Appendix 6.~~F~~ G One–particle knockout within DWBA

6.~~F~~ G.1 Spinless particles

We are going to consider the reaction $A + a \to a + b + c$, in which the cluster b is knocked out from the nucleus $A(= c + b)$. Cluster b is thus initially bound~~ed~~, while the final states of a, b and the initial state of a are all in the continuum, and can be described with distorted waves defined as scattering solutions of ~~a (as a rule, complex) suitable~~ an optical potential. A schematic depiction of the situation is shown in Fig. ~~7.4.1~~ 6.G.1. While the derivation presented below is quite general, special emphasis is set to one–particle knock–out processes. ~~We will begin by considering~~ the simplified case in which the ~~clusters a, b, c~~ are spinless ~~is considered~~.

Within this scenario,

Transition amplitude

~~We consider~~ optical potentials $U(r_{aA}), U(r_{cb}), U(r_{ac})$ ~~which will be central potentials~~ without a spin–orbit term. In addition, the interaction $v(r_{ab})$ between a and b is taken to be ~~an arbitrary~~ a function of the distance r_{ab}. ~~Then,~~ the transition amplitude which is at the basis of the evaluation of the multi–differential cross section is the 6–dimensional integral

$$T_{m_b} = \int d\mathbf{r}_{aA} d\mathbf{r}_{bc} \chi^{(-)*}(\mathbf{r}_{ac}) \chi^{(-)*}(\mathbf{r}_{bc}) v(r_{ab}) \chi^{(+)}(\mathbf{r}_{aA}) u_{l_b}(r_{bc}) Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{bc}). \qquad (6.\text{G}.1)$$

Coordinates

The vectors $\mathbf{r}_{ab}, \mathbf{r}_{ac}$ can easily be written in function of the integration variables $\mathbf{r}_{aA}, \mathbf{r}_{bc}$ (see Fig. ~~7.4.1~~ 6.G.1), namely

Making use of

$$\mathbf{r}_{ac} = \mathbf{r}_{aA} + \frac{b}{A}\mathbf{r}_{bc}, \qquad \mathbf{r}_{ab} = \mathbf{r}_{aA} - \frac{c}{A}\mathbf{r}_{bc}, \qquad (6.\text{G}.2)$$

where b, c, A stand for the number of nucleons of the species b, c and A respectively.

Distorted waves in the continuum

A standard way to reduce the dimensionality of the integral ~~(7.3.110)~~ (6.G.1) consists in expanding the continuum wave functions $\chi^{(+)}(\mathbf{r}_{aA}), \chi^{(-)*}(\mathbf{r}_{ac}), \chi^{(-)*}(\mathbf{r}_{bc})$ in a basis of eigen-states of the angular momentum operator (partial waves). Then ~~we~~ one can exploit the transformation properties of these eigenstates under rotations to ~~perform~~ conveniently carry out the angular integrations. ~~With~~ time–reversed ~~phase convention~~ phasing, that is

$$Y^l_m(\theta,\phi) = i^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta) e^{im\phi}, \qquad (6.\text{G}.3)$$

one can write ~~the general form of these expansions is~~

in the reaction process

$$\chi^{(+)}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^l \sqrt{2l+1} e^{i\sigma^l} F_l(r) \left[Y^l(\hat{\mathbf{r}}) Y^l(\hat{\mathbf{k}})\right]^0_0, \qquad (6.\text{G}.4)$$

suitable

A first derivation will be given in which, for simplicity, all the "particles" (nuclei) involved are spinless, and inert. ~~Integrating with this last ansatz,~~ Use is made of central, complex ~~optical potentials~~

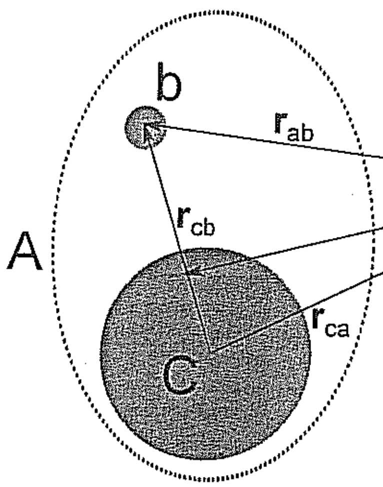

Figure 6.F.1: ~~Sketch of the system considered~~ system of coordinates used to describe the reaction $A+a \rightarrow a+b+c$. The nucleus A is viewed as an inert cluster b bounded to an inert core c.

and

$$\chi^{(-)*}(\mathbf{k},\mathbf{r}) = \sum_l \frac{4\pi}{kr} i^{-l}\sqrt{2l+1}e^{i\sigma^l}F_l(r)\left[Y^l(\hat{\mathbf{r}})Y^l(\hat{\mathbf{k}})\right]^0_0, \qquad (6.\cancel{F}G.5)$$

(6.G.1)

~~where~~ being σ_l ~~is~~ the Coulomb phase shift. The radial functions ~~$F_l(r)$~~ are regular (finite at $r=0$) solutions of the one–dimensional Schrödinger ~~equation~~ with an effective potential $U(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}$ and suitable asymptotic behaviour at $r \rightarrow \infty$ as boundary conditions. ~~So~~ Thus, the distorted waves appearing in (~~7.2.110~~) are,

$$\chi^{(+)}(\mathbf{k}_a,\mathbf{r}_{aA}) = \sum_{l_a} \frac{4\pi}{k_a r_{aA}} i^{l_a}\sqrt{2l_a+1}e^{i\sigma^{l_a}}F_{l_a}(r_{aA})\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l_a}(\hat{\mathbf{k}}_a)\right]^0_0, \qquad (6.F.6)$$

describing the ~~initial~~ relative motion of ~~between~~ A and a in the entrance channel as determined by ~~defined from~~ the complex optical potential $U(r_{Aa})$, ~~and~~

$$\chi^{(-)*}(\mathbf{k}'_a,\mathbf{r}_{ac}) = \sum_{l'_a} \frac{4\pi}{k'_a r_{ac}} i^{-l'_a}\sqrt{2l'_a+1}e^{i\sigma^{l'_a}}F_{l'_a}(r_{ac})\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^0_0, \qquad (6.F.7)$$

which describe the ~~final~~ relative motion of ~~between~~ c and a in the final channel controlled by ~~defined from~~ the complex optical potential $U(r_{ac})$, and finally

$$\chi^{(-)*}(\mathbf{k}'_b,\mathbf{r}_{bc}) = \sum_{l'_b} \frac{4\pi}{k'_b r_{bc}} i^{-l'_b}\sqrt{2l'_b+1}e^{i\sigma^{l'_b}}F_{l'_b}(r_{bc})\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{k}}'_b)\right]^0_0, \qquad (6.F.8)$$

channel wavefunction describing the relative motion of

final ~~relative motion~~ ~~between~~ b and c, as defined ~~from~~ by the complex optical potential $U(r_{bc})$.

Recoupling of angular momenta

~~We thus need to evaluate the 6–dimensional integral~~ One now proceeds to the evaluation of the six-dimensional integral

$$\frac{64\pi^3}{k_a k'_a k'_b}\int d\mathbf{r}_{aA}d\mathbf{r}_{bc}u_{l_b}(r_{cb})v(r_{ab})\sum_{l_a,l'_a,l'_b}\sqrt{(2l_a+1)(2l'_a+1)(2l'_b+1)}$$
$$\times e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})}\frac{F_{l_a}(r_{aA})F_{l'_a}(r_{ac})F_{l'_b}(r_{bc})}{r_{ac}r_{aA}r_{bc}} \qquad (6.\cancel{F}6.9)$$
$$\times\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l_a}(\hat{\mathbf{k}}_a)\right]^0_0\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^0_0\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{k}}'_b)\right]^0_0 Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{bc}).$$

as determined by (k_a, k'_a, k'_b) and $(\hat{k}_a, \hat{k}'_a, \hat{k}'_b)$ respectively.

~~Note that this~~ expression ~~depends~~ and which depends explicitly on the asymptotic kinetic energies ~~(k_a, k'_a, k'_b)~~ and scattering angles $(\hat{\mathbf{k}}_a, \hat{\mathbf{k}}'_a, \hat{\mathbf{k}}'_b)$ of a, b. ~~Now we will~~ In what follows we take advantage of the partial wave expansion to reduce the ~~dimensions~~ dimensionality of the integral from 6 to 3. A possible strategy ~~to deal with (6.F.9) is to recouple~~ to follow is that of recoupling together all the terms that depend on the integration variables to a global angular momentum and retain only the term coupled to 0 as the only one surviving the integration. Let us start to couple separately the terms corresponding to particles a and ~~particle~~ b. For particle a we write

$$\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l_a}(\hat{\mathbf{k}}_a)\right]^0_0\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^0_0 = \sum_K((l_al_a)_0(l'_al'_a)_0|(l_al'_a)_K(l_al'_a)_K)_0$$
$$\times\left\{\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K\left[Y^{l_a}(\hat{\mathbf{k}}_a)Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^K\right\}^0_0. \qquad (6.F.10)$$

~~We can~~ One now evaluates ~~evaluate~~ the $9j$ symbol,

$$((l_al_a)_0(l'_al'_a)_0|(l_al'_a)_K(l_al'_a)_K)_0 = \sqrt{\frac{2K+1}{(2l'_a+1)(2l_a+1)}}, \qquad (6.F.11)$$

and expand the coupling,

$$\left\{\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K\left[Y^{l_a}(\hat{\mathbf{k}}_a)Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^K\right\}^0_0 = \sum_M\langle K\ K\ M\ -M|0\ 0\rangle$$
$$\times\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_a}(\hat{\mathbf{k}}_a)Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^K_{-M} = \sum_M\frac{(-1)^{K+M}}{\sqrt{2K+1}} \qquad (6.F.12)$$
$$\times\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_a}(\hat{\mathbf{k}}_a)Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^K_{-M}.$$

Thus,

$$\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l_a}(\hat{\mathbf{k}}_a)\right]^0_0\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^0_0 = \sqrt{\frac{1}{(2l'_a+1)(2l_a+1)}}$$
$$\times\sum_{KM}(-1)^{K+M}\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_a}(\hat{\mathbf{k}}_a)Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^K_{-M}. \qquad (6.F.13)$$

~~We~~ One can further simplify the above expression ~~if we take~~ by choosing the direction of the initial momentum to be parallel to the z axis, so that $Y^{l_a}_m(\hat{\mathbf{k}}_a) = Y^{l_a}_m(\hat{\mathbf{z}}) = \sqrt{\frac{2l_a+1}{4\pi}}\delta_{m,0}$. Then,

$$\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l_a}(\hat{\mathbf{k}}_a)\right]^0_0\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l'_a}(\hat{\mathbf{k}}'_a)\right]^0_0 = \sqrt{\frac{1}{4\pi(2l'_a+1)}}\sum_{KM}(-1)^{K+M} \times \langle l_a\, 0\, l'_a\, -M|K\, -M\rangle \left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M Y^{l'_a}_{-M}(\hat{\mathbf{k}}'_a). \tag{6.$\cancel{F}$G.14}$$

For particle b we have

$$Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{bc})\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{k}}'_b)\right]^0_0 = Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{cb})\sum_m \frac{(-1)^{l'_b+m}}{\sqrt{2l'_b+1}}Y^{l'_b}_m(\hat{\mathbf{r}}_{bc})Y^{l'_b}_{-m}(\hat{\mathbf{k}}'_b), \tag{6.$\cancel{F}$G.15}$$

and ~~but we~~ can write

$$Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}_m(\hat{\mathbf{r}}_{bc}) = \sum_{K'}\langle l_b\, m_b\, l'_b\, m|K'\, m_b+m\rangle\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^{K'}_{m_b+m}. \tag{6.$\cancel{F}$G.16}$$

In order to couple to 0 angular momentum with (6.~~F~~G.14) we must only keep the term with $K' = K,\ m = -M - m_b$ so

$$Y^{l_b}_{m_b}(\hat{\mathbf{r}}_{bc})\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{k}}'_b)\right]^0_0 = \frac{(-1)^{l'_b-M-m_b}}{\sqrt{2l'_b+1}}\langle l_b\, m_b\, l'_b\, -M-m_b|K\, -M\rangle \times \left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M} Y^{l'_b}_{-M-m_b}(\hat{\mathbf{k}}'_b), \tag{6.$\cancel{F}$G.17}$$

and (6.~~F~~G.9) becomes

$$\frac{32\pi^2}{k_a k'_a k'_b}\sum_{KM}(-1)^{K+l'_b-m_b}\langle l_a\, 0\, l'_a\, -M|K\, -M\rangle\langle l_b\, m_b\, l'_b\, -M-m_b|K\, -M\rangle \times \sum_{l_a,l'_a,l'_b}\sqrt{(2l_a+1)}e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})}Y^{l'_b}_{-M-m_b}(\hat{\mathbf{k}}'_b)Y^{l'_a}_{-M}(\hat{\mathbf{k}}'_a)\int d\mathbf{r}_{aA}d\mathbf{r}_{bc}u_{l_b}(r_{bc})v(r_{ab}) \times \frac{F_{l_a}(r_{aA})F_{l'_a}(r_{ac})F_{l'_b}(r_{bc})}{r_{ac}r_{aA}r_{bc}}\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M}. \tag{6.$\cancel{F}$G.18}$$

Note that

$$\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M} = \sum_P\langle K\, M\, K\, -M|P\, 0\rangle \times \left\{\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K\right\}^P_0, \tag{6.$\cancel{F}$G.19}$$

and that to survive the integration the rotational tensors must be coupled to $P = 0$. Keeping only this term in the sum over P, we get

$$\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M} = \frac{(-1)^{K+M}}{\sqrt{2K+1}}\left\{\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K\right\}^0_0. \tag{6.$\cancel{F}$G.20}$$

The coordinate–dependent part of the latter expression is a rotationally invariant scalar,

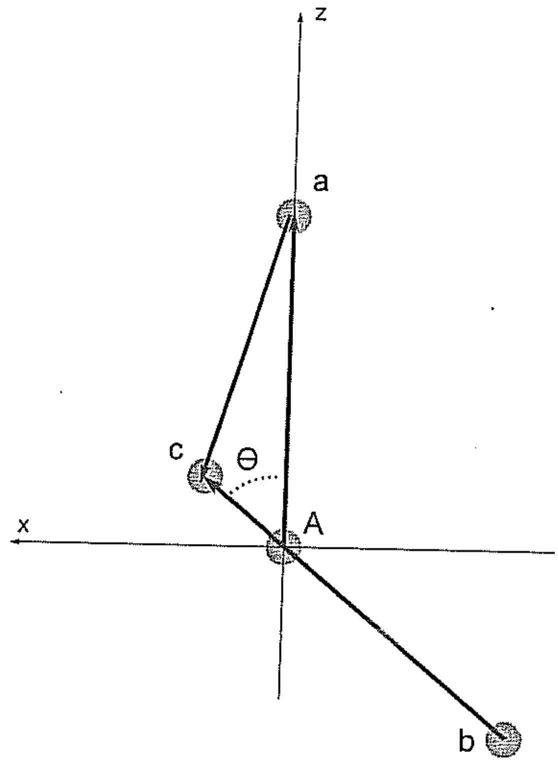

Figure 6.~~F~~6.2: Coordinates in the “standard” configuration.

6.G.2

from

so it can be evaluated in any conventional "standard" configuration such as the one depicted in Fig. ~~7.1.2~~. It must then be multiplied by a factor resulting of the integration of the remaining angular variables, which accounts for the rigid rotations needed to connect any arbitrary configuration to one of this type. This factor turns out to be $8\pi^2$ (a 4π factor for all possible orientations of, say, $\mathbf{r}_{aA}$ and a 2π factor for a complete rotation around its direction). According to Fig. ~~7.1.2~~ 6.F.2

$$
\begin{aligned}
\mathbf{r}_{bc} &= r_{bc}\left(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}\right), \\
\mathbf{r}_{aA} &= -r_{aA}\,\hat{z}, \\
\mathbf{r}_{ac} &= \frac{b}{A}r_{bc}\sin\theta\,\hat{x} + \left(\frac{b}{A}r_{bc}\cos\theta - r_{aA}\right)\hat{z}.
\end{aligned}
\tag{6.F.21}
$$

As $\mathbf{r}_{aA}$ lies parallel to the z axis, $Y^{l_a}_{M_K}(\hat{\mathbf{r}}_{aA}) = \sqrt{\frac{2l_a+1}{4\pi}}\delta_{M_K,0}$ and

$$
\begin{aligned}
&\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})\;Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_{M_K} = \sum_m \langle l_a\; m\; l'_a\; M_K - m|K\; M_K\rangle Y^{l_a}_m(\hat{\mathbf{r}}_{aA})Y^{l'_a}_{M_K-m}(\hat{\mathbf{r}}_{ac}) = \\
&\sqrt{\frac{2l_a+1}{4\pi}}\langle l_a\; 0\; l'_a\; M_K|K\; M_K\rangle Y^{l'_a}_{M_K}(\hat{\mathbf{r}}_{ac}).
\end{aligned}
\tag{6.F.22}
$$

Then

$$
\begin{aligned}
&\left\{\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K\right\}^0_0 = \\
&\sum_{M_K}\langle K\; M_K\; K\; -M_K|0\;0\rangle\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_{M_K}\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M_K} = \\
&\sqrt{\frac{2l_a+1}{4\pi}}\sum_{M_K}\frac{(-1)^{K+M_K}}{\sqrt{2K+1}}\langle l_a\; 0\; l'_a\; M_K|K\; M_K\rangle \\
&\times\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M_K}\;Y^{l'_a}_{M_K}(\hat{\mathbf{r}}_{ac}).
\end{aligned}
\tag{6.F.23}
$$

Remembering the $8\pi^2$ factor, the term arising from (6.F.20) to be considered in the integral is

$$
\begin{aligned}
4\pi^{3/2}\frac{\sqrt{2l_a+1}}{2K+1}(-1)^K\sum_{M_K}(-1)^{M_K}\langle l_a\; 0\; l'_a\; M_K|K\; M_K\rangle& \\
\times\left[Y^{l_b}(\cos\theta,0)Y^{l'_b}(\cos\theta,0)\right]^K_{-M_K}\;Y^{l'_a}_{M_K}(\cos\theta_{ac},0)&,
\end{aligned}
\tag{6.F.24}
$$

6.G.21

with

$$
\cos\theta_{ac} = \frac{\frac{b}{A}r_{bc}\cos\theta - r_{aA}}{\sqrt{\left(\frac{b}{A}r_{bc}\sin\theta\right)^2 + \left(\frac{b}{A}r_{bc}\cos\theta - r_{aA}\right)^2}},
\tag{6.F.25}
$$

(see (~~8.1.2~~)). The final expression of the transition amplitude is

$$
\begin{aligned}
T_{m_b}(\mathbf{k}'_a,\mathbf{k}'_b) = \frac{128\pi^{7/2}}{k_a k'_a k'_b}\sum_{KM}\frac{(-1)^{l'_b+m_b}}{2K+1}\langle l_a\; 0\; l'_a\; -M|K\; -M\rangle\langle l_b\; m_b\; l'_b\; -M-m_b|K\; -M\rangle& \\
\times\sum_{l_a,l'_a,l'_b}(2l_a+1)e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})}Y^{l'_b}_{-M-m_b}(\hat{\mathbf{k}}'_b)Y^{l'_a}_{-M}(\hat{\mathbf{k}}'_a)\,\mathcal{I}(l_a,l'_a,l'_b,K)&,
\end{aligned}
\tag{6.F.26}
$$

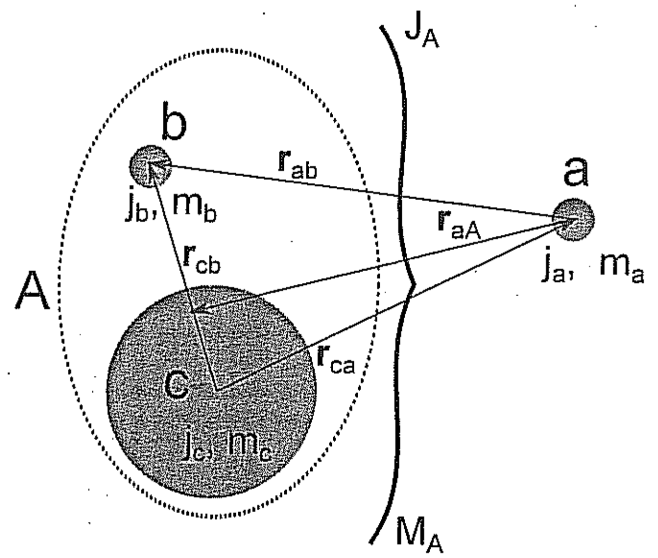

Figure 6.~~F~~G.3: Now all three clusters a, b, c have definite spins and projections. The nucleus A is coupled to total spin J_A, M_A.

where

$$\mathcal{I}(l_a, l'_a, l'_b, K) = \int dr_{aA}\, dr_{bc}\, d\theta\, r_{aA} r_{bc} \frac{\sin\theta}{r_{ac}} u_{l_b}(r_{bc}) v(r_{ab}) F_{l_a}(r_{aA}) F_{l'_a}(r_{ac}) F_{l'_b}(r_{bc}) \times \sum_{M_K} (-1)^{M_K} \langle l_a\; 0\; l'_a\; M_K | K\; M_K \rangle \left[Y^{l_b}(\cos\theta, 0) Y^{l'_b}(\cos\theta, 0) \right]^{K}_{-M_K} Y^{l'_a}_{M_K}(\cos\theta_{ac}, 0) \quad (6.F.27)$$

is a 3–dimensional integral that can be numerically evaluated with the help of e.g., Gaussian integration.

6.F.2 Particles with spin

We ~~will~~ now ~~turn to~~ treat the case in which the clusters ~~have~~ contain a definite spin (see Fig. 6.~~F~~G.3), and the complex optical potentials $U(r_{aA})$, $U(r_{cb})$, $U(r_{ac})$ ~~are~~ now ~~central potentials with~~ a spin–orbit term proportional to the ~~usual~~ product $\mathbf{l} \cdot \mathbf{s} = 1/2(j(j+1) - l(l+1) - 3/4)$ for particles with spin 1/2. In addition, the interaction $V(r_{ab}, \sigma_a, \sigma_b)$ between a and b is taken to be a separable function of the distance r_{ab} and of the spin orientations, $V(r_{ab}, \sigma_a, \sigma_b) = v(r_{ab}) v_\sigma(\sigma_a, \sigma_b)$. Note that this ansatz rules out a spin–orbit ~~term and terms~~ as well as ~~proportional to $\mathbf{r} \cdot \sigma$, such as the~~ tensor terms in the NN-interaction. For the ~~moment~~ time being we will assume that the spin–dependent interaction is rotationally invariant (scalar with respect to rotations), such as, e.g., $v_\sigma(\sigma_a, \sigma_b) \propto \sigma_a \cdot \sigma_b$. Again, ~~that~~ this assumption excludes ~~from our formalism~~ tensor terms in the interaction. The transition amplitude is then,

G

$$T^{m'_a, m'_b}_{m_a, m_b} = \sum_{\sigma_a, \sigma_b} \int d\mathbf{r}_{aA} d\mathbf{r}_{bc} \chi^{(-)*}_{m'_a}(\mathbf{r}_{ac}, \sigma_a) \chi^{(-)*}_{m'_b}(\mathbf{r}_{bc}, \sigma_b)$$
$$\times\, v(r_{ab}) v_\sigma(\sigma_a, \sigma_b) \chi^{(+)}_{m_a}(\mathbf{r}_{aA}, \sigma_a) \psi^{l_b, j_b}_{m_b}(\mathbf{r}_{bc}, \sigma_b). \qquad (6.\cancel{F}.28)$$

Distorted waves

G

involved in the reaction process

The distorted waves in (6.~~F~~G.28) $\chi_m(\mathbf{r}, \sigma) = \chi(\mathbf{r}) \phi^{1/2}_m(\sigma)$ have a spin dependence contained in the actual spinor $\phi^{1/2}_m(\sigma)$, where σ is the spin degree of freedom and m the ~~projection~~ do of the spin along the quantization axis. The superscript 1/2 reminds ~~us that~~ we are considering spin 1/2 particles, which have important consequences ~~when~~ dealing with the spin–orbit term of the optical potentials. As for the spin–dependent term $v_\sigma(\sigma_a, \sigma_b)$, the value of the spin of ~~the involved~~ particles ~~does~~ not make much difference, *as long as this term is rotationally invariant.* ~~After (7.2.111)~~ Following (6.G.4),

G

$$\chi^{(+)}(\mathbf{k}, \mathbf{r}) \phi_m(\sigma) = \sum_{l,j} \frac{4\pi}{kr} i^l \sqrt{2l+1} e^{i\sigma^l} F_{l,j}(r) \left[Y^l(\hat{\mathbf{r}}) Y^l(\hat{\mathbf{k}}) \right]^0_0 \phi^{1/2}_m(\sigma). \qquad (6.\cancel{F}.29)$$

Note that now ~~the sum is~~ one sums also ~~over the~~ both total angular momentum j, ~~because~~ as the radial functions $F_{l,j}(r)$ depend ~~now~~ on j as well as on l, ~~being~~ in keeping with the fact that they are ~~solutions~~ solutions of an optical potential ~~with~~ containing a spin–orbit term proportional to $1/2\,(j(j+1) - l(l+1) - 3/4)$. ~~We~~ One must then couple the radial and spin functions to total angular momentum j, noting that

G

$$\left[Y^l(\hat{\mathbf{r}})\ Y^l(\hat{\mathbf{k}}) \right]^0_0 \phi^{1/2}_m(\sigma) = \sum_{m_l} \langle l\ m_l\ l\ -m_l | 0\ 0 \rangle Y^l_{m_l}(\hat{\mathbf{r}}) Y^l_{-m_l}(\hat{\mathbf{k}}) \phi^{1/2}_m(\sigma) =$$
$$\sum_{m_l} \frac{(-1)^{l-m_l}}{\sqrt{2l+1}} Y^l_{m_l}(\hat{\mathbf{r}}) Y^l_{-m_l}(\hat{\mathbf{k}}) \phi^{1/2}_m(\sigma), \qquad (6.\cancel{F}.30)$$

and

G

$$Y^l_{m_l}(\hat{\mathbf{r}}) \phi^{1/2}_m(\sigma) = \sum_j \langle l\ m_l\ 1/2\ m | j\ m_l + m \rangle \left[Y^l(\hat{\mathbf{r}}) \phi^{1/2}(\sigma) \right]^j_{m_l+m}, \qquad (6.\cancel{F}.31)$$

we can write

G

$$\left[Y^l(\hat{\mathbf{r}})\ Y^l(\hat{\mathbf{k}}) \right]^0_0 \phi^{1/2}_m(\sigma) = \sum_{m_l, j} \frac{(-1)^{l+m_l}}{\sqrt{2l+1}} \langle l\ m_l\ 1/2\ m | j\ m_l + m \rangle$$
$$\times \left[Y^l(\hat{\mathbf{r}}) \phi^{1/2}(\sigma) \right]^j_{m_l+m} Y^l_{-m_l}(\hat{\mathbf{k}}), \qquad (6.\cancel{F}.32)$$

and the distorted waves in (6.~~F~~G.28) are

G

$$\chi^{(+)}_{m_a}(\mathbf{r}_{aA}, \mathbf{k}_a, \sigma_a) = \sum_{l_a, m_{l_a}, j_a} \frac{4\pi}{k_a r_{aA}} i^{l_a} (-1)^{l_a + m_{l_a}} e^{i\sigma^{l_a}} F_{l_a, j_a}(r_{aA})$$
$$\times \langle l_a\ m_{l_a}\ 1/2\ m_a | j_a\ m_{l_a} + m_a \rangle \left[Y^{l_a}(\hat{\mathbf{r}}_{aA}) \phi^{1/2}(\sigma_a) \right]^{j_a}_{m_{l_a} + m_a} Y^{l_a}_{-m_{l_a}}(\hat{\mathbf{k}}_a), \qquad (6.\cancel{F}.33)$$

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$$\chi^{(-)*}_{m'_b}(\mathbf{r}_{bc}, \mathbf{k}'_b, \sigma_b) = \sum_{l'_b, m_{l'_b}, j'_b} \frac{4\pi}{k'_b r_{bc}} i^{-l'_b} (-1)^{l'_b + m_{l'_b}} e^{i\sigma^{l'_b}} F_{l'_b, j'_b}(r_{bc})$$
$$\times \langle l'_b\ m_{l'_b}\ 1/2\ m'_b | j'_b\ m_{l'_b} + m'_b \rangle \left[Y^{l'_b}(\hat{\mathbf{r}}_{bc}) \phi^{1/2}(\sigma_b) \right]^{j'_b *}_{m_{l'_b} + m'_b} Y^{l'_b *}_{-m_{l'_b}}(\hat{\mathbf{k}}'_b), \qquad (6.\cancel{F}.34)$$

$$\chi^{(-)*}_{m'_a}(\mathbf{r}_{ac},\mathbf{k}'_a,\sigma_a) = \sum_{l'_a,m_{l'_a},j'_a} \frac{4\pi}{k'_a r_{ac}} i^{-l'_a}(-1)^{l'_a+m_{l'_a}} e^{i\sigma^{l'_a}} F_{l'_a,j'_a}(r_{ac})$$
$$\times \langle l'_a\; m_{l'_a}\; 1/2\; m'_a | j'_a\; m_{l'_a}+m'_a\rangle \left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})\phi^{1/2}(\sigma_a)\right]^{j'_a *}_{m_{l'_a}+m'_a} Y^{l'_a *}_{-m_{l'_a}}(\hat{\mathbf{k}}'_a). \qquad (6.F.35)$$

The initial bound particle b wavefunction ~~of particle b~~ is

$$\psi^{l_b,j_b}_{m_b}(\mathbf{r}_{bc},\sigma_b) = u_{l_b,j_b}(r_{bc})\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})\phi^{1/2}(\sigma_b)\right]^{j_b}_{m_b}, \qquad (6.F.36)$$

Substituting in (6.F.28), one obtains,

$$T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b) = \frac{64\pi^3}{k_a k'_a k'_b}\sum_{\sigma_a,\sigma_b}\sum_{l_a,m_{l_a},j_a}\sum_{l'_a,m_{l'_a},j'_a}\sum_{l'_b,m_{l'_b},j'_b} e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})} i^{l_a-l'_a-l'_b}(-1)^{l_a-m_{l_a}+l'_a-j'_a+l'_b-j'_b}$$
$$\times \langle l'_a\; m_{l'_a}\; 1/2\; m'_a|j'_a\; m_{l'_a}+m'_a\rangle\langle l_a\; m_{l_a}\; 1/2\; m_a|j_a\; m_{l_a}+m_a\rangle\langle l'_b\; m_{l'_b}\; 1/2\; m'_b|j'_b\; m_{l'_b}+m'_b\rangle$$
$$\times Y^{l_a}_{-m_{l_a}}(\hat{\mathbf{k}}_a)Y^{l'_b}_{-m_{l'_b}}(\hat{\mathbf{k}}'_b)Y^{l'_a}_{-m_{l'_a}}(\hat{\mathbf{k}}'_a)\int d\mathbf{r}_{aA}d\mathbf{r}_{bc}\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})\phi^{1/2}(\sigma_a)\right]^{j'_a}_{-m_{l'_a}-m'_a}\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})\phi^{1/2}(\sigma_b)\right]^{j'_b}_{-m_{l'_b}-m'_b}$$
$$\times \frac{F_{l_a,j_a}(r_{aA})F_{l'_a,j'_a}(r_{ac})F_{l'_b,j'_b}(r_{bc})}{r_{ac}r_{aA}r_{bc}} u_{l_b,j_b}(r_{bc})v(r_{ab})v_\sigma(\sigma_a,\sigma_b)$$
$$\times\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})\phi^{1/2}(\sigma_a)\right]^{j_a}_{m_{l_a}+m_a}\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})\phi^{1/2}(\sigma_b)\right]^{j_b}_{m_b}, \qquad (6.F.37)$$

where ~~we have~~ used was made of the relation

$$\left[Y^{l}(\hat{\mathbf{r}})\phi^{1/2}(\sigma)\right]^{j*}_{m} = (-1)^{j-m}\left[Y^{l}(\hat{\mathbf{r}})\phi^{1/2}(\sigma)\right]^{j}_{-m}. \qquad (6.F.38)$$

Recoupling of angular momenta

Let us now separate spatial and spin coordinates, noting that the spin functions must be coupled to S=0, ~~0 (this is~~ a consequence of the fact that the interaction $v_\sigma(\sigma_a,\sigma_b)$ ~~being~~ is rotationally invariant. Starting with particle a,

$$\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})\phi^{1/2*}(\sigma_a)\right]^{j'_a}_{-m_{l'_a}-m'_a}\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})\phi^{1/2}(\sigma_a)\right]^{j_a}_{m_{l_a}+m_a} =$$
$$\sum_K((l'_a\tfrac{1}{2})_{j'_a}(l_a\tfrac{1}{2})_{j_a}|(l_a l'_a)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K$$
$$\times\left[Y^{l'_a}(\hat{\mathbf{r}}_{ac})Y^{l_a}(\hat{\mathbf{r}}_{aA})\right]^{K}_{-m_{l'_a}-m'_a+m_{l_a}+m_a}\left[\phi^{1/2*}(\sigma_a)\phi^{1/2}(\sigma_a)\right]^0_0. \qquad (6.F.39)$$

For particle b,

$$\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})\phi^{1/2*}(\sigma_b)\right]^{j'_b}_{-m_{l'_b}-m'_b}\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})\phi^{1/2}(\sigma_b)\right]^{j_b}_{m_b} =$$
$$\sum_{K'}((l'_b\tfrac{1}{2})_{j'_b}(l_b\tfrac{1}{2})_{j_b}|(l_b l'_b)_{K'}(\tfrac{1}{2}\tfrac{1}{2})_0)_{K'}$$
$$\times\left[Y^{l'_b}(\hat{\mathbf{r}}_{bc})Y^{l_b}(\hat{\mathbf{r}}_{bc})\right]^{K'}_{-m_{l'_b}-m'_b+m_b}\left[\phi^{1/2*}(\sigma_b)\phi^{1/2}(\sigma_b)\right]^0_0. \qquad (6.F.40)$$

The spin summation yields a constant factor,

$$\sum_{\sigma_a,\sigma_b}\left[\phi^{1/2*}(\sigma_a)\phi^{1/2}(\sigma_a)\right]^0_0\left[\phi^{1/2*}(\sigma_b)\phi^{1/2}(\sigma_b)\right]^0_0 v_\sigma(\sigma_a,\sigma_b)\equiv T_\sigma, \qquad (6.F.41)$$

and what we have yet to do is very similar to what we have done ~~for~~ in the case of spinless particles. First of all note that the ~~necessity to couple~~ constrain of coupling all angular momenta to 0 imposes $K' = K$ and $m_{l_a} + m_a - m_{l'_a} - m'_a = m_{l'_b} + m'_b - m_b$ (see (6.F.39) and (6.F.40)). If we set $M = m_{l_a} + m_a - m_{l'_a} - m'_a$ and take, as before, $\hat{\mathbf{k}}_a \equiv \hat{z}$

$$
\begin{aligned}
T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b) &= \frac{32\pi^{5/2}}{k_a k'_a k'_b} T_\sigma \sum_{l_a,j_a}\sum_{l'_a,j'_a}\sum_{l'_b,j'_b}\sum_{K,M} e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})} i^{l_a-l'_a-l'_b}(-1)^{l_a+l'_a+l'_b-j'_a-j'_b} \\
&\times \sqrt{2l_a+1}\,((l'_a\tfrac{1}{2})_{j'_a}(l_a\tfrac{1}{2})_{j_a}|(l_a l'_a)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\,((l'_b\tfrac{1}{2})_{j'_b}(l_b\tfrac{1}{2})_{j_b}|(l_b l'_b)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K \\
&\times\langle l'_a\; m_a-m'_a-M\; 1/2\; m'_a|j'_a\; m_a-M\rangle\langle l_a\; 0\; 1/2\; m_a|j_a\; m_a\rangle\langle l'_b\; m_b-m'_b+M\; 1/2\; m'_b|j'_b\; M+m_b\rangle \\
&\times Y^{l'_b}_{m'_b-m_b-M}(\hat{\mathbf{k}}'_b)Y^{l'_a}_{m'_a-m_a+M}(\hat{\mathbf{k}}'_a)\int d\mathbf{r}_{aA}d\mathbf{r}_{bc}\frac{F_{l_a,j_a}(r_{aA})F_{l'_a,j'_a}(r_{ac})F_{l'_b,j'_b}(r_{bc})}{r_{ac}r_{aA}r_{bc}} \\
&\times u_{l_b,j_b}(r_{bc})v(r_{ab})\left[Y^{l_a}(\hat{\mathbf{r}}_{aA})Y^{l'_a}(\hat{\mathbf{r}}_{ac})\right]^K_M\left[Y^{l_b}(\hat{\mathbf{r}}_{bc})Y^{l'_b}(\hat{\mathbf{r}}_{bc})\right]^K_{-M}. \qquad (6.F.42)
\end{aligned}
$$

The integral of the above expression is similar to the one in (6.F.18), so we obtain

$$
\begin{aligned}
T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b) &= \frac{128\pi^4}{k_a k'_a k'_b} T_\sigma \sum_{l_a,j_a}\sum_{l'_a,j'_a}\sum_{l'_b,j'_b}\sum_{K,M} e^{i(\sigma^{l_a}+\sigma^{l'_a}+\sigma^{l'_b})} i^{l_a-l'_a-l'_b}(-1)^{l_a+l'_a+l'_b-j'_a-j'_b} \\
&\times \frac{2l_a+1}{2K+1}((l'_a\tfrac{1}{2})_{j'_a}(l_a\tfrac{1}{2})_{j_a}|(l_a l'_a)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K\,((l'_b\tfrac{1}{2})_{j'_b}(l_b\tfrac{1}{2})_{j_b}|(l_b l'_b)_K(\tfrac{1}{2}\tfrac{1}{2})_0)_K \\
&\times\langle l'_a\; m_a-m'_a-M\; 1/2\; m'_a|j'_a\; m_a-M\rangle\langle l'_b\; m_b-m'_b+M\; 1/2\; m'_b|j'_b\; M+m_b\rangle \\
&\times\langle l_a\; 0\; 1/2\; m_a|j_a\; m_a\rangle Y^{l'_b}_{m'_b-m_b-M}(\hat{\mathbf{k}}'_b)Y^{l'_a}_{m_a-m'_a+M}(\hat{\mathbf{k}}'_a)\mathcal{I}(l_a,l'_a,l'_b,j_a,j'_a,j'_b,K), \qquad (6.F.43)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{I}(l_a,l'_a,l'_b,j_a,j'_a,j'_b,K) &= \int dr_{aA}dr_{bc}d\theta\, r_{aA}r_{bc}\frac{\sin\theta}{r_{ac}}u_{l_b}(r_{bc})v(r_{ab}) \\
&\times F_{l_a,j_a}(r_{aA})F_{l'_a,j'_a}(r_{ac})F_{l'_b,j'_b}(r_{bc}) \\
&\times\sum_{M_K}\langle l_a\; 0\; l'_a\; M_K|K\; M_K\rangle\left[Y^{l_b}(\cos\theta,0)Y^{l'_b}(\cos\theta,0)\right]^K_{-M_K}Y^{l'_a}_{M_K}(\cos\theta_{ac},0). \qquad (6.F.44)
\end{aligned}
$$

Again, this is a 3–dimensional integral that can be evaluated with the method of Gaussian quadratures. The transition amplitude $T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b)$ depends explicitly on the initial (m_a,m'_a) and final (m'_a,m'_b) polarizations of a,b. If the particle b is initially coupled to core c to total angular momentum J_A, M_A, the amplitude to be considered is rather

$$
T^{m'_a,m'_b}_{m_a}(\mathbf{k}'_a,\mathbf{k}'_b) = \sum_{m_b}\langle j_b\; m_b\; j_c\; M_A-m_b|J_A\; M_A\rangle\, T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b), \qquad (6.F.45)
$$

and the multi–differential cross section for detecting particle c (or a) is

$$
\left.\frac{d\sigma}{d\mathbf{k}'_a d\mathbf{k}'_b}\right]^{m'_a,m'_b}_{m_a} = \frac{k'_a}{k_a}\frac{\mu_{aA}\mu_{ac}}{4\pi^2\hbar^4}\left|\sum_{m_b}\langle j_b\; m_b\; j_c\; M_A-m_b|J_A\; M_A\rangle\, T^{m'_a,m'_b}_{m_a,m_b}(\mathbf{k}'_a,\mathbf{k}'_b)\right|^2. \qquad (6.F.46)
$$

All spin–polarization observables (analysing powers, etc.,) can be derived from this expression. But let us now work out the expression of the cross section for an unpolarized