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Chapter 1

Chapter 1

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Chapter 7

Two–particle transfer

Cooper pairs are the building blocks of pairing correlations in many-body fermionic systems. In particular in atomic nuclei. As a consequence, nuclear superfluidity can be specifically probed through Cooper pair tunneling. In the simultaneous transfer of two nucleons, one nucleon goes over from target to projectile, or viceversa, under the influence of the nuclear interaction responsible of the existence of a mean field potential, while the other follows suit by profiting of: 1) pairing correlations (simultaneous transfer); 2) the fact that the single-particle wavefunctions describing the motion of Cooper pair partners in both target and projectile are solutions of different single-particle potentials (non-orthogonality transfer). In the limit of independent particle motion, in which all of the nucleon-nucleon interaction is used up in generating a mean field, both contributions to the transfer process (simultaneous and non-orthogonality) cancel out exactly (cf. App. 7.C)

In keeping with the fact that nuclear Cooper pairs are weakly bound ($E_{corr} \ll \epsilon_F$), this cancellation is, in actual nuclei, quite strong. Consequently, successive transfer, a process in which the nuclear interaction acts twice is, as a rule, the main mechanism at the basis of Cooper pair transfer. Because of the same reason (weak binding), the correlation length of Cooper pairs is larger than nuclear dimensions ($\xi = \hbar v_F/E_{corr} \gg R$), a fact which allows the two members of a Cooper pair to move between target and projectile, essentially as a whole, also in the case of successive transfer. In other words, because of its (intrinsic, virtual extension) Cooper pair transfer display equivalent pairing correlations both in simultaneous as in successive transfer [1].

[1] In order for a nucleon to display independent particle motion, all other nucleons must act coherently so as to leave the way free making feel their pullings and pushings only when the nucleon in question tries to leave the self–bound system,thus acting as a reflecting ~~surface which inverts the momentum of the particle~~. It is then natural to consider the nuclear ~~mean~~ field the most striking and fundamental collective feature in all nuclear phenomena. A close second is provided by the BCS mean field, resulting from the condensation of strongly overlapping Cooper pairs (i.e. $\langle BCS|\sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar{\nu}}|BCS\rangle = \alpha_0 \neq 0$) and leading to independent quasiparticle motion. It is a rather unfortunate perversity of popular terminology that regards these collective fields (HF and HFB) as well as successive transfer, as in some sense an antithesis to the nuclear collective modes (Mottelson (1962)) and to simultaneous transfer respectively. Within this context it is of notice that the two–

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The present Chapter is structured in the following way. In section 7.1 we present a summary of two–nucleon transfer reaction theory. It provides the elements needed to calculate the absolute two–nucleon transfer differential cross sections in second order DWBA, and thus to compare theory with experiment. Within this context one can, after reading this section, move directly to Chapter 8 containing examples of applications of this formalism. For the more theoretically oriented reader we provide in section 7.2 a detailed derivation of the equations presented in section 7.1. These equations are implemented and made operative in the softwares used in the applications (cf. App. 8.A). to

In the present Chapter a number of Appendices are provided. ~~In particular~~ one (App. 7.B) in which the derivation of first order DWBA simultaneous transfer is worked out within a formalism tailored to focus the attention on the nuclear structure correlations aspects of the process leading to effective two–nucleon transfer form factors. Another one (App. 7.C) in which the variety of contributions to two–nucleon transfer amplitudes (successive, simultaneous and non–orthogonality) are discussed in detail within the framework of the semi–classical approximation, and other two (App. 7.D and App. 7.E) ~~one~~ in which simple estimates of the relative ~~importance of successive and of simultaneous transfer are worked out.~~ Appendices ~~7.F~~, 7.G, 7.H and 7:I provide elements to be used in the order of magnitude estimates mentioned above, while 7.A deals with nuclear structure processes renormalizing the properties of single particle and collective states and their relation with two–nucleon transfer processes. Appendix 7.F provides simple estimates of the relative importance of final state interactions, while Appendices 7.K and 7.L contain ~~details of the~~ relations used in the derivation presented in Sect. 7.2 while App. 7.N gives a detailed derivation of two-nucleon transfer spectroscopic amplitudes. Finally appendix M provides a glimpse of material which was instrumental to render quantitative studies of two-nucleon transfer which can now be carry out in terms of absolute cross sections and not relative cross sections as done previously (Cf. e.g. Broglia, Riedel and Hansen (1973), Potel et al (2013) and refs. therein))

Within this context, App. 7.J ~~briefly~~ provides an example of coherent state.

7.1 Summary of second order DWBA

Let us illustrate the theory of second order DWBA two–nucleon transfer reactions with the $A + t \to B(\equiv A + 2) + p$ reaction, in which $A + 2$ and A are even nuclei in their 0^+ ground state. The extension of the expressions to the transfer of pairs coupled to arbitrary angular momentum is discussed in subsection 7.2.10.

The wavefunction of the nucleus $A + 2$ can be written as

$$\Psi_{A+2}(\xi_A, \mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \psi_A(\xi_A) \sum_{l_i, j_i} [\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^0_0, \tag{7.1.1}$$

where

$$[\phi^{A+2}_{l_i, j_i}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]^0_0 = \sum_{nm} a_{nm} \left[\varphi^{A+2}_{n, l_i, j_i}(\mathbf{r}_{A1}, \sigma_1)\varphi^{A+2}_{m, l_i, j_i}(\mathbf{r}_{A2}, \sigma_2)\right]^0_0, \tag{7.1.2}$$

while the wavefunctions $\varphi^{A+2}_{n, l_i, j_i}(\mathbf{r})$ are eigenfunctions of a Saxon–Woods potential

$$U(r) = -\frac{V_0}{1 + \exp\left[\frac{r - R_0}{a}\right]}, \qquad R_0 = r_0 A^{1/3}, \tag{7.1.3}$$

nucleon differential cross section associated with the transition between the ground state of superfluid nuclei is proportional to α_0^2 and not to Δ^2. In fact, Cooper pairs partners remain correlated even over regions in which $G = 0$.

Gregory, no sería conveniente tener una figura con las coordenadas usadas en el 2-particle transfer ?

$$T^{(2)}_{succ} = 2\sum_{l_i,j_i}\sum_{l_f,j_f,m_f}\sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'}}\int d\mathbf{r}_{dF}d\mathbf{r}_{p1}d\mathbf{r}_{A2}[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^{0*}_0\chi^{(-)*}_{pB}(\mathbf{r}_{pB})v(\mathbf{r}_{p1})$$
$$\times\phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A2})\int d\mathbf{r}'_{dF}d\mathbf{r}'_{p1}d\mathbf{r}'_{A2}G(\mathbf{r}_{dF},\mathbf{r}'_{dF})$$
$$\times\phi_d(\mathbf{r}'_{p1})^*\varphi^{A+1*}_{l_f,j_f,m_f}(\mathbf{r}'_{A2})\frac{2\mu_{dF}}{\hbar^2}v(\mathbf{r}'_{p2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}), \quad (7.1.5b)$$

$$T^{(2)}_{NO} = 2\sum_{l_i,j_i}\sum_{l_f,j_f,m_f}\sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'}}\int d\mathbf{r}_{dF}d\mathbf{r}_{p1}d\mathbf{r}_{A2}[\phi^{A+2}_{l_i,j_i}(\mathbf{r}_{A1},\sigma_1,\mathbf{r}_{A2},\sigma_2)]^{0*}_0\chi^{(-)*}_{pB}(\mathbf{r}_{pB})v(\mathbf{r}_{p1})$$
$$\times\phi_d(\mathbf{r}_{p1})\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A2})\int d\mathbf{r}'_{p1}d\mathbf{r}'_{A2}d\mathbf{r}'_{dF}$$
$$\times\phi_d(\mathbf{r}'_{p1})^*\varphi^{A+1*}_{l_f,j_f,m_f}(\mathbf{r}'_{A2})\phi_d(\mathbf{r}'_{p1})\phi_d(\mathbf{r}'_{p2})\chi^{(+)}_{tA}(\mathbf{r}'_{tA}). \quad (7.1.5c)$$

The quantities $\mu_i, \mu_f(k_i, k_f)$ are the reduced masses (relative linear momenta) in both entrance (initial, i) and exit (final, f) channels, respectively. In the above expressions, $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ are the wavefunctions describing the intermediate states of the nucleus $F(\equiv (A+1))$, generated as solutions of a Woods–Saxon potential, $\phi_d(\mathbf{r}_{p2})$ being the the deuteron bound wavefunction (see Fig. 7.1.2). Note that some or all of the single–particle states described by the wavefunctions $\varphi^{A+1}_{l_f,j_f,m_f}(\mathbf{r}_{A1})$ may lie in the continuum (case in which the nucleus F is loosely bound or unbound). Although there are a number of ways to exactly treat such states, discretization processes may be sufficiently accurate. They can be implemented by, for example, embedding the Woods–Saxon potential in a spherical box of sufficiently large radius. In actual calculations involving the halo nucleus ^{11}Li, and where $|F\rangle = |^{10}\text{Li}\rangle$, one achieved convergence making use of approximately 20 continuum states and a box of 30 fm of radius. Concerning the components of the triton wavefunction describing the relative motion of the dineutron, it was generated with the $p-n$ interaction (Tang and Herndon, 1965)

$$v(r) = -v_0\exp(-k(r-r_c)) \quad r > r_c \quad (7.1.6)$$
$$v(r) = \infty \quad r < r_c, \quad (7.1.7)$$

where $k = 2.5$ fm^{-1} and $r_c = 0.45$ fm, the depth v_0 being adjusted to reproduce the experimental separation energies. The positive–energy wavefunctions $\chi^{(+)}_{tA}(\mathbf{r}_{tA})$ and $\chi^{(-)}_{pB}(\mathbf{r}_{pB})$ are the ingoing distorted wave in the initial channel and the outgoing distorted wave in the final channel respectively. They are continuum solutions of the Schrödinger equation associated with the corresponding optical potentials.

The transition potential responsible for the transfer of the pair is, in the *post* representation, (cf. Fig. 7.C.1),

$$V_\beta = v_{pB} - U_\beta, \quad (7.1.8)$$

where v_{pB} is the interaction between the proton and nucleus B, and U_β is the optical potential in the final channel. We make the assumption that v_{pB} can be decomposed into a term containing the interaction between A and p and the potential describing the interaction between the proton and each of the transferred nucleons, namely

$$v_{pB} = v_{pA} + v_{p1} + v_{p2}, \tag{7.1.9}$$

where v_{p1} and v_{p2} is the hard–core potential (7.1.6). The transition potential is

$$V_\beta = v_{pA} + v_{p1} + v_{p2} - U_\beta. \tag{7.1.10}$$

Assuming that $\langle\beta|v_{pA}|\alpha\rangle \approx \langle\beta|U_\beta|\alpha\rangle$ (i.e, assuming that the matrix element of the core–core interaction between the initial and final states is very similar to the matrix element of the real part of the optical potential), one obtains the final expression of the transfer potential in the *post* representation, namely,

and

$$V_\beta \simeq v_{p1} + v_{p2} = v(\mathbf{r}_{p1}) + v(\mathbf{r}_{p2}). \tag{7.1.11}$$

We make the further approximation of using the same interaction potential in all (i.e. initial, intermediate and final) the channels.

The extension to a heavy–ion reaction $A + a(\equiv b+2) \longrightarrow B(\equiv A+2) + b$ imply no essential modifications in the formalism. The deuteron and triton wavefunctions appearing in Eqs. (7.1.5a), (7.1.5b) and (7.1.5c) are to be substituted with the corresponding wavefunctions $\Psi_{b+2}(\xi_b, \mathbf{r}_{b1}, \sigma_1, \mathbf{r}_{b2}, \sigma_2)$, constructed in a similar way as those appearing in (7.1.1–7.1.2). The interaction potential used in Eqs. (7.1.5a), (7.1.5b) and (7.1.5c) will now be the Saxon–Woods used to define the initial (final) state in the post (prior) representation, instead of the proton–neutron interaction (7.1.6).

The Green's function $G(\mathbf{r}_{dF}, \mathbf{r}'_{dF})$ appearing in (7.1.5b) propagates the intermediate channel d, F. It can be expanded in partial waves as,

$$G(\mathbf{r}_{dF}, \mathbf{r}'_{dF}) = i\sum_l \sqrt{2l+1}\,\frac{f_l(k_{dF}, r_<)\,g_l(k_{dF}, r_>)}{k_{dF}\, r_{dF}\, r'_{dF}}\left[Y^l(\hat{r}_{dF})Y^l(\hat{r}'_{dF})\right]^0_0. \tag{7.1.12}$$

The $f_l(k_{dF}, r)$ and $g_l(k_{dF}, r)$ are the regular and the irregular solutions of a Schrödinger equation for a suitable optical potential and an energy equal to the kinetic energy of the intermediate state. In most cases of interest, the result is hardly altered if we use the same energy of relative motion for all the intermediate states. This representative energy is calculated when both intermediate nuclei are in their corresponding ground states. It is of notice that the validity of this approximation can break down in some particular cases. If, for example, some relevant intermediate state become off shell, its contribution is significantly quenched. An interesting situation can arise when this happens to all possible intermediate states, so they can only be virtually populated.

7.2 Detailed derivation of second order DWBA

7.2.1 Simultaneous transfer: distorted waves

For a (t, p) reaction, the triton is represented by an incoming distorted wave. We make the assumption that the two neutrons are in an $S = L = 0$ state, and that the

(cf. also App. 7.0)

relative motion of the proton with respect to the dineutron is also $l = 0$. Consequently, the total spin of the triton is entirely due to the spin of the proton. We will explicitly treat it, as we will consider a spin–orbit term in the optical potential acting between the triton and the target. In what follows we will follow the notation of Bayman (1971).

1

Following (7.L.9), we can write the triton distorted wave as

$$\psi^{(+)}_{m_t}(\mathbf{R},\mathbf{k}_i,\sigma_p) = \sum_{l_t} \exp\left(i\sigma^t_{l_t}\right) g_{l_t j_t} Y^{l_t}_0(\hat{\mathbf{R}}) \frac{\sqrt{4\pi(2l_t+1)}}{k_i R} \chi_{m_t}(\sigma_p), \tag{7.2.1}$$

where use was made of $Y^{l_t}_0(\hat{\mathbf{k}}_i) = i^{l_t}\sqrt{\frac{2l_t+1}{4\pi}}\delta_{m_t,0}$, in keeping with the fact that $\mathbf{k}_i$ is oriented along the z–axis. Note the phase difference with eq. (7) of Bayman (1971), due to the use of time–reversal rather than Condon–Shortley phase convention. Making use of the relation

$$Y^{l_t}_0(\hat{\mathbf{R}})\chi_{m_t}(\sigma_p) = \sum_{j_t} \langle l_t\ 0\ 1/2\ m_t | j_t\ m_t\rangle \left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}, \tag{7.2.2}$$

we have

$$\begin{aligned}\psi^{(+)}_{m_t}(\mathbf{R},\mathbf{k}_i,\sigma_p) = \sum_{l_t,j_t} &\exp\left(i\sigma^t_{l_t}\right) \frac{\sqrt{4\pi(2l_t+1)}}{k_i R} g_{l_t j_t}(R) \\ &\times \langle l_t\ 0\ 1/2\ m_t | j_t\ m_t\rangle \left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}.\end{aligned} \tag{7.2.3}$$

We now turn our attention to the outgoing proton distorted wave, which, following (7.L.3) can be written as

$$\psi^{(-)}_{m_p}(\zeta,\mathbf{k}_f,\sigma_p) = \sum_{l_p j_p} \frac{4\pi}{k_f \zeta} i^{l_p} \exp\left(-i\sigma^p_{l_p}\right) f^*_{l_p j_p}(\zeta) \sum_m Y^{l_p}_m(\hat{\zeta}) Y^{l_p *}_m(\hat{\mathbf{k}}_f)\chi_{m_p}(\sigma_p). \tag{7.2.4}$$

Making use of the relation

$$\begin{aligned}\sum_m Y^{l_p}_m(\hat{\zeta}) Y^{l_p *}_m(\hat{\mathbf{k}}_f)\chi_{m_p}(\sigma_p) &= \sum_{m,j_p} Y^{l_p *}_m(\hat{\mathbf{k}}_f)\langle l_p\ m\ 1/2\ m_p | j_p\ m+m_p\rangle \\ &\times \left[Y^{l_p}(\hat{\zeta})\chi_{m_p}(\sigma_p)\right]^{j_p}_{m+m_p} \\ &= \sum_{m,j_p} Y^{l_p *}_{m-m_p}(\hat{\mathbf{k}}_f)\langle l_p\ m-m_p\ 1/2\ m_p | j_p\ m\rangle \left[Y^{l_p}(\hat{\zeta})\chi_{m_p}(\sigma_p)\right]^{j_p}_{m},\end{aligned} \tag{7.2.5}$$

one obtains

$$\begin{aligned}\psi^{(-)}_{m_p}(\zeta,\mathbf{k}_f,\sigma_p) = \frac{4\pi}{k_f\zeta} \sum_{l_p j_p, m} &i^{l_p} \exp\left(-i\sigma^p_{l_p}\right) f^*_{l_p j_p}(\zeta) Y^{l_p *}_{m-m_p}(\hat{\mathbf{k}}_f) \\ &\times \langle l_p\ m-m_p\ 1/2\ m_p | j_p\ m\rangle \left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p}_{m}.\end{aligned} \tag{7.2.6}$$

7.2.2 matrix element for the transition amplitude

We now turn our attention to the evaluation of

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle = \frac{(4\pi)^{3/2}}{k_ik_f}&\sum_{l_pl_tj_pj_tm}((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sqrt{2l_t+1}\\
&\times\langle l_p\; m-m_p\; 1/2\; m_p|j_p\; m\rangle\langle l_t\; 0\; 1/2\; m_t|j_t\; m_t\rangle\, i^{-l_p}\exp[i(\sigma^p_{l_p}+\sigma^t_{l_t})]\\
&\times 2Y^{l_p}_{m-m_p}(\hat{\mathbf{k}}_f)\sum_{\sigma_1\sigma_2\sigma_p}\int\frac{d\zeta d\mathbf{r}d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]^{0*}_0\\
&\times f_{l_pj_p}(\zeta)g_{l_tj_t}(R)\left[\chi(\sigma_1)\chi(\sigma_2)\right]^{0*}_0\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m V(r_{1p})\\
&\times\theta^0_0(\mathbf{r},\mathbf{s})\left[\chi(\sigma_1)\chi(\sigma_2)\right]^0_0\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t},\end{aligned}\tag{7.2.7}$$

where

$$\begin{aligned}\mathbf{r} &=\mathbf{r}_2-\mathbf{r}_1,\\
\mathbf{s} &=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)-\mathbf{r}_p,\\
\eta &=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\\
\zeta &=\mathbf{r}_p-\frac{\mathbf{r}_1+\mathbf{r}_2}{A+2}.\end{aligned}\tag{7.2.8}$$

The sum over σ_1,σ_2 in (7.2.7) is found to be equal to 1. We will now simplify the term $\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}$, noting that, (7.K.13)

$$\left[Y^{l_p}(\hat{\zeta})\chi(\sigma_p)\right]^{j_p*}_m=(-1)^{1/2-\sigma_p+j_p-m}\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}_{-m}.\tag{7.2.9}$$

and that

$$\begin{aligned}\left[Y^{l_p}(\hat{\zeta})\,\chi(-\sigma_p)\right]^{j_p}_{-m}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}_{m_t}&=\sum_{JM}\langle j_p\; -m\; j_t\; m_t|J\; M\rangle\\
&\times\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^J_M\end{aligned}\tag{7.2.10}$$

The only term which does not vanish after the integration is performed is the one in which the angular and spin functions are coupled to $L=0, S=0, J=0$. Thus,

$$\begin{aligned}&\langle j_p\; -m\; j_t\; m_t|0\;0\rangle\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^0_0\delta_{l_pl_t}\delta_{j_pj_t}\delta_{mm_t}\\
&=\frac{(-1)^{j_p+m_t}}{\sqrt{2j_p+1}}\left\{\left[Y^{l_p}(\hat{\zeta})\chi(-\sigma_p)\right]^{j_p}\left[Y^{l_t}(\hat{\mathbf{R}})\chi(\sigma_p)\right]^{j_t}\right\}^0_0\delta_{l_pl_t}\delta_{j_pj_t}\delta_{mm_t}.\end{aligned}\tag{7.2.11}$$

Coupling separately the spin and angular functions, one obtains

$$\left\{\left[Y^l(\hat{\zeta})\chi(-\sigma_p)\right]^j\left[Y^l(\hat{\mathbf{R}})\chi(\sigma_p)\right]^j\right\}_0^0$$
$$= ((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\left[\chi(-\sigma_p)\chi(\sigma_p)\right]_0^0\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0. \tag{7.2.12}$$

We substitute (7.2.9),(7.2.30),(7.2.31) in (7.2.7) to obtain

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle = &-\frac{(4\pi)^{3/2}}{k_ik_f}\sum_{lj}((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sqrt{\frac{2l+1}{2j+1}}\\ &\times\langle l\ m_t-m_p\ 1/2\ m_p|j\ m_t\rangle\langle l\ 0\ 1/2\ m_t|j\ m_t\rangle\, i^{-l}\exp[i(\sigma_l^p+\sigma_l^t)]\\ &\times 2Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int\frac{d\zeta d\mathbf{r}d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*}\\ &\times f_{lj}(\zeta)g_{lj}(R)\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})\\ &\times((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\sum_{\sigma_p}(-1)^{1/2-\sigma_p}\left[\chi(-\sigma_p)\chi(\sigma_p)\right]_0^0.\end{aligned} \tag{7.2.13}$$

The last sum over σ_p leads to

$$\begin{aligned}\sum_{\sigma_p}(-1)^{1/2-\sigma_p}\left[\chi(-\sigma_p)\chi(\sigma_p)\right]_0^0 &= \sum_{\sigma_pm}(-1)^{1/2-\sigma_p}\langle 1/2\ m\ 1/2\ -m|0\ 0\rangle\\ &\times\chi_m(-\sigma_p)\chi_{-m}(\sigma_p)\\ &= \frac{1}{\sqrt{2}}\sum_{\sigma_pm}(-1)^{1/2-\sigma_p}(-1)^{1/2-m}\delta_{m,-\sigma_p}\delta_{-m,\sigma_p} = -\sqrt{2}.\end{aligned} \tag{7.2.14}$$

The $9j$–symbols can be evaluated to find

$$\begin{aligned}((\lambda\tfrac{1}{2})_k(\lambda\tfrac{1}{2})_k|(\lambda\lambda)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 &= \sqrt{\frac{2k+1}{2(2\lambda+1)}}\\ ((l\tfrac{1}{2})_j(l\tfrac{1}{2})_j|(ll)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 &= \sqrt{\frac{2j+1}{2(2l+1)}},\end{aligned} \tag{7.2.15}$$

and consequently,

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle = &\frac{(4\pi)^{3/2}}{k_ik_f}\sum_{lj}\sqrt{\frac{2k+1}{2\lambda+1}}\\ &\times\langle l\ m_t-m_p\ 1/2\ m_p|j\ m_t\rangle\langle l\ 0\ 1/2\ m_t|j\ m_t\rangle\, i^{-l}\exp[i(\sigma_l^p+\sigma_l^t)]\\ &\times\sqrt{2}Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int\frac{d\zeta d\mathbf{r}d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*}\\ &\times f_{lj}(\zeta)g_{lj}(R)\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s}).\end{aligned} \tag{7.2.16}$$

The values of the Clebsh–Gordan coefficients are, for $j = l - 1/2$,

$$\langle l\ m_t - m_p\ 1/2\ m_p | l - 1/2\ m_t\rangle\langle l\ 0\ 1/2\ m_t | l - 1/2\ m_t\rangle = \begin{cases} \dfrac{l}{2l+1} & \text{if } m_t = m_p \\ -\dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases} \tag{7.2.17}$$

and, for $j = l + 1/2$:

$$\langle l\ m_t - m_p\ 1/2\ m_p | l + 1/2\ m_t\rangle\langle l\ 0\ 1/2\ m_t | l + 1/2\ m_t\rangle = \begin{cases} \dfrac{l+1}{2l+1} & \text{if } m_t = m_p \\ \dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases} \tag{7.2.18}$$

One thus can write,

$$\begin{aligned} \langle \Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= \frac{(4\pi)^{3/2}}{k_i k_f}\sum_l \frac{1}{(2l+1)}\sqrt{\frac{(2k+1)}{(2\lambda+1)}}\exp[i(\sigma_l^p + \sigma_l^t)]i^{-l} \\ &\times \sqrt{2}Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int \frac{d\zeta d\mathbf{r} d\eta}{\zeta R}u_{\lambda k}(r_1)u_{\lambda k}(r_2)\left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*} \\ &\times V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0 \\ &\times\Big[\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)(l+1) + f_{ll-1/2}(\zeta)g_{ll-1/2}(R)l\big)\delta_{m_p,m_t} \\ &+\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\sqrt{l(l+1)} - f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\sqrt{l(l+1)}\big)\delta_{m_p,-m_t}\Big]. \end{aligned} \tag{7.2.19}$$

We can further simplify this expression using

$$\begin{aligned} \left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^{0*} = \left[Y^\lambda(\hat{\mathbf{r}}_1)Y^\lambda(\hat{\mathbf{r}}_2)\right]_0^0 &= \sum_m \langle \lambda\ m\ \lambda\ -m|0\ 0\rangle Y^\lambda_m(\hat{\mathbf{r}}_1)Y^\lambda_{-m}(\hat{\mathbf{r}}_2) \\ &= \sum_m (-1)^{\lambda-m}\langle \lambda\ m\ \lambda\ -m|0\ 0\rangle Y^\lambda_m(\hat{\mathbf{r}}_1)Y^{\lambda*}_m(\hat{\mathbf{r}}_2) \\ &= \frac{1}{\sqrt{2\lambda+1}}\sum_m Y^\lambda_m(\hat{\mathbf{r}}_1)Y^{\lambda*}_m(\hat{\mathbf{r}}_2) \\ &= \frac{\sqrt{(2\lambda+1)}}{4\pi}P_\lambda(\cos\theta_{12}). \end{aligned} \tag{7.2.20}$$

Note that when using Condon–Shortley phases this last expression is to be multiplied by $(-1)^\lambda$, and that

$$\begin{aligned} \left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0 &= \sum_m \langle l\ m\ l\ -m|0\ 0\rangle Y^l_m(\hat{\zeta})Y^l_{-m}(\hat{\mathbf{R}}) \\ &= \frac{1}{\sqrt{(2l+1)}}\sum_m (-1)^{l+m}Y^l_m(\hat{\zeta})Y^l_{-m}(\hat{\mathbf{R}}). \end{aligned} \tag{7.2.21}$$

Because the integral of the above expression is independent of m, one can eliminate the m–sum and multiply by $2l+1$ the $m=0$ term, leading to

$$\begin{aligned}\left[Y^l(\hat{\zeta})Y^l(\hat{\mathbf{R}})\right]_0^0 &\Rightarrow (-1)^l\sqrt{(2l+1)}\,Y_0^l(\hat{\zeta})_0 Y^l(\hat{\mathbf{R}}) \\ &= \sqrt{(2l+1)}Y_0^l(\hat{\zeta})Y_0^{l*}(\hat{\mathbf{R}}).\end{aligned} \tag{7.2.22}$$

We now change the integration variables from $(\zeta,\mathbf{r},\eta)$ to $(\mathbf{R},\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p})$, the quantity

$$\left|\frac{\partial(\mathbf{r},\eta,\zeta)}{\partial(\mathbf{R},\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p})}\right| = r_{12}r_{1p}r_{2p}\sin\beta, \tag{7.2.23}$$

being the Jacobian of the transformation. Finally,

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= \frac{\sqrt{8\pi}}{k_ik_f}\sum_l\sqrt{\frac{2k+1}{2l+1}}\exp[i(\sigma_l^p+\sigma_l^t)]i^{-l} \\ &\times Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int d\mathbf{R}Y_0^{l*}(\hat{\mathbf{R}})\int\frac{d\alpha\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta}{\zeta R}Y_0^l(\hat{\zeta}) \\ &\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})\theta_0^0(\mathbf{r},\mathbf{s})P_\lambda(\cos\theta_{12})r_{12}r_{1p}r_{2p} \\ &\times\Big[\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)(l+1)+f_{ll-1/2}(\zeta)g_{ll-1/2}(R)l\big)\delta_{m_p,m_t} \\ &+\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\sqrt{l(l+1)}-f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\sqrt{l(l+1)}\big)\delta_{m_p,-m_t}\Big].\end{aligned} \tag{7.2.24}$$

It is noted that the second integral is a function of solely $\mathbf{R}$ transforming under rotations as $Y_0^l(\hat{\mathbf{R}})$, in keeping with the fact that the full dependence on the orientation of $\mathbf{R}$ is contained in the spherical harmonic $Y_0^l(\hat{\zeta})$. The second integral can thus be cast into the form

$$\begin{aligned}A(R)Y_0^l(\hat{\mathbf{R}}) &= \int d\alpha\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta \\ &\qquad\times F(\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p},R_x,R_y,R_z).\end{aligned} \tag{7.2.25}$$

To evaluate $A(R)$, we set $\mathbf{R}$ along the z–axis

$$\begin{aligned}A(R) &= 2\pi i^{-l}\sqrt{\frac{4\pi}{2l+1}}\int d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,\sin\beta \\ &\qquad\times F(\alpha,\beta,\gamma,r_{12},r_{1p},r_{2p},0,0,R),\end{aligned} \tag{7.2.26}$$

where a factor 2π results from the integration over α, the integrand not depending on α. Substituting (7.2.25) and (7.2.26) in (7.2.24) and, after integration over the angular variables of $\mathbf{R}$, we obtain

$$\begin{aligned}
\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle &= 2\frac{(2\pi)^{3/2}}{k_ik_f}\sum_l\sqrt{\frac{2k+1}{2l+1}}\exp[i(\sigma_l^p+\sigma_l^t)]i^{-l} \\
&\times Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\int dR\,d\beta\,d\gamma\,dr_{12}\,dr_{1p}\,dr_{2p}\,R\sin\beta\,r_{12}r_{1p}r_{2p} \\
&\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})\theta_0^0(\mathbf{r},\mathrm{s})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta) \\
&\times\Big[\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)(l+1)+f_{ll-1/2}(\zeta)g_{ll-1/2}(R)l\big)\delta_{m_p,m_t} \\
&+\big(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\sqrt{l(l+1)}-f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\sqrt{l(l+1)}\big)\delta_{m_p,-m_t}\Big]/\zeta,
\end{aligned} \tag{7.2.27}$$

where use was made of the relation

$$Y_0^l(\hat{\zeta}) = i^l\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta_\zeta). \tag{7.2.28}$$

The final expression of the differential cross section involves a sum over the spin orientations:

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{k_f}{k_i}\frac{\mu_i\mu_f}{(2\pi\hbar^2)^2}\frac{1}{2}\sum_{m_tm_p}|\langle\Psi_f^{(-)}(\mathbf{k}_f)|V(r_{1p})|\Psi_i^{(+)}(k_i,\hat{\mathbf{z}})\rangle|^2. \tag{7.2.29}$$

When $m_p = 1/2, m_t = 1/2$ or $m_p = -1/2, m_t = -1/2$, the terms proportional to δ_{m_p,m_t} including the factor

$$|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,m_t}| = |Y_0^l(\hat{\mathbf{k}}_f)| = \left|i^l\sqrt{\frac{2l+1}{4\pi}}P_l^0(\cos\theta)\right|, \tag{7.2.30}$$

in the case in which $m_p = -1/2, m_t = 1/2$

$$|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y_1^l(\hat{\mathbf{k}}_f)| = \left|i^l\sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}}P_l^1(\cos\theta)\right|, \tag{7.2.31}$$

and

$$|Y^l_{m_t-m_p}(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y^l_{-1}(\hat{\mathbf{k}}_f)| = |Y_1^l(\hat{\mathbf{k}}_f)| = \left|i^l\sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}}P_l^1(\cos\theta)\right|, \tag{7.2.32}$$

when $m_p = 1/2, m_t = -1/2$ Taking the squared modulus of (7.2.27), the sum over m_t and m_p yields a factor 2 multiplying each one of the 2 different terms of the sum ($m_t = m_p$ and $m_t = -m_p$). This is equivalent to multiply each amplitude by $\sqrt{2}$, so the final constant that multiply the amplitudes is

$$\frac{8\pi^{3/2}}{k_ik_f}. \tag{7.2.33}$$

Now, for the triton wavefunction we use

$$\theta_0^0(\mathbf{r},\mathbf{s}) = \rho(r_{1p})\rho(r_{2p})\rho(r_{12}), \tag{7.2.34}$$

$\rho(r)$ being a Tang–Herndon (1965) wave function also used by Bayman (1971). We obtain

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{1}{2E_i^{3/2}E_f^{1/2}}\sqrt{\frac{\mu_f}{\mu_i}}\left(|I_{\lambda k}^{(0)}(\theta)|^2 + |I_{\lambda k}^{(1)}(\theta)|^2\right), \tag{7.2.35}$$

where

$$\begin{aligned} I_{\lambda k}^{(0)}(\theta) = {} & \sum_l P_l^0(\cos\theta)\sqrt{2k+1}\exp[i(\sigma_l^p + \sigma_l^t)] \\ & \times \int dR\, d\beta\, d\gamma\, dr_{12}\, dr_{1p}\, dr_{2p}\, R\sin\beta\, \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ & \times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ & \times \left(f_{ll+1/2}(\zeta)g_{ll+1/2}(R)\,(l+1) + f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\, l\right)/\zeta, \end{aligned} \tag{7.2.36}$$

and

$$\begin{aligned} I_{\lambda k}^{(1)}(\theta) = {} & \sum_l P_l^1(\cos\theta)\sqrt{2k+1}\exp[i(\sigma_l^p + \sigma_l^t)] \\ & \times \int dR\, d\beta\, d\gamma\, dr_{12}\, dr_{1p}\, dr_{2p}\, R\sin\beta\, \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ & \times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ & \times \left(f_{ll+1/2}(\zeta)g_{ll+1/2}(R) - f_{ll-1/2}(\zeta)g_{ll-1/2}(R)\right)/\zeta. \end{aligned} \tag{7.2.37}$$

Note that the absence of the $(-1)^\lambda$ factor with respect to what is found in Bayman (1971), is due to the use of time–reversed phases instead of Condon–Shortley phasing. This is compensated in the total result by a similar difference in the expression of the spectroscopic amplitudes. This ensures that, in either case, the contribution of all the single particle transitions tend to have the same phase for superfluid nuclei, adding coherently to enhance the transfer cross section.

Heavy–ion Reactions A – A

For control, in what follows we work out the same transition amplitude but starting from the distorted waves for a reaction taking place between spinless nuclei, namely

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$$\psi^{(+)}(\mathbf{r}_{Aa},\mathbf{k}_{Aa}) = \sum_l \exp\left(i\sigma_l^i\right) g_l Y_0^l(\hat{\mathbf{r}}_{aA})\frac{\sqrt{4\pi(2l+1)}}{k_{aA}r_{aA}}, \tag{7.2.$\cancel{38}$}$$

and

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$$\psi^{(-)}(\mathbf{r}_{bB},\mathbf{k}_{bB}) = \frac{4\pi}{k_{bB}r_{bB}}\sum_{\bar{l}} i^{\bar{l}}\exp\left(-i\sigma_{\bar{l}}^f\right) f_{\bar{l}}^*(r_{bB})\sum_m Y_m^{\bar{l}*}(\hat{\mathbf{k}}_{bB})Y_m^{\bar{l}}(\hat{\mathbf{r}}_{bB}). \tag{7.2.$\cancel{39}$}$$

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One can then write,

$$
\begin{aligned}
T_{2N}^{1step} = \langle \Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}}\sum_{l\tilde{l}m}((l_f\tfrac{1}{2})_{j_f}(l_f\tfrac{1}{2})_{j_f}|(l_fl_f)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 \\
&\times((l_i\tfrac{1}{2})_{j_i}(l_i\tfrac{1}{2})_{j_i}|(l_il_i)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0\,\sqrt{2l+1}\,i^{-l_p}\exp[i(\sigma_l^f+\sigma_l^i)] \\
&\times 2Y_m^{\tilde{l}}(\hat{\mathbf{k}}_{bB})\sum_{\sigma_1\sigma_2}\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
&\times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]_0^{0*}\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]_0^{0} \\
&\times f_{\tilde{l}}(r_{bB})g_l(r_{aA})\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^{0*}Y_m^{\tilde{l}*}(\hat{\mathbf{r}}_{bB})V(r_{1p}) \\
&\times\left[\chi(\sigma_1)\chi(\sigma_2)\right]_0^{0}Y_0^{l}(\hat{\mathbf{r}}_{aA}),
\end{aligned}
\qquad (7.2.\cancel{40}) \quad 44
$$

w
~~W~~hich, after a number of simplifications becomes

$$
\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}}\sum_{l\tilde{l}m}\sqrt{\frac{(2j_f+1)(2j_i+1)}{(2l_f+1)(2l_i+1)}} \\
&\times\sqrt{2l+1}\,i^{-\tilde{l}}\exp[i(\sigma_l^f+\sigma_l^i)] \\
&\times Y_m^{\tilde{l}}(\hat{\mathbf{k}}_{bB})\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
&\times\left[Y^{l_f}(\hat{\mathbf{r}}_{A1})Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]_0^{0*}\left[Y^{l_i}(\hat{\mathbf{r}}_{b1})Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]_0^{0} \\
&\times f_{\tilde{l}}(r_{bB})g_l(r_{aA})Y_m^{\tilde{l}*}(\hat{\mathbf{r}}_{bB})V(r_{1p})Y_0^{l}(\hat{\mathbf{r}}_{aA}),
\end{aligned}
\qquad (7.2.\cancel{41}) \quad 45
$$

where $l = \tilde{l}$ and $m = 0$. Making use of Legendre polynomials, leads to,

$$
\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{(4\pi)^{-1/2}}{k_{aA}k_{bB}}\sum_{l}\sqrt{(2j_f+1)(2j_i+1)} \\
&\times\sqrt{2l+1}\,i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]Y_0^{l}(\hat{\mathbf{k}}_{bB}) \\
&\times\int\frac{d\mathbf{r}_{bB}d\mathbf{r}d\eta}{r_{bB}r_{aA}}u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\
&\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b) \\
&\times f_l(r_{bB})g_l(r_{aA})Y_0^{l*}(\hat{\mathbf{r}}_{bB})V(r_{1p})Y_0^{l}(\hat{\mathbf{r}}_{aA}).
\end{aligned}
\qquad (7.2.\cancel{42}) \quad 46
$$

Changing the integration variables and proceeding as in last section (implying the multiplicative factor $2\pi\sqrt{\frac{4\pi}{2l+1}}$), the above expression becomes

$$\begin{aligned}\langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{2\pi}{k_{aA}k_{bB}}\sum_l\sqrt{(2j_f+1)(2j_i+1)}\\ &\times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]Y_0^l(\hat{\mathbf{k}}_{bB})\\ &\times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2}\\ &\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})\\ &\times f_l(r_{bB})g_l(r_{aA})Y_0^{l*}(\hat{\mathbf{r}}_{bB})V(r_{1p})/r_{bB},\end{aligned} \qquad (7.2.\cancel{43})\ 47$$

~~W~~ w hich eventually can be recasted, through the use of Legendre polynomials, in the ~~form~~ expression,

$$\begin{aligned}T_{2N}^{1step} = \langle\Psi_f^{(-)}(\mathbf{k}_{bB})|V(r_{1p})|\Psi_i^{(+)}(k_{aA},\hat{\mathbf{z}})\rangle &= \frac{1}{2k_{aA}k_{bB}}\sum_l\sqrt{(2j_f+1)(2j_i+1)}\\ &\times i^{-l}\exp[i(\sigma_l^f+\sigma_l^i)]P_l(\cos\theta)(2l+1)\\ &\times\int dr_{aA}\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,r_{aA}\sin\beta\,r_{12}r_{b1}r_{b2}\\ &\times P_{l_f}(\cos\theta_A)P_{l_i}(\cos\theta_b)u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})V(r_{1p})\\ &\times u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})f_l(r_{bB})g_l(r_{aA})P_l(\cos\theta_{if})/r_{bB},\end{aligned} \qquad (7.2.\cancel{44})\ 48$$

expression which gives the same results as (7.2.41)

Thus

7.2.3 Coordinates for the calculation of simultaneous transfer

In what follows we explicit the coordinates used in the calculation of the above equations. Making use of the notation of Bayman (1971), we find the expression of the variables appearing in the integral as functions of the integration variables $r_{1p}, r_{2p}, r_{12}, R, \beta, \gamma$ (remember that $\mathbf{R} = R\,\hat{\mathbf{z}}$, see last section). $\mathbf{R}$ being the center of mass coordinate, one can write

Gregory

Put the correspon-ding Figure

$$\mathbf{R} = \frac{1}{3}\left(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_p\right) = \frac{1}{3}\left(\mathbf{R}+\mathbf{d}_1+\mathbf{R}+\mathbf{d}_2+\mathbf{R}+\mathbf{d}_p\right), \qquad (7.2.\cancel{45})\ 49$$

so

$$\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_p = 0. \qquad (7.2.\cancel{46})\ 50$$

Together with

$$\mathbf{d}_1+\mathbf{r}_{12} = \mathbf{d}_2 \qquad \mathbf{d}_2+\mathbf{r}_{2p} = \mathbf{d}_p, \qquad (7.2.\cancel{47})\ 51$$

we find

$$\mathbf{d}_1 = \frac{1}{3}\left(2\mathbf{r}_{12}+\mathbf{r}_{2p}\right), \qquad (7.2.\cancel{48})\ 52$$

and

$$d_1^2 = \frac{1}{9}\left(4r_{12}^2 + r_{2p}^2 + 4\mathbf{r}_{12}\mathbf{r}_{2p}\right). \qquad \overset{53}{(7.2.\cancel{49})}$$

Making use of

$$\begin{aligned} \mathbf{r}_{12} + \mathbf{r}_{2p} &= \mathbf{r}_{1p} \\ r_{1p}^2 &= r_{12}^2 + r_{2p}^2 + 2\mathbf{r}_{12}\mathbf{r}_{2p} \\ 2\mathbf{r}_{12}\mathbf{r}_{2p} &= r_{1p}^2 - r_{12}^2 - r_{2p}^2. \end{aligned} \qquad \overset{54}{(7.2.\cancel{50})}$$

one obtains

$$d_1 = \frac{1}{3}\sqrt{2r_{12}^2 + 2r_{1p}^2 - r_{2p}^2}. \qquad \overset{55}{(7.2.\cancel{51})}$$

Similarly,

$$d_2 = \frac{1}{3}\sqrt{2r_{12}^2 + 2r_{2p}^2 - r_{1p}^2} \qquad d_p = \frac{1}{3}\sqrt{2r_{2p}^2 + 2r_{1p}^2 - r_{12}^2}. \qquad \overset{56}{(7.2.\cancel{52})}$$

We now express the angle α between $\mathbf{d}_1$ and $\mathbf{r}_{12}$. We have

$$-\mathbf{d}_1\mathbf{r}_{12} = r_{12}d_1\cos(\alpha), \qquad \overset{57}{(7.2.\cancel{53})}$$

and

$$\begin{aligned} \mathbf{d}_1 + \mathbf{r}_{12} &= \mathbf{d}_2 \\ d_1^2 + r_{12}^2 + 2\mathbf{d}_1\mathbf{r}_{12} &= d_2^2. \end{aligned} \qquad \overset{58}{(7.2.\cancel{54})}$$

Consequently,

$$\cos(\alpha) = \frac{d_1^2 + r_{12}^2 - d_2^2}{2r_{12}d_1}. \qquad \overset{59}{(7.2.\cancel{55})}$$

The complete determination of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{12}$ can be made by writing their expression in a simple configuration, in which the triangle lies in the xz–plane with $\mathbf{d}_1$ pointing along the positive z–direction, and $\mathbf{R} = 0$. Then, a first rotation $\mathcal{R}_z(\gamma)$ of an angle γ around the z–axis, a second rotation $\mathcal{R}_y(\beta)$ of an angle β around the y–axis, and a translation along $\mathbf{R}$ will bring the vectors to the most general configuration. In other words,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_1, \\ \mathbf{r}_{12} &= \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_{12}, \\ \mathbf{r}_2 &= \mathbf{r}_1 + \mathbf{r}_{12}, \end{aligned} \qquad \overset{60}{(7.2.\cancel{56})}$$

with

$$\mathbf{r}'_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \qquad \overset{61}{(7.2.\cancel{57})}$$

$$\mathbf{r}'_{12} = r_{12}\begin{bmatrix} \sin(\alpha) \\ 0 \\ -\cos(\alpha) \end{bmatrix}, \qquad \overset{62}{(7.2.\cancel{58})}$$

and the rotation matrixes are

$$\mathcal{R}_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \qquad (7.2.\cancel{59})\ 63$$

and

$$\mathcal{R}_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (7.2.\cancel{60})\ 64$$

then

$$\mathbf{r}_1 = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \qquad (7.2.\cancel{61})\ 65$$

$$\mathbf{r}_{12} = \begin{bmatrix} r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ -r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}, \qquad (7.2.\cancel{62})\ 66$$

$$\mathbf{r}_2 = \begin{bmatrix} d_1\sin(\beta) + r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ R + d_1\cos(\beta) - r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}. \qquad (7.2.\cancel{63})\ 67$$

We also need $\cos(\theta_{12})$, ζ and $\cos(\theta_\zeta)$, θ_{12} being the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, $\zeta = \mathbf{r}_p - \frac{\mathbf{r}_1+\mathbf{r}_2}{A+2}$ the position of the proton with respect to the final nucleus, and θ_ζ the angle between ζ and the z–axis:

$$\cos(\theta_{12}) = \frac{\mathbf{r}_1\mathbf{r}_2}{r_1 r_2}, \qquad (7.2.\cancel{64})\ 68$$

and

$$\zeta = 3\mathbf{R} - \frac{A+3}{A+2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad (7.2.\cancel{65})\ 69$$

where we have used (7.2.45).

For heavy ions, we find instead

$$\mathbf{R} = \frac{1}{m_a}\left(\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b\mathbf{r}_{Ab}\right), \qquad (7.2.\cancel{66})\ 70$$

$$\mathbf{d}_1 = \frac{1}{m_a}\left(m_b\mathbf{r}_{b2} - (m_b+1)\mathbf{r}_{12}\right), \qquad (7.2.\cancel{67})\ 71$$

$$d_1 = \frac{1}{m_a}\sqrt{(m_b+1)r_{12}^2 + m_b(m_b+1)r_{b1}^2 - m_b r_{b2}^2}, \qquad (7.2.\cancel{68})\ 72$$

$$d_2 = \frac{1}{m_a}\sqrt{(m_b+1)r_{12}^2 + m_b(m_b+1)r_{b2}^2 - m_b r_{b1}^2}, \qquad (7.2.\cancel{69})\ 73$$

and

$$\zeta = \frac{m_a}{m_b}\mathbf{R} - \frac{m_B+m_b}{m_b m_B}(\mathbf{r}_{A1} + \mathbf{r}_{A2}). \qquad (7.2.\cancel{70})\ 74$$

The rest of the formulae are identical to the (t, p) ones. We list them for convenience,

$$\mathbf{r}_{A1} = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \qquad (7.2.\cancel{71})\ 75$$

$$\mathbf{r}_{A2} = \begin{bmatrix} d_1 \sin(\beta) + r_{12}\cos(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\sin(\beta)\cos(\alpha) \\ r_{12}\sin(\gamma)\sin(\alpha) \\ R + d_1\cos(\beta) - r_{12}\sin(\beta)\cos(\gamma)\sin(\alpha) - r_{12}\cos(\alpha)\cos(\beta) \end{bmatrix}. \qquad (7.2.\cancel{72})\ 76$$

We we also find

$$\mathbf{r}_{b1} = \frac{1}{m_b}(\mathbf{r}_{A2} + (m_b + 1)\mathbf{r}_{A1} - m_a\mathbf{R}), \qquad (7.2.\cancel{73})\ 77$$

and

$$\mathbf{r}_{b2} = \frac{1}{m_b}(\mathbf{r}_{A1} + (m_b + 1)\mathbf{r}_{A2} - m_a\mathbf{R}). \qquad (7.2.\cancel{74})\ 78$$

One can readily obtain

$$\cos\theta_{12} = \frac{r_{A1}^2 + r_{A2}^2 - r_{12}^2}{2r_{A1}r_{A2}}, \qquad (7.2.\cancel{75})\ 79$$

and

$$\cos\theta_i = \frac{r_{b1}^2 + r_{b2}^2 - r_{12}^2}{2r_{b1}r_{b2}}. \qquad (7.2.\cancel{76})\ 80$$

A

to p. 63

~~If we are~~ In dealing with a heavy ion reaction, $\theta_0^0(\mathbf{r},\mathrm{s})$ ~~will be~~ are the spatial part of the wavefunction

$$\begin{aligned}\Psi(\mathbf{r}_{b1},\mathbf{r}_{b2},\sigma_1,\sigma_2) &= \left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]_0^0 \\ &= \theta_0^0(\mathbf{r},\mathrm{s})\left[\chi(\sigma_1)\chi(\sigma_2)\right]_{0'}^0,\end{aligned} \qquad (7.2.\cancel{77})\ 38$$

where $\mathbf{r}_{b1}, \mathbf{r}_{b2}$ are the positions of the two neutrons with respect to the b core. It can be shown to be

$$\theta_0^0(\mathbf{r},\mathrm{s}) = \frac{u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2})}{4\pi}\sqrt{\frac{2j_i+1}{2}}P_{l_i}(\cos\theta_i), \qquad (7.2.\cancel{78})\ 39$$

where θ_i is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. Neglecting the spin–orbit term in the optical potential, as is usually done for heavy ion reactions, one obtains

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{\mu_f\mu_i}{16\pi^2\hbar^4k_i^3k_f}|T_l^{j_i,j_f}(\theta)|^2, \qquad (7.2.\cancel{79})\ 40$$

$|T_{2N}^{1\,step}(\theta)|^2$

where

$$\begin{aligned}T_l^{j_i,j_f}(\theta) = &\sum_l (2l+1)P_l(\cos\theta)\sqrt{(2j_i+1)(2j_f+1)}\exp[i(\sigma_l^p + \sigma_l^t)] \\ &\times \int dR\,d\beta\,d\gamma\,dr_{12}\,dr_{b1}\,dr_{b2}\,R\sin\beta\, u_{l_ij_i}(r_{b1})u_{l_ij_i}(r_{b2}) \\ &\times u_{l_fj_f}(r_{A1})u_{l_fj_f}(r_{A2})V(r_{b1})P_A(\cos\theta_{12})P_l(\cos\theta_\zeta) \\ &\times r_{12}r_{b1}r_{b2}P_{l_i}(\cos\theta_i)\frac{f_l(\zeta)g_l(R)}{\zeta},\end{aligned} \qquad (7.2.\cancel{80})\ 41$$

$T_{2N}^{1step}(\theta)$

obtained by using (7,2,39) in (7,2,7) instead of (7,2,34);

$\mathbf{r}_{A1}, \mathbf{r}_{A2}$ being the coordinates of the two transferred of the two neutrons with respect to the A core.

(A)

7.2.4 Matrix element for the transition amplitude (alternative derivation)

In what follows we work an alternative derivation of T_{2N}^{1step}, more closely related to heavy ion reactions. Following Bayman and Chen (1982) it can be written as

$$\begin{aligned}
T_{2NT}^{1step} =& 2\frac{(4\pi)^{3/2}}{k_{Aa}k_{Bb}} \sum_{l_p j_p m l_t j_p} i^{-l_p} \exp[i(\sigma_{l_p}^{p} + \sigma_{l_t}^{t})] \sqrt{2l_t+1} \\
& \times \langle l_p\; m-m_p\; 1/2\; m_p | j_p\; m \rangle \langle l_t\; 0\; 1/2\; m_t | j_t\; m_t \rangle Y_{m-m_p}^{l_p}(\hat{\mathbf{k}}_{Bb}) \\
& \times \sum_{\sigma_1\sigma_2\sigma_p} \int d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2} \left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]_0^{0*} \\
& \times v(r_{b1}) \left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]_0^{0} \frac{g_{l_t j_t}(r_{Aa}) f_{l_p j_p}(r_{Bb})}{r_{Aa} r_{Bb}} \\
& \times \left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]_{m_t}^{j_t} \left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]_{m}^{j_p *}.
\end{aligned} \tag{7.2.81}$$

As shown above one can write,

$$\begin{aligned}
&\sum_{\sigma_p} \langle l_p\; m-m_p\; 1/2\; m_p | j_p\; m \rangle \langle l_t\; 0\; 1/2\; m_t | j_t\; m_t \rangle \left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]_{m_t}^{j_t} \left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]_{m}^{j_p *} \\
&= -\frac{\delta_{l_p,l_t}\delta_{j_p,j_t}\delta_{m,m_t}}{\sqrt{2l+1}} \left[Y^{l}(\hat{\mathbf{r}}_{Aa})Y^{l}(\hat{\mathbf{r}}_{Bb})\right]_0^0 \begin{cases} \dfrac{l}{2l+1} & \text{if } m_t = m_p \\ -\dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\end{aligned} \tag{7.2.82}$$

when $j = l - 1/2$ and

$$\begin{aligned}
&\sum_{\sigma_p} \langle l_p\; m-m_p\; 1/2\; m_p | j_p\; m \rangle \langle l_t\; 0\; 1/2\; m_t | j_t\; m_t \rangle \left[Y^{l_t}(\hat{\mathbf{r}}_{Aa})\chi(\sigma_p)\right]_{m_t}^{j_t} \left[Y^{l_p}(\hat{\mathbf{r}}_{Bb})\chi(\sigma_p)\right]_{m}^{j_p *} \\
&= -\frac{\delta_{l_p,l_t}\delta_{j_p,j_t}\delta_{m,m_t}}{\sqrt{2l+1}} \left[Y^{l}(\hat{\mathbf{r}}_{Aa})Y^{l}(\hat{\mathbf{r}}_{Bb})\right]_0^0 \begin{cases} \dfrac{l+1}{2l+1} & \text{if } m_t = m_p \\ \dfrac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\end{aligned} \tag{7.2.83}$$

if $j = l + 1/2$. One ~~gets~~ then

$$
\begin{aligned}
T_{2NT}^{1step} =& 2\frac{(4\pi)^{3/2}}{k_{Aa}k_{Bb}} \sum_l i^{-l} \frac{\exp[i(\sigma_l^p + \sigma_l^t)]}{2l+1} Y^l_{m_t-m_p}(\hat{\mathbf{k}}_{Bb}) \\
& \times \sum_{\sigma_1\sigma_2} \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} \left[\psi^{j_f}(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0 \\
& \times v(r_{b1}) \left[\psi^{j_i}(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^0_0 \left[Y^l(\hat{\mathbf{r}}_{Aa}) Y^l(\hat{\mathbf{r}}_{Bb})\right]^0_0 \\
& \times \Big[\Big(f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa})(l+1) + f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa}) l\Big)\delta_{m_p,m_t} \\
& + \Big(f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa})\sqrt{l(l+1)} - f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa})\sqrt{l(l+1)}\Big)\delta_{m_p,-m_t}\Big].
\end{aligned}
\tag{7.2.84}
$$

Making use of the relations,

$$
\begin{aligned}
\left[\psi^{j_f}\right. & \left.(\mathbf{r}_{A1},\sigma_1)\psi^{j_f}(\mathbf{r}_{A2},\sigma_2)\right]^{0*}_0 \\
&= ((l_f\tfrac{1}{2})_{j_f}(l_f\tfrac{1}{2})_{j_f}|(l_f l_f)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
&\times \left[Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]^{0*}_0 [\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0 \\
&= \sqrt{\frac{2j_f+1}{2(2l_f+1)}} u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
&\times \left[Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2})\right]^{0*}_0 [\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0 \\
&= \sqrt{\frac{2j_f+1}{2}} \frac{u_{l_f}(r_{A1}) u_{l_f}(r_{A2})}{4\pi} P_{l_f}(\cos\omega_A)\, [\chi(\sigma_1)\chi(\sigma_2)]^{0*}_0,
\end{aligned}
\tag{7.2.85}
$$

and

$$
\begin{aligned}
\left[\psi^{j_i}\right. & \left.(\mathbf{r}_{b1},\sigma_1)\psi^{j_i}(\mathbf{r}_{b2},\sigma_2)\right]^{0}_0 \\
&= ((l_i\tfrac{1}{2})_{j_i}(l_i\tfrac{1}{2})_{j_i}|(l_i l_i)_0(\tfrac{1}{2}\tfrac{1}{2})_0)_0 u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) \\
&\times \left[Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^{0}_0 [\chi(\sigma_1)\chi(\sigma_2)]^{0}_0 \\
&= \sqrt{\frac{2j_i+1}{2(2l_i+1)}} u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) \\
&\times \left[Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2})\right]^{0}_0 [\chi(\sigma_1)\chi(\sigma_2)]^{0}_0 \\
&= \sqrt{\frac{2j_i+1}{2}} \frac{u_{l_i}(r_{b1}) u_{l_i}(r_{b2})}{4\pi} P_{l_i}(\cos\omega_b)\, [\chi(\sigma_1)\chi(\sigma_2)]^{0}_0,
\end{aligned}
\tag{7.2.86}
$$

where ω_A is the angle between $\mathbf{r}_{A1}$ and $\mathbf{r}_{A2}$, and ω_b is the angle between $\mathbf{r}_{b1}$ and $\mathbf{r}_{b2}$. Consequently

$$
\begin{aligned}
T_{2NT}^{1step} =& (4\pi)^{-3/2} \frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}} \sum_l i^{-l} \frac{\exp[i(\sigma_l^p + \sigma_l^t)]}{\sqrt{2l+1}} Y^l_{m_t-m_p}(\hat{\mathbf{k}}_{Bb}) \\
& \times \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} P_{l_f}(\cos\omega_A) P_{l_i}(\cos\omega_b) P_l(\cos\omega_{if}) \\
& \times v(r_{b1}) u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
& \times \Big[\Big(f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa})(l+1) + f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa}) l \Big) \delta_{m_p,m_t} \\
& + \Big(f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa}) \sqrt{l(l+1)} - f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa}) \sqrt{l(l+1)} \Big) \delta_{m_p,-m_t} \Big],
\end{aligned}
\tag{7.2.87}
$$

where ω_{if} is the angle between $\mathbf{r}_{Aa}$ and $\mathbf{r}_{Bb}$. For heavy ions, we can consider that the the optical potential does not have a spin–orbit term, and the distorted waves are independent of j. We thus have

$$
\begin{aligned}
T_{2NT}^{1step} =& (4\pi)^{-3/2} \frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}} \sum_l i^{-l} \exp[i(\sigma_l^p + \sigma_l^t)] Y^l_0(\hat{\mathbf{k}}_{Bb}) \sqrt{2l+1} \\
& \times \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} P_{l_f}(\cos\omega_A) P_{l_i}(\cos\omega_b) P_l(\cos\omega_{if}) \\
& \times v(r_{b1}) u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) f_l(r_{Bb}) g_l(r_{Aa}).
\end{aligned}
\tag{7.2.88}
$$

Changing variables one obtains,

$$
\begin{aligned}
T_{2NT}^{1step} =& (4\pi)^{-1} \frac{\sqrt{(2j_i+1)(2j_f+1)}}{k_{Aa}k_{Bb}} \sum_l \exp[i(\sigma_l^p + \sigma_l^t)] P_l(\cos\theta)(2l+1) \\
& \times \int dr_{1A}\, dr_{2A}\, dr_{Aa}\, d(\cos\beta)\, d(\cos\omega_A)\, d\gamma\, r_{1A}^2 r_{2A}^2 r_{Aa}^2 \\
& \times P_{l_f}(\cos\omega_A) P_{l_i}(\cos\omega_b) P_l(\cos\omega_{if}) v(r_{b1}) \\
& \times u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) f_l(r_{Bb}) g_l(r_{Aa}).
\end{aligned}
\tag{7.2.89}
$$

(Eq. 7.2.89)

7.2.5 Coordinates used to derive Eq. (7.2.89)

We determine the relation between the integration variables in (7.2.87) and the coordinates needed to evaluate the quantities in the integrand. Noting that

$$
\mathbf{r}_{Aa} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b \mathbf{r}_{Ab}}{m_b + 2},
\tag{7.2.90}
$$

one has

$$
\mathbf{r}_{b1} = \mathbf{r}_{bA} + \mathbf{r}_{A1} = \frac{(m_b+1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b+2)\mathbf{r}_{Aa}}{m_b},
\tag{7.2.91}
$$

$$\mathbf{r}_{b2} = \mathbf{r}_{bA} + \mathbf{r}_{A2} = \frac{(m_b+1)\mathbf{r}_{A2} + \mathbf{r}_{A1} - (m_b+2)\mathbf{r}_{Aa}}{m_b}, \tag{7.2.92}$$

and

$$\begin{aligned}\mathbf{r}_{Cc} = \mathbf{r}_{CA} + \mathbf{r}_{A1} + \mathbf{r}_{1c} &= -\frac{1}{m_A+1}\mathbf{r}_{A2} + \mathbf{r}_{A1} - \frac{m_b}{m_b+1}\mathbf{r}_{b1} \\ &= \frac{m_b+2}{m_b+1}\mathbf{r}_{Aa} - \frac{m_b+2+m_A}{(m_b+1)(m_A+1)}\mathbf{r}_{A2}\end{aligned} \tag{7.2.93}$$

Since,

$$\mathbf{r}_{AB} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2}}{m_A+2}, \tag{7.2.94}$$

one obtains

$$\mathbf{r}_{Bb} = \mathbf{r}_{BA} + \mathbf{r}_{Ab} = \frac{m_b+2}{m_b}\mathbf{r}_{Aa} - \frac{m_A+m_b+2}{(m_A+2)m_b}(\mathbf{r}_{A1} + \mathbf{r}_{A2}). \tag{7.2.95}$$

Using the same rotations as those used in Section 7.2.3 one gets,

$$\mathbf{r}_{A1} = r_{A1}\begin{bmatrix}\sin\alpha \\ 0 \\ \cos\alpha\end{bmatrix}, \tag{7.2.96}$$

and

$$\mathbf{r}_{A2} = r_{A2}\begin{bmatrix}-\cos\alpha\cos\gamma\sin\omega_A + \sin\alpha\cos\omega_A \\ -\sin\gamma\sin\omega_A \\ \sin\alpha\cos\gamma\sin\omega_A + \cos\alpha\cos\omega_A\end{bmatrix}, \tag{7.2.97}$$

with

$$\cos\alpha = \frac{r_{A1}^2 - d_1^2 + r_{Aa}^2}{2r_{A1}r_{Aa}}, \tag{7.2.98}$$

and

$$d_1 = \sqrt{r_{A1}^2 - r_{Aa}^2\sin^2\beta} - r_{Aa}\cos\beta. \tag{7.2.99}$$

Note that though β, r_{1A}, r_{Aa} are independent integration variables, they have to fulfill the condition

$$r_{Aa}\sin\beta \leq r_{A1}, \quad \text{for } 0 \leq \beta \leq \pi. \tag{7.2.100}$$

The expression of the remaining quantities appearing in the integral ~~is~~ are now straightforward:

$$\begin{aligned} r_{b1} &= m_b^{-1}\left|(m_b+1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b+2)\mathbf{r}_{Aa}\right| \\ &= m_b^{-1}\Big((m_b+2)^2 r_{Aa}^2 + (m_b+1)^2 r_{A1}^2 + r_{A2}^2 \\ &\quad - 2(m_b+2)(m_b+1)\mathbf{r}_{Aa}\mathbf{r}_{A1} - 2(m_b+2)\mathbf{r}_{Aa}\mathbf{r}_{A2} + 2(m_b+1)\mathbf{r}_{A1}\mathbf{r}_{A2}\Big)^{1/2}, \end{aligned} \tag{7.2.101}$$