

## 0.1 Details of the Calculation

Let us illustrate the calculation with a  $A+t \rightarrow B(\equiv A+2)+p$  reaction, in which  $A+2$  and  $A$  are even nuclei in their  $0^+$  ground state. The extension of the following expressions to the transfer of pairs coupled to arbitrary angular momentum is straightforward. The wavefunction of the nucleus  $A+2$  is

$$\Psi_{A+2}(\xi_A, \mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \psi_A(\xi_A) \sum_{l_i, j_i} [\phi_{l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]_0^0, \quad (1)$$

where

$$\phi_{l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2) = \sum_{nm} a_{nm} \left[ \varphi_{n, l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1) \varphi_{m, l_i, j_i}^{A+2}(\mathbf{r}_{A2}, \sigma_2) \right]_0^0, \quad (2)$$

and the wavefunctions  $\varphi_{n, l_i, j_i}^{A+2}(\mathbf{r})$  are eigenfunctions of a Woods–Saxon potential

$$U(r) = -\frac{V_0}{1 + \exp\left[\frac{r-R_0}{a}\right]}, \quad R_0 = r_0 A^{1/3}. \quad (3)$$

The depth  $V_0$  is adjusted to reproduce the experimental single-particles energies. The spatial part of the wavefunction of the two neutrons in the tritium is  $\phi_t(\mathbf{r}_{p1}, \mathbf{r}_{p2}) = \rho(r_{p1})\rho(r_{p2})\rho(r_{12})$ , where  $r_{p1}, r_{p2}, r_{12}$  are the distances between neutron 1 and the proton, neutron 2 and the proton and between neutrons 1 and 2 respectively, and  $\rho(r)$  is a wavefunction with hard core at  $r = 0.45$  fm, as depicted in Fig 1.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\mu_i \mu_f}{(4\pi\hbar^2)^2} \frac{k_f}{k_i} |T^{(1)} + T_{succ}^{(2)} - T_{NO}^{(2)}|^2, \quad (4)$$

where the three amplitudes contributing to the transfer are (see also [1]):

$$T^{(1)} = 2 \sum_{l_i, j_i} \sum_{\sigma_1 \sigma_2} \int d\mathbf{r}_{tA} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi_{l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]_0^{0*} \chi_{pB}^{(-)*}(\mathbf{r}_{pB}) \times v(\mathbf{r}_{p1}) \phi_t(\mathbf{r}_{p1}, \mathbf{r}_{p2}) \chi_{tA}^{(+)}(\mathbf{r}_{tA}), \quad (5a)$$

$$T_{succ}^{(2)} = 2 \sum_{l_i, j_i} \sum_{l_f, j_f, m_f} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma_1' \sigma_2'}} \int d\mathbf{r}_{dF} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi_{l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]_0^{0*} \chi_{pB}^{(-)*}(\mathbf{r}_{pB}) v(\mathbf{r}_{p1}) \times \phi_d(\mathbf{r}_{p1}) \varphi_{l_f, j_f, m_f}^{A+1}(\mathbf{r}_{A2}) \int d\mathbf{r}'_{dF} d\mathbf{r}'_{p1} d\mathbf{r}'_{A2} G(\mathbf{r}_{dF}, \mathbf{r}'_{dF}) \times \phi_d(\mathbf{r}'_{p1})^* \varphi_{l_f, j_f, m_f}^{A+1*}(\mathbf{r}'_{A2}) \frac{2\mu_{dF}}{\hbar^2} v(\mathbf{r}'_{p2}) \phi_d(\mathbf{r}'_{p1}) \phi_d(\mathbf{r}'_{p2}) \chi_{tA}^{(+)}(\mathbf{r}'_{tA}), \quad (5b)$$

$$T_{NO}^{(2)} = 2 \sum_{l_i, j_i} \sum_{l_f, j_f, m_f} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma_1' \sigma_2'}} \int d\mathbf{r}_{dF} d\mathbf{r}_{p1} d\mathbf{r}_{A2} [\phi_{l_i, j_i}^{A+2}(\mathbf{r}_{A1}, \sigma_1, \mathbf{r}_{A2}, \sigma_2)]_0^{0*} \chi_{pB}^{(-)*}(\mathbf{r}_{pB}) v(\mathbf{r}_{p1}) \times \phi_d(\mathbf{r}_{p1}) \varphi_{l_f, j_f, m_f}^{A+1}(\mathbf{r}_{A2}) \int d\mathbf{r}'_{p1} d\mathbf{r}'_{A2} d\mathbf{r}'_{dF} \times \phi_d(\mathbf{r}'_{p1})^* \varphi_{l_f, j_f, m_f}^{A+1*}(\mathbf{r}'_{A2}) \phi_d(\mathbf{r}'_{p1}) \phi_d(\mathbf{r}'_{p2}) \chi_{tA}^{(+)}(\mathbf{r}'_{tA}). \quad (5c)$$

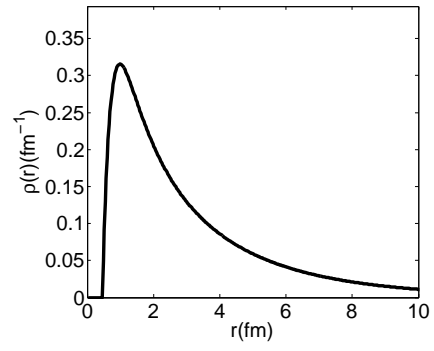


Figure 1: Tritium wavefunction

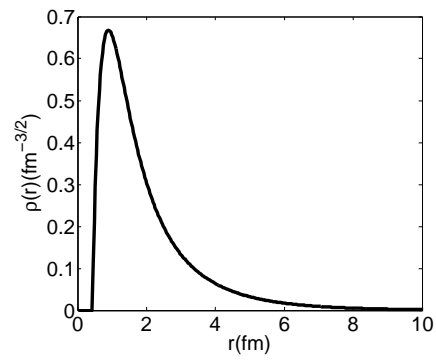


Figure 2: Deuteron wavefunction

In these expressions,  $\varphi_{l_f, j_f, m_f}^{A+1}(\mathbf{r}_{A1})$  are the wavefunctions describing the intermediate states of the nucleus  $F(\equiv A + 1)$ , generated as solutions of a Woods–Saxon potential, and  $\phi_d(\mathbf{r}_{p2})$  is the wavefunction of the deuteron bound state (see Fig. 2). Note that some or all of the  $\varphi_{l_f, j_f, m_f}^{A+1}(\mathbf{r}_{A1})$  may be in the continuum for unbound or loosely bound  $F$ , and some discretization procedure is required in order to deal with these states. In this case, they are generated by embedding the Woods–Saxon potential in a spherical box of large enough radius. In actual calculations we got convergence with less than 20 continuum states in a 30 fm radius box. As for the wavefunction of the neutrons in the tritium, it is generated with the  $p - n$  Tang–Herndon interaction

$$v(r) = -v_0 \exp(-k(r - r_c)) \quad r > r_c \quad (6)$$

$$v(r) = \infty \quad r < r_c, \quad (7)$$

where  $k = 2.5 \text{ fm}^{-1}$  and  $r_c = 0.45 \text{ fm}$ , and the depth  $v_0$  is adjusted to reproduce the experimental separation energies. The positive-energy wavefunctions  $\chi_{tA}^{(+)}(\mathbf{r}_{tA})$  and  $\chi_{pB}^{(-)}(\mathbf{r}_{pB})$  are the ingoing distorted wave in the initial channel and the outgoing distorted wave in the final channel respectively. They are continuum solutions of the Schrödinger equation associated with the corresponding optical potentials.

The transition potential responsible for the transfer of the pair is, in the *post* representation,

$$V_\beta = v_{pB} - U_\beta, \quad (8)$$

where  $v_{pB}$  is the interaction between the proton and nucleus  $B$ , and  $U_\beta$  is the optical potential in the final channel. We make the assumption that  $v_{pB}$  can be decomposed into a term containing the interaction between  $A$  and  $p$  and the potential describing the interaction between the proton and each of the transferred nucleons, namely

$$v_{pB} = v_{pA} + v_{p1} + v_{p2}, \quad (9)$$

where  $v_{p1}$  and  $v_{p2}$  is the hard-core potential (6). The transition potential is

$$V_\beta = v_{pA} + v_{p1} + v_{p2} - U_\beta. \quad (10)$$

Assuming that  $\langle \beta | v_{pA} | \alpha \rangle \simeq \langle \beta | U_\beta | \alpha \rangle$  (i.e, assuming that the matrix element of the core-core interaction between the initial and final states is very similar to the matrix element of the real part of the optical potential), one obtains the final expression of the transfer potential in the *post* representation,

$$V_\beta \simeq v_{p1} + v_{p2} = v(\mathbf{r}_{p1}) + v(\mathbf{r}_{p2}). \quad (11)$$

We make the further approximation of using the same interaction potential in all (e.g. initial, intermediate and final) the channels.

The extension to a heavy-ion reaction  $A + a(\equiv b + 2) \longrightarrow B(\equiv A + 2) + b$  imply no essential modifications in the formalism. The deuteron and triton states in (5a,5b,5c) must be substituted with the corresponding wavefunctions  $\Psi_{b+2}(\xi_b, \mathbf{r}_{b1}, \sigma_1, \mathbf{r}_{b2}, \sigma_2)$ , constructed in a similar way as in (1,2). The interaction potential used in (5a,5b,5c) will now be the Woods–Saxon used to define the initial (final) state in the post (prior) representation, instead of the proton–neutron interaction (6).

The Green function  $G(\mathbf{r}_{dF}, \mathbf{r}'_{dF})$  propagates the intermediate channel  $d, F$ , and can be expanded in partial waves

$$G(\mathbf{r}_{dF}, \mathbf{r}'_{dF}) = i \sum_l \sqrt{2l+1} \frac{f_l(k_{dF}, r_<) P_l(k_{dF}, r_>)}{k_{dF} r_{dF} r'_{dF}} \left[ Y^l(\hat{\mathbf{r}}_{dF}) Y^l(\hat{\mathbf{r}}'_{dF}) \right]_0^0. \quad (12)$$

The  $f_l(k_{dF}, r)$  and  $P_l(k_{dF}, r)$  are the regular and the irregular solutions of a Schrödinger equation with a suitable optical potential and an energy equal to the kinetic energy in the intermediate state. In most cases of interest, the result is hardly altered if we use the same energy of the relative motion between nuclei for all the intermediate states. This representative energy is calculated when both intermediate nuclei are in their corresponding ground states. However, the validity of this approximation can break down in some particular cases. If, for example, some relevant intermediate state become off shell, its contribution is significantly quenched. An interesting situation can arise when this happens to all possible intermediate states, so they can only be virtually populated.

# Bibliography

- [1] B. F. Bayman and J. Chen. One-step and two-step contributions to two-nucleon transfer reactions. *Phys. Rev. C*, 26:1509, 1982.

## Chapter 7

# Two-particle transfer

Cooper pairs are the building blocks of pairing correlations in many-body fermionic systems. In particular in atomic nuclei. As a consequence, nuclear superfluidity can be specifically probed through Cooper pair tunneling.

In the simultaneous transfer of two nucleons, one nucleon goes over from target to projectile, or viceversa, under the influence of the nuclear interaction responsible of the existence of a mean field potential, while the other follows suit by profiting of: 1) pairing correlations (simultaneous transfer); 2) the fact that the single-particle wavefunctions describing the motion of Cooper pair partners in both target and projectile are solutions of different single-particle potentials (non-orthogonality transfer). In the limit of independent particle motion, in which all of the nucleon-nucleon interaction is used up in generating a mean field, both contributions to the transfer process (simultaneous and non-orthogonality) cancel out exactly (cf. App. A).

In keeping with the fact that nuclear Cooper pairs are weakly bound, this cancellation is, in actual nuclei, quite strong. Consequently, successive transfer, a process in which the mean field acts twice is, as a rule, the main mechanism at the basis of Cooper pair transfer. Because of the same reason (weak binding), the correlation length of Cooper pairs is larger than nuclear dimensions, a fact which allows the two members of a Cooper pair, to move between target and projectile, essentially as a whole, also in the case of successive transfer.

Three appendixes are provided. One in which the cancellations existing between the different contributions to the two-nucleon transfer spectroscopic amplitudes (successive, simultaneous and non-orthogonality) are discussed in detail within the framework of the semi-classical approximation. Another one in which simple estimates of the relative importance of successive and of simultaneous transfer are worked out. Finally a derivation of first order DWBA simultaneous transfer is worked out within a formalism tailored to focus the attention on the nuclear structure correlations aspects of the process leading to effective two-nucleon transfer form factors.

The present Chapter is structured in the following way. In section ?? we present a summary of two-nucleon transfer reaction theory. These are all the elements needed to calculate the absolute two-nucleon transfer differential cross sections in second order DWBA, and thus to compare theory with experiment. In this way, after reading this section one can go directly to the next Chapter which contains examples of the applications of this formalism.

For the more theoretically oriented readers we provide in section ?? a detailed derivation of the equations presented in section ?? and which are implemented in the

software COOPS+... used in the applications

## 7.1 simultaneous transfer

### 7.1.1 distorted waves

For a  $(t, p)$  reaction, the triton is represented by an incoming wave. We make the assumption that the two transferred neutrons are in the  $S = 0$  singlet state and that the triton has orbital angular momentum  $L = 0$ , so the spin is entirely due to the spin of the proton. We will explicitly treat it, as, unlike in [?], we will consider a spin-orbit term in the optical potential between the triton and the heavy ion. We use the notation of [?].

After (??) we can write the triton distorted wave as

$$\psi_{m_t}^{(+)}(\mathbf{R}, \mathbf{k}_i, \sigma_p) = \sum_{l_t} \exp(i\sigma_{l_t}^t) g_{l_t, j_t} Y_0^{l_t}(\hat{\mathbf{R}}) \frac{\sqrt{4\pi(2l_t + 1)}}{k_i R} \chi_{m_t}(\sigma_p), \quad (7.1)$$

where we have used  $Y_0^{l_t}(\hat{\mathbf{k}}_i) = i^{l_t} \sqrt{\frac{2l_t+1}{4\pi}} \delta_{m_t, 0}$ , as  $\mathbf{k}_i$  is oriented along the  $z$ -axis. Note the phase difference with eq. (7) of [?], due to the use of time-reversal rather than Condon-Shortley phase convention. If we write

$$Y_0^{l_t}(\hat{\mathbf{R}}) \chi_{m_t}(\sigma_p) = \sum_{j_t} \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle \left[ Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p) \right]_{m_t}^{j_t}, \quad (7.2)$$

we have

$$\begin{aligned} \psi_{m_t}^{(+)}(\mathbf{R}, \mathbf{k}_i, \sigma_p) &= \sum_{l_t, j_t} \exp(i\sigma_{l_t}^t) \frac{\sqrt{4\pi(2l_t + 1)}}{k_i R} g_{l_t, j_t}(R) \\ &\quad \times \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle \left[ Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p) \right]_{m_t}^{j_t}. \end{aligned} \quad (7.3)$$

We now turn our attention to the outgoing proton distorted wave, which, after (??), is

$$\psi_{m_p}^{(-)}(\zeta, \mathbf{k}_f, \sigma_p) = \sum_{l_p, j_p} \frac{4\pi}{k_f \zeta} i^{l_p} \exp(-i\sigma_{l_p}^p) f_{l_p, j_p}^*(\zeta) \sum_m Y_m^{l_p}(\hat{\zeta}) Y_m^{l_p*}(\hat{\mathbf{k}}_f) \chi_{m_p}(\sigma_p). \quad (7.4)$$

Now,

$$\begin{aligned} \sum_m Y_m^{l_p}(\hat{\zeta}) Y_m^{l_p*}(\hat{\mathbf{k}}_f) \chi_{m_p}(\sigma_p) &= \sum_{m, j_p} Y_m^{l_p*}(\hat{\mathbf{k}}_f) \langle l_p \ m \ 1/2 \ m_p | j_p \ m + m_p \rangle \\ &\quad \times \left[ Y^{l_p}(\hat{\zeta}) \chi_{m_p}(\sigma_p) \right]_{m+m_p}^{j_p} \\ &= \sum_{m, j_p} Y_{m-m_p}^{l_p*}(\hat{\mathbf{k}}_f) \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \left[ Y^{l_p}(\hat{\zeta}) \chi_{m_p}(\sigma_p) \right]_m^{j_p}, \end{aligned} \quad (7.5)$$

and, finally,

$$\begin{aligned} \psi_{m_p}^{(-)}(\zeta, \mathbf{k}_f, \sigma_p) &= \frac{4\pi}{k_f \zeta} \sum_{l_p, j_p, m} i^{l_p} \exp(-i\sigma_{l_p}^p) f_{l_p, j_p}^*(\zeta) Y_{m-m_p}^{l_p*}(\hat{\mathbf{k}}_f) \\ &\quad \times \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \left[ Y^{l_p}(\hat{\zeta}) \chi(\sigma_p) \right]_m^{j_p}. \end{aligned} \quad (7.6)$$

### 7.1.2 matrix element for the transition amplitude (1)

We now turn our attention to the evaluation of

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{3/2}}{k_i k_f} \sum_{l_p l_t j_p j_t m} ((\lambda \frac{1}{2})_k (\lambda \frac{1}{2})_k | (\lambda \lambda)_0 (\frac{1}{2} \frac{1}{2})_0 )_0 \sqrt{2l_t + 1} \\
&\times \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle i^{-l_p} \exp[i(\sigma_{l_p}^p + \sigma_{l_t}^t)] \\
&\times 2Y_{m-m_p}^{l_p}(\hat{\mathbf{k}}_f) \sum_{\sigma_1 \sigma_2 \sigma_p} \int \frac{d\zeta d\mathbf{r} d\boldsymbol{\eta}}{\zeta R} u_{\lambda k}(r_1) u_{\lambda k}(r_2) [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^{0*} \\
&\times f_{l_p j_p}(\zeta) g_{l_t j_t}(\mathbf{R}) [\chi(\sigma_1) \chi(\sigma_2)]_0^{0*} [Y^{l_p}(\hat{\zeta}) \chi(\sigma_p)]_m^{j_p*} V(r_{1p}) \\
&\times \theta_0^0(\mathbf{r}, \mathbf{s}) [\chi(\sigma_1) \chi(\sigma_2)]_0^0 [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]_{m_t}^{j_t},
\end{aligned} \tag{7.7}$$

where

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\
\mathbf{s} &= \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) - \mathbf{r}_p \\
\boldsymbol{\eta} &= \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \\
\boldsymbol{\zeta} &= \mathbf{r}_p - \frac{\mathbf{r}_1 + \mathbf{r}_2}{A + 2}.
\end{aligned} \tag{7.8}$$

The sum over  $\sigma_1, \sigma_2$  in (7.7) is readily found to be 1. We will now simplify the term  $[Y^{l_p}(\hat{\zeta}) \chi(\sigma_p)]_m^{j_p*} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]_{m_t}^{j_t}$ , first noting that, after (??)

$$[Y^{l_p}(\hat{\zeta}) \chi(\sigma_p)]_m^{j_p*} = (-1)^{1/2 - \sigma_p + j_p - m} [Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]_{-m}^{j_p}. \tag{7.9}$$

On the other hand,

$$\begin{aligned}
[Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]_{-m}^{j_p} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]_{m_t}^{j_t} &= \sum_{JM} \langle j_p \ -m \ j_t \ m_t | J \ M \rangle \\
&\times \left\{ [Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]^{j_p} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]^{j_t} \right\}_M^J
\end{aligned} \tag{7.10}$$

In order to survive the integration, the angular and spin functions must couple to  $L = 0, S = 0, J = 0$ , so the only term that remains is

$$\begin{aligned}
\langle j_p \ -m \ j_t \ m_t | 0 \ 0 \rangle &\left\{ [Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]^{j_p} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]^{j_t} \right\}_0^0 \delta_{l_p l_t} \delta_{j_p j_t} \delta_{m m_t} \\
&= \frac{(-1)^{j_p + m_t}}{\sqrt{2j_p + 1}} \left\{ [Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]^{j_p} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]^{j_t} \right\}_0^0 \delta_{l_p l_t} \delta_{j_p j_t} \delta_{m m_t}.
\end{aligned} \tag{7.11}$$

We couple separately the spin and spatial functions:

$$\begin{aligned}
&\left\{ [Y^{l_p}(\hat{\zeta}) \chi(-\sigma_p)]^{j_p} [Y^{l_t}(\hat{\mathbf{R}}) \chi(\sigma_p)]^{j_t} \right\}_0^0 \\
&= ((l \frac{1}{2})_j (l \frac{1}{2})_j | (ll)_0 (\frac{1}{2} \frac{1}{2})_0 )_0 [\chi(-\sigma_p) \chi(\sigma_p)]_0^0 [Y^{l_p}(\hat{\zeta}) Y^{l_t}(\hat{\mathbf{R}})]_0^0.
\end{aligned} \tag{7.12}$$



We substitute (7.9),(7.30),(7.31) in (7.7) to obtain

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= -\frac{(4\pi)^{3/2}}{k_i k_f} \sum_{lj} ((\lambda \frac{1}{2})_k (\lambda \frac{1}{2})_k | (\lambda \lambda)_0 (\frac{1}{2} \frac{1}{2})_0)_0 \sqrt{\frac{2l+1}{2j+1}} \\
&\times \langle l m_t - m_p \ 1/2 \ m_p | j \ m_t \rangle \langle l \ 0 \ 1/2 \ m_t | j \ m_t \rangle i^{-l} \exp[i(\sigma_l^p + \sigma_l^t)] \\
&\times 2Y_{m_t-m_p}^l(\hat{\mathbf{k}}_f) \int \frac{d\zeta d\mathbf{r} d\boldsymbol{\eta}}{\zeta R} u_{\lambda k}(r_1) u_{\lambda k}(r_2) [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^{0*} \\
&\times f_{lj}(\zeta) g_{lj}(R) [Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}})]_0^0 V(r_{1p}) \theta_0^0(\mathbf{r}, \mathbf{s}) \\
&\times ((l \frac{1}{2})_j (l \frac{1}{2})_j | (ll)_0 (\frac{1}{2} \frac{1}{2})_0)_0 \sum_{\sigma_p} (-1)^{1/2-\sigma_p} [\chi(-\sigma_p) \chi(\sigma_p)]_0^0.
\end{aligned} \tag{7.13}$$

The last sum over  $\sigma_p$  is

$$\begin{aligned}
\sum_{\sigma_p} (-1)^{1/2-\sigma_p} [\chi(-\sigma_p) \chi(\sigma_p)]_0^0 &= \sum_{\sigma_p m} (-1)^{1/2-\sigma_p} \langle 1/2 \ m \ 1/2 \ -m | 0 \ 0 \rangle \\
&\times \chi_m(-\sigma_p) \chi_{-m}(\sigma_p) \\
&= \frac{1}{\sqrt{2}} \sum_{\sigma_p m} (-1)^{1/2-\sigma_p} (-1)^{1/2-m} \delta_{m, -\sigma_p} \delta_{-m, \sigma_p} = -\sqrt{2}.
\end{aligned} \tag{7.14}$$

The  $9j$  symbols can be evaluated to find

$$\begin{aligned}
((\lambda \frac{1}{2})_k (\lambda \frac{1}{2})_k | (\lambda \lambda)_0 (\frac{1}{2} \frac{1}{2})_0)_0 &= \sqrt{\frac{2k+1}{2(2\lambda+1)}} \\
((l \frac{1}{2})_j (l \frac{1}{2})_j | (ll)_0 (\frac{1}{2} \frac{1}{2})_0)_0 &= \sqrt{\frac{2j+1}{2(2l+1)}},
\end{aligned} \tag{7.15}$$

so

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{3/2}}{k_i k_f} \sum_{lj} \sqrt{\frac{2k+1}{2\lambda+1}} \\
&\times \langle l m_t - m_p \ 1/2 \ m_p | j \ m_t \rangle \langle l \ 0 \ 1/2 \ m_t | j \ m_t \rangle i^{-l} \exp[i(\sigma_l^p + \sigma_l^t)] \\
&\times \sqrt{2} Y_{m_t-m_p}^l(\hat{\mathbf{k}}_f) \int \frac{d\zeta d\mathbf{r} d\boldsymbol{\eta}}{\zeta R} u_{\lambda k}(r_1) u_{\lambda k}(r_2) [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^{0*} \\
&\times f_{lj}(\zeta) g_{lj}(R) [Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}})]_0^0 V(r_{1p}) \theta_0^0(\mathbf{r}, \mathbf{s}).
\end{aligned} \tag{7.16}$$

We now check the possible values of the Clebsh–Gordan coefficients, finding, for  $j = l - 1/2$ :

$$\begin{aligned}
\langle l m_t - m_p \ 1/2 \ m_p | l - 1/2 \ m_t \rangle \langle l \ 0 \ 1/2 \ m_t | l - 1/2 \ m_t \rangle \\
= \begin{cases} \frac{l}{2l+1} & \text{if } m_t = m_p \\ -\frac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\end{aligned} \tag{7.17}$$

and, for  $j = l + 1/2$ :

$$\begin{aligned} & \langle l m_t - m_p \ 1/2 \ m_p | l + 1/2 \ m_t \rangle \langle l \ 0 \ 1/2 \ m_t | l + 1/2 \ m_t \rangle \\ &= \begin{cases} \frac{l+1}{2l+1} & \text{if } m_t = m_p \\ \frac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases} \end{aligned} \quad (7.18)$$

Substituting, we get

$$\begin{aligned} & \langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle = \frac{(4\pi)^{3/2}}{k_i k_f} \sum_l \frac{1}{(2l+1)} \sqrt{\frac{(2k+1)}{(2\lambda+1)}} \exp[i(\sigma_l^p + \sigma_l^t)] i^{-l} \\ & \times \sqrt{2} Y_{m_t - m_p}^l(\hat{\mathbf{k}}_f) \int \frac{d\zeta d\mathbf{r} d\boldsymbol{\eta}}{\zeta R} u_{\lambda k}(r_1) u_{\lambda k}(r_2) [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^{0*} \\ & \times V(r_{1p}) \theta_0^l(\mathbf{r}, \mathbf{s}) [Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}})]_0^0 \\ & \times \left[ (f_{l+1/2}(\zeta) g_{l+1/2}(R) (l+1) + f_{l-1/2}(\zeta) g_{l-1/2}(R) l) \delta_{m_p, m_t} \right. \\ & \left. + (f_{l+1/2}(\zeta) g_{l+1/2}(R) \sqrt{l(l+1)} - f_{l-1/2}(\zeta) g_{l-1/2}(R) \sqrt{l(l+1)}) \delta_{m_p, -m_t} \right]. \end{aligned} \quad (7.19)$$

We can further simplify this expression using

$$\begin{aligned} & [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^{0*} = [Y^\lambda(\hat{\mathbf{r}}_1) Y^\lambda(\hat{\mathbf{r}}_2)]_0^0 = \sum_m \langle \lambda \ m \ \lambda \ -m | 0 \ 0 \rangle Y_m^\lambda(\hat{\mathbf{r}}_1) Y_{-m}^\lambda(\hat{\mathbf{r}}_2) \\ &= \sum_m (-1)^{\lambda-m} \langle \lambda \ m \ \lambda \ -m | 0 \ 0 \rangle Y_m^\lambda(\hat{\mathbf{r}}_1) Y_m^{\lambda*}(\hat{\mathbf{r}}_2) \\ &= \frac{1}{\sqrt{2\lambda+1}} \sum_m Y_m^\lambda(\hat{\mathbf{r}}_1) Y_m^{\lambda*}(\hat{\mathbf{r}}_2) \\ &= \frac{\sqrt{(2\lambda+1)}}{4\pi} P_\lambda(\cos \theta_{12}). \end{aligned} \quad (7.20)$$

Note that when using Condon–Shortley phases this last expression would be multiplied by  $(-1)^l$ . Now,

$$\begin{aligned} & [Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}})]_0^0 = \sum_m \langle l \ m \ l \ -m | 0 \ 0 \rangle Y_m^l(\hat{\zeta}) Y_{-m}^l(\hat{\mathbf{R}}) \\ &= \frac{1}{\sqrt{(2l+1)}} \sum_m (-1)^{l+m} Y_m^l(\hat{\zeta}) Y_{-m}^l(\hat{\mathbf{R}}). \end{aligned} \quad (7.21)$$

We can see that the integral of the above expression is independent of  $m$ , so we can drop the sum and multiply by  $2l+1$  the  $m=0$  term, leaving

$$\begin{aligned} & [Y^l(\hat{\zeta}) Y^l(\hat{\mathbf{R}})]_0^0 \Rightarrow (-1)^l \sqrt{(2l+1)} Y_0^l(\hat{\zeta}) Y_0^l(\hat{\mathbf{R}}) \\ &= \sqrt{(2l+1)} Y_0^l(\hat{\zeta}) Y_0^{l*}(\hat{\mathbf{R}}). \end{aligned} \quad (7.22)$$

We now change the integration variables from  $(\zeta, \mathbf{r}, \boldsymbol{\eta})$  to  $(\mathbf{R}, \alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p})$ ,

$$\left| \frac{\partial(\mathbf{r}, \boldsymbol{\eta}, \zeta)}{\partial(\mathbf{R}, \alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p})} \right| = r_{12} r_{1p} r_{2p} \sin \beta \quad (7.23)$$

being the Jacobian of the transformation. Finally,

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= \frac{\sqrt{8\pi}}{k_i k_f} \sum_l \sqrt{\frac{2k+1}{2l+1}} \exp[i(\sigma_l^p + \sigma_l^t)] i^{-l} \\
&\times Y_{m_i-m_p}^l(\hat{\mathbf{k}}_f) \int d\mathbf{R} Y_0^{l*}(\hat{\mathbf{R}}) \int \frac{d\alpha d\beta d\gamma dr_{12} dr_{1p} dr_{2p} \sin\beta}{\zeta R} Y_0^l(\hat{\zeta}) \\
&\times u_{\lambda k}(r_1) u_{\lambda k}(r_2) V(r_{1p}) \theta_0^0(\mathbf{r}, \mathbf{s}) P_\lambda(\cos\theta_{12}) r_{12} r_{1p} r_{2p} \\
&\times \left[ (f_{l+1/2}(\zeta) g_{l+1/2}(R)(l+1) + f_{l-1/2}(\zeta) g_{l-1/2}(R)l) \delta_{m_p, m_t} \right. \\
&\left. + (f_{l+1/2}(\zeta) g_{l+1/2}(R) \sqrt{l(l+1)} - f_{l-1/2}(\zeta) g_{l-1/2}(R) \sqrt{l(l+1)}) \delta_{m_p, -m_t} \right].
\end{aligned} \tag{7.24}$$

We note that the inner integral is a function of  $\mathbf{R}$  alone, and that it transforms as  $Y_0^l(\hat{\mathbf{R}})$  under rotations, because all the dependence on the orientation of  $\mathbf{R}$  is contained in the term  $Y_0^l(\hat{\zeta})$ . The inner integral can thus be cast into the form

$$\begin{aligned}
A(R) Y_0^l(\hat{\mathbf{R}}) &= \int d\alpha d\beta d\gamma dr_{12} dr_{1p} dr_{2p} \sin\beta \\
&\times F(\alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p}, R_x, R_y, R_z).
\end{aligned} \tag{7.25}$$

To evaluate  $A(R)$ , we put  $\mathbf{R}$  on the  $z$ -axis

$$\begin{aligned}
A(R) &= 2\pi i^{-l} \sqrt{\frac{4\pi}{2l+1}} \int d\beta d\gamma dr_{12} dr_{1p} dr_{2p} \sin\beta \\
&\times F(\alpha, \beta, \gamma, r_{12}, r_{1p}, r_{2p}, 0, 0, R),
\end{aligned} \tag{7.26}$$

where a factor  $2\pi$  has been included as the result of the integral over  $\alpha$ , since the integrand clearly does not depend on  $\alpha$ . We substitute (7.25) and (7.26) in (7.24), and after integrating over the angular variables of  $\mathbf{R}$ , we obtain

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle &= 2 \frac{(2\pi)^{3/2}}{k_i k_f} \sum_l \sqrt{\frac{2k+1}{2l+1}} \exp[i(\sigma_l^p + \sigma_l^t)] i^{-l} \\
&\times Y_{m_i-m_p}^l(\hat{\mathbf{k}}_f) \int dR d\beta d\gamma dr_{12} dr_{1p} dr_{2p} R \sin\beta r_{12} r_{1p} r_{2p} \\
&\times u_{\lambda k}(r_1) u_{\lambda k}(r_2) V(r_{1p}) \theta_0^0(\mathbf{r}, \mathbf{s}) P_\lambda(\cos\theta_{12}) P_l(\cos\theta_\zeta) \\
&\times \left[ (f_{l+1/2}(\zeta) g_{l+1/2}(R)(l+1) + f_{l-1/2}(\zeta) g_{l-1/2}(R)l) \delta_{m_p, m_t} \right. \\
&\left. + (f_{l+1/2}(\zeta) g_{l+1/2}(R) \sqrt{l(l+1)} - f_{l-1/2}(\zeta) g_{l-1/2}(R) \sqrt{l(l+1)}) \delta_{m_p, -m_t} \right] / \zeta,
\end{aligned} \tag{7.27}$$

where we have used

$$Y_0^l(\hat{\zeta}) = i^l \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta_\zeta). \tag{7.28}$$

The final expression of the differential cross section involves a sum over the spin orientations:

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{k_f}{k_i} \frac{\mu_i \mu_f}{(2\pi\hbar^2)^2} \frac{1}{2} \sum_{m_i m_p} |\langle \Psi_f^{(-)}(\mathbf{k}_f) | V(r_{1p}) | \Psi_i^{(+)}(k_i, \hat{\mathbf{z}}) \rangle|^2. \tag{7.29}$$

When  $m_p = 1/2, m_t = 1/2$  or  $m_p = -1/2, m_t = -1/2$ , the terms proportional to  $\delta_{m_p, m_t}$  will include the factor

$$|Y_{m_i-m_p}^l(\hat{\mathbf{k}}_f) \delta_{m_p, m_t}| = |Y_0^l(\hat{\mathbf{k}}_f)| = \left| i^l \sqrt{\frac{2l+1}{4\pi}} P_l^0(\cos\theta) \right|, \tag{7.30}$$

when  $m_p = -1/2, m_t = 1/2$

$$|Y_{m_t-m_p}^l(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y_1^l(\hat{\mathbf{k}}_f)| = \left| i^l \sqrt{\frac{2l+1}{4\pi}} \frac{1}{l(l+1)} P_l^1(\cos\theta) \right|, \quad (7.31)$$

and when  $m_p = 1/2, m_t = -1/2$

$$|Y_{m_t-m_p}^l(\hat{\mathbf{k}}_f)\delta_{m_p,-m_t}| = |Y_{-1}^l(\hat{\mathbf{k}}_f)| = |Y_1^l(\hat{\mathbf{k}}_f)| = \left| i^l \sqrt{\frac{2l+1}{4\pi}} \frac{1}{l(l+1)} P_l^1(\cos\theta) \right|. \quad (7.32)$$

It is easily checked that, after taking the squared modulus of (7.27), the sum over  $m_t$  and  $m_p$  yields a factor 2 multiplying each one of the 2 different terms of the sum ( $m_t = m_p$  and  $m_t = -m_p$ ). This is equivalent to multiply each amplitude by  $\sqrt{2}$ , so the final constant that multiply the amplitudes is

$$\frac{8\pi^{3/2}}{k_i k_f}. \quad (7.33)$$

Now, for the tritium we can take

$$\theta_0^0(\mathbf{r}, \mathbf{s}) = \rho(r_{1p})\rho(r_{2p})\rho(r_{12}), \quad (7.34)$$

$\rho(r)$  being a Tang–Herndon wave function as done in [?]. We obtain

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{1}{2E_i^{3/2}E_f^{1/2}} \sqrt{\frac{\mu_f}{\mu_i}} \left( |I_{\lambda k}^{(0)}(\theta)|^2 + |I_{\lambda k}^{(1)}(\theta)|^2 \right), \quad (7.35)$$

with

$$\begin{aligned} I_{\lambda k}^{(0)}(\theta) &= \sum_l P_l^0(\cos\theta) \sqrt{2k+1} \exp[i(\sigma_l^p + \sigma_l^t)] \\ &\times \int dR d\beta d\gamma dr_{12} dr_{1p} dr_{2p} R \sin\beta \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ &\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ &\times (f_{l+1/2}(\zeta)g_{l+1/2}(R)(l+1) + f_{l-1/2}(\zeta)g_{l-1/2}(R)l)/\zeta, \end{aligned} \quad (7.36)$$

and

$$\begin{aligned} I_{\lambda k}^{(1)}(\theta) &= \sum_l P_l^1(\cos\theta) \sqrt{2k+1} \exp[i(\sigma_l^p + \sigma_l^t)] \\ &\times \int dR d\beta d\gamma dr_{12} dr_{1p} dr_{2p} R \sin\beta \rho(r_{1p})\rho(r_{2p})\rho(r_{12}) \\ &\times u_{\lambda k}(r_1)u_{\lambda k}(r_2)V(r_{1p})P_\lambda(\cos\theta_{12})P_l(\cos\theta_\zeta)r_{12}r_{1p}r_{2p} \\ &\times (f_{l+1/2}(\zeta)g_{l+1/2}(R) - f_{l-1/2}(\zeta)g_{l-1/2}(R))/\zeta. \end{aligned} \quad (7.37)$$

Note the absence of the  $(-1)^l$  factor with respect to what can be found in [?], due to the use of time-reversed phases instead of Condon–Shortley. This is compensated in the total result with the same difference in the expression of the spectroscopic factors. This ensures that, in either case, the contribution of all the single particle transitions tend

to have the same phase for superfluid nuclei, adding coherently to enhance the transfer cross section.

If we are dealing with a heavy ion reaction,  $\theta_0^0(\mathbf{r}, \mathbf{s})$  will be the spatial part of the wavefunction

$$\begin{aligned}\Psi(\mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2) &= \left[ \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \psi^{j_i}(\mathbf{r}_{b2}, \sigma_2) \right]_0^0 \\ &= \theta_0^0(\mathbf{r}, \mathbf{s}) \left[ \chi(\sigma_1) \chi(\sigma_2) \right]_0^0,\end{aligned}\quad (7.38)$$

where  $\mathbf{r}_{b1}, \mathbf{r}_{b2}$  are the positions of the two neutrons with respect to the  $b$  core. It can be shown to be

$$\theta_0^0(\mathbf{r}, \mathbf{s}) = \frac{u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2})}{4\pi} \sqrt{\frac{2j_i + 1}{2}} P_{l_i}(\cos \theta_i), \quad (7.39)$$

where  $\theta_i$  is the angle between  $\mathbf{r}_{b1}$  and  $\mathbf{r}_{b2}$ . If we neglect the spin-orbit term in the optical potential, as is usually done for heavy ion reactions, we obtain

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_f) = \frac{\mu_f \mu_i}{16\pi^2 \hbar^4 k_i^3 k_f} |T_l^{j_i, j_f}(\theta)|^2, \quad (7.40)$$

with

$$\begin{aligned}T_l^{j_i, j_f}(\theta) &= \sum_l (2l + 1) P_l(\cos \theta) \sqrt{(2j_i + 1)(2j_f + 1)} \exp[i(\sigma_l^p + \sigma_l^f)] \\ &\times \int dR d\beta d\gamma dr_{12} dr_{b1} dr_{b2} R \sin \beta u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) \\ &\times u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) V(r_{b1}) P_\lambda(\cos \theta_{12}) P_l(\cos \theta_\zeta) \\ &\times r_{12} r_{b1} r_{b2} P_{l_i}(\cos \theta_i) \frac{f_l(\zeta) g_l(R)}{\zeta},\end{aligned}\quad (7.41)$$

where  $\mathbf{r}_{A1}, \mathbf{r}_{A2}$  are the positions of the two neutrons with respect to the  $A$  core.

### heavy-ion version

The distorted waves for a reaction between spinless nuclei are

$$\psi^{(+)}(\mathbf{r}_{Aa}, \mathbf{k}_{Aa}) = \sum_l \exp(i\sigma_l^i) g_l Y_l^l(\hat{\mathbf{r}}_{aA}) \frac{\sqrt{4\pi(2l + 1)}}{k_{aA} r_{aA}}, \quad (7.42)$$

and

$$\psi^{(-)}(\mathbf{r}_{bB}, \mathbf{k}_{bB}) = \frac{4\pi}{k_{bB} r_{bB}} \sum_{\bar{l}} i^{\bar{l}} \exp(-i\sigma_{\bar{l}}^f) f_{\bar{l}}^*(r_{bB}) \sum_m Y_{\bar{l}}^{\bar{l}*}(\hat{\mathbf{k}}_{bB}) Y_{\bar{l}}^{\bar{l}}(\hat{\mathbf{r}}_{bB}). \quad (7.43)$$

Matrix element,

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB}) | V(r_{1p}) | \Psi_i^{(+)}(k_{aA}, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}} \sum_{\tilde{l}m} ((l_f \frac{1}{2})_{j_f} (l_f \frac{1}{2})_{j_f} | (l_f l_f)_0 (\frac{1}{2} \frac{1}{2})_0 \rangle_0 \\
&\times ((l_i \frac{1}{2})_{j_i} (l_i \frac{1}{2})_{j_i} | (l_i l_i)_0 (\frac{1}{2} \frac{1}{2})_0 \rangle_0 \sqrt{2l+1} i^{-l_p} \exp[i(\sigma_l^f + \sigma_l^i)] \\
&\times 2Y_m^{\tilde{l}}(\hat{\mathbf{k}}_{bB}) \sum_{\sigma_1 \sigma_2} \int \frac{d\mathbf{r}_{bB} d\mathbf{r} d\boldsymbol{\eta}}{r_{bB} r_{aA}} u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) \\
&\times [Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2})]_0^{0*} [Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2})]_0^0 \\
&\times f_{\tilde{l}}(r_{bB}) g_l(r_{aA}) [\chi(\sigma_1) \chi(\sigma_2)]_0^{0*} Y_m^{\tilde{l}*}(\hat{\mathbf{r}}_{bB}) V(r_{1p}) \\
&\times [\chi(\sigma_1) \chi(\sigma_2)]_0^0 Y_0^l(\hat{\mathbf{r}}_{aA}).
\end{aligned} \tag{7.44}$$

Simplifying...

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB}) | V(r_{1p}) | \Psi_i^{(+)}(k_{aA}, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{3/2}}{k_{aA}k_{bB}} \sum_{\tilde{l}m} \sqrt{\frac{(2j_f+1)(2j_i+1)}{(2l_f+1)(2l_i+1)}} \\
&\times \sqrt{2l+1} i^{-\tilde{l}} \exp[i(\sigma_l^f + \sigma_l^i)] \\
&\times Y_m^{\tilde{l}}(\hat{\mathbf{k}}_{bB}) \int \frac{d\mathbf{r}_{bB} d\mathbf{r} d\boldsymbol{\eta}}{r_{bB} r_{aA}} u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) \\
&\times [Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2})]_0^{0*} [Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2})]_0^0 \\
&\times f_{\tilde{l}}(r_{bB}) g_l(r_{aA}) Y_m^{\tilde{l}*}(\hat{\mathbf{r}}_{bB}) V(r_{1p}) Y_0^l(\hat{\mathbf{r}}_{aA}).
\end{aligned} \tag{7.45}$$

We clearly need  $l = \tilde{l}$  and  $m = 0$ . We also introduce the Legendre polynomials,

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB}) | V(r_{1p}) | \Psi_i^{(+)}(k_{aA}, \hat{\mathbf{z}}) \rangle &= \frac{(4\pi)^{-1/2}}{k_{aA}k_{bB}} \sum_l \sqrt{(2j_f+1)(2j_i+1)} \\
&\times \sqrt{2l+1} i^{-l} \exp[i(\sigma_l^f + \sigma_l^i)] Y_0^l(\hat{\mathbf{k}}_{bB}) \\
&\times \int \frac{d\mathbf{r}_{bB} d\mathbf{r} d\boldsymbol{\eta}}{r_{bB} r_{aA}} u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) \\
&\times P_{l_f}(\cos \theta_A) P_{l_i}(\cos \theta_b) \\
&\times f_l(r_{bB}) g_l(r_{aA}) Y_0^{l*}(\hat{\mathbf{r}}_{bB}) V(r_{1p}) Y_0^l(\hat{\mathbf{r}}_{aA}).
\end{aligned} \tag{7.46}$$

We change the integration variables and proceed as in last section, what involves multiplying by  $2\pi \sqrt{\frac{4\pi}{2l+1}}$

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB}) | V(r_{1p}) | \Psi_i^{(+)}(k_{aA}, \hat{\mathbf{z}}) \rangle &= \frac{2\pi}{k_{aA}k_{bB}} \sum_l \sqrt{(2j_f+1)(2j_i+1)} \\
&\times i^{-l} \exp[i(\sigma_l^f + \sigma_l^i)] Y_0^l(\hat{\mathbf{k}}_{bB}) \\
&\times \int dr_{aA} d\beta d\gamma dr_{12} dr_{b1} dr_{b2} r_{aA} \sin \beta r_{12} r_{b1} r_{b2} \\
&\times P_{l_f}(\cos \theta_A) P_{l_i}(\cos \theta_b) u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) \\
&\times f_l(r_{bB}) g_l(r_{aA}) Y_0^{l*}(\hat{\mathbf{r}}_{bB}) V(r_{1p}) / r_{bB}.
\end{aligned} \tag{7.47}$$

We introduce some more polynomials,

$$\begin{aligned}
\langle \Psi_f^{(-)}(\mathbf{k}_{bB}) | V(r_{1p}) | \Psi_i^{(+)}(k_{aA}, \hat{\mathbf{z}}) \rangle &= \frac{1}{2k_{aA}k_{bB}} \sum_l \sqrt{(2j_f + 1)(2j_i + 1)} \\
&\times i^{-l} \exp[i(\sigma_l^f + \sigma_l^i)] P_l(\cos \theta)(2l + 1) \\
&\times \int dr_{aA} d\beta d\gamma dr_{12} dr_{b1} dr_{b2} r_{aA} \sin \beta r_{12} r_{b1} r_{b2} \\
&\times P_{l_f}(\cos \theta_A) P_{l_i}(\cos \theta_b) u_{l_f j_f}(r_{A1}) u_{l_f j_f}(r_{A2}) V(r_{1p}) \\
&\times u_{l_i j_i}(r_{b1}) u_{l_i j_i}(r_{b2}) f_l(r_{bB}) g_l(r_{aA}) P_l(\cos \theta_{if}) / r_{bB}.
\end{aligned} \tag{7.48}$$

### 7.1.3 coordinates for the simultaneous calculation

We refer to the notation used in [?]. We must find the expression of the variables appearing in the integral as functions of the integration variables  $r_{1p}, r_{2p}, r_{12}, R, \beta, \gamma$  (remember that  $\mathbf{R} = R \hat{\mathbf{z}}$ , see last section).  $\mathbf{R}$  being the center of mass coordinate, we have

$$\mathbf{R} = \frac{1}{3} (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_p) = \frac{1}{3} (\mathbf{R} + \mathbf{d}_1 + \mathbf{R} + \mathbf{d}_2 + \mathbf{R} + \mathbf{d}_p), \tag{7.49}$$

so

$$\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_p = 0. \tag{7.50}$$

Together with

$$\mathbf{d}_1 + \mathbf{r}_{12} = \mathbf{d}_2 \quad \mathbf{d}_2 + \mathbf{r}_{2p} = \mathbf{d}_p, \tag{7.51}$$

we find

$$\mathbf{d}_1 = \frac{1}{3} (2\mathbf{r}_{12} + \mathbf{r}_{2p}), \tag{7.52}$$

so

$$d_1^2 = \frac{1}{9} (4r_{12}^2 + r_{2p}^2 + 4\mathbf{r}_{12}\mathbf{r}_{2p}). \tag{7.53}$$

but

$$\begin{aligned}
\mathbf{r}_{12} + \mathbf{r}_{2p} &= \mathbf{r}_{1p} \\
r_{1p}^2 &= r_{12}^2 + r_{2p}^2 + 2\mathbf{r}_{12}\mathbf{r}_{2p} \\
2\mathbf{r}_{12}\mathbf{r}_{2p} &= r_{1p}^2 - r_{12}^2 - r_{2p}^2.
\end{aligned} \tag{7.54}$$

Finally,

$$d_1 = \frac{1}{3} \sqrt{2r_{12}^2 + 2r_{1p}^2 - r_{2p}^2}. \tag{7.55}$$

Similarly, we find

$$d_2 = \frac{1}{3} \sqrt{2r_{12}^2 + 2r_{2p}^2 - r_{1p}^2} \quad d_p = \frac{1}{3} \sqrt{2r_{2p}^2 + 2r_{1p}^2 - r_{12}^2}. \tag{7.56}$$

We now express the angle  $\alpha$  between  $\mathbf{d}_1$  and  $\mathbf{r}_{12}$ . We have

$$-\mathbf{d}_1 \mathbf{r}_{12} = r_{12} d_1 \cos(\alpha), \tag{7.57}$$

and

$$\begin{aligned}
\mathbf{d}_1 + \mathbf{r}_{12} &= \mathbf{d}_2 \\
d_1^2 + r_{12}^2 + 2\mathbf{d}_1 \mathbf{r}_{12} &= d_2^2,
\end{aligned} \tag{7.58}$$

so

$$\cos(\alpha) = \frac{d_1^2 + r_{12}^2 - d_2^2}{2r_{12}d_1}. \quad (7.59)$$

The complete determination of  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{12}$  can be made by writing their expression in a simple configuration, in which the triangle lies in the  $xz$ -plane with  $\mathbf{d}_1$  pointing along the positive  $z$ -direction, and  $\mathbf{R} = 0$ . Then, a first rotation  $\mathcal{R}_z(\gamma)$  of an angle  $\gamma$  around the  $z$ -axis, a second rotation  $\mathcal{R}_y(\beta)$  of an angle  $\beta$  around the  $y$ -axis, and a translation along  $\mathbf{R}$  will bring the vectors to the most general configuration. In other words,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_1, \\ \mathbf{r}_{12} &= \mathcal{R}_y(\beta)\mathcal{R}_z(\gamma)\mathbf{r}'_{12}, \\ \mathbf{r}_2 &= \mathbf{r}_1 + \mathbf{r}_{12}, \end{aligned} \quad (7.60)$$

with

$$\mathbf{r}'_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \quad (7.61)$$

$$\mathbf{r}'_{12} = r_{12} \begin{bmatrix} \sin(\alpha) \\ 0 \\ -\cos(\alpha) \end{bmatrix}, \quad (7.62)$$

and the rotation matrixes are

$$\mathcal{R}_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \quad (7.63)$$

and

$$\mathcal{R}_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (7.64)$$

We obtain

$$\mathbf{r}_1 = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \quad (7.65)$$

$$\mathbf{r}_{12} = \begin{bmatrix} r_{12} \cos(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \sin(\beta) \cos(\alpha) \\ r_{12} \sin(\gamma) \sin(\alpha) \\ -r_{12} \sin(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \cos(\alpha) \cos(\beta) \end{bmatrix}, \quad (7.66)$$

$$\mathbf{r}_2 = \begin{bmatrix} d_1 \sin(\beta) + r_{12} \cos(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \sin(\beta) \cos(\alpha) \\ r_{12} \sin(\gamma) \sin(\alpha) \\ R + d_1 \cos(\beta) - r_{12} \sin(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \cos(\alpha) \cos(\beta) \end{bmatrix}. \quad (7.67)$$

We also need  $\cos(\theta_{12})$ ,  $\zeta$  and  $\cos(\theta_\zeta)$ ,  $\theta_{12}$  being the angle between  $\mathbf{r}_1$  and  $\mathbf{r}_2$ ,  $\zeta = \mathbf{r}_p - \frac{\mathbf{r}_1 + \mathbf{r}_2}{A+2}$  the position of the proton with respect to the final nucleus, and  $\theta_\zeta$  the angle between  $\zeta$  and the  $z$ -axis:

$$\cos(\theta_{12}) = \frac{\mathbf{r}_1 \mathbf{r}_2}{r_1 r_2}, \quad (7.68)$$

and

$$\zeta = 3\mathbf{R} - \frac{A+3}{A+2}(\mathbf{r}_1 + \mathbf{r}_2), \quad (7.69)$$



where we have used (7.49).

For heavy ions, we find instead

$$\mathbf{R} = \frac{1}{m_a} (\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b \mathbf{r}_{Ab}), \quad (7.70)$$

$$\mathbf{d}_1 = \frac{1}{m_a} (m_b \mathbf{r}_{b2} - (m_b + 1) \mathbf{r}_{12}), \quad (7.71)$$

$$d_1 = \frac{1}{m_a} \sqrt{(m_b + 1)r_{12}^2 + m_b(m_b + 1)r_{b1}^2 - m_b r_{b2}^2}, \quad (7.72)$$

$$d_2 = \frac{1}{m_a} \sqrt{(m_b + 1)r_{12}^2 + m_b(m_b + 1)r_{b2}^2 - m_b r_{b1}^2}, \quad (7.73)$$

and

$$\boldsymbol{\zeta} = \frac{m_a}{m_b} \mathbf{R} - \frac{m_b + m_p}{m_b m_B} (\mathbf{r}_{A1} + \mathbf{r}_{A2}). \quad (7.74)$$

The rest of the formulae are identical to  $(t, p)$  ones, we list them for reference:

$$\mathbf{r}_{A1} = \begin{bmatrix} d_1 \sin(\beta) \\ 0 \\ R + d_1 \cos(\beta) \end{bmatrix}, \quad (7.75)$$

$$\mathbf{r}_{A2} = \begin{bmatrix} d_1 \sin(\beta) + r_{12} \cos(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \sin(\beta) \cos(\alpha) \\ r_{12} \sin(\gamma) \sin(\alpha) \\ R + d_1 \cos(\beta) - r_{12} \sin(\beta) \cos(\gamma) \sin(\alpha) - r_{12} \cos(\alpha) \cos(\beta) \end{bmatrix}. \quad (7.76)$$

We we also find

$$\mathbf{r}_{b1} = \frac{1}{m_b} (\mathbf{r}_{A2} + (m_b + 1) \mathbf{r}_{A1} - m_a \mathbf{R}), \quad (7.77)$$

and

$$\mathbf{r}_{b2} = \frac{1}{m_b} (\mathbf{r}_{A1} + (m_b + 1) \mathbf{r}_{A2} - m_a \mathbf{R}). \quad (7.78)$$

We easily obtain

$$\cos \theta_{12} = \frac{r_{A1}^2 + r_{A2}^2 - r_{12}^2}{2r_{A1}r_{A2}}, \quad (7.79)$$

and

$$\cos \theta_i = \frac{r_{b1}^2 + r_{b2}^2 - r_{12}^2}{2r_{b1}r_{b2}}. \quad (7.80)$$

#### 7.1.4 matrix element for the transition amplitude (2)

The simultaneous amplitude can be written as (see [?])

$$\begin{aligned} T_{2NT}^{1step} = & 2 \frac{(4\pi)^{3/2}}{k_{Aa} k_{Bb}} \sum_{l_p j_p m_l j_p} i^{-l_p} \exp[i(\sigma_{l_p}^p + \sigma_{l_t}^t)] \sqrt{2l_t + 1} \\ & \times \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle Y_{m-m_p}^{l_p}(\hat{\mathbf{k}}_{Bb}) \\ & \times \sum_{\sigma_1 \sigma_2 \sigma_p} \int d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2} [\psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2)]_0^{0*} \\ & \times v(r_{b1}) [\psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \psi^{j_i}(\mathbf{r}_{b2}, \sigma_2)]_0^0 \frac{g_{l_t j_t}(r_{Aa}) f_{l_p j_p}(r_{Bb})}{r_{Aa} r_{Bb}} \\ & \times [Y^{l_t}(\hat{\mathbf{r}}_{Aa}) \chi(\sigma_p)]_{m_t}^{j_i} [Y^{l_p}(\hat{\mathbf{r}}_{Bb}) \chi(\sigma_p)]_m^{j_p*}. \end{aligned} \quad (7.81)$$

As we have shown before, we can write

$$\begin{aligned}
& \sum_{\sigma_p} \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle \left[ Y^{l_t}(\hat{\mathbf{r}}_{Aa}) \chi(\sigma_p) \right]_{m_t}^{j_t} \left[ Y^{l_p}(\hat{\mathbf{r}}_{Bb}) \chi(\sigma_p) \right]_m^{j_p*} \\
&= -\frac{\delta_{l_p, l_t} \delta_{j_p, j_t} \delta_{m, m_t}}{\sqrt{2l+1}} \left[ Y^l(\hat{\mathbf{r}}_{Aa}) Y^l(\hat{\mathbf{r}}_{Bb}) \right]_0^0 \begin{cases} \frac{l}{2l+1} & \text{if } m_t = m_p \\ -\frac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\end{aligned} \tag{7.82}$$

when  $j = l - 1/2$  and

$$\begin{aligned}
& \sum_{\sigma_p} \langle l_p \ m - m_p \ 1/2 \ m_p | j_p \ m \rangle \langle l_t \ 0 \ 1/2 \ m_t | j_t \ m_t \rangle \left[ Y^{l_t}(\hat{\mathbf{r}}_{Aa}) \chi(\sigma_p) \right]_{m_t}^{j_t} \left[ Y^{l_p}(\hat{\mathbf{r}}_{Bb}) \chi(\sigma_p) \right]_m^{j_p*} \\
&= -\frac{\delta_{l_p, l_t} \delta_{j_p, j_t} \delta_{m, m_t}}{\sqrt{2l+1}} \left[ Y^l(\hat{\mathbf{r}}_{Aa}) Y^l(\hat{\mathbf{r}}_{Bb}) \right]_0^0 \begin{cases} \frac{l+1}{2l+1} & \text{if } m_t = m_p \\ \frac{\sqrt{l(l+1)}}{2l+1} & \text{if } m_t = -m_p \end{cases}
\end{aligned} \tag{7.83}$$

if  $j = l + 1/2$ . We get

$$\begin{aligned}
T_{2NT}^{1step} &= 2 \frac{(4\pi)^{3/2}}{k_{Aa} k_{Bb}} \sum_l i^{-l} \frac{\exp[i(\sigma_l^p + \sigma_l^t)]}{2l+1} Y_{m_t - m_p}^l(\hat{\mathbf{k}}_{Bb}) \\
&\times \sum_{\sigma_1 \sigma_2} \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \\
&\times v(r_{b1}) \left[ \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \psi^{j_i}(\mathbf{r}_{b2}, \sigma_2) \right]_0^0 \left[ Y^l(\hat{\mathbf{r}}_{Aa}) Y^l(\hat{\mathbf{r}}_{Bb}) \right]_0^0 \\
&\times \left[ \left( f_{l+1/2}(r_{Bb}) g_{l+1/2}(r_{Aa}) (l+1) + f_{l-1/2}(r_{Bb}) g_{l-1/2}(r_{Aa}) l \right) \delta_{m_p, m_t} \right. \\
&\left. + \left( f_{l+1/2}(r_{Bb}) g_{l+1/2}(r_{Aa}) \sqrt{l(l+1)} - f_{l-1/2}(r_{Bb}) g_{l-1/2}(r_{Aa}) \sqrt{l(l+1)} \right) \delta_{m_p, -m_t} \right].
\end{aligned} \tag{7.84}$$

Now,

$$\begin{aligned}
& \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \\
&= \left( (l_f \frac{1}{2})_{j_f} (l_f \frac{1}{2})_{j_f} | (l_f l_f)_0 (\frac{1}{2} \frac{1}{2})_0 \right) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
&\times \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2}) \right]_0^{0*} [\chi(\sigma_1) \chi(\sigma_2)]_0^{0*} \\
&= \sqrt{\frac{2j_f+1}{2(2l_f+1)}} u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
&\times \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_f}(\hat{\mathbf{r}}_{A2}) \right]_0^{0*} [\chi(\sigma_1) \chi(\sigma_2)]_0^{0*} \\
&= \sqrt{\frac{2j_f+1}{2}} \frac{u_{l_f}(r_{A1}) u_{l_f}(r_{A2})}{4\pi} P_{l_f}(\cos \omega_A) [\chi(\sigma_1) \chi(\sigma_2)]_0^{0*}
\end{aligned} \tag{7.85}$$

and

$$\begin{aligned}
& \left[ \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \psi^{j_i}(\mathbf{r}_{b2}, \sigma_2) \right]_0^0 \\
&= \left( (l_i \frac{1}{2})_{j_i} (l_i \frac{1}{2})_{j_i} (l_i l_i)_0 (\frac{1}{2} \frac{1}{2})_0 \right) u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) \\
&\times \left[ Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2}) \right]_0^0 [\chi(\sigma_1) \chi(\sigma_2)]_0^0 \\
&= \sqrt{\frac{2j_i + 1}{2(2l_i + 1)}} u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) \\
&\times \left[ Y^{l_i}(\hat{\mathbf{r}}_{b1}) Y^{l_i}(\hat{\mathbf{r}}_{b2}) \right]_0^0 [\chi(\sigma_1) \chi(\sigma_2)]_0^0 \\
&= \sqrt{\frac{2j_i + 1}{2}} \frac{u_{l_i}(r_{b1}) u_{l_i}(r_{b2})}{4\pi} P_{l_i}(\cos \omega_b) [\chi(\sigma_1) \chi(\sigma_2)]_0^0,
\end{aligned} \tag{7.86}$$

where  $\omega_A$  is the angle between  $\mathbf{r}_{A1}$  and  $\mathbf{r}_{A2}$ , and  $\omega_b$  is the angle between  $\mathbf{r}_{b1}$  and  $\mathbf{r}_{b2}$ . So

$$\begin{aligned}
T_{2NT}^{1step} &= (4\pi)^{-3/2} \frac{\sqrt{(2j_i + 1)(2j_f + 1)}}{k_{Aa} k_{Bb}} \sum_l i^{-l} \frac{\exp[i(\sigma_l^p + \sigma_l^t)]}{\sqrt{2l + 1}} Y_{m_t - m_p}^l(\hat{\mathbf{k}}_{Bb}) \\
&\times \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} P_{l_f}(\cos \omega_A) P_{l_i}(\cos \omega_b) P_l(\cos \omega_{if}) \\
&\times v(r_{b1}) u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) \\
&\times \left[ (f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa})(l + 1) + f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa})l) \delta_{m_p, m_t} \right. \\
&\left. + (f_{ll+1/2}(r_{Bb}) g_{ll+1/2}(r_{Aa}) \sqrt{l(l + 1)} - f_{ll-1/2}(r_{Bb}) g_{ll-1/2}(r_{Aa}) \sqrt{l(l + 1)}) \delta_{m_p, -m_t} \right],
\end{aligned} \tag{7.87}$$

where  $\omega_{if}$  is the angle between  $\mathbf{r}_{Aa}$  and  $\mathbf{r}_{Bb}$ . For heavy ions, we can consider that the the optical potential does not have a spin-orbit term, and the distorted waves are independent of  $j$ . We thus have

$$\begin{aligned}
T_{2NT}^{1step} &= (4\pi)^{-3/2} \frac{\sqrt{(2j_i + 1)(2j_f + 1)}}{k_{Aa} k_{Bb}} \sum_l i^{-l} \exp[i(\sigma_l^p + \sigma_l^t)] Y_0^l(\hat{\mathbf{k}}_{Bb}) \sqrt{2l + 1} \\
&\times \int \frac{d\mathbf{r}_{Cc} d\mathbf{r}_{b1} d\mathbf{r}_{A2}}{r_{Aa} r_{Bb}} P_{l_f}(\cos \omega_A) P_{l_i}(\cos \omega_b) P_l(\cos \omega_{if}) \\
&\times v(r_{b1}) u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) f_l(r_{Bb}) g_l(r_{Aa}).
\end{aligned} \tag{7.88}$$

We change the variables:

$$\begin{aligned}
T_{2NT}^{1step} &= (4\pi)^{-1} \frac{\sqrt{(2j_i + 1)(2j_f + 1)}}{k_{Aa} k_{Bb}} \sum_l \exp[i(\sigma_l^p + \sigma_l^t)] P_l(\cos \theta) (2l + 1) \\
&\times \int dr_{1A} dr_{2A} dr_{Aa} d(\cos \beta) d(\cos \omega_A) d\gamma r_{1A}^2 r_{2A}^2 r_{Aa}^2 \\
&\times P_{l_f}(\cos \omega_A) P_{l_i}(\cos \omega_b) P_l(\cos \omega_{if}) v(r_{b1}) \\
&\times u_{l_i}(r_{b1}) u_{l_i}(r_{b2}) u_{l_f}(r_{A1}) u_{l_f}(r_{A2}) f_l(r_{Bb}) g_l(r_{Aa}).
\end{aligned} \tag{7.89}$$

### 7.1.5 coordinates

We determine the relation between the integration variables in (7.87) and the coordinates needed to evaluate the quantities in the integrand. Noting that

$$\mathbf{r}_{Aa} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2} + m_b \mathbf{r}_{Ab}}{m_b + 2}, \quad (7.90)$$

We have

$$\mathbf{r}_{b1} = \mathbf{r}_{bA} + \mathbf{r}_{A1} = \frac{(m_b + 1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b + 2)\mathbf{r}_{Aa}}{m_b}, \quad (7.91)$$

$$\mathbf{r}_{b2} = \mathbf{r}_{bA} + \mathbf{r}_{A2} = \frac{(m_b + 1)\mathbf{r}_{A2} + \mathbf{r}_{A1} - (m_b + 2)\mathbf{r}_{Aa}}{m_b}, \quad (7.92)$$

and

$$\begin{aligned} \mathbf{r}_{Cc} = \mathbf{r}_{CA} + \mathbf{r}_{A1} + \mathbf{r}_{1c} &= -\frac{1}{m_A + 1} \mathbf{r}_{A2} + \mathbf{r}_{A1} - \frac{m_b}{m_b + 1} \mathbf{r}_{b1} \\ &= \frac{m_b + 2}{m_b + 1} \mathbf{r}_{Aa} - \frac{m_b + 2 + m_A}{(m_b + 1)(m_A + 1)} \mathbf{r}_{A2} \end{aligned} \quad (7.93)$$

Now, since

$$\mathbf{r}_{AB} = \frac{\mathbf{r}_{A1} + \mathbf{r}_{A2}}{m_A + 2}, \quad (7.94)$$

$$\mathbf{r}_{Bb} = \mathbf{r}_{BA} + \mathbf{r}_{Ab} = \frac{m_b + 2}{m_b} \mathbf{r}_{Aa} - \frac{m_A + m_b + 2}{(m_A + 2)m_b} (\mathbf{r}_{A1} + \mathbf{r}_{A2}). \quad (7.95)$$

We use the same rotations as in section 7.1.3 to get

$$\mathbf{r}_{A1} = r_{A1} \begin{bmatrix} \sin \alpha \\ 0 \\ \cos \alpha \end{bmatrix}, \quad (7.96)$$

and

$$\mathbf{r}_{A2} = r_{A2} \begin{bmatrix} -\cos \alpha \cos \gamma \sin \omega_A + \sin \alpha \cos \omega_A \\ -\sin \gamma \sin \omega_A \\ \sin \alpha \cos \gamma \sin \omega_A + \cos \alpha \cos \omega_A \end{bmatrix}, \quad (7.97)$$

with

$$\cos \alpha = \frac{r_{A1}^2 - d_1^2 + r_{Aa}^2}{2r_{A1}r_{Aa}}, \quad (7.98)$$

and

$$d_1 = \sqrt{r_{A1}^2 - r_{Aa}^2 \sin^2 \beta - r_{Aa} \cos \beta}. \quad (7.99)$$

Note that though  $\beta, r_{1A}, r_{Aa}$  are independent integration variables, they have to fulfill the condition

$$r_{Aa} \sin \beta \leq r_{A1}, \quad \text{for } 0 \leq \beta \leq \pi. \quad (7.100)$$

The expression of the other quantities appearing in the integral is now straightforward:

$$\begin{aligned} r_{b1} &= m_b^{-1} |(m_b + 1)\mathbf{r}_{A1} + \mathbf{r}_{A2} - (m_b + 2)\mathbf{r}_{Aa}| \\ &= m_b^{-1} \left( (m_b + 2)^2 r_{Aa}^2 + (m_b + 1)^2 r_{A1}^2 + r_{A2}^2 \right. \\ &\quad \left. - 2(m_b + 2)(m_b + 1)\mathbf{r}_{Aa} \cdot \mathbf{r}_{A1} - 2(m_b + 2)\mathbf{r}_{Aa} \cdot \mathbf{r}_{A2} + 2(m_b + 1)\mathbf{r}_{A1} \cdot \mathbf{r}_{A2} \right)^{1/2}, \end{aligned} \quad (7.101)$$

$$\begin{aligned}
r_{b2} &= m_b^{-1} |(m_b + 1)\mathbf{r}_{A2} + \mathbf{r}_{A1} - (m_b + 2)\mathbf{r}_{Aa}| \\
&= m_b^{-1} \left( (m_b + 2)^2 r_{Aa}^2 + (m_b + 1)^2 r_{A2}^2 + r_{A1}^2 \right. \\
&\quad \left. - 2(m_b + 2)(m_b + 1)\mathbf{r}_{Aa} \cdot \mathbf{r}_{A2} - 2(m_b + 2)\mathbf{r}_{Aa} \cdot \mathbf{r}_{A1} + 2(m_b + 1)\mathbf{r}_{A2} \cdot \mathbf{r}_{A1} \right)^{1/2},
\end{aligned} \tag{7.102}$$

$$\begin{aligned}
r_{Bb} &= \left| \frac{m_b + 2}{m_b} \mathbf{r}_{Aa} - \frac{m_A + m_b + 2}{(m_A + 2)m_b} (\mathbf{r}_{A1} + \mathbf{r}_{A2}) \right| \\
&= \left[ \left( \frac{m_b + 2}{m_b} \right)^2 r_{Aa}^2 + \left( \frac{m_A + m_b + 2}{(m_A + 2)m_b} \right)^2 (r_{A1}^2 + r_{A2}^2 + 2\mathbf{r}_{A1} \cdot \mathbf{r}_{A2}) \right. \\
&\quad \left. - 2 \frac{(m_b + 2)(m_A + m_b + 2)}{(m_A + 2)m_b^2} \mathbf{r}_{Aa} \cdot (\mathbf{r}_{A1} + \mathbf{r}_{A2}) \right]^{1/2},
\end{aligned} \tag{7.103}$$

$$\begin{aligned}
r_{Cc} &= \left| \frac{m_b + 2}{m_b + 1} \mathbf{r}_{Aa} - \frac{m_b + 2 + m_A}{(m_b + 1)(m_A + 1)} \mathbf{r}_{A2} \right| \\
&= \left[ \left( \frac{m_b + 2}{m_b + 1} \right)^2 r_{Aa}^2 + \left( \frac{m_A + m_b + 2}{(m_A + 1)(m_b + 1)} \right)^2 r_{A2}^2 \right. \\
&\quad \left. - 2 \frac{m_A m_b + m_b^2 + m_A^2}{(m_A + 1)(m_b + 1)^2} \mathbf{r}_{Aa} \cdot \mathbf{r}_{A2} \right]^{1/2},
\end{aligned} \tag{7.104}$$

$$\cos \omega_b = \frac{\mathbf{r}_{b1} \cdot \mathbf{r}_{b2}}{r_{b1} r_{b2}}, \tag{7.105}$$

$$\cos \omega_{if} = \frac{\mathbf{r}_{Aa} \cdot \mathbf{r}_{Bb}}{r_{Aa} r_{Bb}}, \tag{7.106}$$

with

$$\mathbf{r}_{Aa} \cdot \mathbf{r}_{A1} = r_{Aa} r_{A1} \cos \alpha, \tag{7.107}$$

$$\mathbf{r}_{Aa} \cdot \mathbf{r}_{A2} = r_{Aa} r_{A2} (\sin \alpha \cos \gamma \sin \omega_A + \cos \alpha \cos \omega_A), \tag{7.108}$$

$$\mathbf{r}_{A1} \cdot \mathbf{r}_{A2} = r_{A1} r_{A2} \cos \omega_A. \tag{7.109}$$

## 7.2 successive transfer

Note that we use time-reversed phases for the spherical harmonics (see (??)) throughout. We write the successive transition amplitude (see [?]):

$$\begin{aligned}
T_{2NT}^{VV} &= \frac{4\mu_{Cc}}{\hbar^2} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma'_1 \sigma'_2 \\ KM}} \int d^3 r_{Cc} d^3 r_{b1} d^3 r_{A2} d^3 r'_{Cc} d^3 r'_{b1} d^3 r'_{A2} \chi^{(-)*}(\mathbf{k}_{Bb}, \mathbf{r}_{Bb}) \\
&\quad \times \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} v(r_{b1}) \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&\quad \times G(\mathbf{r}_{Cc}, \mathbf{r}'_{Cc}) \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} v(r'_{c2}) \\
&\quad \times \left[ \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2) \right]_0^0 \chi^{(+)}(\mathbf{r}'_{Aa})
\end{aligned} \tag{7.110}$$

Expansion of the Green function and distorted waves in a basis of angular momentum eigenstates:

$$\chi^{(-)*}(\mathbf{k}_{Bb}, \mathbf{r}_{Bb}) = \sum_{\tilde{l}} \frac{4\pi}{k_{Bb} r_{Bb}} i^{-\tilde{l}} e^{i\sigma_{\tilde{l}}} F_{\tilde{l}} \sum_m Y_{\tilde{l}}^m(\hat{\mathbf{r}}_{Bb}) Y_{\tilde{l}}^{m*}(\hat{\mathbf{k}}_{Bb}) \tag{7.111}$$

but the sum over  $m$  is

$$\sum_m (-1)^{\tilde{l}-m} Y_m^{\tilde{l}}(\hat{r}_{Bb}) Y_{-m}^{\tilde{l}}(\hat{k}_{Bb}) = \sqrt{2\tilde{l}+1} \left[ Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb}) \right]_0^0, \quad (7.112)$$

where we have used (??) and (??), so

$$\chi^{(-)*}(\mathbf{k}_{Bb}, \mathbf{r}_{Bb}) = \sum_{\tilde{l}} \sqrt{2\tilde{l}+1} \frac{4\pi}{k_{Bb} r_{Bb}} i^{-\tilde{l}} e^{i\sigma_{\tilde{l}}^j} F_{\tilde{l}}(r_{Bb}) \left[ Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb}) \right]_0^0 \quad (7.113)$$

similarly:

$$\chi^{(+)}(\mathbf{r}'_{Aa}) = \sum_l i^l \sqrt{2l+1} \frac{4\pi}{k_{Aa} r'_{Aa}} e^{i\sigma_l^j} F_l(r'_{Aa}) \left[ Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa}) \right]_0^0 \quad (7.114)$$

where we have taken into account that  $\hat{k}_{Aa} \equiv \hat{z}$ . And the Green function is

$$G(\mathbf{r}_{Cc}, \mathbf{r}'_{Cc}) = i \sum_{l_c} \sqrt{2l_c+1} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{k_{Cc} r_{Cc} r'_{Cc}} \left[ Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc}) \right]_0^0. \quad (7.115)$$

Finally

$$\begin{aligned} T_{2NT}^{VV} &= \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \sum_{l, l_c, \tilde{l}} e^{i(\sigma_l^j + \sigma_{\tilde{l}}^j)} i^{l-\tilde{l}} \sqrt{(2l+1)(2l_c+1)(2\tilde{l}+1)} \\ &\times \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma'_1 \sigma'_2}} \int d^3 r_{Cc} d^3 r_{b1} d^3 r_{A2} d^3 r'_{Cc} d^3 r'_{b1} d^3 r'_{A2} v(r_{b1}) v(r'_{c2}) \left[ Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb}) \right]_0^0 \\ &\times \left[ Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa}) \right]_0^0 \left[ Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc}) \right]_0^0 \frac{F_{\tilde{l}}(r_{Bb})}{r_{Bb}} \frac{F_l(r'_{Aa})}{r'_{Aa}} \\ &\times \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r_{Cc} r'_{Cc}} \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \\ &\times \left[ \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2) \right]_0^0 \sum_{KM} \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\ &\times \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \end{aligned} \quad (7.116)$$

Let us now perform the integration over  $\mathbf{r}_{A2}$

$$\begin{aligned}
& \sum_{\sigma_1, \sigma_2} \int d\mathbf{r}_{A2} \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&= \sum_{\sigma_1, \sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \int d\mathbf{r}_{A2} \left[ \psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2) \right]_0^0 \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&= - \sum_{\sigma_1, \sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \int d\mathbf{r}_{A2} \left[ \psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2) \psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1) \right]_0^0 \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&= -((j_f j_f)_0(j_f j_i)_K | (j_f j_f)_0(j_f j_i)_K)_K \sum_{\sigma_1, \sigma_2} (-1)^{1/2-\sigma_1+1/2-\sigma_2} \\
&\times \int d\mathbf{r}_{A2} \left[ \psi^{j_f}(\mathbf{r}_{A2}, -\sigma_2) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^0 \left[ \psi^{j_f}(\mathbf{r}_{A1}, -\sigma_1) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&= \frac{1}{2j_f + 1} \sqrt{2j_f + 1} ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0)_K \\
&\times u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K \sum_{\sigma_1} (-1)^{1/2-\sigma_1} \left[ \chi^{1/2}(-\sigma_1) \chi^{1/2}(\sigma_1) \right]_0^0 \\
&= -\sqrt{\frac{2}{2j_f + 1}} ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0)_K \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K u_{l_f}(r_{A1}) u_{l_i}(r_{b1}),
\end{aligned} \tag{7.117}$$

where we have evaluated the  $9j$  symbol

$$((j_f j_f)_0(j_f j_i)_K | (j_f j_f)_0(j_f j_i)_K)_K = \frac{1}{2j_f + 1}, \tag{7.118}$$

and have also used (??). We proceed in a similar way to evaluate the integral over  $\mathbf{r}'_{b1}$

$$\begin{aligned}
& \sum_{\sigma'_1, \sigma'_2} \int d\mathbf{r}'_{b1} \left[ \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2) \right]_0^0 \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \\
&= -(-1)^{K-M} \sum_{\sigma'_1, \sigma'_2} \int d\mathbf{r}'_{b1} \left[ \psi^{j_f}(\mathbf{r}'_{A2}, -\sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, -\sigma'_1) \right]_{-M}^K \\
&\times \left[ \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \right]_0^0 (-1)^{1/2-\sigma'_1+1/2-\sigma'_2} \\
&= -(-1)^{K-M} ((j_f j_i)_K(j_i j_i)_0 | (j_f j_i)_K(j_i j_i)_0)_K (-\sqrt{2j_i + 1}) \\
&\times ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0)_K (-\sqrt{2}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}) \left[ Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2}) \right]_{-M}^K \\
&= -\sqrt{\frac{2}{2j_i + 1}} ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0)_K \left[ Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2}) \right]_{-M}^{K*} u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}).
\end{aligned} \tag{7.119}$$

Putting all together

$$\begin{aligned}
T_{2NT}^{VV} = & \frac{4\mu_{Cc}(4\pi)^2 i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \frac{2}{\sqrt{(2j_i+1)(2j_f+1)}} \sum_{K,M} ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} |(l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0|_K^2 \\
& \times \sum_{l_c, l, \tilde{l}} e^{i(\sigma_l^i + \sigma_{\tilde{l}}^f)} \sqrt{(2l_c+1)(2l+1)(2\tilde{l}+1)} i^{l-\tilde{l}} \\
& \times \int d^3 r_{Cc} d^3 r_{b1} d^3 r'_{Cc} d^3 r'_{A2} v(r_{b1}) v(r'_{c2}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}) \\
& \times [Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2})]_M^{K*} [Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1})]_M^K \frac{F_l(r'_{Aa}) F_{\tilde{l}}(r'_{Bb}) f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r'_{Aa} r_{Bb} r_{Cc} r'_{Cc}} \\
& \times [Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb})]_0^0 [Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa})]_0^0 [Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc})]_0^0.
\end{aligned} \tag{7.120}$$

We can write

$$\begin{aligned}
[Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb})]_0^0 [Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa})]_0^0 = \\
((l l)_0 (\tilde{l} \tilde{l})_0 | (l \tilde{l})_0 (l \tilde{l})_0)_0 [Y^{\tilde{l}}(\hat{r}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Bb})]_0^0 [Y^l(\hat{r}'_{Aa}) Y^l(\hat{k}_{Aa})]_0^0 \\
= \frac{\delta_{\tilde{l}l}}{2l+1} [Y^l(\hat{r}_{Bb}) Y^l(\hat{r}'_{Aa})]_0^0 [Y^{\tilde{l}}(\hat{k}_{Bb}) Y^{\tilde{l}}(\hat{k}_{Aa})]_0^0.
\end{aligned} \tag{7.121}$$

Taking into account that

$$[Y^l(\hat{k}_{Bb}) Y^l(\hat{k}_{Aa})]_0^0 = \frac{(-1)^l}{\sqrt{4\pi}} Y_0^l(\hat{k}_{Bb}) i^l, \tag{7.122}$$

and

$$\begin{aligned}
[Y^l(\hat{r}_{Bb}) Y^l(\hat{r}'_{Aa})]_0^0 [Y^{l_c}(\hat{r}_{Cc}) Y^{l_c}(\hat{r}'_{Cc})]_0^0 = \\
((l l)_0 (l_c l_c)_0 | (l l)_K (l_c l_c)_K)_0 \left\{ [Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc})]_0^K [Y^l(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc})]_0^K \right\}_0^0 \\
= \sqrt{\frac{2K+1}{(2l+1)(2l_c+1)}} \\
\times \sum_{M'} \frac{(-1)^{K+M'}}{\sqrt{2K+1}} [Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc})]_{-M'}^K [Y^l(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc})]_{M'}^K \\
= \sqrt{\frac{1}{(2l+1)(2l_c+1)}} \\
\times \sum_{M'} [Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc})]_{M'}^{K*} [Y^l(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc})]_{M'}^K.
\end{aligned} \tag{7.123}$$

It is important to note that the integrals

$$\int d\hat{r}_{Cc} d\hat{r}_{b1} [Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc})]_M^{K*} [Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1})]_M^K, \tag{7.124}$$

and

$$\int d\hat{r}'_{Cc} d\hat{r}'_{A2} [Y^l(\hat{r}'_{Aa}) Y^{l_c}(\hat{r}'_{Cc})]_M^K [Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2})]_M^{K*}, \tag{7.125}$$



over the angular variables do not depend on  $M$ . Let us see why with (7.124),

$$\begin{aligned} \left[ Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]_M^{K*} \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K &= (-1)^{K-M} \left[ Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]_{-M}^K \\ &\times \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K = (-1)^{K-M} \sum_J \langle K K M -M | J 0 \rangle \quad (7.126) \\ &\times \left\{ \left[ Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^K \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]^K \right\}_0^J. \end{aligned}$$

After integration, only the term

$$\begin{aligned} (-1)^{K-M} \langle K K M -M | 0 0 \rangle \left\{ \left[ Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^K \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]^K \right\}_0^0 &= . \quad (7.127) \\ \frac{1}{\sqrt{2K+1}} \left\{ \left[ Y^l(\hat{r}_{Bb}) Y^{l_c}(\hat{r}_{Cc}) \right]^K \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]^K \right\}_0^0 \end{aligned}$$

corresponding to  $J = 0$  survives, which is indeed independent of  $M$ . We can thus omit the sum over  $M$  and multiply by  $(2K+1)$ , obtaining

$$\begin{aligned} T_{2NT}^{VV} &= \frac{64\mu_{Cc}(\pi)^{3/2}i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \frac{i^{-l}}{\sqrt{(2j_i+1)(2j_f+1)}} \\ &\times \sum_K (2K+1) ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0)_K^2 \quad (7.128) \\ &\times \sum_{l_c, l} \frac{e^{i(\sigma_i^l + \sigma_f^l)}}{\sqrt{(2l+1)}} Y_0^l(\hat{k}_{Bb}) S_{K, l, l_c}, \end{aligned}$$

with

$$\begin{aligned} S_{K, l, l_c} &= \int d^3 r_{Cc} d^3 r_{b1} v(r_{b1}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \frac{s_{K, l, l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \quad (7.129) \\ &\times \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K \left[ Y^{l_c}(\hat{r}_{Cc}) Y^l(\hat{r}_{Bb}) \right]_M^{K*}, \end{aligned}$$

and

$$\begin{aligned} s_{K, l, l_c}(r_{Cc}) &= \int_{r_{Cc} \text{ fixed}} d^3 r'_{Cc} d^3 r'_{A2} v(r'_{c2}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{b2}) \frac{F_l(r'_{Aa})}{r'_{Aa}} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r'_{Cc}} \\ &\times \left[ Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{b2}) \right]_M^{K*} \left[ Y^{l_c}(\hat{r}'_{Cc}) Y^l(\hat{r}'_{Aa}) \right]_M^K. \quad (7.130) \end{aligned}$$

The integrand in (7.129) can easily seen to be independent of  $M$ , so we can sum over  $M$  and divide by  $(2K+1)$ , to get the integrand

$$\begin{aligned} \frac{1}{2K+1} v(r_{b1}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \frac{s_{K, l, l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \quad (7.131) \\ \times \sum_M \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\hat{r}_{b1}) \right]_M^K \left[ Y^{l_c}(\hat{r}_{Cc}) Y^l(\hat{r}_{Bb}) \right]_M^{K*}. \end{aligned}$$

This integrand is rotationally invariant (it is proportional to a  $T_M^L$  spherical tensor with  $L = 0, M = 0$ ), so we can just evaluate it in the “standard” configuration in

which  $\mathbf{r}_{Cc}$  is directed along the  $z$ -axis and multiply by  $8\pi^2$  (see [?]), obtaining the final expression for  $S_{K,l,l_c}$ :

$$\begin{aligned} S_{K,l,l_c} &= \frac{4\pi^{3/2} \sqrt{2l_c + 1}}{2K + 1} i^{-l_c} \\ &\times \int r_{Cc}^2 dr_{Cc} r_{b1}^2 dr_{b1} \sin \theta d\theta v(r_{b1}) u_{l_f}(r_{A1}) u_{l_i}(r_{b1}) \\ &\times \frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \\ &\times \sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{r}_{A1}) Y^{l_i}(\theta + \pi, 0) \right]_M^K Y_M^{l*}(\hat{r}_{Bb}). \end{aligned} \quad (7.132)$$

Similarly, we have

$$\begin{aligned} s_{K,l,l_c}(r_{Cc}) &= \frac{4\pi^{3/2} \sqrt{2l_c + 1}}{2K + 1} i^{l_c} \\ &\times \int r_{Cc}'^2 dr_{Cc}' r_{A2}'^2 dr_{A2}' \sin \theta' d\theta' v(r_{c2}') u_{l_f}(r_{A2}') u_{l_i}(r_{b2}') \\ &\times \frac{F_l(r_{Aa}')}{r_{Aa}'} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r_{Cc}'} \\ &\times \sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{r}_{A2}') Y^{l_i}(\hat{r}_{b2}') \right]_M^{K*} Y_M^l(\hat{r}_{Aa}'). \end{aligned} \quad (7.133)$$

If we do the further approximations  $\mathbf{r}_{A1} \approx \mathbf{r}_{C1}$  and  $\mathbf{r}_{b2} \approx \mathbf{r}_{c2}$ , we obtain the final expression

$$\begin{aligned} T_{2NT}^{VV} &= \frac{1024\mu_{Cc}\pi^{9/2}i}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \frac{1}{\sqrt{(2j_i + 1)(2j_f + 1)}} \\ &\times \sum_K \frac{1}{2K + 1} ((l_f \frac{1}{2})_{j_f} (l_i \frac{1}{2})_{j_i} | (l_f l_i)_K (\frac{1}{2} \frac{1}{2})_0 )_K^2 \\ &\times \sum_{l_c, l} e^{i(\sigma_i' + \sigma_f')} \frac{(2l_c + 1)}{\sqrt{2l + 1}} Y_0^l(\hat{k}_{Bb}) S_{K,l,l_c}, \end{aligned} \quad (7.134)$$

with

$$\begin{aligned} S_{K,l,l_c} &= \int r_{Cc}^2 dr_{Cc} r_{b1}^2 dr_{b1} \sin \theta d\theta v(r_{b1}) u_{l_f}(r_{C1}) u_{l_i}(r_{b1}) \\ &\times \frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \\ &\times \sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{r}_{C1}) Y^{l_i}(\theta + \pi, 0) \right]_M^K Y_M^{l*}(\hat{r}_{Bb}), \end{aligned} \quad (7.135)$$

and

$$\begin{aligned} s_{K,l,l_c}(r_{Cc}) &= \int r_{Cc}'^2 dr_{Cc}' r_{A2}'^2 dr_{A2}' \sin \theta' d\theta' v(r_{c2}') u_{l_f}(r_{A2}') u_{l_i}(r_{c2}') \\ &\times \frac{F_l(r_{Aa}')}{r_{Aa}'} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r_{Cc}'} \\ &\times \sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{r}_{A2}') Y^{l_i}(\hat{r}_{c2}') \right]_M^{K*} Y_M^l(\hat{r}_{Aa}'). \end{aligned} \quad (7.136)$$

### 7.2.1 coordinates for the successive transfer

In the standard configuration in which the integrals (7.135) and (7.136) are to be evaluated, we have

$$\mathbf{r}_{Cc} = r_{Cc} \hat{\mathbf{z}}, \quad \mathbf{r}_{b1} = r_{b1}(-\cos \theta \hat{\mathbf{z}} - \sin \theta \hat{\mathbf{x}}). \quad (7.137)$$

Now,

$$\begin{aligned} \mathbf{r}_{C1} &= \mathbf{r}_{Cc} + \mathbf{r}_{c1} = \mathbf{r}_{Cc} + \frac{m_b}{m_b + 1} \mathbf{r}_{b1} \\ &= \left( r_{Cc} - \frac{m_b}{m_b + 1} r_{b1} \cos \theta \right) \hat{\mathbf{z}} - \frac{m_b}{m_b + 1} r_{b1} \sin \theta \hat{\mathbf{x}}, \end{aligned} \quad (7.138)$$

and

$$\mathbf{r}_{Bb} = \mathbf{r}_{BC} + \mathbf{r}_{Cb} = -\frac{1}{m_B} \mathbf{r}_{C1} + \mathbf{r}_{Cb}. \quad (7.139)$$

But

$$\mathbf{r}_{Cb} = \mathbf{r}_{Cc} + \mathbf{r}_{cb} = \mathbf{r}_{Cc} - \frac{1}{m_b + 1} \mathbf{r}_{b1}, \quad (7.140)$$

so, substituting in (7.139) we get

$$\mathbf{r}_{Bb} = \left( \frac{m_B - 1}{m_B} r_{Cc} + \frac{m_b + m_B}{m_B(m_b + 1)} r_{b1} \cos \theta \right) \hat{\mathbf{z}} + \frac{m_b + m_B}{m_B(m_b + 1)} r_{b1} \sin \theta \hat{\mathbf{x}}. \quad (7.141)$$

The primed variables are arranged in a similar fashion,

$$\mathbf{r}'_{Cc} = r'_{Cc} \hat{\mathbf{z}}, \quad \mathbf{r}'_{A2} = r'_{A2}(-\cos \theta' \hat{\mathbf{z}} - \sin \theta' \hat{\mathbf{x}}). \quad (7.142)$$

And we get

$$\mathbf{r}'_{c2} = \left( -r'_{Cc} - \frac{m_A}{m_A + 1} r'_{A2} \cos \theta' \right) \hat{\mathbf{z}} - \frac{m_A}{m_A + 1} r'_{A2} \sin \theta' \hat{\mathbf{x}}, \quad (7.143)$$

and

$$\mathbf{r}'_{Aa} = \left( \frac{m_a - 1}{m_a} r'_{Cc} - \frac{m_A + m_a}{m_a(m_A + 1)} r'_{A2} \cos \theta' \right) \hat{\mathbf{z}} - \frac{m_A + m_a}{m_a(m_A + 1)} r'_{A2} \sin \theta' \hat{\mathbf{x}}. \quad (7.144)$$

### 7.2.2 simplification of the vector coupling

We will now turn our attention to the vector-coupled quantities in (7.135) and (7.136),

$$\sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{\mathbf{r}}_{C1}) Y^{l_i}(\theta + \pi, 0) \right]_M^K Y_M^{l*}(\hat{\mathbf{r}}_{Bb}), \quad (7.145)$$

and

$$\sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_i}(\hat{\mathbf{r}}'_{c2}) \right]_M^{K*} Y_M^l(\hat{\mathbf{r}}'_{Aa}). \quad (7.146)$$

We will simplify these expressions in order to ease the computational numerical evaluation. We can express them as

$$\sum_M f(M), \quad (7.147)$$

where for, e.g., (7.145), we have

$$\langle l_c \ 0 \ l \ M | K \ M \rangle \left[ Y^{l_f}(\hat{r}_{C1}) Y^{l_i}(\theta + \pi, 0) \right]_M^K Y_M^{l*}(\hat{r}_{Bb}). \quad (7.148)$$

Note that all the vectors that come into play in the above expressions are in the  $xz$ -plane, and thus the azimuthal angle  $\phi$  is always equal to zero. Under these circumstances and for time-reversed phases,  $Y_M^{L*}(\theta, 0) = (-1)^L Y_M^L(\theta, 0)$ , so it is easy to verify that

$$f(-M) = (-1)^{l_c + l_f + l_i + l} f(M). \quad (7.149)$$

From (7.149), we have

$$\begin{aligned} \sum_M \langle l_c \ 0 \ l \ M | K \ M \rangle f(M) &= \langle l_c \ 0 \ l \ 0 | K \ 0 \rangle f(0) \\ &+ \sum_{M>0} \langle l_c \ 0 \ l \ M | K \ M \rangle f(M) \left( 1 + (-1)^{l_c + l + l_i + l_f} \right). \end{aligned} \quad (7.150)$$

We see that when  $l_c + l + l_i + l_f$  is odd we only have to evaluate the  $M = 0$  contribution. This consideration is useful to restrict the number of numerical operations needed to calculate the transition amplitude.

### 7.3 non-orthogonality term

We write the non-orthogonality contribution to the transition amplitude (see [?]):

$$\begin{aligned} T_{2NT}^{NO} &= 2 \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma'_1 \sigma'_2 \\ KM}} \int d^3 r_{Cc} d^3 r_{b1} d^3 r_{A2} d^3 r'_{b1} d^3 r'_{A2} \chi^{(-)*}(\mathbf{k}_{Bb}, \mathbf{r}_{Bb}) \\ &\times \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} v(r_{b1}) \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_i}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\ &\times \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^K \left[ \psi^{j_i}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_i}(\mathbf{r}'_{b2}, \sigma'_2) \right]_0^0 \chi^{(+)}(\mathbf{r}'_{Aa}). \end{aligned} \quad (7.151)$$

This expression is equivalent to (7.110) if we make the replacement

$$\frac{2\mu_{Cc}}{\hbar^2} G(\mathbf{r}_{Cc}, \mathbf{r}'_{Cc}) v(r'_{A2}) \rightarrow \delta(\mathbf{r}_{Cc} - \mathbf{r}'_{Cc}). \quad (7.152)$$

Looking at the partial-wave expansions of  $G(\mathbf{r}_{Cc}, \mathbf{r}'_{Cc})$  and  $\delta(\mathbf{r}_{Cc} - \mathbf{r}'_{Cc})$  (see Section ??), we find that we can use the above expressions for the successive transfer with the replacement

$$i \frac{2\mu_{Cc}}{\hbar^2} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{k_{Cc}} \rightarrow \delta(r_{Cc} - r'_{Cc}). \quad (7.153)$$

We thus have

$$\begin{aligned} T_{2NT}^{NO} &= \frac{512\pi^{9/2}}{k_{Aa} k_{Bb}} \frac{1}{\sqrt{(2j_i + 1)(2j_f + 1)}} \\ &\times \sum_K ((l_f \tfrac{1}{2})_{j_f} (l_i \tfrac{1}{2})_{j_i} |(l_f l_i)_K (\tfrac{1}{2} \tfrac{1}{2})_0|_K^2 \\ &\times \sum_{l_c, l} e^{i(\sigma'_l + \sigma'_f)} \frac{(2l_c + 1)}{\sqrt{2l + 1}} Y_0^l(\hat{k}_{Bb}) S_{K, l, l_c}, \end{aligned} \quad (7.154)$$

with

$$\begin{aligned}
S_{K,l,l_c} &= \int r_{Cc}^2 dr_{Cc} r_{b1}^2 dr_{b1} \sin \theta d\theta v(r_{b1}) u_{l_f}(r_{C1}) u_{l_i}(r_{b1}) \\
&\times \frac{s_{K,l,l_c}(r_{Cc})}{r_{Cc}} \frac{F_l(r_{Bb})}{r_{Bb}} \\
&\times \sum_M \langle l_c 0 l M | K M \rangle \left[ Y^{l_f}(\hat{r}_{C1}) Y^{l_i}(\theta + \pi, 0) \right]_M^K Y_M^{l_*}(\hat{r}_{Bb}),
\end{aligned} \tag{7.155}$$

and

$$\begin{aligned}
s_{K,l,l_c}(r_{Cc}) &= r_{Cc} \int dr'_{A2} r'^2_{A2} \sin \theta' d\theta' u_{l_f}(r'_{A2}) u_{l_i}(r'_{c2}) \frac{F_l(r'_{Aa})}{r'_{Aa}} \\
&\times \sum_M \langle l_c 0 l M | K M \rangle \left[ Y^{l_f}(\hat{r}'_{A2}) Y^{l_i}(\hat{r}'_{c2}) \right]_M^{K*} Y_M^{l_*}(\hat{r}'_{Aa}).
\end{aligned} \tag{7.156}$$

## 7.4 Arbitrary orbital momentum transfer

We will now examine the case in which the two transferred nucleons carry an angular momentum  $\Lambda$  different from 0. Let us assume that two nucleons coupled to angular momentum  $\Lambda$  in the initial nucleus  $a$  are transferred into a final state of zero angular momentum in nucleus  $B$ . The transition amplitude is given by the integral

$$\begin{aligned}
&2 \sum_{\sigma_1 \sigma_2} \int d\mathbf{r}_{cC} d\mathbf{r}_{A2} d\mathbf{r}_{b1} \chi^{(-)*}(\mathbf{r}_{bB}) \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \\
&\times v(r_{b1}) \Psi^{(+)}(\mathbf{r}_{aA}, \mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2).
\end{aligned} \tag{7.157}$$

If we neglect core excitations, the above expression is exact as long as  $\Psi^{(+)}(\mathbf{r}_{aA}, \mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2)$  is the exact wavefunction. We can instead obtain an approximation for the transfer amplitude using

$$\begin{aligned}
\Psi^{(+)}(\mathbf{r}_{aA}, \mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2) &\approx \chi^{(+)}(\mathbf{r}_{aA}) \left[ \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \psi^{j_{i2}}(\mathbf{r}_{b2}, \sigma_2) \right]_\mu^\Lambda \\
&+ \sum_{K,M} \mathcal{U}_{K,M}(\mathbf{r}_{cC}) \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K
\end{aligned} \tag{7.158}$$

as an approximation for the incoming state. The first term of (7.158) gives rise to the simultaneous amplitude, while from second one we get the successive and the non-orthogonality contributions. To extract the amplitude  $\mathcal{U}_{K,M}(\mathbf{r}_{cC})$ , we define  $f_{KM}(\mathbf{r}_{cC})$  as the scalar product

$$f_{KM}(\mathbf{r}_{cC}) = \left\langle \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \middle| \Psi^{(+)}(\mathbf{r}_{aA}, \mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2) \right\rangle \tag{7.159}$$

for fixed  $\mathbf{r}_{cC}$ , which can be seen to obey the equation

$$\begin{aligned}
&\left( \frac{\hbar^2}{2\mu_{cC}} k_{cC}^2 + \frac{\hbar^2}{2\mu_{cC}} \nabla_{r_{cC}}^2 - U(r_{cC}) \right) f_{KM}(\mathbf{r}_{cC}) \\
&= \left\langle \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \middle| v(r_{c2}) \middle| \Psi^{(+)}(\mathbf{r}_{aA}, \mathbf{r}_{b1}, \mathbf{r}_{b2}, \sigma_1, \sigma_2) \right\rangle.
\end{aligned} \tag{7.160}$$

The solution can be written in terms of the Green function  $G(\mathbf{r}_{cC}, \mathbf{r}'_{cC})$  defined by

$$\left( \frac{\hbar^2}{2\mu_{cC}} k_{cC}^2 + \frac{\hbar^2}{2\mu_{cC}} \nabla_{r_{cC}}^2 - U(r_{cC}) \right) G(\mathbf{r}_{cC}, \mathbf{r}'_{cC}) = \frac{\hbar^2}{2\mu_{cC}} \delta(\mathbf{r}_{cC} - \mathbf{r}'_{cC}). \quad (7.161)$$

Thus,

$$\begin{aligned} f_{KM}(\mathbf{r}_{cC}) &= \frac{2\mu_{cC}}{\hbar^2} \int d\mathbf{r}'_{cC} G(\mathbf{r}_{cC}, \mathbf{r}'_{cC}) \left\langle \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^K \left| v(r_{c2}) \right| \Psi^{(+)}(\mathbf{r}'_{aA}, \mathbf{r}'_{b1}, \mathbf{r}'_{b2}, \sigma'_1, \sigma'_2) \right\rangle \\ &\approx \frac{2\mu_{cC}}{\hbar^2} \sum_{\sigma'_1 \sigma'_2} \int d\mathbf{r}'_{cC} d\mathbf{r}'_{A2} d\mathbf{r}'_{b1} G(\mathbf{r}_{cC}, \mathbf{r}'_{cC}) \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \\ &\quad \times v(r'_{c2}) \chi^{(+)}(\mathbf{r}'_{aA}) \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda = \mathcal{U}_{K,M}(\mathbf{r}_{cC}) \\ &\quad + \left\langle \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma_1) \right]_M^K \left| \chi^{(+)}(\mathbf{r}'_{aA}) \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \right\rangle. \end{aligned} \quad (7.162)$$

Therefore

$$\begin{aligned} \mathcal{U}_{K,M}(\mathbf{r}_{cC}) &= \frac{2\mu_{cC}}{\hbar^2} \sum_{\sigma'_1 \sigma'_2} \int d\mathbf{r}'_{cC} d\mathbf{r}'_{A2} d\mathbf{r}'_{b1} G(\mathbf{r}_{cC}, \mathbf{r}'_{cC}) \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \\ &\quad \times v(r'_{c2}) \chi^{(+)}(\mathbf{r}'_{aA}) \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \\ &\quad - \left\langle \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma_1) \right]_M^K \left| \chi^{(+)}(\mathbf{r}'_{aA}) \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \right\rangle. \end{aligned} \quad (7.163)$$

When we substitute  $\mathcal{U}_{K,M}(\mathbf{r}_{cC})$  into (7.158) and (7.157), the first term gives rise to the successive amplitude for the two-particle transfer, while the second term is responsible for the non-orthogonal contribution.

### 7.4.1 Successive contribution

We need to evaluate the integral

$$\begin{aligned} T_\mu^{succ} &= \frac{4\mu_{cC}}{\hbar^2} \sum_{\sigma_1 \sigma_2} \sum_{KM} \int d\mathbf{r}_{cC} d\mathbf{r}_{A2} d\mathbf{r}_{b1} d\mathbf{r}'_{cC} d\mathbf{r}'_{A2} d\mathbf{r}'_{b1} \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \\ &\quad \times \chi^{(-)*}(\mathbf{r}_{bB}) G(\mathbf{r}_{cC}, \mathbf{r}'_{cC}) \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \chi^{(+)}(\mathbf{r}'_{aA}) v(r'_{c2}) v(r_{b1}) \\ &\quad \times \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K, \end{aligned} \quad (7.164)$$

where we must substitute the Green function and the distorted waves by their partial wave expansions (see Appendix). The integral over  $\mathbf{r}'_{b1}$  is:

$$\begin{aligned}
& \sum_{\sigma'_1} \int d\mathbf{r}'_{b1} \left[ \psi^{j_f}(\mathbf{r}'_{A2}, \sigma'_2) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_M^{K*} \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \\
&= (-1)^{-M+j_f+j_{i1}-\sigma_1-\sigma_2} \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, -\sigma'_1) \psi^{j_f}(\mathbf{r}'_{A2}, -\sigma'_2) \right]_{-M}^K \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_\mu^\Lambda \\
&= \sum_{\sigma'_1} \int (-1)^{-M+j_f+j_{i1}-\sigma_1-\sigma_2} \sum_P \langle K \Lambda -M \mu | P \mu -M \rangle \langle (j_{i1} j_f)_K (j_{i1} j_{i2})_\Lambda | (j_{i1} j_{i1})_0 (j_f j_{i2})_P \rangle_P \\
&\quad \times \left[ \psi^{j_{i1}}(\mathbf{r}'_{b1}, -\sigma'_1) \psi^{j_{i1}}(\mathbf{r}'_{b1}, \sigma'_1) \right]_0^0 \left[ \psi^{j_f}(\mathbf{r}'_{A2}, -\sigma'_2) \psi^{j_{i2}}(\mathbf{r}'_{b2}, \sigma'_2) \right]_{\mu-M}^P \\
&= (-1)^{-M+j_f+j_{i1}} \sqrt{2j_{i1}+1} u_{l_f}(r_{A2}) u_{l_2}(r'_{b2}) \sum_P \langle K \Lambda -M \mu | P \mu -M \rangle \\
&\quad \times \langle (j_{i1} j_f)_K (j_{i1} j_{i2})_\Lambda | (j_{i1} j_{i1})_0 (j_f j_{i2})_P \rangle_P \langle (l_f \frac{1}{2})_{j_f} (l_{i2} \frac{1}{2})_{j_{i2}} | (l_f l_{i2})_P (\frac{1}{2} \frac{1}{2})_0 \rangle_P \\
&\quad \times \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]_{\mu-M}^P u_{l_f}(r_{A2}) u_{l_2}(r_{b2}). \quad (7.165)
\end{aligned}$$

Integral over  $\mathbf{r}_{A2}$  (see (7.117)):

$$\begin{aligned}
& \sum_{\sigma_2} \int d\mathbf{r}_{A2} \left[ \psi^{j_f}(\mathbf{r}_{A1}, \sigma_1) \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \right]_0^{0*} \left[ \psi^{j_f}(\mathbf{r}_{A2}, \sigma_2) \psi^{j_{i1}}(\mathbf{r}_{b1}, \sigma_1) \right]_M^K \\
&= -\sqrt{\frac{2}{2j_f+1}} \langle (l_f \frac{1}{2})_{j_f} (l_{i1} \frac{1}{2})_{j_{i1}} | (l_f l_{i1})_K (\frac{1}{2} \frac{1}{2})_0 \rangle_K \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]_M^K u_{l_f}(r_{A1}) u_{l_{i1}}(r_{b1}). \quad (7.166)
\end{aligned}$$

Let us examine the term

$$\sum_M (-1)^M \langle K \Lambda -M \mu | P \mu -M \rangle \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]_M^K \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]_{\mu-M}^P. \quad (7.167)$$

By virtue of the property of Clebsh–Gordan coefficients

$$\langle l_1 l_2 m_1 m_2 | L M_L \rangle = (-1)^{l_2-m_2} \sqrt{\frac{2L+1}{2l_1+1}} \langle L l_2 -M_L m_2 | l_1 -m_1 \rangle, \quad (7.168)$$

the expression (7.168) is equivalent to

$$(-1)^K \sqrt{\frac{2P+1}{2\Lambda+1}} \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]_M^K \right\}_\mu^\Lambda. \quad (7.169)$$

We re-couple the following terms arising from the partial wave expansion of the incoming and outgoing distorted waves:

$$\left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_a}(\hat{\mathbf{k}}_{aA}) \right]_0^0 \left[ Y^{l_b}(\hat{\mathbf{r}}_{bB}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]_0^0 \quad (7.170)$$

to have

$$((l_a l_a)_0 (l_b l_b)_0 | (l_a l_b)_\Lambda (l_a l_b)_\Lambda)_0 \left\{ \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_b}(\hat{\mathbf{r}}_{bB}) \right]^\Lambda \left[ Y^{l_a}(\hat{\mathbf{k}}_{aA}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]^\Lambda \right\}_0^0. \quad (7.171)$$

The only term that survives the integration is

$$\frac{(-1)^{\Lambda-\mu}}{\sqrt{(2l_a+1)(2l_b+1)}} \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_b}(\hat{\mathbf{r}}_{bB}) \right]_{-\mu}^{\Lambda} \left[ Y^{l_a}(\hat{\mathbf{k}}_{aA}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]_{\mu}^{\Lambda}. \quad (7.172)$$

Again, the only term surviving

$$\left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \right\}_{\mu}^{\Lambda} \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_b}(\hat{\mathbf{r}}_{bB}) \right]_{-\mu}^{\Lambda} \quad (7.173)$$

is

$$\frac{(-1)^{\Lambda+\mu}}{\sqrt{2\Lambda+1}} \left[ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \right]_{\mu}^{\Lambda} \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_b}(\hat{\mathbf{r}}_{bB}) \right]_{\mu}^{\Lambda}. \quad (7.174)$$

Now we couple this last term with  $\left[ Y^{l_c}(\hat{\mathbf{r}}'_{cC}) Y^{l_c}(\hat{\mathbf{r}}_{cC}) \right]_0^0$ , which arises from the partial wave expansion of the Green function,

$$\begin{aligned} & \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \right\}_{\mu}^{\Lambda} \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_b}(\hat{\mathbf{r}}_{bB}) \right]_{\mu}^{\Lambda} \left[ Y^{l_c}(\hat{\mathbf{r}}'_{cC}) Y^{l_c}(\hat{\mathbf{r}}_{cC}) \right]_0^0 \\ &= ((l_a l_b)_{\Lambda} (l_c l_c)_0 | (l_a l_c)_P (l_b l_c)_K)_{\Lambda} \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \right\}_{\mu}^{\Lambda} \\ & \left\{ \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_c}(\hat{\mathbf{r}}'_{cC}) \right]^P \left[ Y^{l_b}(\hat{\mathbf{r}}_{bB}) Y^{l_c}(\hat{\mathbf{r}}_{cC}) \right]^K \right\}_{\mu}^{\Lambda} = ((l_a l_b)_{\Lambda} (l_c l_c)_0 | (l_a l_c)_P (l_b l_c)_K)_{\Lambda} \\ & \times ((PK)_{\Lambda} (PK)_{\Lambda} | (PP)_0 (KK)_0)_0 \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_c}(\hat{\mathbf{r}}'_{cC}) \right]^P \right\}_0^0 \\ & \times \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \left[ Y^{l_b}(\hat{\mathbf{r}}_{bB}) Y^{l_c}(\hat{\mathbf{r}}_{cC}) \right]^K \right\}_0^0 = ((l_a l_b)_{\Lambda} (l_c l_c)_0 | (l_a l_c)_P (l_b l_c)_K)_{\Lambda} \\ & \times \sqrt{\frac{2\Lambda+1}{(2K+1)(2P+1)}} \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}'_{A2}) Y^{l_{i2}}(\hat{\mathbf{r}}'_{b2}) \right]^P \left[ Y^{l_a}(\hat{\mathbf{r}}'_{aA}) Y^{l_c}(\hat{\mathbf{r}}'_{cC}) \right]^P \right\}_0^0 \\ & \times \left\{ \left[ Y^{l_f}(\hat{\mathbf{r}}_{A1}) Y^{l_{i1}}(\hat{\mathbf{r}}_{b1}) \right]^K \left[ Y^{l_b}(\hat{\mathbf{r}}_{bB}) Y^{l_c}(\hat{\mathbf{r}}_{cC}) \right]^K \right\}_0^0. \end{aligned} \quad (7.175)$$

When we collect all the pieces (including the constants and phases coming from the partial wave expansion of the distorted waves and the Green function), we finally get

$$\begin{aligned} T_{\mu}^{succ} &= (-1)^{j_f+j_{i1}} \frac{2048\pi^5 \mu_{Cc}}{\hbar^2 k_{Aa} k_{Bb} k_{Cc}} \sqrt{\frac{(2j_{i1}+1)}{(2\Lambda+1)(2j_f+1)}} \sum_{K,P} ((l_f \frac{1}{2})_{j_f} (l_{i2} \frac{1}{2})_{j_{i2}} | (l_f l_{i2})_P (\frac{1}{2} \frac{1}{2})_0)_P \\ & \times ((l_f \frac{1}{2})_{j_f} (l_{i1} \frac{1}{2})_{j_{i1}} | (l_f l_{i1})_K (\frac{1}{2} \frac{1}{2})_0)_K ((j_{i1} j_f)_K (j_{i1} j_{i2})_{\Lambda} | (j_{i1} j_{i1})_0 (j_f j_{i2})_P)_P \\ & \times \frac{(-1)^K}{(2K+1) \sqrt{2P+1}} \sum_{l_c, l_a, l_b} ((l_a l_b)_{\Lambda} (l_c l_c)_0 | (l_a l_c)_P (l_b l_c)_K)_{\Lambda} e^{i(\sigma_i^{l_a} + \sigma_f^{l_b})} i^{l_a-l_b} \\ & \times (2l_c+1)^{3/2} \left[ Y^{l_a}(\hat{\mathbf{k}}_{aA}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]_{\mu}^{\Lambda} S_{K,P,l_a,l_b,l_c} \end{aligned} \quad (7.176)$$



with (note that we have reduced the dimensionality of the integrals in the same fashion as for the 0-angular momentum transfer calculation, see (7.132))

$$\begin{aligned}
 S_{K, P, l_a, l_b, l_c} &= \int r_{Cc}^2 dr_{Cc} r_{b1}^2 dr_{b1} \sin \theta d\theta v(r_{b1}) u_{l_f}(r_{C1}) u_{l_i}(r_{b1}) \\
 &\times \frac{s_{P, l_a, l_c}(r_{Cc})}{r_{Cc}} \frac{F_{l_b}(r_{Bb})}{r_{Bb}} \\
 &\times \sum_M \langle l_c 0 l_b M | K M \rangle \left[ Y^{l_f}(\hat{r}_{C1}) Y^{l_{i1}}(\theta + \pi, 0) \right]_M^K Y_{-M}^{l_b}(\hat{r}_{Bb}),
 \end{aligned} \tag{7.177}$$

and

$$\begin{aligned}
 s_{P, l_a, l_c}(r_{Cc}) &= \int r_{Cc}^{\prime 2} dr_{Cc}' r_{A2}^{\prime 2} dr_{A2}' \sin \theta' d\theta' v(r_{c2}') u_{l_f}(r_{A2}') u_{l_i}(r_{c2}') \\
 &\times \frac{F_{l_a}(r_{Aa}')}{r_{Aa}'} \frac{f_{l_c}(k_{Cc}, r_{<}) P_{l_c}(k_{Cc}, r_{>})}{r_{Cc}'} \\
 &\times \sum_M \langle l_c 0 l_a M | P M \rangle \left[ Y^{l_f}(\hat{r}_{A2}') Y^{l_{i2}}(\hat{r}_{c2}') \right]_M^P Y_{-M}^{l_a}(\hat{r}_{Aa}').
 \end{aligned} \tag{7.178}$$

Note that we have evaluated the transition matrix element for a particular projection  $\mu$  of the initial angular momentum of the two transferred nucleons. If they are coupled to a core of angular momentum  $J_f$  to total angular momentum  $J_i, M_i$ , the fraction of the initial wavefunction with projection  $\mu$  is  $\langle \Lambda \mu J_f M_i - \mu | J_i M_i \rangle$ , and the cross section will be

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA} \mu_{bB}}{(2\pi\hbar^2)^2} \left| \sum_{\mu} \langle \Lambda \mu J_f M_i - \mu | J_i M_i \rangle T_{\mu} \right|^2. \tag{7.179}$$

For a non polarized incident beam,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA} \mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2J_i + 1} \sum_{M_i} \left| \sum_{\mu} \langle \Lambda \mu J_f M_i - \mu | J_i M_i \rangle T_{\mu} \right|^2. \tag{7.180}$$

This would be the differential cross section for a transition to a definite final state  $M_f$ . If we don't measure  $M_f$  we have to sum for all  $M_f$ ,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA} \mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2J_i + 1} \sum_{\mu} |T_{\mu}|^2 \sum_{M_i, M_f} |\langle \Lambda \mu J_f M_f | J_i M_i \rangle|^2. \tag{7.181}$$

The sum over  $M_i, M_f$  of the Clebsch–Gordan coefficients is  $(2J_i + 1)/(2\Lambda + 1)$  (see ??), so we finally get

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA} \mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{(2\Lambda + 1)} \sum_{\mu} |T_{\mu}|^2. \tag{7.182}$$

where we can write

$$\begin{aligned}
 T_{\mu} &= \sum_{l_a, l_b} C_{l_a, l_b} \left[ Y^{l_a}(\hat{\mathbf{k}}_{aA}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]_{\mu}^{\Lambda} \\
 &= \sum_{l_a, l_b} C_{l_a, l_b} i^{l_a} \sqrt{\frac{2l_a + 1}{4\pi}} \langle l_a l_b 0 \mu | \Lambda \mu \rangle Y_{\mu}^{l_b}(\hat{\mathbf{k}}_{bB}).
 \end{aligned} \tag{7.183}$$

Note that (7.182) takes into account only the spins of the heavy nucleus. In a  $(t, p)$  or  $(p, t)$  reaction, we have to sum over the spins of the proton and of the triton and divide by 2. If a spin orbit term is present in the optical potential, the sum yields the combination of terms shown in section (7.1):

$$\frac{d\sigma}{d\Omega}(\mathbf{k}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2(2\Lambda + 1)} \sum_{\mu} |A_{\mu}|^2 + |B_{\mu}|^2. \quad (7.184)$$

## 7.5 Two-nucleon transfer reactions

Again we assume that the reaction is direct, and that it is adequately described by first-order distorted-wave Born approximation.

To be specific, we will concentrate on  $(t, p)$  reaction, namely reactions of the type  $A(\alpha, \beta)B$  where  $\alpha = \beta + 2$  and  $B = A + 2$ .

The intrinsic wave functions are in this case

$$\begin{aligned} \psi_{\alpha} &= \psi_{M_i}^{J_i}(\xi_A) \sum_{ss'} [\chi^s(\sigma_{\alpha}) \chi^{s'}(\sigma_{\beta})]_{M_{s_i}}^{s_i} \phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|) \\ &= \psi_{M_i}^{J_i}(\xi_A) \sum_{M_s M_{s_f}} (s M_{s_f}' s_f' M_{s_f}' | s_i M_{s_i}) \chi_{M_s}^s(\sigma_{\alpha}) \chi_{M_{s_f}}^{s_f'}(\sigma_{\beta}) \\ &\quad \times \phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|) \end{aligned} \quad (7.185)$$

while

$$\begin{aligned} \psi_{\beta} &= \psi_{M_f}^{J_f}(\xi_{A+2}) \chi_{M_{s_f}}^{s_f'}(\sigma_{\beta}) \\ &= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) [\phi^J(j_1 j_2) \phi^{J_i'}(\xi_A)]_{M_f}^{J_f} \\ &\quad \times \chi_{M_{s_f}}^{s_f'}(\sigma_{\beta}) \end{aligned} \quad (7.186)$$

But from eq. (7.186) is easy to see that the spectroscopic amplitude  $B$  is equal to

$$\begin{aligned} &B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \\ &= \left\langle \psi^{J_f}(\xi_{A+2}) \left| [\phi^J(j_1 j_2) \phi^{J_i'}(\xi_A)]^{J_f} \right. \right\rangle \end{aligned} \quad (7.187)$$

where

$$\phi^J(j_1 j_2) = \frac{[\phi_{j_1}(\vec{r}_1) \phi_{j_2}(\vec{r}_2)]^J - [\phi_{j_1}(\vec{r}_2) \phi_{j_2}(\vec{r}_1)]^J}{\sqrt{1 + \delta(j_1, j_2)}} \quad (7.188)$$

is an antisymmetrized, normalized wave function of the two transferred particles. The function  $\chi_{M_s}^s(\sigma_{\beta})$  appearing both in eq. (7.185) and (7.186) is the spin wave function of the proton while  $\chi^s(\sigma_{\alpha})$  is equal to

$$\chi^s(\sigma_{\alpha}) = [\chi^{s_1}(\sigma_{n_1}) \chi^{s_2}(\sigma_{n_2})]^s \quad (7.189)$$

is the spin function of the two-neutron system.

The function  $\phi_t^{L=0}$  describes the internal degree of freedom of the triton. A good description of this system is obtained by using a wave function symmetric in the coordinates of all particles, i.e.

$$\begin{aligned}\phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|) &= N_t e^{[(r_1-r_2)^2 + (r_1-r_p)^2 + (r_2-r_p)^2]} \\ &= \phi_{000}(\vec{r})\phi_{000}(\vec{\rho}) \\ \phi_{000}(\vec{r}) &= R_{nl}(v^{1/2}r)Y_{lm}(\hat{r})\end{aligned}\quad (7.190)$$

The coordinate  $\vec{\rho}$  is the radius vector which measures the distance between the center of mass of the dineutron and the proton. The vector  $\vec{r}$  is the dineutron relative coordinate.

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad (\text{relative distance between the neutrons}) \quad (7.191a)$$

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \quad (\text{coord. of the CM of the dineutron}) \quad (7.191b)$$

$$\vec{\rho} = \vec{r}_p - \frac{\vec{r}_1 + \vec{r}_2}{2} \quad (\text{distance between the CM of the dineutron and the proton}) \quad (7.191c)$$

$$\vec{R}_2 = \vec{r}_p - \frac{\vec{r}_1 + \vec{r}_2}{A+2} \quad (\text{distance of the proton from the CM of the system } A+2) \quad (7.191d)$$

$$\vec{R}_1 = \frac{\vec{r}_p + \vec{r}_1 + \vec{r}_2}{3} \quad (\text{coord. of the CM of the triton}) \quad (7.191e)$$

To obtain the DWBA cross section we have to calculate the integral

$$T = \int d\xi_A d\vec{r}_1 d\vec{r}_2 d\vec{r}_p \chi_p^{(-)}(\vec{R}_2) \psi_\beta^*(\xi_{A+2}, \sigma_\beta) V'_\beta \psi_\alpha(\xi_A, \sigma_\alpha, \sigma_\beta) \psi_t^{(+)}(\vec{R}_1) \quad (7.192)$$

Instead of integrating over  $\xi_A, \vec{r}_1, \vec{r}_2$  and  $\vec{r}_p$  we would integrate over  $\xi_A, \vec{r}, \vec{\rho}$  and  $\vec{r}_p$ . The Jacobian of the transformation is equal to 1, i.e.  $\partial(\vec{r}_1, \vec{r}_2)/\partial(\vec{r}, \vec{\rho}) = 1$ .

To carry out the integral (7.192) we transform the wave function (7.188) into center of mass and relative coordinates. If we assume that both  $\phi_{j_1}(\vec{r}_1)$  and  $\phi_{j_2}(\vec{r}_2)$  are harmonic oscillator wave functions, this transformation can easily be carried with the aid of the Moshinsky brackets. If  $|n_1 l_1, n_2 l_2; \lambda \mu\rangle$  is a complete system of wave functions in the harmonic oscillator basis, depending on  $\vec{r}_1$  and  $\vec{r}_2$  and  $|nl, NL; \lambda \mu\rangle$  is the corresponding one depending on  $\vec{r}$  and  $\vec{R}$ , we can write

$$\begin{aligned}|n_1 l_1, n_2 l_2; \lambda \mu\rangle &= \left( \sum_{nlNL} |nl, NL; \lambda \mu\rangle \langle nl, NL; \lambda \mu| \right) |n_1 l_1, n_2 l_2; \lambda \mu\rangle \\ &= \sum_{nlNL} |nl, NL; \lambda \mu\rangle \langle nl, NL; \lambda \mu| n_1 l_1, n_2 l_2; \lambda \mu\rangle\end{aligned}\quad (7.193)$$

The labels  $n, l$  are the principal and angular momentum quantum numbers of the relative motion, while  $N, L$  are the corresponding ones corresponding to the center of mass motion of the two-neutron system. Because of energy and parity conservation we have

$$\begin{aligned}
2n_1 + l_1 + 2n_2 + l_2 &= 2n + l + 2N + L \\
(-1)^{l_1+l_2} &= (-1)^{l+L}
\end{aligned}
\tag{7.194}$$

The coefficients  $\langle nl, NL, L | n_1 l_1, n_2 l_2, L \rangle$  are tabulated and were first discussed by M. Moshinsky in Nucl. Physics, 13 (1959) 104.

With the help of eq.(7.193) we can write the wave function  $\psi_{M_f}^{J_f}(\xi_{A+2})$  as

$$\begin{aligned}
\psi_{M_f}^{J_f}(\xi_{A+2}) &= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2 \\ J J_i}} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \left[ \phi^J(j_1 j_2) \phi^{J_i'}(\xi_A) \right]_{M_f}^{J_f} \\
&= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} \sum_{J J_i} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \\
&\times \sum_{M_J M_{J_i}} \langle J M_J J_i' M_{J_i} | J_f M_{J_f} \rangle \psi_{M_{J_i}}^{J_i'}(\xi_A) \\
&\times \sum_{L S'} \langle S' L J | j_1 j_2 J \rangle \sum_{M_L M_S'} \langle L M_L S' M_S' | J M_J \rangle \chi_{M_S'}^{S'}(\sigma_\alpha) \\
&\times \sum_{n l N \Lambda} \langle n l, N \Lambda, L | n_1 l_1, n_2 l_2, L \rangle \\
&\times \sum_{m_l M_\Lambda} \langle l m_l \Lambda M_\Lambda | L M_L \rangle \phi_{n l m_l}(\vec{r}) \phi_{N \Lambda M_\Lambda}(\vec{R})
\end{aligned}
\tag{7.195}$$

Integration over  $\vec{r}$  gives

$$\langle \phi_{n l m_l}(\vec{r}) | \phi_{000}(\vec{r}) \rangle = \delta(l, 0) \delta(m_l, 0) \Omega_n
\tag{7.196}$$

where

$$\Omega_n = \int R_{nl}(v_1^{1/2} r) R_{00}(v_2^{1/2} r) r^2 dr
\tag{7.197}$$

Note that there is no selection rule in the principal quantum number  $n$ , as the potential in which the two neutrons move in the triton has a frequency  $v_2$  which is different from the one that the two neutrons are subjected to, when moving in the system  $A$ .

Integration over  $\xi_A$  and multiplication of the spin functions gives

$$\begin{aligned}
(\psi_{M_{J_i}}^{J_i}, (V(\vec{r}_1) + V(\vec{r}_2) + V(\vec{r}_p) - U) \psi_{M_{J_i}'}^{J_i'}) &= \delta(J_i, J_i') \delta(M_{J_i}, M_{J_i}') V_{eff}(\vec{\rho}) \\
(\chi_{M_S}^S(\sigma_\alpha), \chi_{M_{S'}}^{S'}(\sigma_\alpha)) &= \delta(S, S') \delta(M_S, M_{S'}) \\
(\chi_{M_{S_f}}^{S_f}(\sigma_\beta), \chi_{M_{S_f}'}^{S_f'}(\sigma_\beta)) &= \delta(S_f, S_f') \delta(M_{S_f}, M_{S_f}')
\end{aligned}
\tag{7.198}$$

The integral (7.192) can now be written as

$$\begin{aligned}
T = & \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} \sum_{JM_J} \sum_{nN} \sum_S B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \\
& \times \langle J M_J J_i M_{J_i} | J_f M_{J_f} \rangle \langle S L J | j_1 j_2 J \rangle \\
& \times \langle L M_L S M_S | J M_J \rangle \langle n 0, N L, L | n_1 l_1, n_2 l_2, L \rangle \\
& \times \langle S M_S S_f M_{S_f} | S_i M_{S_i} \rangle \Omega_n \\
& \times \int d\vec{R} d\vec{r}_p \chi_i^{(+)*}(\vec{R}_1) \phi_{NL M_L}^*(\vec{R}) V_{eff}(\vec{\rho}) \phi_{000}(\vec{\rho}) \chi_i^{(+)}(\vec{R}_1)
\end{aligned} \tag{7.199}$$

where we have approximated  $V'_\beta$  by an effective interaction  $V_{eff}$  depending only on  $\rho = |\vec{\rho}|$ . It is important to point out that the two-body interaction would act on the two-particle system at once, but the single particle potential would act on each particle independently. The reason why we can neglect the successive transfer of the nucleons (two-step process) is because the two neutrons in the triton are very strongly correlated and they build to a large extent a unity.

We now define the two-nucleon transfer form factor as

$$\begin{aligned}
u_{LSJ}^{j_i J_f}(R) = & \sum_{n_1 l_1 j_1} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i J_f) \langle S L J | j_1 j_2 J \rangle \\
& \langle n 0, N L, L | n_1 l_1, n_2 l_2; L \rangle \Omega_n R_{nL}(R)
\end{aligned} \tag{7.200}$$

We can now rewrite eq. (7.199) as

$$\begin{aligned}
T = & \sum_J \sum_L \sum_S (J M_J J_i M_{J_i} | J_f M_{J_f}) (S M_S S_f M_{S_f} | S_i M_{S_i}) (L M_L S M_S | J M_J) \\
& \times \int d\vec{R} d\vec{r}_p \chi_p^{*(-)}(\vec{R}_2) u_{LSJ}^{j_i J_f}(R) Y_{LM_L}^* V(\rho) \phi_{000}(\vec{\rho}) \chi_i^{(+)}(\vec{R}_1)
\end{aligned} \tag{7.201}$$

Because the di-neutron has  $S = 0$ , we have that

$$(L M_L 0 0 | J M_J) = \delta(J, L) \delta(M_L, M_J) \tag{7.202}$$

and the summations over  $S$  and  $L$  disappear from eq. (7.201).

The integral to be carried out in eq. (7.201) is six-dimensional, and is a formidable task to calculate it exactly (one of these integrals takes  $\approx 5$  hs in a CDC 6600 computer).

One can make also here the zero range approximation, i.e.

$$V(\rho) \phi_{000}(\vec{\rho}) = D_0 \delta(\vec{\rho}) \tag{7.203}$$

This means that the proton interacts with the center of mass of the di-neutron, only when they are at the same point in space.

From eqs. (??) we obtain

$$\begin{aligned}
\vec{R} = \vec{R}_1 = \vec{r} \\
\vec{R}_2 = \frac{A}{A+2} \vec{R}
\end{aligned} \tag{7.204}$$

Then eq. (7.199) can be written as

$$T = D_0 \sum_L (LM_L J_i M_{J_i} | J_f M_{J_f}) \times \int d\vec{R} \chi_p^{*(-)} \left( \frac{A}{A+2} \vec{R} \right) u_L^{J_i J_f}(R) Y_{LM_L}^*(\hat{R}) \chi_i^{(+)}(\vec{R}) \quad (7.205)$$

From eq. (7.205) it is seen that the change in parity implied by the reaction is given by  $\Delta\Pi = (-1)^L$ . Consequently, the selection rules for  $(t, p)$  and  $(p, t)$  reactions are

$$\begin{aligned} \Delta S &= 0 \\ \Delta J &= \Delta L = L \\ \Delta\Pi &= (-1)^L \end{aligned} \quad (7.206)$$

i.e. only normal parity states are excited.

The integral appearing in eq. (7.205) has the same structure as the DWBA integral (??) (Fourth Lecture) which was derived in the case of one-nucleon transfer reactions.

The difference between the two processes manifest itself through the different structure of the two form factors. While  $u_l(r)$  appearing in equation (??) is a single-particle bound state wave function,  $u_L^{J_i J_f}$  is a coherent summation over the center of mass states of motion of the two transferred neutrons.

## 7.6 On the relative importance of successive, simultaneous, and pairing induced two-particle transfer

Let us denote

$$H = T + V, \quad (7.207)$$

the total hamiltonian describing the nuclear system, where  $V$  is the nuclear two-body interaction.

The fact that the nuclear quantality parameter has a value of  $Q \approx 0.4$  testifies to the validity of independent particle motion in nuclei. This is tantamount to saying that there exist a single-particle potential  $U$ , such that

$$\langle \Psi_0 | U | \Psi_0 \rangle \ll \langle \Psi_0 | (V - U) | \Psi_0 \rangle, \quad (7.208)$$

where  $\Psi_0$  is the exact ground state wavefunction, that is,  $H\Psi_0 = E_0\Psi_0$ . One can then write 7.207 as

$$H = T + V_{eff}, \quad (7.209)$$

where

$$V_{eff} = U + (V - U). \quad (7.210)$$

Let us now consider a reaction in which two nucleons are transferred between target and projectile, that is,

$$a(= b + 2) + A \rightarrow b + B(= A + 2). \quad (7.211)$$

The transfer cross section is proportional to the square of the amplitude

$$\sqrt{\sigma} \sim \langle bB | V_{eff} | aA \rangle = \langle bB | U | aA \rangle + \langle bB | (V - U) | aA \rangle. \quad (7.212)$$

Let us assume that the transferred nucleons are e.g. two neutrons moving in time reversal states lying close to the Fermi energy (Cooper pair). In this case it is natural to assume that the operative component of  $(V - U)$  is the pairing interaction

$$V_p = -GP^\dagger P, \quad (7.213)$$

where

$$P^\dagger = \sum_{\nu>0} a_\nu^\dagger a_{\bar{\nu}}^\dagger, \quad (7.214)$$

is the pair operator, and

$$G \approx \frac{18}{A} \text{ MeV}, \quad (7.215)$$

is the pairing coupling constant for nucleons moving in an extended (2–3 major shell) configuration.

One can then write Eq. 7.212 as

$$\sqrt{\sigma} = \sqrt{\sigma_1} + \sqrt{\sigma_2}, \quad (7.216)$$

where

$$\sqrt{\sigma_1} \sim \langle Bb|U|aA \rangle \approx 2 \left( \frac{|V_0|}{2} \right) O, \text{ (SUCC+NO)} \quad (7.217)$$

and

$$\begin{aligned} \sqrt{\sigma_2} &\sim \langle Bb|V - U|aA \rangle = \langle Bb|H_p|aA \rangle \\ &\approx GU(b)V(B) \approx \frac{G}{2}, \text{ (PAIRING)} \end{aligned} \quad (7.218)$$

In the process described by the transfer amplitude  $\langle Bb|U|aA \rangle$ , one nucleon is transferred under the effect of the single-particle potential of depth  $V_0 (\approx -50 \text{ MeV})$  while, simultaneously, the second nucleon moves over from a single-particle orbit centered around  $b$  to one centered around  $A$  profiting of the non-orthogonality of the corresponding wavefunctions  $\varphi^{(b)}(r_{1b})$  and  $\varphi^{(A)}(r_{1A})$ . Within this context, it is then natural that  $O$  stands for the overlap between these two wavefunctions, that is, (see below simple estimate of  $O$ ),

$$O = \langle \varphi^{(b)} | \varphi^{(A)} \rangle \approx 0.3 \times 10^{-2}, \quad (7.219)$$

and that (7.217) is known as the sum of the simultaneous plus non-orthogonality contributions to the two-nucleon transfer amplitude. Of notice that the prefactor 2 in (7.217) is associated with the fact that two nucleons can choose to jump from one system to the other through non-orthogonality while the factor  $|V_0|/2$  is associated with the fact that transfer takes mainly place at the surface.

The term (7.218) corresponds to the simultaneous  $(t, p)$  transfer via the pairing two-body interaction  $V_p$  (see Eq. (7.213)),  $U(A)$  and  $V(B)$  being the product of two occupation amplitudes:  $U(A)$  measures the availability of free single-particle orbitals around the Fermi energy in the target nucleus, while  $V(B)$  reflects the degree of occupancy of levels in the residual system. Close to the Fermi energy  $U(b)V(B) \approx (1/\sqrt{2})^2 = 1/2$ , leading to the final expression of (7.218).

In keeping with the fact that the ratio of transfer amplitudes

$$\begin{aligned} \left( \frac{\sigma_1}{\sigma_2} \right)^{1/2} &\approx 2 \frac{|V_0|}{2} \times O \frac{1}{G/2} \approx 2 \times A \times 10^{-2} \\ &\approx 2(A \approx 100), \end{aligned} \quad (7.220)$$

is larger than one, and that the correlation length of nuclear Cooper pairs ( $\xi \approx \hbar v_F / 2\Delta \approx 30$  fm) is larger than nuclear dimensions, one can expect that the successive transfer of two nucleons under the influence of the single-particle field, can give an important contribution to the total transfer amplitude  $\sqrt{\sigma}$ . In other words, we expect the process

$$a(=b+2)+A \rightarrow f(=b+1)+F(A+1) \rightarrow b+B(=A+2) \quad (7.221)$$

gives a consistent contribution to  $\sqrt{\sigma}$ . The associated amplitude can be written as

$$\begin{aligned} \sqrt{\sigma_3} &\sim \sum_{fF} \frac{\langle bB|U|fF\rangle \langle fF|U|aA\rangle}{E_{aA} - E_{fF}} \\ &\approx \frac{(V_0/13)(V_0/13)}{\Delta E}, \end{aligned} \quad (7.222)$$

the factor  $1/7$  appears in each of the steps (instead of  $1/2$ , see (7.217)) in keeping with the fact that many other reactions channels and then, absorptive processes will take place at closer distance in two-step channels (of notice that  $1/7$  corresponds to  $r = R_0 + 2.5a$ ).

Typical values of the energy denominator in (7.222) are  $\Delta E = 30$  MeV for medium heavy nuclei lying along the stability valley.

## 7.7 Transfer amplitudes

Making use of a simplified expression for the elastic scattering amplitude, that is

$$\sqrt{\sigma_{el}} \sim \langle aA|U|aA\rangle, \quad (7.223)$$

one can calculate the transfer probabilities associated with the different processes discussed above, namely

$$P_i = \left( \frac{\sigma_i}{\sigma_{el}} \right) = \begin{cases} \left| \frac{\langle bB|U|aA\rangle}{\langle aA|U|aA\rangle} \right|^2 \approx \mathcal{O}^2 \approx 0.9 \times 10^{-5} & (i=1), \\ \left| \frac{\langle bB|V_p|aA\rangle}{\langle aA|U|aA\rangle} \right|^2 \approx \left( \frac{G}{2V_0} \right)^2 \approx 1.4 \times 10^{-6} & (i=2), \\ \left| \frac{\langle bB|U|fF\rangle \langle fF|U|aA\rangle}{\Delta E \langle aA|U|aA\rangle} \right|^2 \approx \left( \frac{V_0}{170\Delta E} \right)^2 \approx 0.96 \times 10^{-4} & (i=3). \end{cases} \quad (7.224)$$

Because all these probabilities are small, one can write

$$\sigma_i = P_i \sigma_{el}, \quad (7.225)$$

where

$$\begin{aligned} \sigma_{el} &= \left( \frac{\mu_\alpha}{2\pi\hbar^2} \right)^2 |\langle aA|U|aA\rangle|^2 \\ &\approx \left( \left( \frac{\mu_\alpha}{2\pi\hbar^2} \right) (V_0) \right)^2 U_0^2 \\ &\approx \left( 1.8 \text{ MeV}^{-1} \text{ fm} \right)^2 (50 \text{ MeV})^2 \\ &\approx (90 \text{ fm})^2 = 0.8 \text{ b} \end{aligned} \quad (7.226)$$

where use has been made of the effective volume associated with the reaction process (see )

$$V_{ol} \approx \frac{4\pi}{3} 3R^2 a \approx 12A^{2/3} \text{ fm}^3 \approx 260 \text{ fm}^3, \quad (7.227)$$



as well as of  $\frac{\mu_\alpha}{2\pi\hbar^2}$  factor of the typical two-nucleon transfer reaction  $^{120}\text{Sn}+p \rightarrow ^{118}\text{Sn}+t$ , that is (see ),

$$\frac{\sqrt{\mu_\alpha\mu_\beta}}{2\pi\hbar^2} \approx \frac{\sqrt{3}M}{2\pi\hbar^2} \approx \frac{1}{145} \text{MeV}^{-1} \text{fm}^{-2}. \quad (7.228)$$

Summing up, one can write

$$\sigma_i = P_i 0.8 \text{b}. \quad (7.229)$$

Making use of (7.224) one obtains

$$\sigma_i = \begin{cases} 0.7 \times 10^{-2} \text{mb} & (i = 1), \\ 1.1 \times 10^{-3} \text{mb} & (i = 2), \\ 8 \text{mb} & (i = 1), \quad (i = 3). \end{cases} \quad (7.230)$$

These numbers, although worked out for  $A=100$  can be rescaled in connection with the reaction  $^{11}\text{Li}(p, t)^9\text{Li}(\text{gs})$ , in which case, microscopic calculation lead to  $d\sigma_1(\theta = 60^\circ)/d\Omega \approx 0.01 \text{mb/sr}$  and  $d\sigma_3(\theta = 60^\circ)/d\Omega \approx 5 \text{mb/sr}$ .

## 7.8 Inelastic scattering following two-particle transfer: final state interaction

This subject is qualitatively discussed in connection with the  $^{11}\text{Li}(p, t)^9\text{Li}(1/2^-; 2.69)$ , but of course is a general question, also in connection with the validity of considering perturbation theory instead of coupled channels.

In keeping with the fact that the first excited state of  $^9\text{Li}$  can be viewed as

$$|^9\text{Li}(1/2^-; 2.69 \text{MeV})\rangle \approx |2^+ (^8\text{Be} \otimes p_{3/2}(\pi))_{1/2^-}\rangle, \quad (7.231)$$

this state can, in principle, be excited in a two-step process, namely

$$\text{gs } (^{11}\text{Li})+t \longrightarrow \text{gs } (^9\text{Li})+p \longrightarrow 1/2^- (^9\text{Li})+p. \quad (7.232)$$

Let us calculate the probability associated with the inelastic scattering of the lowest  $2^+$  of  $^8\text{Li}$ . In this case, we are interested in the component of  $V - U$  corresponding to  $\delta U_C = -KF\alpha = -R_0 \frac{\partial U}{\partial r} \beta_L$ , namely the field associated with the inelastic excitation of multipole vibrations. Making use of the Saxon-Woods potential one obtains

$$R_0 \frac{\partial U}{\partial r} = \frac{R_0}{a} \frac{\exp\left(\frac{(r-R_0)}{a}\right)}{\left(1 + \exp\left(\frac{(r-R_0)}{a}\right)\right)^2}. \quad (7.233)$$

In keeping with the fact that

$$\left\langle R_0 \frac{\partial U}{\partial r} \right|_{r=R_0} \rangle \approx \left\langle \frac{R_0 U_0}{a} \right\rangle \approx 1.2 U_0 \text{MeV} \approx -60 \text{MeV} \quad (7.234)$$

and that the main contributions of surface dominated reactions processes is estimated to arise from distances of the order of  $r \approx R_0 + 2.5a$ , one obtains

$$\begin{aligned} \left\langle \frac{R_0}{a} \frac{e^{2.5} U_0}{(1 + \exp 2.5)^2} \right\rangle &= \left\langle \frac{R_0 U_0}{a} \right\rangle \frac{e^{2.5}}{(1 + \exp 2.5)^2} \\ &\approx 1.2 U_0 \times 0.7 \times 10^{-1} = 0.84 \times 10^{-1} U_0. \end{aligned} \quad (7.235)$$

Thus

$$\langle bB^* | \delta U_C | bB \rangle \approx 0.84 \times 10^{-1} U_0 \beta_L. \quad (7.236)$$

Consequently

$$\begin{aligned} P_{inel} &\approx \left| \frac{\langle bB^* | \delta U_C | aA \rangle}{\langle aA | U | aA \rangle} \right|^2 = (0.84 \times 10^{-1} \beta_L)^2 \\ &\approx 0.7 \times 10^{-2} \beta_L^2. \end{aligned} \quad (7.237)$$

In keeping with the fact that the  $\beta_L$  associated with the lowest  $2^+$  vibrational states of the Sn-isotopes and of  $^{11}\text{Li}$  are  $\approx 0.1$  and  $\approx 1$  respectively one can write

$$P_{inel} = \begin{cases} 0.7 \times 10^{-4} & (\text{Sn-isotopes}), \\ 0.7 \times 10^{-2} & (^{11}\text{Li}). \end{cases} \quad (7.238)$$

Making use of the results collected in (7.224),

$$\begin{aligned} \sqrt{P(p, t)} &= \sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3} \\ &= \sqrt{0.9 \times 10^{-1}} + \sqrt{1.4 \times 10^{-6}} + \sqrt{0.96 \times 10^{-4}} \\ &\approx 3 \times 10^{-3} + 1.2 \times 10^{-3} + 0.98 \times 10^{-2} \\ &\approx 1.4 \times 10^{-2}. \end{aligned} \quad (7.239)$$

Thus

$$P((p, t) \otimes P(\text{inel})) = P(p, t)P(\text{inel}) = \begin{cases} 2 \times 10^{-4} \times 10^{-4} \approx 10^{-8} & (\text{Sn}), \\ 2 \times 10^{-4} \times 10^{-2} \approx 10^{-6} & (^{11}\text{Li}), \end{cases} \quad (7.240)$$

in overall agreement with the result of microscopic calculations (for  $^{11}\text{Li}$ ).

## 7.9 Simple estimate $O$

The nuclear density can be parametrized according to

$$\rho(r) = \frac{\rho_0}{1 + \exp\left(\frac{r-R_0}{a}\right)}. \quad (7.241)$$

Let us calculate this function for

$$r = R_0 + 3a, \quad (7.242)$$

that is

$$\rho(r = R_0 + 3a) = \frac{\rho_0}{1 + \exp 3} = 5 \times 10^{-2} \rho_0. \quad (7.243)$$

In other words, we assume that the main transfer takes place from densities of the order of 5% saturation density

$$O \approx \frac{\rho_A(R_0^A + 3a)\rho_a(R_0^a + 3a)}{\rho_0^2} = 25 \times 10^{-4} \approx 0.3 \times 10^{-2}. \quad (7.244)$$

another estimate

$$r = R_0 + 2.5a, \quad (7.245)$$

for which

$$\rho(r = R_0 + 2.5a) = \frac{\rho_0}{1 + \exp 2.5} \approx 0.76 \times 10^{-1} \rho_0, \quad (7.246)$$

leading to

$$O \approx 0.5 \times 10^{-2}. \quad (7.247)$$

### 7.10 Simple estimate of $\frac{(\mu_\alpha \mu_\beta)^{1/2}}{2\pi\hbar^2}$ .

Let us do it for the case of  $^{120}\text{Sn} + p \rightarrow ^{118}\text{Sn} + t$ . In this case

$$\begin{aligned} \mu_\alpha &= \frac{120}{121} M \approx M, \\ \mu_\beta &= \frac{118 \times 3}{121} \approx 2.9M. \end{aligned} \quad (7.248)$$

Thus

$$\begin{aligned} \frac{\sqrt{\mu_\alpha \mu_\beta}}{2\pi\hbar^2} &\approx \frac{\sqrt{3}M}{2\pi\hbar^2} = \frac{\sqrt{3}}{2\pi 40 \text{ MeV fm}^2} \\ &\approx \frac{1}{145} \times \text{MeV}^{-1} \times \text{fm}^{-2} \end{aligned} \quad (7.249)$$

### 7.11 Simple estimate of $V_{ol}$

In keeping with the assumption that transfer processes are expected to take place at the nuclear surface, the effective volume associate with such processes can be estimated to be

$$\begin{aligned} V_{ol} &= \frac{4\pi}{3} (R^3 - (R + a)^3) \\ &\approx \frac{4\pi}{3} 3aR^2 \approx \frac{4\pi}{3} (2\text{fm})R^2 \\ &\approx \frac{8\pi}{3} (1.2A^{1/3})^2 \text{fm}^3 \\ &\approx 1.2A^{1/3} \text{fm}^3 \approx 260 \text{fm}^3 (A \approx 100) \end{aligned} \quad (7.250)$$

### 7.12 Calculation of the (p,t) strength function

An important component of the interaction which binds the dineutron halo of  $^{11}\text{Li}$  to the core  $^9\text{Li}$  is associated with the exchange, between the two neutrons of dipole (pigmy  $1^-$  resonance of  $^{11}\text{Li}$ ) and quadrupole ( $2^+$  mode of  $^9\text{Li}$ ) vibrations. Consequently, it is expected that resonant effects can be observed in the (p,t) strength function in which this neutrons are picked-up from  $^{11}\text{Li}$ . In keeping with the fact that successive transfer plays a central role in the two-particle pick up process, the corresponding transfer amplitude can be written as

$$\begin{aligned} &\left\langle \chi^{(-)} \sum_{fF} \frac{\langle bB | U | fF \rangle \langle fF | U | aA \rangle}{E_{aA} - E_{fF}} \chi^{(+)} \right\rangle \\ &\approx \left\langle \frac{e^{iqr}}{\hbar\omega_L} \right\rangle \sim \left\langle e^{i(qr - \ln \tau / \omega_L)} \right\rangle \\ &\sim \cos(q(E_{CM})r - \ln \tau / \omega_L). \end{aligned} \quad (7.251)$$

Of notice that (p,t) strength function measurements can be viewed as a frequency dependent single Cooper pair transfer, and thus in some way connected to  $\omega$ -dependent Josephson supercurrent measurement.

### 7.13 Relative importance of successive and simultaneous transfer and non-orthogonality corrections

In what follows we discuss the relative importance of successive and simultaneous two-neutron transfer and of non-orthogonality corrections associated with the reaction

$$\alpha \equiv a(= b + 2) + A \rightarrow b + B(= A + 2) \equiv \beta \quad (7.252)$$

in the limits of independent particles and of strongly correlated Cooper pairs, making use for simplicity of the semiclassical approximation (for details cf. [?],[?] and refs. therein), in which case the two-particle transfer differential cross section can be written as

$$\frac{d\sigma_{\alpha \rightarrow \beta}}{d\Omega} = P_{\alpha \rightarrow \beta}(t = +\infty) \sqrt{\left(\frac{d\sigma_{\alpha}}{d\Omega}\right)_{el}} \sqrt{\left(\frac{d\sigma_{\beta}}{d\Omega}\right)_{el}}, \quad (7.253)$$

where  $P$  is the absolute value squared of a quantum mechanical transition amplitude. It gives the probability that the system at  $t = +\infty$  is found in the final channel. The quantities  $(d\sigma/d\Omega)_{el}$  are the classical elastic cross sections in the center of mass system, calculated in terms of the deflection function, namely the functional relating the impact parameter and the scattering angle.

The transfer amplitude can be written as

$$a(t = +\infty) = a^{(1)}(\infty) - a^{(NO)}(\infty) + \tilde{a}^{(2)}(\infty), \quad (7.254)$$

where  $\tilde{a}^{(2)}(\infty)$  at  $t = +\infty$  labels the successive transfer amplitude expressed in the post-prior representation (see below). The simultaneous transfer amplitude is given by (see Fig. A1(I))

$$\begin{aligned} a^{(1)}(\infty) &= \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt (\psi^b \psi^B, (V_{bB} - < V_{bB} >) \psi^a \psi^A) \times \exp\left[\frac{i}{\hbar}(E^{bB} - E^{aA})t\right] \\ &\approx \frac{2}{i\hbar} \int_{-\infty}^{\infty} dt \left( \phi^{B(A)}(S_{(2n)}^B; \vec{r}_{1A}, \vec{r}_{2A}), U(r_{1b}) e^{i(\sigma_1 + \sigma_2)} \phi^{a(b)}(S_{(2n)}^a; \vec{r}_{1b}, \vec{r}_{2b}) \right) \exp\left[\frac{i}{\hbar}(E^{bB} - E^{aA})t + \gamma(t)\right]. \end{aligned} \quad (7.255)$$

where

$$\sigma_1 + \sigma_2 = \frac{1}{\hbar} \frac{m_n}{m_A} (m_{aA} \vec{v}_{aA}(t) + m_{bB} \vec{v}_{bB}(t)) \cdot (\vec{r}_{1a} + \vec{r}_{2a}), \quad (7.256)$$

in keeping with the fact that  $\exp(i(\sigma_1 + \sigma_2))$  takes care of recoil effects (Galilean transformation associated with the mismatch between entrance and exit channels).

The phase  $\gamma(t)$  is related with the effective  $Q$ -value of the reaction. In the above expression,  $\phi$  indicates an antisymmetrized, correlated two-particle (Cooper pair) wave-function,  $S(2n)$  being the two-neutron separation energy (see Fig. A1),  $U(r_{1b})$  being the single particle potential generated by nucleus  $b$  ( $U(r) = \int d^3r' \rho^b(r') v(|r - r'|)$ ). The contribution arising from non-orthogonality effects can be written as (see Fig. A1(II))

$$a^{(NO)}(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt (\psi^b \psi^B, (V_{bB-} < V_{bB} >) \psi^f \psi^F) (\psi^f \psi^F, \psi^a \psi^A) \exp\left[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)\right] \\ \approx \frac{2}{i\hbar} \int_{-\infty}^{\infty} \phi^{B(F)}(S_{(n)}^B, \vec{r}_{1A}), U(r_{1b}) e^{i\sigma_1} (\phi^{f(b)}(S^f(n), \vec{r}_{1b}) \phi^{F(A)}(S^F(n), \vec{r}_{2A}) e^{i\sigma_2} \phi^{a(f)}(S^a(n), \vec{r}_{2b})) \exp\left[\frac{i}{\hbar}(E^{bB} - E^{aA})t + \gamma_2(t)\right] dt$$

the reaction channel  $f = (b + 1) + F (= A + 1)$  having been introduced, the quantity  $S(n)$  being the one-neutron separation energy (see Fig. A1). The summation over  $f (\equiv a'_1, a'_2)$  and  $F (\equiv a_1, a_2)$  involves a restricted number of states, namely the valence shells in nuclei  $B$  and  $a$ .

The successive transfer amplitude  $\tilde{a}_{\infty}^{(2)}$  written making use of the post-prior representation is equal to (see Fig. A1(III))

$$\tilde{a}^{(2)}(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt (\psi^b \psi^B, (V_{bB-} < V_{bB} >) e^{i\sigma_1} \psi^f \psi^F) \times \exp\left[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)\right] \\ \times \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt' (\psi^f \psi^F, (V_{fF-} < V_{fF} >) e^{i\sigma_2} \psi^a \psi^A) \times \exp\left[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')\right] \quad (7.258)$$

To gain insight into the relative importance of the three terms contributing to Eq. 7.254 we discuss two situations, namely, the independent-particle model and the strong-correlation limits.

### 7.13.1 Independent particle limit

In the independent particle limit, the two transferred particles do not interact among themselves but for antisymmetrization. Thus, the separation energies fulfill the relations (see Fig. A2)

$$S^B(2n) = 2S^B(n) = 2S^F(n), \quad (7.259)$$

and

$$S^a(2n) = 2S^a(n) = 2S^f(n). \quad (7.260)$$

In this case

$$\phi^{B(A)}(S^B(2n), \vec{r}_{1A}, \vec{r}_{2A}) = \sum_{a_1 a_2} \phi_{a_1}^{B(F)}(S^B(n), \vec{r}_{1A}) \phi_{a_2}^{F(A)}(S^F(n), \vec{r}_{2A}), \quad (7.261)$$

and

$$\phi^{a(b)}(S^a(2n), \vec{r}_{1b}, \vec{r}_{2b}) = \sum_{a'_1 a'_2} \phi_{a'_1}^{a(f)}(S^a(n), \vec{r}_{2b}) \phi_{a'_2}^{f(b)}(S^f(n), \vec{r}_{1b}), \quad (7.262)$$

where  $(a_1, a_2) \equiv F$  and  $(a'_1, a'_2) \equiv f$  span, as mentioned above, shells in nuclei  $B$  and  $a$  respectively.

Inserting (A9) and (A10) in (A4) one can show that

$$a^{(1)}(\infty) = a^{(NO)}(\infty). \quad (7.263)$$

It can be demonstrated that within the present approximation,  $\text{Im } \tilde{a}^{(2)} = 0$ , and that

$$\tilde{a}^{(2)}(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt (\psi^b \psi^B, (V_{bB-} < V_{bB} >) e^{i\sigma_1} \psi^f \psi^F) \times \exp\left[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)\right] \\ \times \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt' (\psi^f \psi^F, (V_{fF-} < V_{fF} >) e^{i\sigma_2} \psi^a \psi^A) \times \exp\left[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')\right] \quad (7.264)$$

### 7.13. RELATIVE IMPORTANCE OF SUCCESSIVE AND SIMULTANEOUS TRANSFER AND NON-ORTHOGONALITY

The total absolute differential cross section 7.253, where  $P = |a(\infty)|^2 = |\tilde{a}^{(2)}|^2$ , is then equal to the product of two one-particle transfer cross sections (see Fig. A3), associated with the (virtual) reaction channels

$$\alpha \equiv a + A \rightarrow f + F \equiv \gamma, \quad (7.265)$$

and

$$\gamma \equiv f + F \rightarrow b + B \equiv \beta. \quad (7.266)$$

In fact, Eq.(7.264) involves no time ordering and consequently the two processes above are completely independent of each other. This result was expected because being  $v_{12} = 0$ , the transfer of one nucleon cannot influence, aside from selecting the initial state for the second step, the behaviour of the other nucleon.

#### 7.13.2 Strong correlation (cluster) limit

The second limit to be considered is the one in which the correlation between the two nucleons is so strong that (see Fig. A2)

$$S^a(2n) \approx S^a(n) \gg S^f(n), \quad (7.267)$$

and

$$S^B(2n) \approx S^B(n) \gg S^F(n). \quad (7.268)$$

That is, the magnitude of the one-nucleon separation energy is strongly modified by the pair breaking.

There is a different, although equivalent way to express (7.254) which is the more convenient to discuss the strong coupling limit. In fact, making use of the post-prior representation one can write

$$\begin{aligned} a^{(2)}(t) = \tilde{a}^{(2)}(t) - a^{(NO)}(t) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt (\psi^b \psi^B, (V_{bB-} - < V_{bB} >) e^{i\sigma_1} \psi^f \psi^F) \\ \times \exp\left[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)\right] \times \\ \frac{1}{i\hbar} \int_{-\infty}^t dt' (\psi^f \psi^F, (V_{aA-} - < V_{aA} >) \psi^a \psi^A) \times \exp\left[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')\right]. \end{aligned} \quad (7.269)$$

The relations (7.267), (7.268) imply

$$E^{fF} - E^{aA} = S^a(n) - S^f(n) \gg \frac{\hbar}{\tau}, \quad (7.270)$$

where  $\tau$  is the collision time. Consequently the real part of  $a^{(2)}(\infty)$  vanishes exponentially with the  $Q$ -value of the intermediate transition, while the imaginary part vanishes inversely proportional to this energy. One can thus write,

$$Re a^{(2)}(\infty) \approx 0, \quad (7.271)$$

and

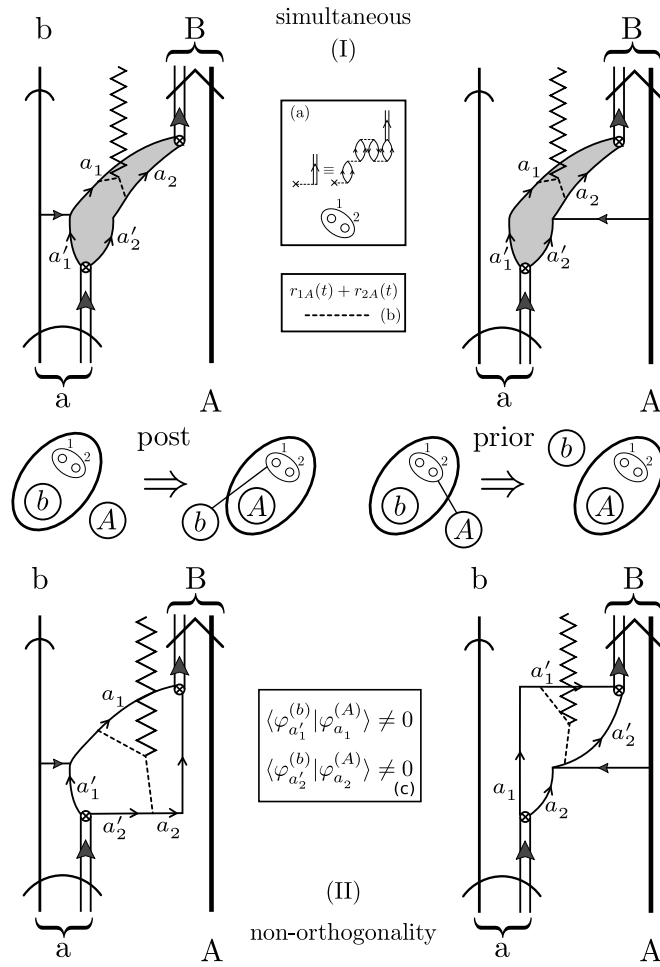
$$a^{(2)}(\infty) \approx \frac{1}{i\hbar} \frac{\tau}{< E^{fF} > - E^{bB}} \sum_{fF} (\psi^b \psi^B, (V_{bB-} - < V_{bB} >) \psi^f \psi^F)_{t=0} \times (\psi^f \psi^F, (V_{aA-} - < V_{aA} >) \psi^a \psi^A)_{t=0}, \quad (7.272)$$

where one has utilized the fact that  $E^{bB} \approx E^{aA}$ . For  $v_{12} \rightarrow \infty$ ,  $(\langle E^{fF} \rangle - E^{bB}) \rightarrow \infty$  and, consequently,

$$\lim_{v_{12} \rightarrow \infty} a^{(2)}(\infty) = 0. \quad (7.273)$$

Thus the total two-nucleon transfer amplitude is equal, in the strong coupling limit, to the amplitude  $a^{(1)}(\infty)$ .

Summing up, only when successive transfer and non-orthogonal corrections are included in the description of the two-nucleon transfer process, does one obtain a consistent description of the process, which correctly converges to the weak and strong correlation limiting values.



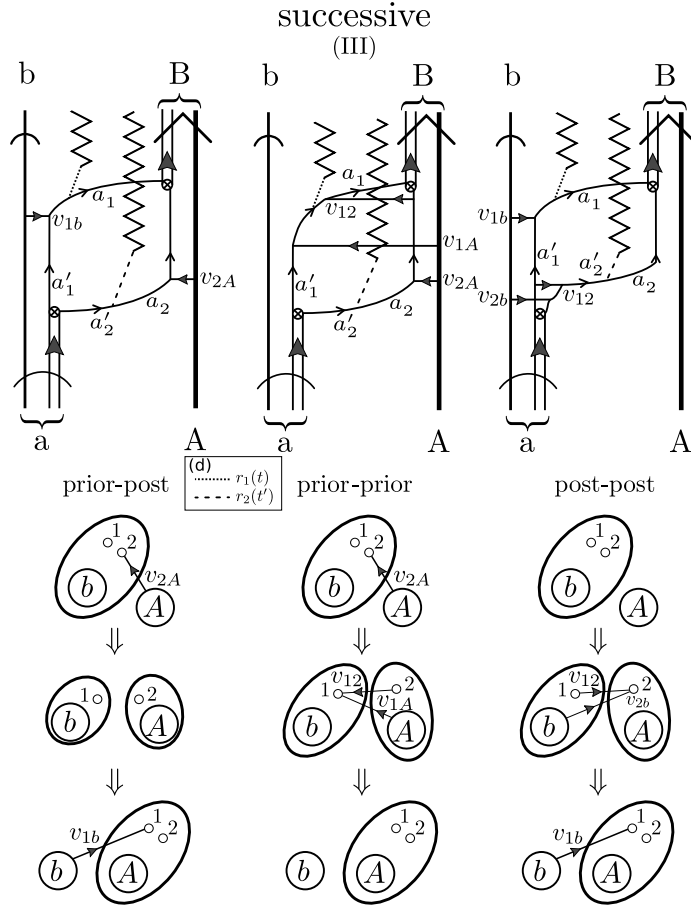


Figure 7.1: Graphical representation of the lowest order ((I),(II) and (III) first and second order in  $v$  respectively), two-nucleon transfer processes, which correctly converge to the strong-correlation (only simultaneous transfer), and to the independent-particle (only successive transfer) limits. The time arrow is assumed to point upwards: (I) Simultaneous transfer, in which one particle is transferred by the nucleon-nucleon interaction (note that  $U(r) = \int d^3r' \rho(r') v(|\vec{r} - \vec{r}'|)$ ) acting either in the entrance  $\alpha \equiv a + A$  channel (prior) or in the final  $\beta \equiv b + B$  channel (post), while the other particle follows suit making use of the particle-particle correlation (grey area) which binds the Cooper pair (see upper inset labelled (a)), represented by a solid arrow on a double line, to the projectile (curved arrowed lines) or to the target (opened arrowed lines). The above argument provides the explanation why when e.g.  $v_{1b}$  acts on one nucleon, the other nucleon also reacts instantaneously. In fact a Cooper pair displays generalized rigidity (emergent property in gauge space). A crossed open circle represents the particle-pair vibration coupling. The associated strength, together with an energy denominator, determines the amplitude  $X_{a'_1 a'_2}$  (cf. Table 1) with which the pair mode (Cooper pair) is in the (time reversed) two particle configuration  $a'_1 a'_2$ . In the transfer process, the orbital of relative motion changes, the readjustment of the corresponding trajectory mismatch being operated by a Galilean operator ( $\exp\{\vec{k} \cdot (\vec{r}_{1A}(t) + \vec{r}_{2A}(t))\}$ ). This phenomenon, known as recoil process, is represented by a jagged line which provides simultaneous information on the two transferred nucleons (single time appearing as argument of both single-particle coordinates  $r_1$  and  $r_2$ ; see inset labeled (b)). In other words, information on the coupling of structure and reaction modes. (II) Non-orthogonality contribution. While one of the nucleons of the Cooper pairs is transferred under the action of  $v$ , the other goes, uncorrelatedly over, profiting of the non-orthogonality of the associated single-particle wavefunctions (see inset (c)). In other words of the non-vanishing values of the overlaps, as shown in the inset. (III) Successive transfer. In this case, there are two time dependences associated with the acting of the nucleon-nucleon interaction twice (see inset (d)).



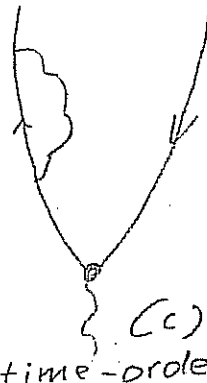
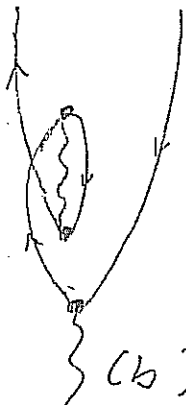
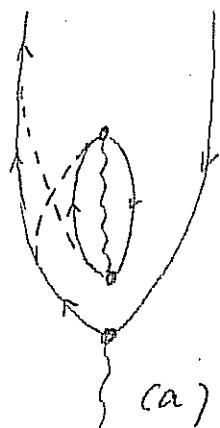
ZPF and Pauli principle at the basis of medium polariz- (1)  
ization effects: self-energy, vertex corrections and induced interaction

In keeping with a central objective of the formulation of quantum mechanics, namely that the ~~central~~ <sup>basic</sup> concepts on which it is based relate directly to experiment, elementary modes of nuclear excitation (single-particle, collective vibrations and rotations), are solidly anchored on observation (inelastic and Coulomb excitations, one- and two-particle transfer reactions).

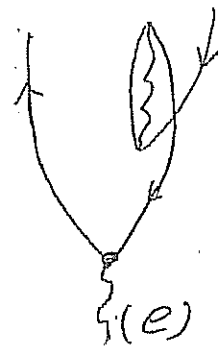
of all quantal phenomena, zero point fluctuations (ZPF), <sup>closely connected with virtual state,</sup> are likely to be most representative of the essential difference existing between quantum and classical mechanics. In fact, ZPF are intimately connected with the complementary principle (Bohr), and thus with the indeterminacy (Heisenberg) and non-commutative (Born, Jordan) relations, and with the probabilistic interpretation (Born) of the (modulus squared) of the wavefunction. ~~These are the essential features of the quantum theory.~~ ~~These are the essential features of the quantum theory.~~ ~~These are the essential features of the quantum theory.~~ of the wavefunctions, solution of Schrödinger's or Dirac's equations.

Pauli principle brings about essential modifications of the virtual fluctuations

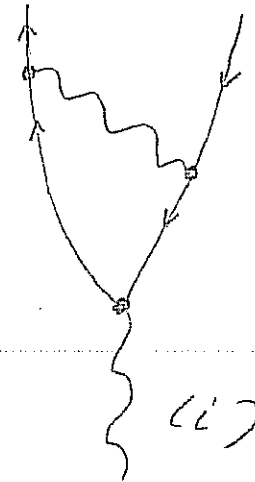
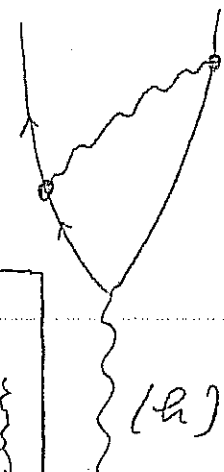
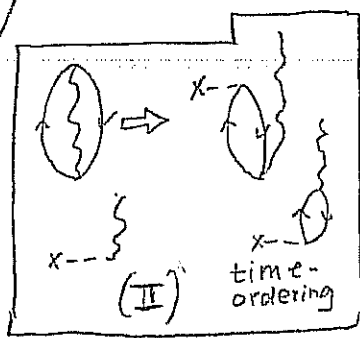
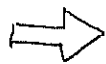
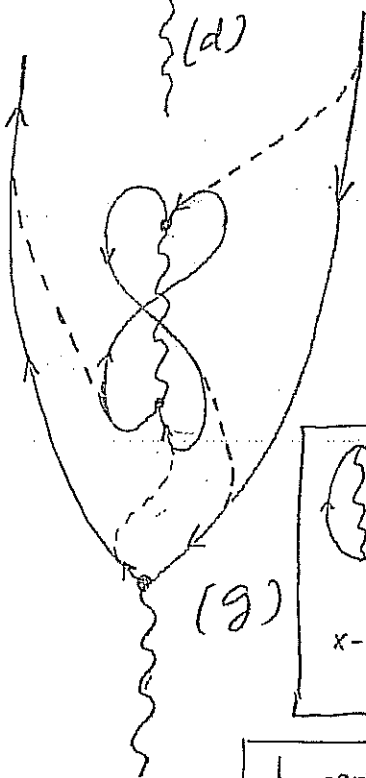
of the many-body ~~system~~ system, 22/04/13 (2)  
modifications which are instrumental  
in the dressing and interweaving  
of the elementary modes of excita-  
tion (see Figs. A and B; ~~within~~ within  
~~the~~ the present context, see also  
J.R. Schrieffer, Theory of superconductivi-  
ty, Benjamin, N.Y (1964) p. 134).



time-ordering  
self-energy



time-ordering  
self-energy



time-ordering

vertex correction  
(reduced interaction)

- ↑ particle
- ↓ hole (I)
- ⋮ (p-ph) vibration
- particle-(ph) vibr. vertex

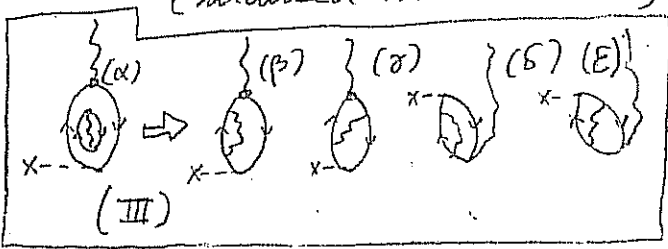


Fig. A

# Caption Fig. A

22/04/13

(4)

Nuclear field theory (NFT) diagrams corresponding to the lowest order medium polarization effects renormalizing the properties of a particle-hole collective mode <sup>(curly line)</sup> linear combination of particle-hole <sup>(up-going) - (down-going) arrowed lines</sup> excitations calculated within the random phase approximation (RPA) in of a bare interaction, and leading to the particle vibration coupling vertex (solid dot, see inset (I), bottom). The action of an external field on the zero point fluctuations (ZPF) of the vacuum (inset (II)), forces a virtual process to become real, leading to a collective vibration by annihilating a (virtual, spontaneous) particle-hole excitation (backwards going RPA amplitude) or, in the time ordered process, by creating a particle-hole excitation which eventually, through the particle-vibration coupling vertex, correlate into the collective (coherent state; forwards going amplitudes). Now, oyster-like diagrams associated with the vacuum ZPF can occur at any time (see inset III). Because the texture of the vacuum is permeated by symmetry rules (while one can violate energy <sup>conservation</sup> in a virtual state one cannot violate e.g. angular momentum <sup>conservation</sup> or the Pauli principle), the process shown in the inset III (a) leads, through Pauli principle

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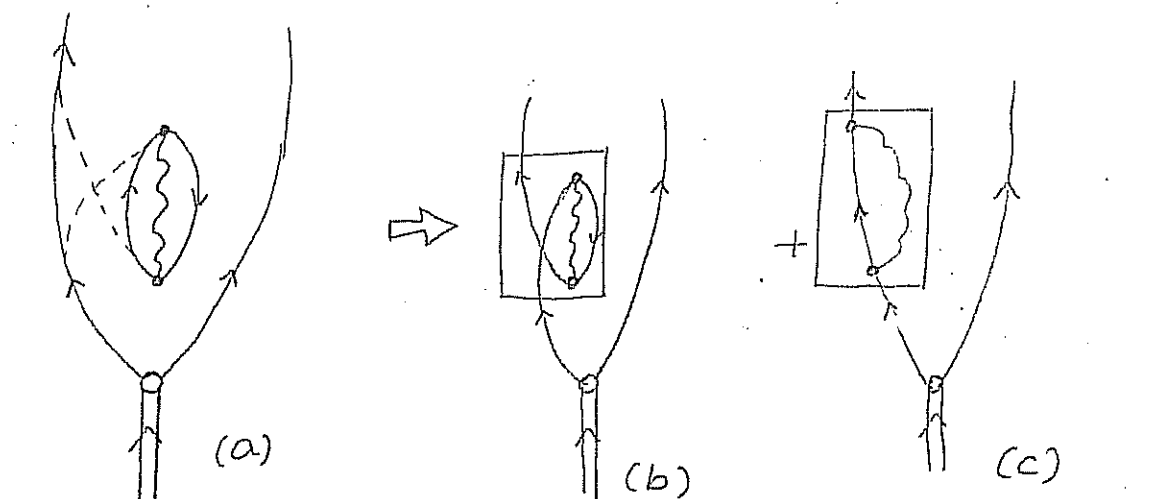
correcting processes (exchange of fermionic arrowed lines) to self-energy (inset III (β), (δ)) and vertex corrections (induced p-h interaction; inset III (γ), (ε)) processes. (5)

The first ones are detailed in graphs (a)-(f), while the second ones in graphs (g)-(i).

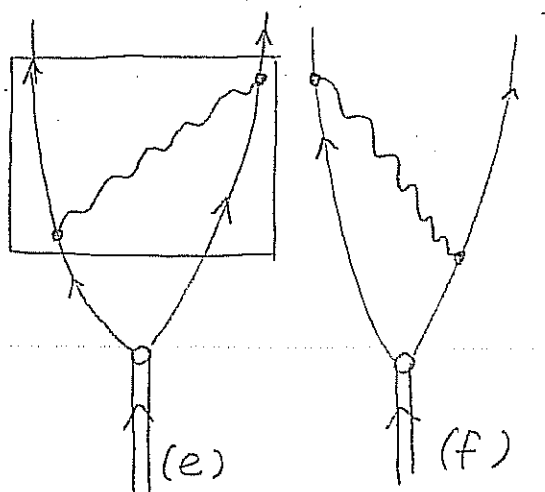
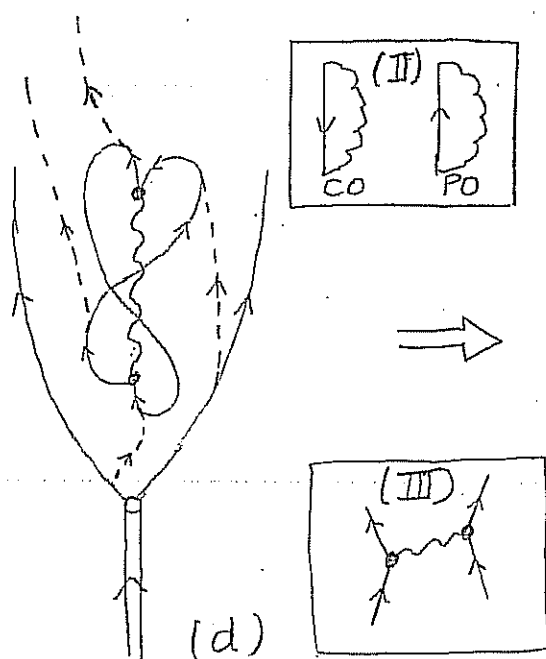
In keeping with the fact that the vibrational states can be viewed as a coherent state exhausting a large fraction of the EWSR (e.g. a Giant Resonance) for which the

associated uncertainty relations in momentum and coordinate fulfill the absolute minimum consistent with quantum mechanics ( $\Delta\alpha_{\lambda\mu} \Delta\pi_{\lambda\mu} = \hbar/2$ ,  $\alpha_{\lambda\mu} = (\hbar\omega_\lambda/2C_\lambda)^{1/2} (\pi_{\lambda\mu}^+ + \pi_{\lambda\mu})$ )

being the (harmonic) collective coordinate,  $\pi_{\lambda\mu}$  being the conjugate momentum; cf. e.g. R. Glauber, in Proceedings of the International School of Physics E. Fermi on Quantum Optics, course XLII, ed. R. Glauber, Ac. Press, N.Y. (1969) p. 15), there is a strong cancellation between the contribution of self-energy and vertex correction diagrams (P.F. Bortignon and R.A. Broglia, Nucl. Phys., ), implying small antihermiticity and long lifetimes ( $\Gamma/E \ll 1$ , where  $\Gamma$  is the width and  $E$  the centroid of the mode  $|\lambda\mu\rangle = \pi_{\lambda\mu}^+ |0\rangle$ ,  $(\hbar\omega_\lambda/2C_\lambda)^{1/2}$  being the 2.P.F amplitude (cf. e.g. Brink and Broglia, Nuclear Superfluidity, pp 185, 298)



time ordering,  
self-energy.



time ordering  
vertex correction  
(induced interaction)

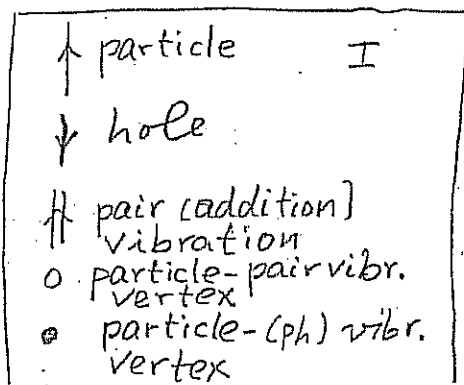


FIG. B

Caption Fig. B

22/04/13

(7)

Pauli effects associated (p-r)  $\pm$  PF dressing  
pairing vibrational modes (see inset  
bottom left) in terms of self-energy  
(graphs (a) - (c); correlation (CO) and pola-  
rization (PO) diagrams, inset II) and  
vertex correction (graph (d) - (f);  
induced particle-particle interaction,  
inset (III)) processes.

even for rather small energy losses. The relation of this open question to the observed "macroscopic" transfer of mass observed in, for example, the  $\text{Ca} + \text{U}$  reaction is a central topic in the field of heavy ion reaction.

The results of the above model, which account for the main features experimentally observed, can be summarized as follows. At an early stage of the collision when the two surfaces get into contact, energy and angular momentum is absorbed at a fast rate by the damped giant resonances. Low-lying modes with small restoring forces are important towards the final stages of the collision where they give rise to large deformations keeping the nuclear surfaces into contact. This "neck" formation is responsible for the experimentally observed fact that the two nuclei often emerge with relative kinetic energy which is below the Coulomb barrier of the corresponding spherical nuclei. The exchange of nucleons between the nuclei removes energy and angular momentum from relative motion throughout the collision.

The central feature of heavy ion collisions seems to be the importance of the coherent response of the different degrees of freedom. Thus in the description of the excitation of the surface modes it is not enough to know the population of the vibrational states ~~assumed~~ but also the relative phases which determine the shape of the nuclei as a function of time (cf. fig. <sup>15</sup> ~~14~~). Although it is difficult to document such a result in a more intuitive way it is possible to obtain a more accurate mathematical description. Thus solving the problem quantum mechanical<sup>ly</sup> we obtain the following total wavefunction

*In a heavy ion collision in which the two nuclei interact through the Coulomb*  
*Example of coherent state*



cf. Sect 4  
and App. B  
Ch II HIR

$$\begin{aligned}
 |\psi(t)\rangle &= e^{-i \frac{H_0 t}{\hbar}} |\phi(t)\rangle \\
 &= \sum_{\{n_\mu\}} \left( \prod_{\mu} e^{-\frac{|I_\mu(t)|^2}{2}} \frac{(I_\mu(t))^{n_\mu}}{\sqrt{n_\mu!}} \right) |\{n_\mu\}\rangle,
 \end{aligned}$$

where

$$I_\mu(t) = \frac{1}{\hbar} \int_0^t f_\mu^*(t') e^{i\omega t'} dt',$$

and where  $H_0$  is the Hamiltonian describing the intrinsic degrees of freedom of each nuclei. The wavefunction  $\phi(t)$  is the solution of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = \tilde{V} \phi$$

where  $\tilde{V} = \exp(i H_0 t/\hbar) V \exp(-i H_0 t/\hbar)$ ,  $V$  being the external field.

The integral  $I_\mu$  is related to the average number of phonons by

$$\langle n_\mu \rangle = |I_\mu(t)|^2,$$

the corresponding values for the reaction Xe + Pb at the instant of maximum deformation are quoted in table 1.

The state  $|\psi(t)\rangle$  is known in quantum mechanics as a coherent state.<sup>14</sup>

Its name stems from the fact that the associated uncertainty relations in momentum and coordinate associated with it fulfills the absolute minimum consistent with quantum mechanics, that is,

$$\Delta x_{\psi} \Delta \pi_{\psi} = \frac{\hbar}{2}$$

Note that this value is normally associated with the ground state. In general states described by a wavefunction of the type  $\exp \{ \frac{i}{\hbar} \hat{O} \} \phi(t)$  exhaust the energy weighted sum rule of the associated operator<sup>15</sup> which in the present case is the Hamiltonian.

Heavy ion collisions seem thus specific to study the nuclear spectroscopy of the coherent nuclear state. Note that we have left behind the field of experiments where the system that is probed can be described as if the probe was not present.

The coherent state which pictorially looks so simple, being almost a classical state, arises from the excitation and delicate phase relation of many collective and non-collective states of the individual nuclei. Thus, the full response function is tested in these reactions in a totally novel way. Note that collective vibrations as those discussed in connection with <sup>e.g.</sup> figs. 3 <sup>and 4</sup> are also coherent states and arise from the correlated efforts of many particle-hole excitations.

It is interesting to speculate whether the coherent excitation of the gas of phonons will lead to new super-collectivities displaying different condensation or phases as a function of the continuous excitation energy.

The behavior of the total energy absorbed by the coherent state in the reaction  $\text{Kr} + \text{Pb}$  as a function of angle (or linear momentum) shown in fig. 16 ~~15~~ is suggestively reminiscent of the behavior of the coherent state excited in liquid helium (cf. fig. 1).

A first step in relating microscopically the coherent state to the nuclear response function is provided by the analysis of the  $^{16}_0 + ^{208}_{\text{Pb}}$  reaction at 340 MeV summarized in fig. <sup>17</sup>~~16~~.

The usefulness of a model is measured by the ratio between the amount of experimental data it correlates and the number of concepts upon which it is based. There is a good chance that the model discussed in this comment can lead to a large value. *The possibility to ~~be~~ make rigorous statements about the different physical processes seem also to be quite large.*

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