ will determine the elastic differential cross section[17].

---

[17] Because of the absence of potential (interaction terms) in the free field Eqs (X.12)–(?) one obtains the same wavefunction by first solving the full Schrödinger equation and then taking the asymptotic limit ($r \to \infty$) or vice versa. In connection with equation (X.34) (see also (X.35)), only the first procedure is correct, in keeping with the presence of short range interactions in the Hamiltonian.

Of notice that the rhs term gives the coupling of the entrance channel with all other channels. These couplings lead to both depopulation and distortion of entrance channel outgoing spherical wave. To the extent that these couplings are weak, the rhs term can be just viewed as a (source – sink)–like term, that is, as an imaginary potential depopulating (populating) the entrance channel. In other words, the above equation can be rewritten as

$$\left(-\nabla^2_{r_\alpha} + \bar{U}(r_\alpha) + i\bar{W}(r_\beta) - k_\alpha^2\right) \chi_\alpha^{(+)}(\vec{k_\alpha}, \vec{r_\alpha}) = 0 \tag{X.35}$$

where $\chi_\alpha^{(+)}$ is the standard notation for $\varphi_\alpha^{(+)}$ when the coupling term $\langle \psi_\alpha(\xi_\alpha), V'_\alpha \psi_\alpha^{(+)} \rangle$ is not considered. This is known as the Distorted Wave Born Approximation (DWBA).

Appendix ~~Y~~ E: Elastic scattering solution

$$\lim \psi_{\text{scatt}} =_{r\to\infty} \frac{e^{ikr}}{r} f(E,\theta,\phi)\psi_\alpha(\xi_\alpha)$$

$$\frac{\partial}{\partial r}\frac{e^{ikr}}{r} = -\frac{1}{r^2}e^{ikr} + \frac{ike^{ikr}}{r}$$

$$r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r} = -e^{ikr} + rike^{ikr}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) = -ike^{ikr} + ike^{ikr} + r(ik)^2 e^{ikr} = -rk^2 e^{ikr}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) = -\frac{k^2 e^{ikr}}{r}$$

$$\lim_{r\to\infty} H = \lim_{r\to\infty}(T + U + iW) = \lim_{r\to\infty} T = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)$$

Thus

$$\begin{aligned}
\lim_{r\to\infty} H\psi_{\text{scatt}} &= -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)\frac{e^{ikr}}{r} f\psi \\
&= -\frac{\hbar^2}{2\mu}\left(-k^2\frac{e^{ikr}}{r}\right) f\psi \\
&= \frac{\hbar^2 k^2}{2\mu}\frac{e^{ikr}}{r} f\psi = E \lim_{r\to\infty}\psi_{\text{scatt}}
\end{aligned}$$

Current

$$\vec{I} = \frac{\hbar}{\mu}\mathfrak{Im}(\psi^\star \vec{\nabla}\psi)$$

$$\psi = \int \mathrm{d}\xi \psi_i \psi(\xi) = \begin{cases} e^{ikz} \\ \frac{e^{ikr}}{r} f \end{cases}$$

where

$$\psi_i = \begin{cases} e^{ikz}\psi(\xi)\hat{z} & \text{(incident)} \\ \frac{e^{ikz}}{r} f(E,\theta,\phi)\psi(\xi) & \text{(scattered)} \end{cases}$$

$$\vec{\nabla} = \begin{cases} \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} & \text{(cartesian coordinates)} \\ \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial \psi}{\partial \phi} & \text{(spehrical coordinates)} \end{cases}$$

Incident flux

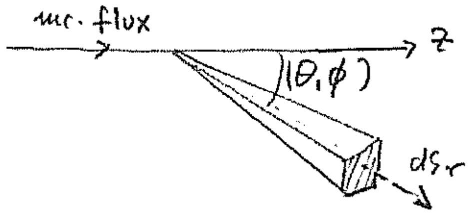
$$\vec{\nabla}\psi_{\text{inc}} = \hat{z}\frac{\partial}{\partial z} e^{ikz} = ike^{ikz}$$

$$\lim_{r\to\infty} \vec{\nabla}\psi_{\text{scatt}} = \lim_{r\to\infty} \hat{r}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r} f\right) = \lim_{r\to\infty}\left(-\frac{e^{ikr}}{r^2} f + ik\frac{e^{ikr}}{r} f\right)\hat{r} \approx ik\frac{e^{ikr}}{r} f\hat{r}$$

$$I_{\text{inc}} = \frac{\hbar}{\mu}\mathfrak{Im}(e^{-ikz} ike^{ikz})\hat{z} = \frac{\hbar k}{\mu}\hat{z} = v_\infty \hat{z} \,,$$

$$\begin{aligned} \vec{I_{\text{scatt}}} &= \frac{\hbar}{\mu}\mathfrak{Im}\left(\frac{e^{-ikr}}{r} f^\star ik\frac{e^{ikr}}{r} f\right)\hat{r} \\ &= \frac{\hbar k}{\mu}\frac{|f|^2}{r^2}\hat{r} = v_\infty \frac{|f|^2}{r^2}\hat{r} \end{aligned}$$

$$\mathrm{d}\vec{s} = r^2 \mathrm{d}\Omega \hat{r}$$



$$\vec{I_{\text{scatt}}} \cdot \mathrm{d}\vec{s} = |f|^2 v_\infty \mathrm{d}\Omega$$

$$\mathrm{d}\sigma_\alpha(\theta,\phi) = \frac{\vec{I_{\text{scatt}}} \cdot \mathrm{d}\vec{s}}{\vec{I_{\text{inc}}} \cdot \hat{z}} = |f_{\alpha\alpha}(E,\theta,\phi)|^2 \mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma_\alpha(\theta,\phi)}{\mathrm{d}\Omega} = |f_{\alpha\alpha}(E,\theta,\phi)|^2$$

Bohr, N (1928a) Nature **121**, 580

Bohr, N (1928b) Naturwissenschaften **16**, 245

Bohr, N (1935)

Heisenberg, E (1927)

Heisenberg, E (1930), *The Physical Principles of Quantum Theory*, Dover, New York

Einstein, A (1905)

de Broglie, L (1925)

Born, M (1935)

Born, M and Jordan, P (1925), Zeit. für Physik **34**; 858

Born, M, Hesienberg, W and Jordan, P (1926) Zeit. für Physik **35**; 557 Mach, E (1923)

Boltzmann (1897a)

Boltzmann (1897b)

Greenberg (2000)

Reid, C (1972) *Hilbert*, Springer Verlag, Berlin, Heidelberg

Schrödinger, E ( )

Weinberg, S (1996) *The Quantum Theory of Fields, Vol II*, Cambridge University Press, Cambridge

Dyson, F (1979)

Feynman, RP (1949) *Space–time approach to Quantum Electrodynamics*, PR, 787

Feynman, RP (1975) *Theory of fundamental processes*, Benjamin, Reading, Mass.

Kemp, (2000)

Feynman, RP (1961) *Quantum Electrodynamics*, Frontiers in Physics, Benjamin, Reading, Mass.

Greiner, W (1998) *Quantum Mechanics*, Special Chapters, Springer Verlag, Heidelberg

Brink, D and Broglia, RA (2005) *Nuclear Superfluidity*, Cambridge University Press, Cambridge

Ring and Schuck (1980) *The Nuclear Many–Body Problem*, Springer, Berlin

Baroni et al ( )

Bertsch et al ( )

Bardeen, Cooper and Schrieffer (1957a) *Microscopic Theory of Superconductivity*, Physical Review **106**, 162

Bardeen, Cooper and Schrieffer (1957b) *Theory of Superconductivity*, Physical Review **108**, 1175

Nilsson, (1955)

Nilsson and Ragnarsson (1995)

Bohr and Mottelson (1975) *Nuclear Structure, Vol.II*, Benjamin, New York