~~Matrix element,~~ The transition matrix element is

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle = {}& \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}}\sum_{l\bar{l}m}((l_f\tfrac{1}{2})_{j_f}(l_f\tfrac{1}{2})_{j_f}|(l_fl_f)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 \\
& \times((l_i\tfrac{1}{2})_{j_i}(l_i\tfrac{1}{2})_{j_i}|(l_il_i)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\,\sqrt{2l+1}\,i^{-\bar{l}}\exp[i(\sigma_{\bar{l}}^f+\sigma_l^i)] \\
& \times 2Y_m^{\bar{l}}(\hat{\mathbf{k}}_{bB})\sum_{\sigma_1\sigma_2}\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
& \times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]_0^{0*}\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]_0^0 \\
& \times f_{\bar{l}}(r_{bB})g_l(r_{aA})\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^{0*}Y_m^{\bar{l}*}(\hat{\mathbf{r}}_{bB})V(r_{1p}) \\
& \times\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^0Y_0^l(\hat{\mathbf{r}}_{aA}),
\end{aligned}
\tag{7.44}
$$

~~Simplifying...~~ which after a number of simplifications becomes

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle = {}& \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}}\sum_{l\bar{l}m}\sqrt{\frac{(2j_f+1)(2j_i+1)}{(2l_f+1)(2l_i+1)}} \\
& \times\sqrt{2l+1}\,i^{-\bar{l}}\exp[i(\sigma_{\bar{l}}^f+\sigma_l^i)] \\
& \times Y_m^{\bar{l}}(\hat{\mathbf{k}}_{bB})\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
& \times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]_0^{0*}\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]_0^0 \\
& \times f_{\bar{l}}(r_{bB})g_l(r_{aA})Y_m^{\bar{l}*}(\hat{\mathbf{r}}_{bB})V(r_{1p})Y_0^l(\hat{\mathbf{r}}_{aA}).
\end{aligned}
\tag{7.45}
$$

One has to set ~~We clearly need~~ $l=\bar{l}$ and $m=0$. making use of ~~We also introduce the~~ Legendre polynomials, leads to,

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle = {}& \frac{(4\pi)^{-1/2}}{k_{aA}k_{bB}}\sum_{l}\sqrt{(2j_f+1)(2j_i+1)} \\
& \times\sqrt{2l+1}\,i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]Y_0^l(\hat{\mathbf{k}}_{bB}) \\
& \times\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
& \times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b) \\
& \times f_l(r_{bB})g_l(r_{aA})Y_0^{l*}(\hat{\mathbf{r}}_{bB})V(r_{1p})Y_0^l(\hat{\mathbf{r}}_{aA}).
\end{aligned}
\tag{7.46}
$$

We use ~~change~~ the integration variables and proceed as in last section. This implies ~~what involves~~ multiplying the above expression by $2\pi\sqrt{\frac{4\pi}{2l+1}}$, resulting in

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle = {}& \frac{2\pi}{k_{aA}k_{bB}}\sum_{l}\sqrt{(2j_f+1)(2j_i+1)} \\
& \times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]Y_0^l(\hat{\mathbf{k}}_{bB}) \\
& \times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2} \\
& \times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
& \times f_l(r_{bB})g_l(r_{aA})Y_0^{l*}(\hat{\mathbf{r}}_{bB})V(r_{1p})/r_{bB}.
\end{aligned}
\tag{7.47}
$$

an expression which can be ~~secondly~~ rewritten as
~~We introduce some more polynomials~~,

$$
\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{1}{2k_{aA}k_{bB}}\sum_l\sqrt{(2j_f+1)(2j_i+1)} \\
&\times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]P_l(\cos\theta)(2l+1) \\
&\times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2} \\
&\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})V(r_{1p}) \\
&\times u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})f_l(r_{bB})g_l(r_{aA})P_l(\cos\theta_{if})/r_{bB}.
\end{aligned}
\tag{7.48}
$$

7.~~1~~2.3 coordinates for the calculation of simultaneous transfer ~~calculation~~

To start we have to find

We refer to the notation used in [2]. ~~We must find~~ the expression of the variables appearing in the integral as functions of the integration variables $r_{1p}, r_{2p}, r_{12}, R, \beta, \gamma$ (remember that $\mathbf{R} = R\,\hat{\mathbf{z}}$, see last section). $\mathbf{R}$ being the center of mass coordinate, ~~we have~~ one can write

$$\mathbf{R} = \frac{1}{3}\left(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_p\right) = \frac{1}{3}\left(\mathbf{R} + \mathbf{d}_1 + \mathbf{R} + \mathbf{d}_2 + \mathbf{R} + \mathbf{d}_p\right), \tag{7.49}$$

so

$$\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_p = 0. \tag{7.50}$$

Together with

$$\mathbf{d}_1 + \mathbf{r}_{12} = \mathbf{d}_2 \qquad \mathbf{d}_2 + \mathbf{r}_{2p} = \mathbf{d}_p, \tag{7.51}$$

we find

$$\mathbf{d}_1 = \frac{1}{3}\left(2\mathbf{r}_{12} + \mathbf{r}_{2p}\right), \tag{7.52}$$

and

$$d_1^2 = \frac{1}{9}\left(4r_{12}^2 + r_{2p}^2 + 4\mathbf{r}_{12}\mathbf{r}_{2p}\right). \tag{7.53}$$

Making use of

$$
\begin{aligned}
\mathbf{r}_{12} + \mathbf{r}_{2p} &= \mathbf{r}_{1p} \\
r_{1p}^2 &= r_{12}^2 + r_{2p}^2 + 2\mathbf{r}_{12}\mathbf{r}_{2p} \\
2\mathbf{r}_{12}\mathbf{r}_{2p} &= r_{1p}^2 - r_{12}^2 - r_{2p}^2.
\end{aligned}
\tag{7.54}
$$

~~Finally,~~ one obtains

$$d_1 = \frac{1}{3}\sqrt{2r_{12}^2 + 2r_{1p}^2 - r_{2p}^2}. \tag{7.55}$$

Similarly, ~~we find~~

$$d_2 = \frac{1}{3}\sqrt{2r_{12}^2 + 2r_{2p}^2 - r_{1p}^2} \qquad d_p = \frac{1}{3}\sqrt{2r_{2p}^2 + 2r_{1p}^2 - r_{12}^2}. \tag{7.56}$$

We now calculate ~~express~~ the angle α between $\mathbf{d}_1$ and $\mathbf{r}_{12}$. We have

$$-\mathbf{d}_1\mathbf{r}_{12} = r_{12}d_1\cos(\alpha), \tag{7.57}$$

and

$$
\begin{aligned}
\mathbf{d}_1 + \mathbf{r}_{12} &= \mathbf{d}_2 \\
d_1^2 + r_{12}^2 + 2\mathbf{d}_1\mathbf{r}_{12} &= d_2^2.
\end{aligned}
\tag{7.58}
$$

~~So~~ Consequently,

$$\cos(\alpha) = \frac{d_1^2 + r_{12}^2 - d_2^2}{2r_{12}d_1}. \tag{7.59}$$

The complete determination of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{12}$ can be made by writing their expression in a simple configuration, in which the triangle lies in the xz–plane with $\mathbf{d}_1$ pointing along the positive z–direction, and $\mathbf{R} = 0$. Then, a first rotation $\mathcal{R}_z(\gamma)$ of an angle γ around the z–axis, a second rotation $\mathcal{R}_y(\beta)$ of an angle β around the y–axis, and a translation along $\mathbf{R}$ will bring the vectors to the most general configuration. In other words,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_1, \\ \mathbf{r}_{12} &= \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_{12}, \\ \mathbf{r}_2 &= \mathbf{r}_1 + \mathbf{r}_{12}, \end{aligned} \tag{7.60}$$

with

$$\mathbf{r}'_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \tag{7.61}$$

$$\mathbf{r}'_{12} = r_{12} \begin{bmatrix} \sin(\alpha) \\ 0 \\ -\cos(\alpha) \end{bmatrix}, \tag{7.62}$$

and the rotation matrixes are

$$\mathcal{R}_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \tag{7.63}$$

and

$$\mathcal{R}_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.64}$$

then

We obtain

$$\mathbf{r}_1 = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \tag{7.65}$$

$$\mathbf{r}_{12} = \begin{bmatrix} r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ -r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}, \tag{7.66}$$

$$\mathbf{r}_2 = \begin{bmatrix} d_1\sin(\beta) + r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ R + d_1\cos(\beta) - r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}. \tag{7.67}$$

We also need $\cos(\theta_{12})$, ζ and $\cos(\theta_\zeta)$, θ_{12} being the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, $\zeta = \mathbf{r}_p - \frac{\mathbf{r}_1+\mathbf{r}_2}{A+2}$ the position of the proton with respect to the final nucleus, and θ_ζ the angle between ζ and the z–axis:

$$\cos(\theta_{12}) = \frac{\mathbf{r}_1\mathbf{r}_2}{r_1 r_2}, \tag{7.68}$$

and

$$\zeta = 3\mathbf{R} - \frac{A+3}{A+2}(\mathbf{r}_1 + \mathbf{r}_2), \tag{7.69}$$

where we have used (7.49).

For heavy ions, we find instead

$$\mathbf{R} = \frac{1}{m_a}\left(\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b \mathbf{r}_{Ab}\right), \tag{7.70}$$

$$\mathbf{d}_1 = \frac{1}{m_a}\left(m_b \mathbf{r}_{b2} - (m_b+1)\mathbf{r}_{12}\right), \tag{7.71}$$

$$d_1 = \frac{1}{m_a}\sqrt{(m_b+1)r_{12}^2 + m_b(m_b+1)r_{b1}^2 - m_b r_{b2}^2}, \tag{7.72}$$

$$d_2 = \frac{1}{m_a}\sqrt{(m_b+1)r_{12}^2 + m_b(m_b+1)r_{b2}^2 - m_b r_{b1}^2}, \tag{7.73}$$

and

$$\zeta = \frac{m_a}{m_b}\mathbf{R} - \frac{m_B + m_b}{m_b m_B}(\mathbf{r}_{A1} + \mathbf{r}_{A2}). \tag{7.74}$$

The rest of the formulae are identical to the (t, p) ones, we list them for ~~reference:~~ convenience;

$$\mathbf{r}_{A1} = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \tag{7.75}$$

$$\mathbf{r}_{A2} = \begin{bmatrix} d_1 \sin(\beta) + r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ R + d_1\cos(\beta) - r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}. \tag{7.76}$$

We we also find

$$\mathbf{r}_{b1} = \frac{1}{m_b}(\mathbf{r}_{A2} + (m_b+1)\mathbf{r}_{A1} - m_a \mathbf{R}), \tag{7.77}$$

and

$$\mathbf{r}_{b2} = \frac{1}{m_b}(\mathbf{r}_{A1} + (m_b+1)\mathbf{r}_{A2} - m_a \mathbf{R}). \tag{7.78}$$

~~We easily obtain~~ One can readily obtain

$$\cos\theta_{12} = \frac{r_{A1}^2 + r_{A2}^2 - r_{12}^2}{2 r_{A1} r_{A2}}, \tag{7.79}$$

and

$$\cos\theta_i = \frac{r_{b1}^2 + r_{b2}^2 - r_{12}^2}{2 r_{b1} r_{b2}}. \tag{7.80}$$

7.~~1~~2.4 matrix element for the transition amplitude (2)

The simultaneous amplitude can be written as (see [?])

?

$$\begin{aligned} T^{(1step)}_{2NT} =& 2\frac{(4\pi)^{3/2}}{k_{Aa}k_{Bb}} \sum_{l_p j_p m l_t j_p} i^{-l_p} \exp[i(\sigma^p_{l_p} + \sigma^t_{l_t})]\sqrt{2l_t+1} \\ & \times \langle l_p\, m-m_p\; 1/2\; m_p | j_p\; m\rangle \langle l_t\; 0\; 1/2\; m_t | j_t\; m_t\rangle Y^{l_p}_{m-m_p}(\hat{\mathbf{k}}_{Bb}) \\ & \times \sum_{\sigma_1\sigma_2\sigma_p} \int d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0 \\ & \times v(r_{b1})\left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^0_0 \frac{g_{l_t j_t}(r_{Aa}) f_{l_p j_p}(r_{Bb})}{r_{Aa} r_{Bb}} \\ & \times \left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]^{j_t}_{m_t} \left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]^{j_p *}_{m}. \end{aligned} \tag{7.81}$$

As ~~we have~~ shown ~~before~~ above, ~~we~~ one can write

$$\sum_{\sigma_p}\langle l_p\ m-m_p\ 1/2\ m_p|j_p\ m\rangle\langle l_t\ 0\ 1/2\ m_t|j_t\ m_t\rangle\left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]^{j_t}_{m_t}\left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]^{j_p*}_{m}$$
$$=-\frac{\delta_{l_p,l_t}\delta_{j_p,j_t}\delta_{m,m_t}}{\sqrt{2l+1}}\left[Y^{l}(\hat{\mathbf{r}}_{Aa})Y^{l}(\hat{\mathbf{r}}_{Bb})\right]^0_0\begin{cases}\dfrac{l}{2l+1} & \text{if } m_t=m_p\\ -\dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t=-m_p\end{cases} \tag{7.82}$$

when $j=l-1/2$ and

$$\sum_{\sigma_p}\langle l_p\ m-m_p\ 1/2\ m_p|j_p\ m\rangle\langle l_t\ 0\ 1/2\ m_t|j_t\ m_t\rangle\left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]^{j_t}_{m_t}\left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]^{j_p*}_{m}$$
$$=-\frac{\delta_{l_p,l_t}\delta_{j_p,j_t}\delta_{m,m_t}}{\sqrt{2l+1}}\left[Y^{l}(\hat{\mathbf{r}}_{Aa})Y^{l}(\hat{\mathbf{r}}_{Bb})\right]^0_0\begin{cases}\dfrac{l+1}{2l+1} & \text{if } m_t=m_p\\ \dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t=-m_p\end{cases} \tag{7.83}$$

if $j=l+1/2$. ~~We get~~ One then gets

$$\begin{aligned}T^{1step}_{2NT}=&2\frac{(4\pi)^{3/2}}{k_{Aa}k_{Bb}}\sum_{l}i^{-l}\frac{\exp[i(\sigma^p_l+\sigma^t_l)]}{2l+1}Y^{l}_{m_t-m_p}(\hat{\mathbf{k}}_{Bb})\\
&\times\sum_{\sigma_1\sigma_2}\int\frac{d\mathbf{r}_{Cc}d\mathbf{r}_{b1}d\mathbf{r}_{A2}}{r_{Aa}r_{Bb}}\left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0\\
&\times v(r_{b1})\left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^0_0\left[Y^{l}(\hat{\mathbf{r}}_{Aa})Y^{l}(\hat{\mathbf{r}}_{Bb})\right]^0_0\\
&\times\Big[\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})(l+1)+f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})l\Big)\delta_{m_p,m_t}\\
&+\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})\sqrt{l(l+1)}-f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})\sqrt{l(l+1)}\Big)\delta_{m_p,-m_t}\Big].\end{aligned} \tag{7.84}$$

~~Note~~ Making use of the relations,

$$\begin{aligned}&\left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0\\
&=((l_f\tfrac{1}{2})_{j_f}(l_f\tfrac{1}{2})_{j_f}|(l_fl_f)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0u_{l_f}(r_{A1})u_{l_f}(r_{A2})\\
&\times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]^{0*}_0[\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0\\
&=\sqrt{\frac{2j_f+1}{2(2l_f+1)}}u_{l_f}(r_{A1})u_{l_f}(r_{A2})\\
&\times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]^{0*}_0[\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0\\
&=\sqrt{\frac{2j_f+1}{2}}\frac{u_{l_f}(r_{A1})u_{l_f}(r_{A2})}{4\pi}P_{l_f}(\cos\omega_A)[\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0,\end{aligned} \tag{7.85}$$

and

$$
\begin{aligned}
&\left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^0_0 \\
&\quad= ((l_i\tfrac{1}{2})_{j_i}(l_i\tfrac{1}{2})_{j_i}|(l_il_i)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 u_{l_i}(r_{b1})u_{l_i}(r_{b2}) \\
&\quad\times\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^0_0\left[\chi(\sigma_1)\chi(\sigma_2)\right]^0_0 \\
&\quad= \sqrt{\frac{2j_i+1}{2(2l_i+1)}}u_{l_i}(r_{b1})u_{l_i}(r_{b2}) \\
&\quad\times\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^0_0\left[\chi(\sigma_1)\chi(\sigma_2)\right]^0_0 \\
&\quad= \sqrt{\frac{2j_i+1}{2}}\frac{u_{l_i}(r_{b1})u_{l_i}(r_{b2})}{4\pi}P_{l_i}(\cos\omega_b)\left[\chi(\sigma_1)\chi(\sigma_2)\right]^0_0,
\end{aligned}
\tag{7.86}
$$

where ω_A is the angle between $\mathbf{r}_{A1}$ and $\mathbf{r}_{A2}$, and ω_b is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$, one can write

$$
\begin{aligned}
T^{1step}_{2NT} =&(4\pi)^{-3/2}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l i^{-l}\frac{\exp[i(\sigma^p_l+\sigma^t_l)]}{\sqrt{2l+1}}Y^l_{m_t-m_p}(\hat{\mathbf{k}}_{Bb}) \\
&\times\int\frac{d\mathbf{r}_{Cc}d\mathbf{r}_{b1}d\mathbf{r}_{A2}}{r_{Aa}r_{Bb}}P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if}) \\
&\times v(r_{b1})u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2}) \\
&\times\Big[\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})(l+1)+f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})l\Big)\delta_{m_p,m_t} \\
&+\Big(f_{ll+1/2}(r_{Bb})g_{ll+1/2}(r_{Aa})\sqrt{l(l+1)}-f_{ll-1/2}(r_{Bb})g_{ll-1/2}(r_{Aa})\sqrt{l(l+1)}\Big)\delta_{m_p,-m_t}\Big],
\end{aligned}
\tag{7.87}
$$

where ω_{if} is the angle between $\mathbf{r}_{Aa}$ and $\mathbf{r}_{Bb}$. For heavy ions, we can consider that the the optical potential does not have a spin–orbit term, and the distorted waves are independent of j. We thus have

$$
\begin{aligned}
T^{1step}_{2NT} =&(4\pi)^{-3/2}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l i^{-l}\exp[i(\sigma^p_l+\sigma^t_l)]Y^l_0(\hat{\mathbf{k}}_{Bb})\sqrt{2l+1} \\
&\times\int\frac{d\mathbf{r}_{Cc}d\mathbf{r}_{b1}d\mathbf{r}_{A2}}{r_{Aa}r_{Bb}}P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if}) \\
&\times v(r_{b1})u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2})f_l(r_{Bb})g_l(r_{Aa}).
\end{aligned}
\tag{7.88}
$$

~~We change the variables:~~ Changing variables, one obtains,

$$
\begin{aligned}
T^{1step}_{2NT} =&(4\pi)^{-1}\frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}}\sum_l\exp[i(\sigma^p_l+\sigma^t_l)]P_l(\cos\theta)(2l+1) \\
&\times\int dr_{1A}\,dr_{2A}\,dr_{Aa}\,d(\cos\beta)\,d(\cos\omega_A)\,d\gamma\,r^2_{1A}r^2_{2A}r^2_{Aa} \\
&\times P_{l_f}(\cos\omega_A)P_{l_i}(\cos\omega_b)P_l(\cos\omega_{if})v(r_{b1}) \\
&\times u_{l_i}(r_{b1})u_{l_i}(r_{b2})u_{l_f}(r_{A1})u_{l_f}(r_{A2})f_l(r_{Bb})g_l(r_{Aa}).
\end{aligned}
\tag{7.89}
$$

7.~~1~~2.5 coordinates

We determine the relation between the integration variables in (7.87) and the coordinates needed to evaluate the quantities in the integrand. Noting that

$$\mathbf{r}_{Aa} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b \mathbf{r}_{Ab}}{m_b + 2}, \tag{7.90}$$

we
~~We~~ have

$$\mathbf{r}_{b1} = \mathbf{r}_{bA} + \mathbf{r}_{A1} = \frac{(m_b + 1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b + 2)\mathbf{r}_{Aa}}{m_b}, \tag{7.91}$$

$$\mathbf{r}_{b2} = \mathbf{r}_{bA} + \mathbf{r}_{A2} = \frac{(m_b + 1)\mathbf{r}_{A2} + \mathbf{r}_{A1} - (m_b + 2)\mathbf{r}_{Aa}}{m_b}, \tag{7.92}$$

and

$$\begin{aligned} \mathbf{r}_{Cc} = \mathbf{r}_{CA} + \mathbf{r}_{A1} + \mathbf{r}_{1c} &= -\frac{1}{m_A + 1}\mathbf{r}_{A2} + \mathbf{r}_{A1} - \frac{m_b}{m_b + 1}\mathbf{r}_{b1} \\ &= \frac{m_b + 2}{m_b + 1}\mathbf{r}_{Aa} - \frac{m_b + 2 + m_A}{(m_b + 1)(m_A + 1)}\mathbf{r}_{A2} \end{aligned} \tag{7.93}$$

~~Now, since~~ Since

$$\mathbf{r}_{AB} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2}}{m_A + 2}, \tag{7.94}$$

one obtains 2

$$\mathbf{r}_{Bb} = \mathbf{r}_{BA} + \mathbf{r}_{Ab} = \frac{m_b + 2}{m_b}\mathbf{r}_{Aa} - \frac{\cancel{m_A} + m_b + 2}{(m_A + 2)m_b}(\mathbf{r}_{A1} + \mathbf{r}_{A2}). \tag{7.95}$$

Using
~~We use~~ the same rotations as in section 7.~~1~~.3 ~~to get~~ one gets ,

as those used in Sect. 7,2,3

$$\mathbf{r}_{A1} = r_{A1}\begin{bmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{bmatrix}, \tag{7.96}$$

and

$$\mathbf{r}_{A2} = r_{A2}\begin{bmatrix} -\cos\alpha\cos\gamma\sin\omega_A + \sin\alpha\cos\omega_A \\ -\sin\gamma\sin\omega_A \\ \sin\alpha\cos\gamma\sin\omega_A + \cos\alpha\cos\omega_A \end{bmatrix}, \tag{7.97}$$

with

$$\cos\alpha = \frac{r_{A1}^2 - d_1^2 + r_{Aa}^2}{2 r_{A1} r_{Aa}}, \tag{7.98}$$

and

$$d_1 = \sqrt{r_{A1}^2 - r_{Aa}^2 \sin^2\beta} - r_{Aa}\cos\beta. \tag{7.99}$$

Note that though β, r_{1A}, r_{Aa} are independent integration variables, they have ~~to fulfill~~ (7.89) the condition

$$r_{Aa}\sin\beta \le r_{A1}, \quad \text{for } 0 \le \beta \le \pi. \tag{7.100}$$

The expression of the ~~other~~ remaining quantities appearing in the integral ~~is~~ are now straightforward e

$$\begin{aligned} r_{b1} &= m_b^{-1}\,|(m_b + 1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b + 2)\mathbf{r}_{Aa}| \\ &= m_b^{-1}\Big((m_b + 2)^2 r_{Aa}^2 + (m_b + 1)^2 r_{A1}^2 + r_{A2}^2 \\ &\quad - 2(m_b + 2)(m_b + 1)\mathbf{r}_{Aa}\,\mathbf{r}_{A1} - 2(m_b + 2)\mathbf{r}_{Aa}\,\mathbf{r}_{A2} + 2(m_b + 1)\mathbf{r}_{A1}\mathbf{r}_{A2}\Big)^{1/2}, \end{aligned} \tag{7.101}$$

~~(7.89)~~

$$
\begin{aligned}
r_{b2} &= m_b^{-1}\left|(m_b+1)\mathbf{r}_{A2}+\mathbf{r}_{A1}-(m_b+2)\mathbf{r}_{Aa}\right| \\
&= m_b^{-1}\Big((m_b+2)^2 r_{Aa}^2+(m_b+1)^2 r_{A2}^2+r_{A1}^2 \qquad (7.102)\\
&\quad -2(m_b+2)(m_b+1)\mathbf{r}_{Aa}\mathbf{r}_{A2}-2(m_b+2)\mathbf{r}_{Aa}\mathbf{r}_{A1}+2(m_b+1)\mathbf{r}_{A2}\mathbf{r}_{A1}\Big)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
r_{Bb} &= \left|\frac{m_b+2}{m_b}\mathbf{r}_{Aa}-\frac{m_A+m_b+2}{(m_A+2)m_b}(\mathbf{r}_{A1}+\mathbf{r}_{A2})\right| \\
&= \left[\left(\frac{m_b+2}{m_b}\right)^2 r_{Aa}^2+\left(\frac{m_A+m_b+2}{(m_A+2)m_b}\right)^2(r_{A1}^2+r_{A2}^2+2\mathbf{r}_{A1}\mathbf{r}_{A2}) \qquad (7.103)\right.\\
&\quad \left.-2\frac{(m_b+2)(m_A+m_b+2)}{(m_A+2)m_b^2}\mathbf{r}_{Aa}(\mathbf{r}_{A1}+\mathbf{r}_{A2})\right]^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
r_{Cc} &= \left|\frac{m_b+2}{m_b+1}\mathbf{r}_{Aa}-\frac{m_b+2+m_A}{(m_b+1)(m_A+1)}\mathbf{r}_{A2}\right| \\
&= \left[\left(\frac{m_a}{(m_a-1)}\right)^2 r_{Aa}^2+\left(\frac{m_A+m_a}{(m_A+1)(m_a-1)}\right)^2 r_{A2}^2 \qquad (7.104)\right.\\
&\quad \left.-2\frac{m_A m_a+m_a^2}{(m_A+1)(m_a-1)^2}\mathbf{r}_{Aa}\mathbf{r}_{A2}\right]^{1/2},
\end{aligned}
$$

$$\cos\omega_b = \frac{\mathbf{r}_{b1}\mathbf{r}_{b2}}{r_{b1}r_{b2}}, \qquad (7.105)$$

$$\cos\omega_{if} = \frac{\mathbf{r}_{Aa}\mathbf{r}_{Bb}}{r_{Aa}r_{Bb}}, \qquad (7.106)$$

with

$$\mathbf{r}_{Aa}\mathbf{r}_{A1} = r_{Aa}r_{A1}\cos\alpha, \qquad (7.107)$$

$$\mathbf{r}_{Aa}\mathbf{r}_{A2} = r_{Aa}r_{A2}(\sin\alpha\cos\gamma\sin\omega_A+\cos\alpha\cos\omega_A), \qquad (7.108)$$

$$\mathbf{r}_{A1}\mathbf{r}_{A2} = r_{A1}r_{A2}\cos\omega_A. \qquad (7.109)$$

OK

7.~~2~~ 2.6 successive transfer

Containing two potentials V, following Bayman [ ])

Note that we use time–reversed phases for the spherical harmonics (see (??)) throughout. We write the successive transition amplitude (see [?]):

22

$$
\begin{aligned}
T^{VV}_{2NT} = \frac{4\mu_{Cc}}{\hbar^2}\sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'\\ KM}}\int & d^3r_{Cc}d^3r_{b1}d^3r_{A2}d^3r'_{Cc}d^3r'_{b1}d^3r'_{A2}\,\chi^{(-)*}(\mathbf{k}_{Bb},\mathbf{r}_{Bb}) \\
&\times\left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0 v(r_{b1})\left[\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]^K_M \qquad (7.110)\\
&\times G(\mathbf{r}_{Cc},\mathbf{r}'_{Cc})\left[\psi^{j_f}(\mathbf{r}'_{A2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]^{K*}_M v(r'_{c2}) \\
&\times\left[\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\right]^0_0\chi^{(+)}(\mathbf{r}'_{Aa})
\end{aligned}
$$

Expanding

~~Expansion of~~ the Green function and the distorted waves in a basis of angular momentum eigenstates, one can write,

Bayman 82

$$\chi^{(-)*}(\mathbf{k}_{Bb},\mathbf{r}_{Bb}) = \sum_{\tilde l}\frac{4\pi}{k_{Bb}r_{Bb}}i^{-\tilde l}e^{i\sigma^{\tilde l}_f}F_{\tilde l}\sum_m Y^{\tilde l}_m(\hat r_{Bb})Y^{\tilde l*}_m(\hat k_{Bb})\,) \qquad (7.111)$$

~~As~~ the sum over m ~~is~~ being

$$\sum_m (-1)^{\bar{l}-m} Y^{\bar{l}}_m(\hat{r}_{Bb}) Y^{\bar{l}}_{-m}(\hat{k}_{Bb}) = \sqrt{2\bar{l}+1}\left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{\bar{l}}(\hat{k}_{Bb})\right]^0_0, \qquad (7.112)$$

where we have used (??) and (??), so that

$$\chi^{(-)*}(\mathbf{k}_{Bb}, \mathbf{r}_{Bb}) = \sum_{\bar{l}} \sqrt{2\bar{l}+1}\frac{4\pi}{k_{Bb} r_{Bb}} i^{-\bar{l}} e^{i\sigma^{\bar{l}}_f} F_{\bar{l}}(r_{Bb})\left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{\bar{l}}(\hat{k}_{Bb})\right]^0_0 . \qquad (7.113)$$

Similarly
~~similarly:~~

$$\chi^{(+)}(\mathbf{r}'_{Aa}) = \sum_l i^l \sqrt{2l+1}\frac{4\pi}{k_{Aa} r'_{Aa}} e^{i\sigma^l_i} F_l(r'_{Aa})\left[Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa})\right]^0_0 \qquad (7.114)$$

vector

where we have taken into account that $\hat{k}_{Aa} \equiv \hat{z}$. ~~And~~ The Green function ~~is~~ becomes

$$G(\mathbf{r}_{Cc}, \mathbf{r}'_{Cc}) = i \sum_{l_c} \sqrt{2l_c+1}\frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{k_{Cc} r_{Cc} r'_{Cc}}\left[Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc})\right]^0_0 . \qquad (7.115)$$

Finally

OK

Cosa significa ?

$$\begin{aligned}
T^{(VV)}_{2NT} =& \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \sum_{l, l_c, \bar{l}} e^{i(\sigma^l_i + \sigma^{\bar{l}}_f)} i^{l-\bar{l}} \sqrt{(2l+1)(2l_c+1)(2\bar{l}+1)} \\
&\times \sum_{\substack{\sigma_1\sigma_2 \\ \sigma'_1\sigma'_2}} \int d^3r_{Cc} d^3r_{b1} d^3r_{A2} d^3r'_{Cc} d^3r'_{b1} d^3r'_{A2} v(r_{b1}) v(r'_{c2}) \left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{\bar{l}}(\hat{k}_{Bb})\right]^0_0 \\
&\times \left[Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}'_{Aa})\right]^0_0 \left[Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc})\right]^0_0 \frac{F_{\bar{l}}(r_{Bb})}{r_{Bb}} \frac{F_l(r'_{Aa})}{r'_{Aa}} \\
&\times \frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r_{Cc} r'_{Cc}} \left[\psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)\right]^{0*}_0 \\
&\times \left[\psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2)\right]^0_0 \sum_{KM} \left[\psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1)\right]^K_M \\
&\times \left[\psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1)\right]^{K*}_M
\end{aligned} \qquad (7.116)$$

Let us now perform the integration over $\mathbf{r}_{A2}$

$$
\begin{aligned}
&\sum_{\sigma_1,\sigma_2}\int d\mathbf{r}_{A2}\left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]_0^{0*}\left[\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]_M^K\\
&=\sum_{\sigma_1,\sigma_2}(-1)^{1/2-\sigma_1+1/2-\sigma_2}\int d\mathbf{r}_{A2}\left[\psi^{j_f}(\mathbf{r}_{A1},-\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},-\sigma_2)\right]_0^0\left[\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]_M^K\\
&=-\sum_{\sigma_1,\sigma_2}(-1)^{1/2-\sigma_1+1/2-\sigma_2}\int d\mathbf{r}_{A2}\left[\psi^{j_f}(\mathbf{r}_{A2},-\sigma_2)\psi^{j_f}(\mathbf{r}_{A1},-\sigma_1)\right]_0^0\left[\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]_M^K\\
&=-((j_fj_f)_0(j_fj_i)_K|(j_fj_f)_0(j_fj_i)_K)_K\sum_{\sigma_1,\sigma_2}(-1)^{1/2-\sigma_1+1/2-\sigma_2}\\
&\times\int d\mathbf{r}_{A2}\left[\psi^{j_f}(\mathbf{r}_{A2},-\sigma_2)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]_0^0\left[\psi^{j_f}(\mathbf{r}_{A1},-\sigma_1)\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\right]_M^K\\
&=\frac{1}{2j_f+1}\sqrt{2j_f+1}((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_fl_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\\
&\times u_{l_f}(r_{A1})u_{l_i}(r_{b1})\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]_M^K\sum_{\sigma_1}(-1)^{1/2-\sigma_1}\left[\chi^{1/2}(-\sigma_1)\chi^{1/2}(\sigma_1)\right]_0^0\\
&=-\sqrt{\frac{2}{2j_f+1}}((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_fl_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]_M^K u_{l_f}(r_{A1})u_{l_i}(r_{b1}),
\end{aligned}
\tag{7.117}
$$

used
where we have ~~evaluated~~ the $9j$ symbol

$$
((j_fj_f)_0(j_fj_i)_K|(j_fj_f)_0(j_fj_i)_K)_K=\frac{1}{2j_f+1},\tag{7.118}
$$

as well as ( )
~~and have also used (??)~~. We proceed in a similar way to evaluate the integral over $\mathbf{r}'_{b1}$

$$
\begin{aligned}
&\sum_{\sigma'_1,\sigma'_2}\int d\mathbf{r}'_{b1}\left[\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\right]_0^0\left[\psi^{j_f}(\mathbf{r}'_{A2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]_M^{K*}\\
&=-(-1)^{K-M}\sum_{\sigma'_1,\sigma'_2}\int d\mathbf{r}'_{b1}\left[\psi^{j_f}(\mathbf{r}'_{A2},-\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},-\sigma'_1)\right]_{-M}^K\\
&\times\left[\psi^{j_i}(\mathbf{r}'_{b2},\sigma'_2)\psi^{j_i}(\mathbf{r}'_{b1},\sigma'_1)\right]_0^0(-1)^{1/2-\sigma'_1+1/2-\sigma'_2}\\
&=-(-1)^{K-M}((j_fj_i)_K(j_ij_i)_0|(j_fj_i)_K(j_ij_i)_0)_K(-\sqrt{2j_i+1})\\
&\times((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_fl_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K(-\sqrt{2})u_{l_f}(r'_{A2})u_{l_i}(r'_{b2})\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]_{-M}^K\\
&=-\sqrt{\frac{2}{2j_i+1}}((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_fl_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]_M^{K*}u_{l_f}(r'_{A2})u_{l_i}(r'_{b2}).
\end{aligned}
\tag{7.119}
$$

setting the different contributions ~~Putting all~~ together one obtains,

$$
\begin{aligned}
T^{VV}_{2NT} = & \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \frac{2}{\sqrt{(2j_i+1)(2j_f+1)}} \sum_{K,M} ((l_f \tfrac{1}{2})_{j_f} (l_i \tfrac{1}{2})_{j_i} | (l_f l_i)_K (\tfrac{1}{2}\tfrac{1}{2})_0)^2_K \\
& \times \sum_{l_c, l, \bar{l}} e^{i(\sigma_i^l + \sigma_f^{\bar{l}})} \sqrt{(2l_c+1)(2l+1)(2\bar{l}+1)}\, i^{l-\bar{l}} \\
& \times \int d^3 r_{Cc} d^3 r_{b1} d^3 r'_{Cc} d^3 r'_{A2} v(r_{b1}) v(r'_{c2}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}) \\
& \times \left[Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2}) \right]^{K*}_M \left[Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]^K_M \frac{F_l(r'_{Aa}) F_{\bar{l}}(r'_{Bb}) f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r'_{Aa} r_{Bb} r_{Cc} r'_{Cc}} \\
& \times \left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{\bar{l}}(\hat{k}_{Bb}) \right]^0_0 \left[Y^{l}(\hat{r}'_{Aa}) Y^{l}(\hat{k}_{Aa}) \right]^0_0 \left[Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc}) \right]^0_0 .
\end{aligned}
\tag{7.120}
$$

~~We can write~~ To be used in obtaining (7.128) we note that

$$
\begin{aligned}
& \left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{\bar{l}}(\hat{k}_{Bb}) \right]^0_0 \left[Y^{l}(\hat{r}'_{Aa}) Y^{l}(\hat{k}_{Aa}) \right]^0_0 = \\
& \qquad ((l\,l)_0 (\bar{l}\,\bar{l})_0 | (l\,\bar{l})_0 (l\,\bar{l})_0)_0 \left[Y^{\bar{l}}(\hat{r}_{Bb}) Y^{l}(\hat{r}'_{Aa}) \right]^0_0 \left[Y^{\bar{l}}(\hat{k}_{Bb}) Y^{l}(\hat{k}_{Aa}) \right]^0_0 \\
& \qquad = \frac{\delta_{\bar{l}l}}{2l+1} \left[Y^{l}(\hat{r}_{Bb}) Y^{l}(\hat{r}'_{Aa}) \right]^0_0 \left[Y^{l}(\hat{k}_{Bb}) Y^{l}(\hat{k}_{Aa}) \right]^0_0 .
\end{aligned}
\tag{7.121}
$$

~~Taking into account that~~

And →
$$
\left[Y^{l}(\hat{k}_{Bb}) Y^{l}(\hat{k}_{Aa}) \right]^0_0 = \frac{(-1)^l}{\sqrt{4\pi}} Y^l_0(\hat{k}_{Bb}) i^l , \tag{7.122}
$$

?

and

$$
\begin{aligned}
& \left[Y^{l}(\hat{r}_{Bb}) Y^{l}(\hat{r}'_{Aa}) \right]^0_0 \left[Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc}) \right]^0_0 = \\
& \qquad ((l\,l)_0 (l_c\,l_c)_0 | (l\,l_c)_K (l\,l_c)_K)_0 \left\{ \left[Y^{l}(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^K \left[Y^{l}(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc}) \right]^K \right\}^0_0 \\
& \qquad = \sqrt{\frac{2K+1}{(2l+1)(2l_c+1)}} \\
& \qquad \times \sum_{M'} \frac{(-1)^{K+M'}}{\sqrt{2K+1}} \left[Y^{l}(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^K_{-M'} \left[Y^{l}(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc}) \right]^K_{M'} \\
& \qquad = \sqrt{\frac{1}{(2l+1)(2l_c+1)}} \\
& \qquad \times \sum_{M'} \left[Y^{l}(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^{K*}_{M'} \left[Y^{l}(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc}) \right]^K_{M'} .
\end{aligned}
\tag{7.123}
$$

It is ~~important to note~~ of notice that the integrals

$$
\int d\hat{r}_{Cc} d\hat{r}_{b1} \left[Y^{l}(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^{K*}_M \left[Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]^K_M , \tag{7.124}
$$

and

$$
\int d\hat{r}'_{Cc} d\hat{r}'_{A2} \left[Y^{l}(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc}) \right]^K_M \left[Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2}) \right]^{K*}_M , \tag{7.125}
$$

over the angular variables do not depend on M. Let us see ~~why~~ this with the help of (7.124),

$$\left[Y^{l}(\hat{r}_{Bb})Y^{l_c}(\hat{r}_{Cc})\right]^{K*}_{M}\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}_{M} = (-1)^{K-M}\left[Y^{l}(\hat{r}_{Bb})Y^{l_c}(\hat{r}_{Cc})\right]^{K}_{-M}$$
$$\times\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}_{M} = (-1)^{K-M}\sum_{J}\langle K\,K\,M\ -M|J\,0\rangle \tag{7.126}$$
$$\times\left\{\left[Y^{l}(\hat{r}_{Bb})Y^{l_c}(\hat{r}_{Cc})\right]^{K}\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}\right\}^{J}_{0}.$$

After integration, only the term

$$(-1)^{K-M}\langle K\,K\,M\ -M|0\,0\rangle\left\{\left[Y^{l}(\hat{r}_{Bb})Y^{l_c}(\hat{r}_{Cc})\right]^{K}\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}\right\}^{0}_{0} =$$
$$\frac{1}{\sqrt{2K+1}}\left\{\left[Y^{l}(\hat{r}_{Bb})Y^{l_c}(\hat{r}_{Cc})\right]^{K}\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}\right\}^{0}_{0} \tag{7.127}$$

corresponding to $J = 0$ survives, the corresponding expression ~~which~~ is indeed independent of M. We can thus omit the sum over M and multiply ~~by~~ by $(2K + 1)$, obtaining

$$T^{VV}_{2NT} = \frac{64\mu_{Cc}(\pi)^{3/2}i}{\hbar^2 k_{Aa}k_{Bb}k_{Cc}}\frac{i^{-l}}{\sqrt{(2j_i+1)(2j_f+1)}}$$
$$\times\sum_{K}(2K+1)((l_f\tfrac{1}{2})_{j_f}(l_i\tfrac{1}{2})_{j_i}|(l_f l_i)_K(\tfrac{1}{2}\tfrac{1}{2})_0)^2_K \tag{7.128}$$
$$\times\sum_{l_c,l}\frac{e^{i(\sigma^l_i+\sigma^l_f)}}{\sqrt{(2l+1)}}Y^l_0(\hat{k}_{Bb})S_{K,l,l_c},$$

where ~~with~~

$$S_{K,l,l_c} = \int d^3r_{Cc}d^3r_{b1}v(r_{b1})u_{l_f}(r_{A1})u_{l_i}(r_{b1})\frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}}\frac{F_l(r_{Bb})}{r_{Bb}}$$
$$\times\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}_{M}\left[Y^{l_c}(\hat{r}_{Cc})Y^{l}(\hat{r}_{Bb})\right]^{K*}_{M}, \tag{7.129}$$

and

$$s_{K,l,l_c}(r_{Cc}) = \int_{r_{Cc}\,fixed} d^3r'_{Cc}d^3r'_{A2}v(r'_{c2})u_{l_f}(r'_{A2})u_{l_i}(r'_{b2})\frac{F_l(r'_{Aa})}{r'_{Aa}}\frac{f_{l_c}(k_{Cc},r_<)P_{l_c}(k_{Cc},r_>)}{r'_{Cc}}$$
$$\times\left[Y^{l_f}(\hat{r}'_{A2})Y^{l_i}(\hat{r}'_{b2})\right]^{K*}_{M}\left[Y^{l_c}(\hat{r}'_{Cc})Y^{l}(\hat{r}'_{Aa})\right]^{K}_{M}. \tag{7.130}$$

It can be shown ~~that~~ the ~~The~~ integrand in (7.129) ~~can easily seen to be~~ is independent of M. Consequently, one ~~so we~~ can sum over M and divide by $(2K + 1)$, to get ~~the integrand~~

$$\frac{1}{2K+1}v(r_{b1})u_{l_f}(r_{A1})u_{l_i}(r_{b1})\frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}}\frac{F_l(r_{Bb})}{r_{Bb}}$$
$$\times\sum_{M}\left[Y^{l_f}(\hat{r}_{A1})Y^{l_i}(\hat{r}_{b1})\right]^{K}_{M}\left[Y^{l_c}(\hat{r}_{Cc})Y^{l}(\hat{r}_{Bb})\right]^{K*}_{M}. \tag{7.131}$$

This integrand is rotationally invariant (it is proportional to a T^L_M spherical tensor with $L = 0$, $M = 0$), so we can just evaluate it in the "standard" configuration in which $\mathbf{r}_{Cc}$ is directed along the z–axis and multiply by $8\pi^2$ (see [?]), obtaining the final expression for S_{K,l,l_c}:

$$\begin{aligned}
S_{K,l,l_c} =& \frac{4\pi^{3/2}\sqrt{2l_c+1}}{2K+1} i^{-l_c} \\
&\times \int r_{Cc}^2\, dr_{Cc}\, r_{b1}^2\, dr_{b1}\, \sin\theta\, d\theta\, v(r_{b1}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \\
&\times \frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \\
&\times \sum_M \langle l_c\ 0\ l\ M | K\ M\rangle \left[Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\theta+\pi, 0)\right]^K_M Y^{l*}_M(\hat{r}_{Bb}).
\end{aligned} \tag{7.132}$$

Similarly, we have

$$\begin{aligned}
s_{K,l,l_c}(r_{Cc}) =& \frac{4\pi^{3/2}\sqrt{2l_c+1}}{2K+1} i^{l_c} \\
&\times \int r'^2_{Cc}\, dr'_{Cc}\, r'^2_{A2}\, dr'_{A2}\, \sin\theta'\, d\theta'\, v(r'_{c2}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}) \\
&\times \frac{F_l(r'_{Aa})}{r'_{Aa}} \frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r'_{Cc}} \\
&\times \sum_M \langle l_c\ 0\ l\ M | K\ M\rangle \left[Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2})\right]^{K*}_M Y^l_M(\hat{r}'_{Aa}).
\end{aligned} \tag{7.133}$$

If we do the further approximations $\mathbf{r}_{A1} \approx \mathbf{r}_{C1}$ and $\mathbf{r}_{b2} \approx \mathbf{r}_{c2}$, we obtain the final expression

$$\begin{aligned}
T^{VV}_{2NT} =& \frac{1024\mu_{Cc}\pi^{9/2} i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \frac{1}{\sqrt{(2j_i+1)(2j_f+1)}} \\
&\times \sum_K \frac{1}{2K+1} ((l_f \tfrac{1}{2})_{j_f} (l_i \tfrac{1}{2})_{j_i} | (l_f l_i)_K (\tfrac{1}{2}\tfrac{1}{2})_0)^2_K \\
&\times \sum_{l_c, l} e^{i(\sigma^l_i + \sigma^l_f)} \frac{(2l_c+1)}{\sqrt{2l+1}} Y^l_0(\hat{k}_{Bb}) S_{K,l,l_c},
\end{aligned} \tag{7.134}$$

with

$$\begin{aligned}
S_{K,l,l_c} =& \int r_{Cc}^2\, dr_{Cc}\, r_{b1}^2\, dr_{b1}\, \sin\theta\, d\theta\, v(r_{b1}) u_{l_f}(r_{C1}) u_{l_i}(r_{b1}) \\
&\times \frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \\
&\times \sum_M \langle l_c\ 0\ l\ M | K\ M\rangle \left[Y^{l_f}(\hat{r}_{C1}) Y^{l_i}(\theta+\pi, 0)\right]^K_M Y^{l*}_M(\hat{r}_{Bb}),
\end{aligned} \tag{7.135}$$

and

$$\begin{aligned}
s_{K,l,l_c}(r_{Cc}) =& \int r'^2_{Cc}\, dr'_{Cc}\, r'^2_{A2}\, dr'_{A2}\, \sin\theta'\, d\theta'\, v(r'_{c2}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{c2}) \\
&\times \frac{F_l(r'_{Aa})}{r'_{Aa}} \frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r'_{Cc}} \\
&\times \sum_M \langle l_c\ 0\ l\ M | K\ M\rangle \left[Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{c2})\right]^{K*}_M Y^l_M(\hat{r}'_{Aa}).
\end{aligned} \tag{7.136}$$