Chapter 1

Introduction

1.1 Elementary modes of excitation

Subject to external probes which couple weakly to the nucleus, that is in such a way that the system can be expressed in terms of the properties of the excitation in the absence of probes (Pines, D. and Nozières (1966)), the nucleus reacts (Bohr, A. and Mottelson (1975)) in terms of single–particle (–hole) motion (one–particle transfer), vibrations (surface, spin, etc.) and rotations (Coulomb excitation and inelastic scattering) and pairing vibration and rotation (two–nucleon transfer reactions) (cf. Figs. 1.1.1, 1.1.2 and 1.1.3) ~~also App. 1.A~~).

Echoing Heisenberg's requirement (Heisenberg (1949)) that no concept enters the quantal description of a physical system which has no direct relation to experiments, and Landau's result that any weakly excited state of a quantal many–body system may be regarded as a gas of weakly interacting elementary modes of excitation (Landau (1941)), Bohr, Mottelson and coworkers (Bohr and Mottelson (1969), Bohr (1976), Mottelson (1976), Bohr, A. and Mottelson (1975), Bohr et al. (1958), Belyaev (1959), Nilsson (1955), Bès,D. R. and Broglia (1966), Bès and Broglia (1977) and references therein) developed a unified field theoretical description of the nuclear structure in terms of quasiparticles, vibrations and rotations, both in 3D as well as in gauge space, which was eventually extended to direct nuclear reactions (Alder et al. (1956), Alder and Winther (1975), Broglia and Winther (2004)).

In keeping with the fact that all the nuclear degrees of freedom are exhausted by those of the nucleons, and that the different reactions, that is Coulomb, inelastic and one– and two– particle transfer reactions project particular, but somewhat overlapping components of the total wavefunction, the nuclear elementary modes of excitation give rise to an overcomplete, non orthogonal, Pauli principle violating basis, both concerning structure as well as reactions. The coupling between unperturbed fermionic and bosonic degrees of freedom is proportional to this overlap ~~between single–particle and collective modes~~. Nuclear Field Theory (Bès et al. (1976b), Bès et al. (1976c), Bès et al. (1976a), Bès and Broglia (1975), Broglia et al. (1976), Bès, D. R. and Broglia, R. A. and Dussel, G. G. and Liotta (1975),

more abstract, but not less real spaces, like e.g. in (NFT), Bès et al (1974),

see new reference bibliography

Mottelson (1976), Bès and Broglia (1977), Bortignon, P. F. et al. (1977), Bortignon, P. F. et al. (1978), Broglia and Winther (2004), Hugo Reinhardt) provides the rules to diagonalize ~~conserving sum rules~~, these couplings to any order of perturbation theory, also infinite if so required for specific processes. The clothed elementary modes resulting from the interweaving of the bare modes are orthogonal to each other, fulfill Pauli principle, and behave like a non interacting gas providing a microscopic solution of the many–body problem. Its properties, embodied in absolute cross sections and transition probabilities, can be compared with the observables whose values are obtained by studying the nuclear system with a variety of specific probes. A schematic presentation of NFT for nuclei-like systems is given in App. A.

1.2 Sum rules

A quantitative measure of the above mentioned overcompleteness is provided by exact and approximate sum rules that the observables (cross sections) associated with the variety of probes to which the nucleus is subject, have to fulfill. An example of the first type (exact) is provided by the Thomas–Reiche–Kuhn (TRK) sum rule. Of the second type (approximate) by some of the two–nucleon transfer (TNTR) sum rules (Broglia, R. A. et al. (1972)), others which relate one– with two–particle transfer processes (Bayman, B. F. and Clement (1972)) being exact. In both cases they embody particle (pair) number conservation. Charged particles in the first case (electrons in atoms and molecules, effective charges of neutrons and protons in nuclei). Number of Cooper pairs in nuclei in the second. Physically, they provide with information concerning: 1) the maximum amount of photons (cross section) which the quantal system can absorb from a beam of light (γ–rays) shined on it; 2) the total two–nucleon transfer cross section (ring area fraction of the total (geometrical) reaction cross section) exhausted by the final ($A \pm 2$) states populated in the transfer process.

In other words, these sum rules provide: a quantitative measure of the single–particle subspace the quantal system under study, in particular the nucleus, uses to correlate particle–hole excitations and thus induce the antenna–like motion of protons against neutrons or, to correlate pairs of nucleons moving in time reversal states around the Fermi energy thus providing a sigmoidal distribution of the associated level occupancy. The TRK sum rule can, in the nuclear case, be written as

$$S(E1) = \sum_n \left|\langle\alpha|F|\tilde{0}\rangle\right|^2 (E_\alpha - E_0) = \frac{9}{4\pi}\frac{\hbar^2 e^2}{2m}\frac{NZ}{A}, \qquad (1.2.1)$$

where $|\alpha\rangle$ labels the complete set of excited dipole states which can be reached operating with the dipole operator F on the initial correlated vacuum state $|\tilde{0}\rangle$. Within this context, each elementary mode of excitation, in the present case of dipole type, define a ground state. This is in keeping with the fact that they induce specific Zero

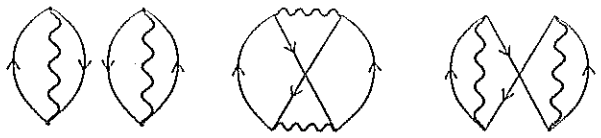

Figure 1.A.1: Oyster diagrams describing the correlation of the nuclear ground state on which a collective particle–hole excitation is built, and Pauli principle correction processes in which fermions are exchanged.

Appendix 1.A Specific probes and elementary modes of excitation

In a classical world, empty space is the absence of physics, and the existence of something, e.g. of light or of an electron is only a clue to eventually learn what the "object" can be: wave or particle. Think only on all the work concerning the vibrations of an hypothetical "aether", or concerning the "divergent" mass (energy) of the electron. Within this last context, the same is somewhat true in quantum mechanics, with a small difference. We do not need to think whether light is a particle or a wave. But yes, whether it is a composite particle or less, in both the sense of finite lifetime or coupling to other particles. Namely, what are the vaues of the parameters characterizing the bare particles (e.g. m_{bare}) so that the dressed ones reproduce the experimental findings.

Within this context, think only on the discussions concerning spectroscopic factors, RPA vibrations (sharp states, no coupling to $2p$–$2h$ states), role of induced interaction in pairing correlations in nuclei, etc.

Now, all these doubts vanish by acting with the variety of specific probes on the quantal vacuum. Namely, by making virtual processes associated with the vacuum ZPF (cf. Fig. 1.A.1) become real, taking properly into account Pauli principles (see e.g. Fig. 1.A.2)

Within the framework of this example, acting with an external, time dependent single–particle hadronic field (e.g. p, p'), one excites properly dressed (p–h)–like vibrations (cf. e.g. Brink, D. and Broglia (2005) pp. 240–242 and references therein).

In other words, if one is in doubt of what properly dressed (nuclear) elementary modes of excitation are, do not study them, or calculate them and then compare the results with the experimental data. This comes after. One should first find out how to specifically excite the mode in question, by acting with an external field on the ZPF of the vacuum. That is, by carrying out a gedanken, NFT–like experiment as in Fig. 1.A.2 for $p-h$–vibrations and in Fig. 1.A.3 regarding pairing vibrations. Because the vacuum contains all the information (right physical degrees of freedom) of each quantal system (also of the whole Universe), forcing virtual processes associated with vacuum ZPF to become real, one is sure to get, in each instance, the real, dressed, physical particle. Of course, once the *gedanken eksperiment* has pro-

Let us now provide a short introduction of NFT for pedestrians. For details we refer to Bortignon et al (1977).

vided this information, one should use such properly renormalized modes, in all the rest of the calculations, at the risk of neglecting relevant physics.

(A)-(A) pp. 107-132 Varenna Bes and Broglia 1977 p. 55

Appendix 1.B Two–nucleon transfer sum rules

We consider the reaction

$$a + A \rightarrow b + B. \tag{1.B.1}$$

In the center–of–mass system, the local Hamiltonian may be written

$$H = T_{aA} + H_a + H_A + V_{aA} = T_{bB} + H_b + H_B + V_{bA}, \tag{1.B.2}$$

in keeping with energy conservation. Within this context other, mixed, representations are possible.

One then solves the time–dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{1.B.3}$$

with the initial conditions that the nuclei a and A are in their ground states, and where the relative motion is described by a narrow wavepacket of rather well defined impact parameter and velocity.

We expend Ψ on (stationary) channel wavefunctions

$$\Psi = \sum_\beta c_\beta\Big(\big(\mathbf{r}_\beta - R_\beta\big)\Big)\Psi_\beta e^{-iE_\beta t/\hbar}, \tag{1.B.4}$$

where

$$\Psi_\beta(t) = \Psi^b_m(\xi_b)\Psi^B_n(\xi_B)\exp(i\delta_\beta). \tag{1.B.5}$$

The index β labels both the partition of nuclei (b, B) as well as the quantal states of the two nucleons (m, n).

The phase δ_β is defined as

$$\delta_\beta = \frac{1}{\hbar}\left\{m_\beta\mathbf{v}_\beta\big(\mathbf{r}_\beta - \mathbf{R}_\beta(t)\big) - \int_0^t\left(U_\beta\big(R_\beta(t')\big) - \frac{1}{2}m_\beta\mathbf{v}_\beta(t')^2\right)\right\}. \tag{1.B.6}$$

Where an extra phase has been added to eliminate, as far as possible, the diagonal matrix elements in the coupled equations. The phase factor $\exp(i\delta_\beta)$ acting on the channel wavefunction is essentially a Galilean transformation (cf. jagged "phonon" in the NFT reaction diagrams displayed in Figs. 4.1.1 (one–particle transfer) and 5.C.1 and 5.C.2 (two–particle transfer); see also Figs. 1.1.2 and 1.1.3).

The function c_β can be expressed as

$$c_\beta = a_\beta(t)\chi_\beta(\mathbf{r}_\beta - \mathbf{R}_\beta(t), t) \tag{1.B.7}$$

p. 127
a) The $3^+/2$ states
b) The ~~multiplet~~

p. 107
ch. 5) 1.A.1 Elementary modes of excitation
sec. 6 p. 110) 1.A.2 Nuclear field theory
c. 6.1 p. 110) 1.A.2.1 Schematic model
c. 6.2 p. 111) 1.A.2.2 Field-theoretical solutions
(7 p. 117) 1.A.3 Spurious states → Fig
+8 p. 125) 1.A.4 Applications

Phys. Lett. 64B, 29 (1976)

Varenna + Phys. Lett. Spurious (1976)

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p. 127
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p. 107
Sect. 5) 1.A.1 Elementary modes of excitation
Sec. 6 p. 110) 1.A.2 Nuclear field theory
Sec. 6.1 p. 110) 1.A.2.1 Schematic model
Sec. 6.2 p. 111) 1.A.2.2 Field-theoretical solutions
Sect 7 p. 117) 1.A.3 Spurious states → Fig
Sect 8 p. 125) 1.A.4 Applications

Phys. Lett. 64B, 29 (1976)

Varenna + Phys. Lett. Spurious Broglia et al (1976) Treatment of the spurious

populated in two-nucleon transfer processes, are weaker than predicted by the harmonic model. The residual pairing interactions tend to produce a condensation of the boson, namely a superconducting state. This tendency drains strength from the excited into the ground state.

II: Field theory of elementary excitations in nuclei.

In the first part of these lectures the concept of elementary modes of excitation was discussed in terms of simple schematic models. The usefulness of such a concept to describe the nuclear structure was demonstrated in the discussion of selected examples involving pairing modes.

Because all the nuclear degrees of freedom are exhausted by the protons and the neutrons, it is an essential feature of descriptions based on elementary modes of excitation to violate the Pauli principle and to require the use of an over-complete basis.

In the second part of these lectures we develop a field theory, in which the nuclear elementary modes of excitation are the fermionic and bosonic free fields, and where their mutual interweaving takes place through the particle-vibration coupling and the bare model four-point vertex interaction.

5. – The concept of elementary modes of excitation.

The Hamiltonian of a many-body system of noninteracting particles, bosons or fermions, can be written as

$$H = \sum_i H_i , \tag{5.1}$$

where the summation is over all the particles of the system and where each H_i depends only on the variables of the i-th particle. The single-particle Schrödinger equation is

$$H_i \psi_k(\boldsymbol{r}_i) = \varepsilon_k \psi_k(\boldsymbol{r}_i) , \tag{5.2}$$

where ε_k is the single-particle energy eigenvalue and

$$\psi_k(\boldsymbol{r}_i) \equiv \langle \boldsymbol{r}_i | a_k^\dagger | 0 \rangle \tag{5.3}$$

is the corresponding wave function. The operator $a_k^\dagger$ creates a particle in the state k when acting in the vacuum state $|0\rangle$. The energy levels of the system

are then given by the equation

$$E_n = \sum_k n_k \varepsilon_k \,, \tag{5.4}$$

the corresponding eigenstates being

$$|n\rangle = \prod_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0\rangle \,, \tag{5.5}$$

where $n_k = 0$ or 1 in the case of fermions and $n_k = 0, 1, 2, \ldots$ in the case of bosons.

Now we consider a system of interacting particles. The Hamiltonian will in this case be

$$H = \sum_i H_i + \tfrac{1}{2} \sum_{i,j} H_{ij} \,, \tag{5.6}$$

where i, j label the co-ordinates of the i-th and j-th particle.

In some cases it is possible to recast the two-body Hamiltonian in the form

$$H = \sum_\tau H'_\tau \,, \tag{5.7}$$

with the associated Schrödinger equation

$$H'_\tau \psi_\tau(\zeta) = \varepsilon_\tau \psi_\tau(\zeta) \,, \tag{5.8}$$

ζ representing a generalized variable (*e.g.* the single-particle co-ordinate, the gap parameter, the shape of the nucleus, etc.). The wave function $\psi_\tau(\zeta)$ is the ζ–co-ordinate representation of the eigenstate $\alpha_\tau^\dagger|\tilde{0}\rangle$. The operator $\alpha_\tau^\dagger$ creates an excitation with quantum number τ when acting in the state $|\tilde{0}\rangle$, the vacuum of all the excitations τ.

The energy of the levels of the system, or at any rate of the most important ones to determine the physical response of it to external probles, can be written in the form

$$E_m = \sum_\tau n_\tau \varepsilon_\tau \,. \tag{5.9}$$

The corresponding eigenstate can be written in the same way as before, *i.e.*

$$|n\rangle = \prod_\tau \frac{(\alpha_\tau^\dagger)^{n_\tau}}{\sqrt{n_\tau!}} |\tilde{0}\rangle \,. \tag{5.10}$$

(e.g. cf. Bohr and Mottelson (1975), Bès and Broglia 1966, Bès and Kurchan (1990) and refs. therein)

Additivity features similar to (5.9) hold for other physical quantities, *i.e.*

$$\langle n|\mathcal{O}|m\rangle = \sum_\tau A_\tau \sqrt{n_\tau}\,\delta(n_\tau, m_\tau + 1), \qquad (1, A.11)$$

where

$$\mathcal{O} = \sum_\tau A_\tau \alpha_\tau^\dagger \qquad (1.A.12)$$

is the operator which specifically excites the eigenstates described by $\psi_\tau(\xi)$. Because the excitation energies E_m and observables $|\langle m'|\mathcal{O}|m\rangle|^2$ (*e.g.* two-particle transfer cross-section, electromagnetic-transition probabilities, etc.) are linear combinations of ε_τ and A_τ, respectively, the eigenstates with energy ε_τ and associated observable A_τ are called the *elementary excitations of the system*.

There lie thus two ideas behind the concept of elementary excitations (*). First, there is the idea that the total binding energy does not have much to do with the behaviour of the physical system. Thus, the state $|\tilde{0}\rangle$ is assumed to exist but to act only as a background whose detailed intrinsic structure one does not need to know to describe the behaviour of the system. What is important is the behaviour of the lower excited states relative to the ground state.

The second idea is that the low-lying states often are of a particular simple character, and are amenable to a simple and rigorous mathematical treatment.

With the help of experimental probes which couple weakly to the nucleus, *i.e.* in such a way that the system can be expressed in terms of the properties of the excitation in the absence of probes, it has been possible to identify the following elementary excitations in systems around closed shells (**):

a) single particle and holes,

b) shape vibrations,

c) spin and isospin vibrations and charge exchange modes,

d) pairing vibrations.

Different probes have been utilized in the process of identification of the different modes. In particular two-neutron transfer reactions induced by tritons and protons have played a central role in unraveling the basic features of the pairing modes. This subject has been discussed in detail in part I of these lectures. → the central subject of the present monography

(*) This concept was first introduced by LANDAU [53] to describe the spectrum of He II. It was subsequently utilized in nuclear physics by BOHR and MOTTELSON [6] to obtain a unified description of the nuclear spectrum.

(**) The restriction to closed nuclei is made to simplify the discussion. The concept of elementary modes of excitation applies equally well to open-shell nuclei (cf. [55]).

6. – Nuclear field theory.

A field theory can be formulated in which the nuclear elementary modes of excitation play the role of the free fields and in which their mutual interweaving takes place through the particle-vibration coupling vertices [6, 54, 55]. This theory provides a graphical perturbative approach to obtain the exact solution of the many-body nuclear-structure problem in the product basis $\psi_\tau(\zeta)\,\psi_\eta(\Delta)\ldots\psi_\gamma(\Gamma)$.

Note that the nuclear bosonic fields are built out by utilizing those degrees of freedom (particles and holes) which already exhaust all the nuclear degrees of freedom. It is thus an essential feature of the product basis to be overcomplete and to violate the Pauli principle. On the other hand, this basis is directly related to observables of the system. The different experiments project out only one or two of its components.

In what follows we state and apply the nuclear-field-theory rules, to calculate the interactions between the nuclear free fields and the reaction processes between the resulting physical states. This is done for a system with one particle outside closed shells and which displays collective vibrations, in the framework of a two-level model.

6.1. *Schematic model.* –

The model considered consists of two single-particle levels, each with degeneracy Ω and with a schematic monopole particle-hole interaction coupling the particles in the two levels.

The total Hamiltonian is equal to

$$H = H_{\text{sp}} + H_{\text{TB}}\,, \tag{6.1}$$

where

$$H_{\text{sp}} = \frac{\varepsilon}{2} N_0\,, \qquad N_0 = \sum_{\sigma=\pm 1, m} \sigma a^\dagger_{m,\sigma} a_{m,\sigma}\,, \tag{6.2}$$

and

$$H_{\text{TB}} = -\frac{V}{2}(A^\dagger A + A A^\dagger)\,, \qquad A^\dagger = \sum_m a^\dagger_{m,1} a_{m,-1}\,. \tag{6.3}$$

The index σ labels the two levels, while m labels the degenerate states within each level. The strength of the monopole coupling is denoted by V and the energy difference between the two levels is ε.

The matrix element of (6.3) is given by

$$\langle m, 1; m', -1 | H_{\text{TB}} | m'', 1; m''', -1\rangle = -V\delta(m, m')\delta(m'', m''')\,. \tag{6.3a}$$

6.2. *Field-theoretical solutions.* – The free nuclear and fermion fields are the elementary modes of excitation comprising surface vibrations and single particles, respectively. The boson fields are defined through the random-phase approximation, in terms of particle-hole excitations. The basis utilized to describe the nuclear systems is a product of the different free fields. This basis is, as a rule, over-complete, nonorthogonal and violates the Pauli principle.

The closed-shell system of the schematic model under consideration corresponds to the lowest ($\sigma = -1$) level filled with Ω particles, while the upper ($\sigma = 1$) level remains empty. The basis particle and hole states are obtained by adding or removing a single particle from this closed-shell configuration. The corresponding wave functions and energies, which should include the Hartree-Fock corrections generated by the residual interaction (2), are

$$\begin{cases} |m,1\rangle = a^{\dagger}_{m,1}|0\rangle\,, & E(m,1) = \frac{1}{2}(\varepsilon + V)\,, \\ |m,-1\rangle = a_{m,-1}|0\rangle\,, & E(m,-1) = \frac{1}{2}(\varepsilon + V)\,. \end{cases} \tag{6.4}$$

Thus the unperturbed energy for producing a particle-hole excitation with respect to the ground state is

$$\varepsilon' = E(m,1) + E(m,-1) = \varepsilon + V\,. \tag{6.5}$$

The contribution V in (6.5) is the Hartree-Fock contribution to the particle-hole excitation.

If we define the creation operator of the normal modes as

$$\beta^{\dagger}_{\nu} = \sum_{m} \lambda^{\nu}_{m} a^{\dagger}_{m,1} a_{m,-1}\,, \tag{6.6}$$

the linearization equation

$$[H, \beta^{\dagger}_{\nu}] = \omega_{\nu}\beta^{\dagger}_{\nu} \tag{6.7}$$

yields

$$\begin{cases} \omega_1 = \varepsilon' - V\Omega\,, \\ \omega_{\nu} = \varepsilon' \end{cases} \quad (\nu = 2, 3, \ldots, \Omega)\,. \tag{6.8}$$

Utilizing (6.7) and the normalization condition

$$[\beta_{\nu}, \beta^{\dagger}_{\nu'}] = \delta(\nu, \nu')\,, \tag{6.9}$$

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2) The Hartree-Fock energy associated with the Hamiltonian (6.1) can be obtained from the linearization relation $[H, a^{\dagger}_{\sigma,m}] = E(m,\sigma)a^{\dagger}_{\sigma,m}$ acting on the Hartree-Fock vacuum, which in this case coincides with the single-particle vacuum defined by $a^{\dagger}_{m,-1}|0\rangle = a_{m,1}|0\rangle = 0$.

we obtain for the amplitudes λ^1_m associated with the lowest mode

$$\lambda^1_m = \frac{1}{\sqrt{\Omega}}. \qquad (1,A,24)$$

One can also write this amplitude as the ratio between a coupling matrix element and an energy denominator, *i.e.*

$$\lambda^1_m = \frac{A_1}{\omega_1 - \varepsilon'}. \qquad (1,A,25)$$

From (6.8), (6.10) and (6.11) we obtain

$$A_1 = -V\sqrt{\Omega}, \qquad (1,A,26)$$

which is the strength with which a particle-hole excitation $(m,1;m,-1)$ couples to the collective phonon (see fig. 18). This can also be seen by calculating the matrix element of the interaction Hamiltonian (6.3) between the normal modes and the single particle-hole state

$$A_\nu = \langle n_\nu = 1|H_{\rm TB}|m,1;m',-1\rangle = -V\sqrt{\Omega}\,\delta(m,m')\delta(\nu,1). \qquad (1,A,26)$$

Note that the particle-vibration coupling strengths associated with the other normal modes lying at an energy ε' are equal to zero.

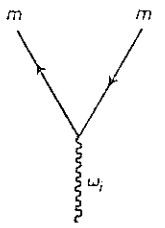


Fig. 18. – Graphical representation of the amplitude of the collective phonon (wavy line) on a given particle-hole excitation $((m,1),(m,-1))$. This amplitude can be written in terms of the interaction vertex denoted by A_i, and the energy denominator $\omega_i - \varepsilon'$. The particles and holes are depicted by arrowed lines.

As shown in ref. [54-55], the exact solution of (6.1) is reproduced by utilizing as the basic degrees of freedom both the vibrations (cf. (6.8)) and the particles (cf. (6.1)) coupled through the interactions (6.3*a*) and (6.13). A significant part of the original interaction has already been included in generating the collective mode (6.8). This implies that the rules for evaluating the effect of the couplings (6.3*a*) and (6.13) between fermions and bosons involve a number

of restrictions as compared with the usual rules of perturbation theory that are to be utilized in evaluating the effect of the original interaction (6.3) acting in a fermion space. They read as follows:

I) In initial and final states, proper diagrams involve collective modes and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. This restriction permits an initial state comprising the configuration $(n_i = 1; m, 1)$, but excludes $(m, 1; m, -1; m', 1)$.

II) The couplings (6.3*a*) and (6.13) are allowed to act in all orders to generate the different diagrams of perturbation theory; the restriction I) does not apply to internal lines of these diagrams.

III) The internal lines of diagrams are, however, restricted by the exclusion of diagrams in which a particle-hole pair is created and subsequently annihilated without having participated in subsequent interactions.

IV) The energies of the uncoupled particle and phonon fields are to be calculated by utilizing the Hartree-Fock approximation (cf. eq. (6.4)) and the RPA (cf. eq. (6.8)), respectively. The contributions of all allowed diagrams are evaluated by the usual rules of perturbation theory.

We note that the external fields acting on the system are allowed to create any state which may generate the different diagrams of perturbation theory. The corresponding matrix elements should be weighted with the amplitude of the component through which the final state is excited.

The above rules are also valid for those situations which cannot be treated in perturbation theory and where a full diagonalization is called for. Thus, *e.g.*, when the system displays a spurious state (cf. sect. 4).

In what follows we discuss the energy of the 2p-1h-like excitations. We distinguish between two types of states, namely

$$|n_i = 1; m, 1\rangle, \quad \begin{cases} \omega_1 = \varepsilon' - V\Omega, & \Lambda_1 = -\sqrt{\Omega}\, V \quad (i = 1;\ m = 1, 2, \ldots, \Omega), \\ \omega_i = \varepsilon', & \Lambda_i = 0 \quad (i = 2, \ldots, \Omega;\ m = 1, 2, \ldots, \Omega), \end{cases}$$

and

$$|m, 1; m, -1; m', 1\rangle, \quad \varepsilon' \quad (m, m' = 1, 2, \ldots, \Omega). \tag{6.15}$$

Since the states (6.15) are restricted to be intermediate states of the perturbation expansion, the configuration $(m, 1; m, -1; m, 1)$ is allowed.

The physical states are to be written as

$$|qm\rangle = \sum_i \xi_{iqm} |n_i = 1; m, 1\rangle , \qquad (6.16)$$

as (6.15) cannot be basis states according to rule I), but only intermediate states. The quantities ξ_{iqm} are the amplitudes of the physical state in the different components of the product basis of elementary excitations.

The model space contains Ω^2 states, while the correct number is $\Omega - 1$. Thus the basis $|n_i = 1; m, 1\rangle$ contains Ω spurious states. Its origin can be traced back to the violation of the Pauli principle (cf. also sect. 5).

To obtain the energy of $|qm\rangle$ we have to allow the states $|n_1 = 1; m, 1\rangle$ to interact through the vertices (6.3a) and (6.13) and generate all the different perturbation theory diagrams (cf. rule II)) except those containing bubbles (cf. rule III)).

The different graphical contributions calculated in the framework of the Brillouin-Wigner perturbation theory are displayed in fig. 1. There is only

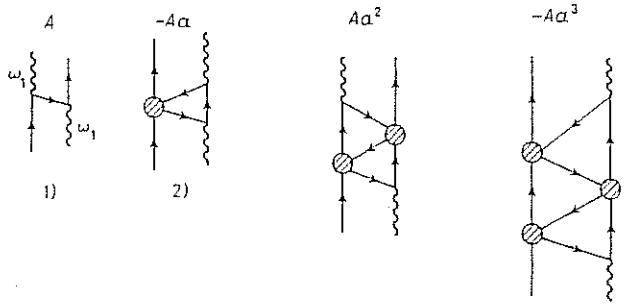


Fig. 1. – Contributions to the interaction of a fermion and a collective boson ω_i to order $1/\Omega^4$. The secular equation $E - E^{(0)} = A \sum_n a^n (-1)^n$ is given in terms of the quantities $A = 4\Omega V^2/(3\varepsilon' - 2E)$ and $a = 2V/(3\varepsilon' - 2E)$.

one (diagonal) matrix element given by a single summation, which can be carried to all orders in the interaction vertices, and can be written as

$$\Sigma_{ii'} = A \sum_n (-1)^n a^n \delta(i, i') = \qquad (6.17)$$

$$= \frac{A}{1 + a} \delta(i, i') \delta(n, 1) = - K(E) (\sqrt{\Omega}\, V)^2 \delta(i, i') \delta(i, 1) ,$$

where a and A are defined in the caption to the figure and

$$K(E) = (\tfrac{3}{2}\varepsilon' - E + V)^{-1} \qquad (6.18)$$

is the effective coupling strength. The associated secular equation

(6.19) $$|(\omega_i - E)\delta(i, i') + \Lambda_{ii'}| = 0$$ (1.A.32)

is equivalent to the dispersion relation

(6.20) $$\frac{1}{K(E)} = \sum_i \frac{(\sqrt{\Omega}\, V)^2}{\omega_i - E}\,\delta(i, 1)\,.$$ (1.A.33)

Thus the energies of the system are determined by the equation

(6.21) $$E = \omega_1 + \frac{\Omega V^2}{\frac{3}{2}\varepsilon' - E + V}\,.$$ (1.A.34)

It admits the two solutions

(6.22) $$E_{qm} = \begin{cases} \frac{3}{2}\varepsilon'\,, \\ \frac{1}{2}\varepsilon' + \omega_1 + V = \frac{3}{2}\varepsilon' - \Omega V + V\,, \end{cases}$$ (1.A.35)

and agree with the exact value [4] (1.A.29)

Because $\Lambda_i = 0$ for $i \neq 1$, there is no summation in (6.16) and

$$|qm\rangle = N_{qm}|n_1 = 1;\, m, 1\rangle\,,$$ (1.A.36)

where

(6.23) $$1 = N_{qm}^2\left(1 - \frac{\partial \Lambda_{11}}{\partial E}\right) = N_{qm}^2\left(1 - \frac{\Omega V^2}{(\frac{3}{2}\varepsilon' - E + V)^2}\right).$$ (1.A.37)

For $E_{qm} = \frac{1}{2}\varepsilon' + \omega_1 + V$ we obtain

(6.24) $$N_{qm}^2 = \frac{\Omega}{\Omega - 1}\,,$$ (1.A.38)

while for $E_{qm} = \frac{3}{2}\varepsilon'$ the state is nonnormalizable as the quantity in parentheses in (6.23) is either negative ($\Omega > 1$) or zero ($\Omega = 1$).

The state defined by

(6.25a) $$|q, m\rangle = \sqrt{\frac{\Omega}{\Omega - 1}}\,|n_1 = 1;\, m, 1\rangle\,,$$ (1.A.39)

and

(6.25b) $$E_{qm} = \frac{1}{2}\varepsilon' + \omega_1 + V = \frac{3}{2}\varepsilon' - V(\Omega - 1)\,,$$ (1.A.40)

[4] The exact solutions can be easily obtained by noting that the operators $A^\dagger$, A and $\frac{1}{2}N_0$ are generators of the SU_2 group.

exhausts the inelastic sum rule in agreement with the exact results. Note that (6.25*a*) is specifically excited in inelastic processes, as can be seen by direct inspection.

The external inelastic field can act in two ways, exciting either a particle-hole pair or a phonon, with amplitudes

$$\langle m, 1; m', -1|A_1^\dagger|0\rangle = \delta(m, m') \tag{6.26}$$

and

$$\langle n_i = 1|A_1^\dagger|0\rangle = \sqrt{\Omega}\,\delta(i, 1)\,, \tag{6.27}$$

respectively. The different graphical contributions to the inelastic-scattering process are displayed in fig. 26, and can again be summed to all orders in the interaction vertices giving

$$\langle n_1 = 1; m, 1|A_1^\dagger|m, 1\rangle = \sqrt{\Omega} + \frac{A_1}{\frac{3}{2}\varepsilon' - E_{qm} + V}\,. \tag{6.28}$$

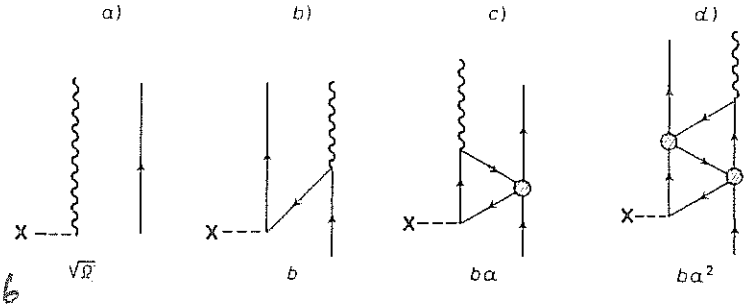


Fig. 26. – Graphical representation of the different terms contributing to the matrix element of the transfer operator $\sqrt{\Omega}A^\dagger$ up to order $1/\Omega^3$. Note that the different contributions *b*), *c*), etc. have a one-to-one correspondence with the different contributions to E (cf. fig. 10). $a = -2V/(3\varepsilon' - 2E)$, $b = 2A_1/(3\varepsilon' - 2E)$.

For $E_{qm} = \frac{3}{2}\varepsilon'$ this quantity is equal to zero. Thus, the corresponding states do not carry any inelastic strength, a feature which is closely related to the fact that they cannot be normalized and that they do not display any correlation energy (*).

(*) Note that, even if $N(E_{qm} = \varepsilon_m) \to \infty$, the matrix elements associated with the different transitions tend to zero more rapidly and the final result converges and is equal to zero as expected.

On the other hand, the matrix element associated with (6.25a) is

$$\langle qm|A^\dagger|m,1\rangle = \sqrt{\frac{\Omega}{\Omega-1}}\,\frac{\Omega-1}{\sqrt{\Omega}} = \sqrt{\Omega-1}\,, \tag{6.29}$$

which agrees with the exact answer.

The results (6.25b) and (6.29) can be traced down to Pauli-principle corrections. In fact, the state $|n_i=1;\ m,1\rangle$ has a nonvanishing matrix element, implying a single particle-vibration coupling vertex, with the state $|m,1;\ m,-1;\ m,1\rangle$. This component, which is spurious, is removed by the different graphs displayed in fig. 19 and 20. The presence of the odd particle $(m,1)$ blocks the particle-hole excitation $(m,1;\ m,-1)$ which was present in the uncoupled system. Thus the system increases its energy by a quantity V. The reduction of the inelastic amplitude from $\sqrt{\Omega}$ to $\sqrt{\Omega-1}$ also indicates that there is one less particle-hole excitation responding to the external probe.

– Spurious states.

While the model space product of elementary modes of excitation discussed in the last section contains Ω^2 states, only $\Omega(\Omega-1)$ are physically possible, the number of spurious states being Ω. On the other hand, the agreement between the exact and the nuclear-field-theoretical results shows that the effects of those spurious states are eliminated from all the matrix elements associated with physical observables.

In what follows we show that, in fact, the spurious states are isolated in an explicit way in the nuclear field theory [5]. Their energy coincides with the initial unperturbed energy, while all physical operators have zero off-diagonal matrix elements between any physical state and a spurious state, in particular the unit operator, which measures the overlap of the two types of states.[6]

For this purpose we use again a schematic model consisting in a number, Ω, of single-particle levels in which particles interact by means of a « monopole » force

$$H = H_{\rm sp} + H_{\rm int}\,, \tag{7.1}$$

where

$$H_{\rm sp} = \tfrac{1}{2}\sum_{m=1}^{\Omega}\varepsilon_m(a^\dagger_{m,1}a_{m,1} + a^\dagger_{m,-1}a_{m,-1}) \tag{7.2}$$

and

$$H_{\rm int} = -\,V A^\dagger A \tag{7.3}$$

6) @-@ p. 117a handwritten next

footnote p. 117

6) It is of notice the essential difference concerning a the origin of a ~~state~~ with zero energy in the present case (overcounting of the degrees of freedom thus leading to a bona fide spurious ~~state~~ whose effect is to be eliminated), and the situation in which one is confronted with a spontaneous breaking of the symmetry. Also in this case one is confronted with a zero frequency mode as a result of the ~~fact that~~ the restoring force vanishes, although displaying a finite inertia. Within the realm of intrinsic excitations of the system, this state gives no contribution. On the other hand, its ZPF divergence in exactly the right way so as to restore symmetry, leading to a collective motion of the system as a whole. Examples are provided by spontaneous breaking of gauge invariance (pairing rotations, cf. Bes and Broglia (2006), cf. also Brink and Broglia (2005) App. I and Ch. 4, Broglia et al (2000)), translational invariance (GDR and galilean motion, cf. eg Bohr and Mottelson (1975) p. 444), etc.

R. A. Broglia, J. Terasaki and N. Giovanardi, The Anderson-Goldstone-Nambu mode in finite and in infinite systems, Phys. Rep. 355, 1 (2000)

with

$$A^{\dagger} = \sum_{m=1}^{\Omega} a^{\dagger}_{m,1} a_{m,-1} . \qquad (7.4)$$

The energy of the i-th phonon is determined by the RPA dispersion relation (cf. rule IV))

$$\sum_{m=1}^{\Omega} \frac{1}{\varepsilon_m - \omega_i} = \frac{1}{V} \qquad (i = 1, 2, \ldots, \Omega) . \qquad (7.5)$$

The eigenfunction corresponding to the different modes is

$$|n_i = 1\rangle = \sum_m \frac{A_i}{\varepsilon_m - \omega_i} a^{\dagger}_{m,1} a_{m,-1} |0\rangle . \qquad (7.6)$$

The particle-vibration coupling constant is given by

$$A_i = -\langle n_i = 1 | H_{\mathrm{int}} | m, 1; m', -1 \rangle = \left[\sum_m \frac{1}{(\varepsilon_m - \omega_i)^2} \right]^{-\frac{1}{2}} \delta(n, n') , \qquad (7.7)$$

where $|n_i = 1\rangle$ denotes a state containing one phonon, while $|m, 1; m, -1\rangle$ is the eigenstate associated with particle-hole excitation.

The other interaction to be included (rule II)) is the four-point vertex which has the value

$$\langle m, 1; m', -1 | H_{\mathrm{int}} | m'', 1; m''', -1 \rangle = -V \delta(m, m') \delta(m'', m''') . \qquad (7.8)$$

The single-particle energies to be used in calculating the different graphs are $\frac{1}{2}\varepsilon_m$, as the Hartree-Fock contribution (cf. rule IV)) of H_{int} is zero.

Similarly to H_{int}, the « inelastic operator » has two different matrix elements, namely

$$\langle n_i = 1 | a^{\dagger}_{m',1} a_{m',-1} | 0 \rangle = \frac{A_i}{\varepsilon_{m'} - \omega_i} \qquad (7.9)$$

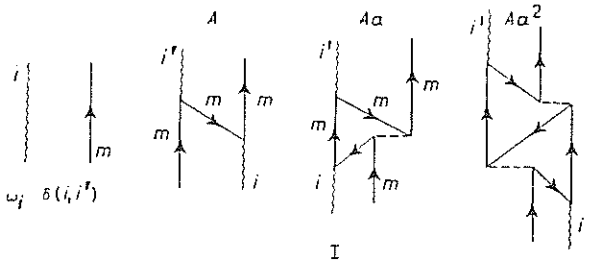
and

$$\langle m', 1; m'', -1 | a^{\dagger}_{m,1} a_{m,-1} | 0 \rangle = \delta(m, m') \delta(m', m'') . \qquad (7.10)$$

In what follows we discuss again the system comprising an odd particle, in the orbit $(m, 1)$, in addition to a single phonon excitation of the vacuum.

According to rule I) initial and final states may involve both collective mode and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. The exclusion of the states $|m, 1; m', 1; m', -1\rangle$ eliminates most of the double counting of two-particle, one-hole states. The Ω « proper » states of the form $|n_i = 1; m, 1\rangle$ are allowed. However,

there are only $\Omega - 1$ (two-particle, one-hole) states in which the odd particle is in the state $(m, 1)$. Therefore, a spurious state remains in the spectrum of the elementary modes of excitation.



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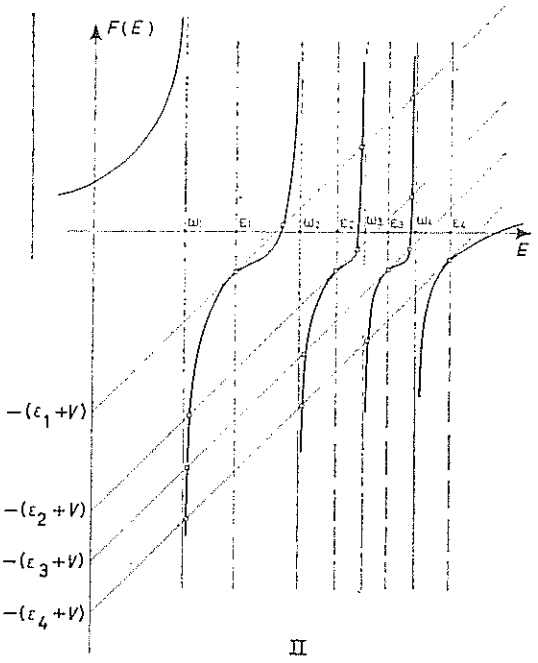


II

Fig. 21. – I) Lower-order contributions to the energy matrix element between the basis states $|n_i = 1; m, 1\rangle$. The dashed line stands for the model bare interaction (cf. eq. (2.8)). The quantity $X_{ii'}(E) = A \sum_n a^n = -A_i A_{i'}/(E - \varepsilon_m - V)$, where $A = -A_i A_{i'}/(E - \varepsilon_m)$ and $a = V/(E - \varepsilon_m)$, is the matrix element iterated to all orders in $1/\Omega$. The secular equation of the problem is $|(\omega_i - E)\,\delta(i, i') + X_{ii'}| = 0$, and is equivalent to the dispersion relation (7.15). II) Graphical solution of the dispersion relation (7.15), for the case $\Omega = 4$. The function $F(E) = \sum_i A_i^2/(\omega_i - E)$ is displayed as a continuous thick line, while the parallel lines $E - \varepsilon_m - V$ have been drawn as thin continuous lines intersecting the ordinate axis at $-(\varepsilon_m + V)$. The intersections between the two functions give the eigenvalues of the secular equation. For each value of ε_m there are $\Omega + 1$ roots, the root at $E = \varepsilon_m$ being double.

If we displace the zero-point energy of the odd system to $\frac{1}{2}\varepsilon_m$, the unperturbed energy of the basis state $|n_i = 1; m, 1\rangle$ is ω_i.

The lower-order corrections to this energy which do not contain bubbles are drawn in fig. 21. Iterating these processes to infinite order we obtain the secular equation

(7.11) $$|(\omega_i - E)\delta(i, i') + X_{ii'}(E)| = 0 \,, \qquad (1, A, 55)$$

where

(7.12) $$X_{ii'} = -\frac{A_i A_{i'}}{E - \varepsilon_m - V} \,. \qquad (1, A, 56)$$

The different contributions calculated in the framework of the Brillouin-Wigner perturbation theory are energy dependent, and take into account renormalization effects of the states not explicitly included in the calculations.

If we utilize the following correspondence

(7.13) $$\sqrt{j + \tfrac{1}{2}} \leftrightarrow A_i \qquad (1, A, 57)$$

and

(7.14) $$G \leftrightarrow (E - \varepsilon_m - V)^{-1} \,, \qquad (1, A, 58)$$

the result (1.3) can be directly utilized. Thus

(7.15) $$E - \varepsilon_m - V = \sum_{i=1}^{\Omega} \frac{A_i^2}{\omega_i - E} = F(E) \qquad (1, A, 59)$$

is the dispersion relation fixing the energies E_{qm} of the physical states. There is one equation for each single-particle level because the monopole force cannot change the m-state of the odd particle. The relation (7.15) can be solved graphically as shown in fig. 21 II. The energy $E = \varepsilon_m$ is always a root of (7.15) (1, A, 59) in fact a double root since

(7.16) $$\left[\frac{\mathrm{d}F(E)}{\mathrm{d}E}\right]_{E=\varepsilon_m} = \sum_i \frac{A_i^2}{(\omega_i - \varepsilon_m)^2} = 1 \qquad (1, A, 60)$$

and the line $E - \varepsilon_m - V$ is at 45°. The remaining intersections of this line and the function $F(E)$ give rise to $\Omega - 1$ additional roots denoted by (qm), whose energy E_{qm} agrees with the physical eigenvalues obtained from the exact solution of the model.

The eigenvectors associated with the physical states (qm) are

(7.17) $$|qm\rangle_F = \sum_i \xi_{iqm} |i; m, 1\rangle \,, \qquad (1, A, 61)$$

$1/G = \Sigma_j (j + 1/2)/(2\varepsilon_j - E) = \sum_j \sum_{m>0} 1/(2\varepsilon_j - E) = F(E)$

nella linea

no como ecuación separada

where

$$\xi_{iqm} = -N_{qm}\frac{A_i}{\omega_i - E_{qm}} = \langle i;\, m, 1|qm\rangle_F\,. \tag{7.18}$$

The normalization condition which determines N_{qm} is

$$\begin{aligned} {}_F\langle qm|qm\rangle_F = 1 &= \sum_{i,i'}\left(\delta(i,i') - \frac{\partial X_{ii'}}{\partial E}\right)\xi^*_{iqm}\xi_{i'qm} = \\ &= N^2_{qm}\left[\sum_i \frac{A_i^2}{(\omega_i - E_{qm})^2} - \frac{1}{(E_{qm}-\varepsilon_m - V)^2}\sum_{i,i'}\frac{A_i^2 A_{i'}^2}{(\omega_i - E_{qm})(\omega_{i'} - E_{qm})}\right] = \\ &= N^2_{qm}\left[\sum_i \frac{A_i^2}{(\omega_i - E_{qm})^2} - 1\right], \end{aligned} \tag{7.19}$$

where the dispersion relation (7.15) has been utilized, and where $X_{ii'}$ is the matrix element appearing in (7.11). For $E_{qm} = \varepsilon_m$ the factor multiplying N^2_{qm} is zero (cf. eq. (7.16)). Thus, there are only $\Omega - 1$ states which can be normalized when solving the Hamiltonian (7.1) in the framework of the nuclear field theory. The full spuriosity of the elementary-mode product basis is concentrated in a single state (*).

The subscript F has been utilized in (7.17) to indicate that we are dealing with the nuclear-field solution of the Hamiltonian (7.1). Note that these eigenvectors are expressed in terms of only the allowed initial or final states (cf. rule I))

$$|i;\, m, 1\rangle \equiv a^\dagger_{m,1}|i\rangle\,, \tag{7.20}$$

which are assumed to form an orthonormal basis, in particular in deriving the relation (7.19). This is equivalent to the basic assumption of the nuclear field theory of the independence of the different modes of excitation, *i.e.*, in the present case,

$$[\Gamma_i, a^\dagger_{m,1}] = 0\,. \tag{7.21}$$

Rules I)-IV) discussed in the last section give the proper mathematical framework to this ansatz, which has played a basic role in developing a unified theory of nuclear structure (cf. [6]).

The above discussion can be illuminated by utilizing a conventional treatment of the residual interaction (cf. also [62]). Expanding the states $|n_i = 1;$

(*) Note that the mathematical relation $N^2 f(E) = 1$, N^2 being the norm of the state with energy E, implies that such state is spurious if $f(E) = 0$ or $f(E) < 0$ (cf. eq. (6.25) and subsequent discussion).

$m, 1$ in terms of particle and hole state, we can write, with the help of (7.6),

$$a^{\dagger}_{m,1}|n_i = 1\rangle = a^{\dagger}_{m,1} \sum_{m' \neq m} \frac{A_i}{\varepsilon_{m'} - \omega_i} a^{\dagger}_{m',1} a_{m',-1}|0\rangle \,. \tag{7.22}$$

The overlap between the states $|n_i = 1;\, m, 1\rangle$ is thus given by

$$Z(i, i') = \langle i'|a_{m,1} a^{\dagger}_{m,1}|i\rangle = \tag{7.23}$$

$$= \sum_{m' \neq m} \frac{A_i A_{i'}}{(\varepsilon_{m'} - \omega_i)(\varepsilon_{m'} - \omega_{i'})} = \delta(i, i') - \frac{A_i A_{i'}}{(\varepsilon_m - \omega_i)(\varepsilon_m - \omega_{i'})} \,,$$

where the orthogonality relation

$$\sum_{m'} \frac{A_i A_{i'}}{(\varepsilon_{m'} - \omega_i)(\varepsilon_{m'} - \omega_{i'})} = \delta(i, i') \tag{7.24}$$

of the RPA solutions in the even system has been utilized. Because of the nonorthogonality of the basis, the eigenvalues of the system are determined by the relation

$$|Z(E)(H - E)| = 0 \,. \tag{7.25}$$

This is fulfilled for

$$|H - E| = 0 \,,$$

which yields the $\Omega - 1$ physical roots, as well as for

$$|Z(E)| = 0 \,. \tag{7.26}$$

This solution corresponds to the spurious root $E_{qm} = \varepsilon_m$. In fact,

$$\lim_{\delta \to 0} \sum_i \xi_{iqm}(E_{qm} = \varepsilon_m + \delta) Z_{ii'} = \lim_{\delta \to 0} N_{qm}(E_{qm} = \varepsilon_m + \delta) \cdot \tag{7.27}$$

$$\cdot \sum_i \frac{A_i}{\omega_i - (\varepsilon_m + \delta)} \sum_{m \neq m'} \frac{A_i A_{i'}}{(\varepsilon_{m'} - \omega_i)(\varepsilon_{m'} - \omega_{i'})} = 0 \,,$$

since

$$\sum_i \frac{A_i^2}{(\omega_i - \varepsilon_m)(\omega_i - \varepsilon_{m'})} = \delta(m, m') \,. \tag{7.28}$$

Note that this solution in terms of the overlap Z gives the exact answer in the present case, because of the simplicity of the model. In a general case which includes ground-state correlations this may not be true any longer.

We now calculate the one-particle stripping process leading to the odd system. This calculation illustrates the explicit concentration of the whole spuriosity into a single state which has zero correlation energy (*) and zero amplitude for the different physical processes exciting the $\Omega - 1$ physical states.

One has first to calculate the amplitude for the transition to a basis component $(n_i = 1; m, 1)$ including only those graphs in which all intermediate states are excluded from appearing as initial or final states. This exclusion reflects the fact that the diagonalization procedure has included all interaction effects that link these allowed states. The final amplitude for the transition to the state (qm) is obtained by summing the amplitudes to $(n_i = 1; m, 1)$ each weighted by the amplitude ξ_{iqm} given by eq. (7.18).

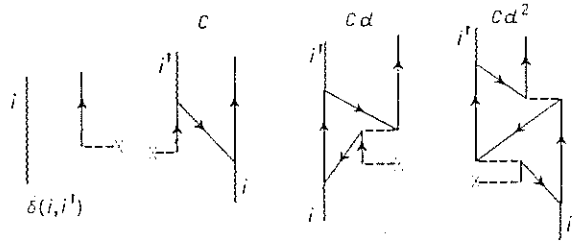


Fig. 22. – Lower-order contributions to the one-particle transfer reaction induced by $a^{\dagger}_{m,1}$. The result of iterating the different contributions to all orders in $1/\Omega$ is equal to $T'_{qm}(ii') = C \sum_n d^n = -\Lambda_i \Lambda_{i'}/(\omega_i - \varepsilon_m)(E_{qm} - \varepsilon_m - V)$, $C = -\Lambda_i \Lambda_{i'}/(\omega_i - \varepsilon_m)(E_{qm} - \varepsilon_m)$, $d = V/(E_{qm} - \varepsilon_m)$.

The lower-order contributions to the one-particle transfer amplitude between the state $|n_i = 1\rangle$ and the state $|qm\rangle$ are displayed in fig. 22. They can be summed up to all orders of $1/\Omega$, the result being equal to

$$
\begin{aligned}
(7.29) \qquad & \langle qm | a^{\dagger}_{m,1} | n_i = 1 \rangle = \\
& = \sum_{i'} \xi_{i'qm} \left\{ \delta(i, i') - \frac{\Lambda_i \Lambda_{i'}}{(\omega_i - \varepsilon_m)(E_{qm} - \varepsilon_m)} \left[\frac{1}{1 - V/(E_{qm} - \varepsilon_m)} \right] \right\} = \\
& = \sum_{i'} \xi_{i'qm} \{ \delta(i, i') - T_{qm}(i, i') \} = \\
& = - N_{qm} \left[\frac{\Lambda_i}{\omega_i - E_{qm}} - \frac{\Lambda_i}{(\omega_i - \varepsilon_m)(E_{qm} - \varepsilon_m - V)} \sum_{i'} \frac{\Lambda^2_{i'}}{\omega_{i'} - E_{qm}} \right] = \\
& = \frac{N_{qm}(E_{qm} - \varepsilon_m)\Lambda_i}{(E_{qm} - \omega_i)(\omega_i - \varepsilon_m)} .
\end{aligned}
$$

(*) This is because the spurious state has zero phase space to correlate.

This quantity is zero for the spurious roots (Ω) (*i.e.* $E_{qm} = \varepsilon_m$) and agrees with the exact result for the $\Omega - 1$ remaining physical roots.

Utilizing the relations

$$\frac{1}{V} = \sum_m \frac{1}{\varepsilon_m - \omega_i} \tag{7.30}$$

and

$$\frac{1}{V} = \sum_{m \neq m'} \frac{1}{\varepsilon_{m'} - E_{qm}}, \tag{7.31}$$

we obtain

$$\sum_{m \neq m'} \frac{1}{(\varepsilon_{m'} - E_{qm})(\varepsilon_{m'} - \omega_i)} = \frac{1}{(E_{qm} - \omega_i)(\varepsilon_m - \omega_i)}. \tag{7.32}$$

Utilizing this relation we can derive the one-particle transfer sum rule. Note that (7.30) is the dispersion relation for the free phonon field. The second relation is, however, alien to the field theory results. Nevertheless, one can show that the solutions E_{qm} of (7.31) and of the nuclear-field-theory dispersion relation (7.15) are identical, except for the root $E_{qm} = \varepsilon_m$. One can, therefore, utilize (7.31) as a mathematical relation without further justifications in the present context. One obtains

$$\sum_{qm} |\langle qm|a^\dagger_{m,1}|n_i = 1\rangle|^2 = \sum_{qm} A^2_{qm} A^2_i \sum_{m \neq m'} \frac{1}{(\varepsilon_{m'} - E_{qm})(\varepsilon_{m'} - \omega_i)} \cdot \sum_{m'' \neq m} \frac{1}{(\varepsilon_{m''} - E_{qm})(\varepsilon_{m''} - \omega_i)}, \tag{7.33}$$

where

$$A_{qm} = -N_{qm}(E_{qm} - \varepsilon_m) = \left[\sum_{m \neq m'} \frac{1}{(\varepsilon_{m'} - E_{qm})^2}\right]^{-\frac{1}{2}}. \tag{7.34}$$

Thus

$$\sum_{qm} |\langle qm|a^\dagger_{m,1}|n_i = 1\rangle|^2 = A^2_i \sum_{m' \neq m} \frac{1}{(\varepsilon_{m'} - \omega_i)^2} = 1 - \frac{A^2_i}{(\varepsilon_m - \omega_i)^2}, \tag{7.35}$$

where use has been made of the orthogonality relation

$$\sum_{qm} \frac{A^2_{qm}}{(\varepsilon_{m'} - E_{qm})(\varepsilon_{m''} - E_{qm})} = \delta(m', m'') \qquad (m', m'' \neq m). \tag{7.36}$$

(*) In fact

$$\lim_{\delta \to 0} [(E_{qm} - \varepsilon_m) N_{qm}]_{E_{qm} = \varepsilon_m + \delta} = \lim_{\delta \to 0} \left\{ \sqrt{2}\, \delta^{\frac{1}{2}} \Big/ \left[\sum_i \frac{A_i}{\omega_i - \varepsilon_m}\right]^{\frac{1}{2}} \right\} = 0.$$

The result (7.35) coincides with the exact result. Physically it means that the single-particle orbital $(m, 1)$ is blocked by the amount $A_i^2/(\varepsilon_m - \omega_i)^2$, which is the probability that the phonon $(n_i = 1)$ is in the particle-hole configuration $(m, 1; m, -1)$, *i.e.* with its particle in the orbital $(m, 1)$.

8. – Applications.

In what follows we discuss some aspects of the low-lying spectrum of the nucleus ^{209}Bi in terms of fermions, surface $(\beta^\dagger(0\lambda))$ and pairing $(\beta^\dagger(2\lambda))$ modes. The application to states containing two or more pairing and surface phonons will be discussed by BORTIGNON [57].

The unperturbed states of the closed-shell-plus-one-particle system can be written in terms of the free fields as

$$|n2\lambda, j; IM\rangle = [\beta_n^\dagger(2\lambda) a_j]_{IM}|0\rangle \tag{8.1}$$

and

$$|n0\lambda, j; IM\rangle = [\beta_n^\dagger(0\lambda) a_j^\dagger]_{IM}|0\rangle . \tag{8.2}$$

This constitutes the basis set of states $\{\alpha_i\}$. All other states give rise to the complementary Hilbert space $\{a_i\}$.

The elementary modes of excitation interact through the particle-vibration and four-point vertices displayed in fig. 23 giving rise to the matrix elements

$$M_1(nj, n'j') \equiv \langle [\beta_n^\dagger(0\lambda) a_{j'}^\dagger]_{IM} | h_{\text{eff}}(E) | [\beta_n^\dagger(0\lambda) a_j^\dagger]_{IM} \rangle , \tag{8.3a}$$

$$M_2(nj, n'j') \equiv \langle [\beta_{n'}^\dagger(2\lambda) a_{j'}]_{IM} | h_{\text{eff}}(E) | [\beta_n^\dagger(2\lambda) a_j]_{IM} \rangle \tag{8.3b}$$

and

$$M_3(nj, n'j') \equiv \langle [\beta_{n'}^\dagger(2\lambda) a_{j'}]_{IM} | h_{\text{eff}}(E) | [\beta_n^\dagger(0\lambda) a_j^\dagger]_{IM} \rangle . \tag{8.3c}$$

They are to be calculated by utilizing the graphical techniques of perturbation theory and the rules discussed in sect. 6.

There are two parameters on which to expand upon in carrying out a perturbative calculation. The first one is the strength of the interaction vertices measured in terms of the average distance between single-particle levels. The second is $1/\Omega$, where $\Omega = \sum_j (j + \frac{1}{2})$ is the effective degeneracy of the valence shells. These two parameters are in general connected through involved expressions. In the schematic model discussed in sect. 6, however, their relation is explicit and can be expressed as

$$\varepsilon = \mathcal{O}(1) , \quad A = \mathcal{O}\left(\frac{1}{\sqrt{\Omega}}\right) \quad \text{and} \quad \Gamma = \mathcal{O}\left(\frac{1}{\Omega}\right) . \tag{8.4}$$

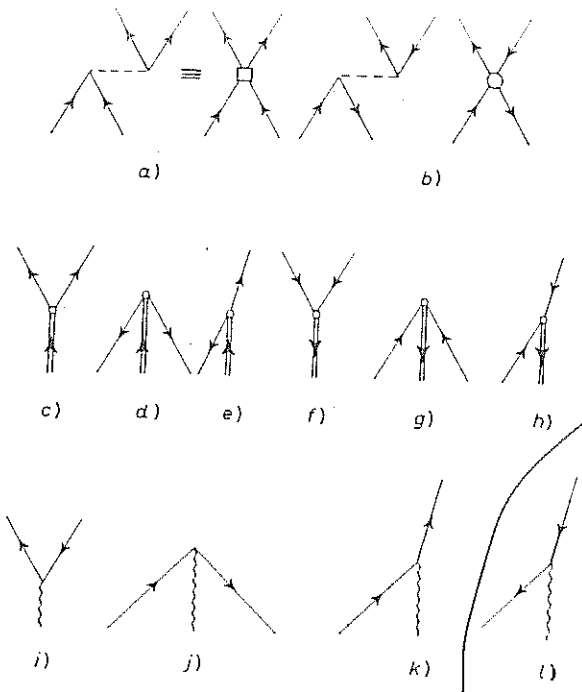


Fig. 23. – Interactions coupling the fermion fields with the pairing and surface vibrations. The different fermion and boson free fields are particle, hole, pairing vibration ($\alpha = 2$), pairing vibration ($\alpha = -2$), surface vibration ($\alpha = 0$). The two possible four-point vertices are given in *a*) and *b*). They correspond to the pairing and particle-hole model bare interactions. In graphs *c*)-*h*) all possible couplings between the fermion fields (arrowed lines) and the pairing vibrational fields (double lines arrowed) are displayed. Graphs *i*)-*l*) are all the coupling vertices between the surface vibrations (wavy line) and the fermion fields. Note that there is no direct coupling between the two boson fields, as the field theory we are dealing with is linear in the different field co-ordinates.

Another feature which determines the family of diagrams to select to a given order of perturbation is the number of internal lines which can be freely summed up. Each of these summations introduces a multiplicative factor Ω.

Because most of the present knowledge on the applicability of the field-theoretical techniques rests upon schematic models, we utilize $1/\Omega$ as the expansion parameter, and assume the relations (8.4) to be valid for more general situations.

The nucleus ^{209}Bi has been investigated by means of high-resolution inelastic scattering [58] and Coulomb excitation [59]. Through these experiments a septuplet of states around 2.6 MeV of excitation was identified, with spins ranging from $\frac{3}{2}^+$ to $\frac{15}{2}^+$.

In zeroth order these states can be interpreted in terms of a proton moving in the $h_{\frac{9}{2}}$ orbital coupled to the lowest octupole vibration of ^{208}Pb. The $\frac{3}{2}^+$ of this multiplet displays also a large parentage based on the proton pair addition and proton hole moving in the $d_{\frac{3}{2}}$ orbital, as revealed by the (t, α) reaction on ^{210}Po [60].

The above results indicate that the (two-particle, one-hole) type of states in ^{209}Bi are amenable to a simple description in term of the basis states

(8.5) $$|2\lambda, j_1^{-1}; IM\rangle \equiv |j^{-1} \otimes \lambda(^{210}\mathrm{Po}); IM\rangle \qquad (\lambda^\pi = 0^+, 2^+, 4^+)$$

and

(8.6) $$|0\lambda, j_2; IM\rangle \equiv |j_2 \otimes \lambda^\pi(^{208}\mathrm{Pb}); IM\rangle \qquad (\lambda^\pi = 3^-) .$$

Only the lowest states of each spin and parity λ^π are included in the basis states, while all the RPA solutions are included in the intermediate states. The quadrupole surface vibration modes were allowed only as intermediate states. The single hole and particle states j_1^{-1} and j_2, respectively, correspond to experimentally known levels around the $Z = 82$ shell closure.

In what follows we discuss the different properties of the states generated by the basis spanned by the eigenvectors $|2\lambda, j_1^{-1}; IM\rangle$ and $|0\lambda, j_2; IM\rangle$. We have divided the discussion in two parts.

In the first part the two $\frac{3}{2}^+$ states built out of the $|d_{\frac{3}{2}}^{-1} \otimes \mathrm{gs}(^{210}\mathrm{Po})\rangle$ and $|h_{\frac{9}{2}} \otimes 3^-(^{208}\mathrm{Pb})\rangle$ configurations are studied in this space. This two-state system provides a rich laboratory to study the interplay of surface and pairing modes.

In the second part the properties of the entire multiplet and of those states strongly excited in either the (t, α) or (d, d′) reactions are studied, in the complete configuration space.

a) The $\frac{3}{2}^+$ states.

The two states

(8.7) $$|1\rangle \equiv |d_{\frac{3}{2}}^{-1} \otimes \mathrm{gs}(^{210}\mathrm{Po}); \tfrac{3}{2}^+\rangle \qquad (2.733\ \mathrm{MeV})$$

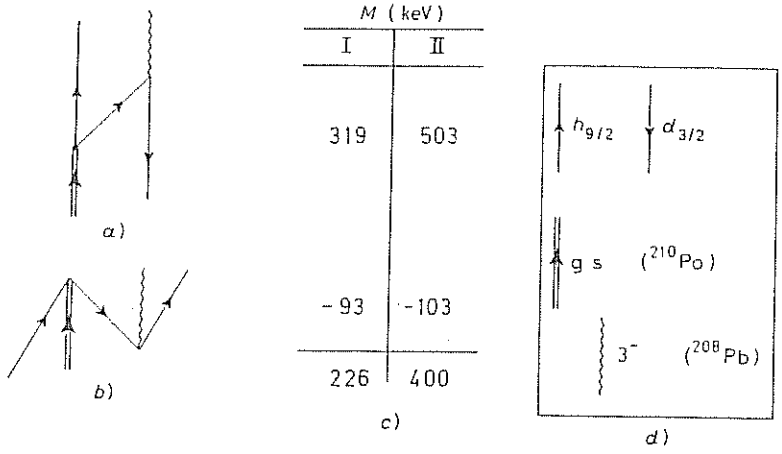
and

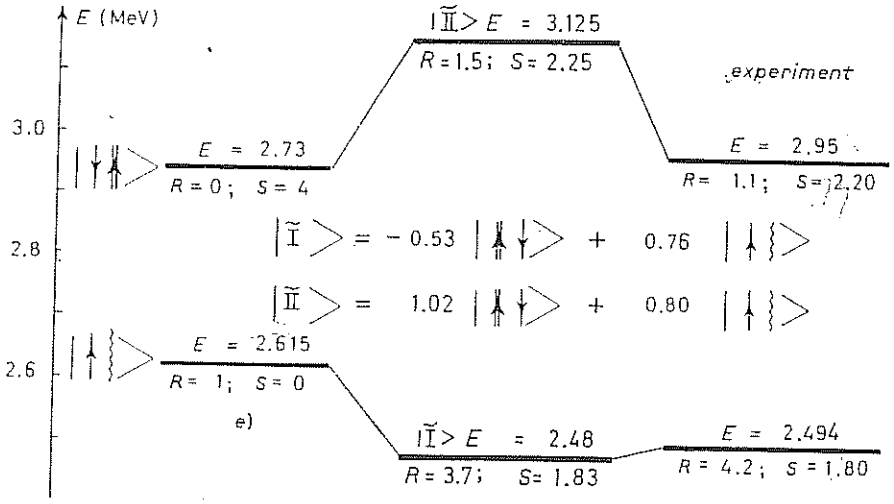
(8.8) $$|2\rangle \equiv |h_{\frac{9}{2}} \otimes 3^-(^{208}\mathrm{Pb}); \tfrac{3}{2}\rangle \qquad (2.615\ \mathrm{MeV})$$

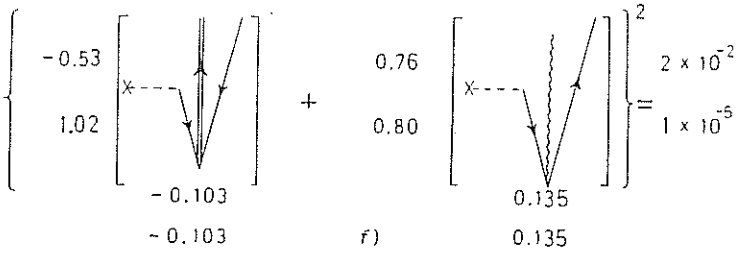
are 118 keV apart. They mix strongly through the couplings depicted by the graphs *a*) and *b*) of fig. 24.

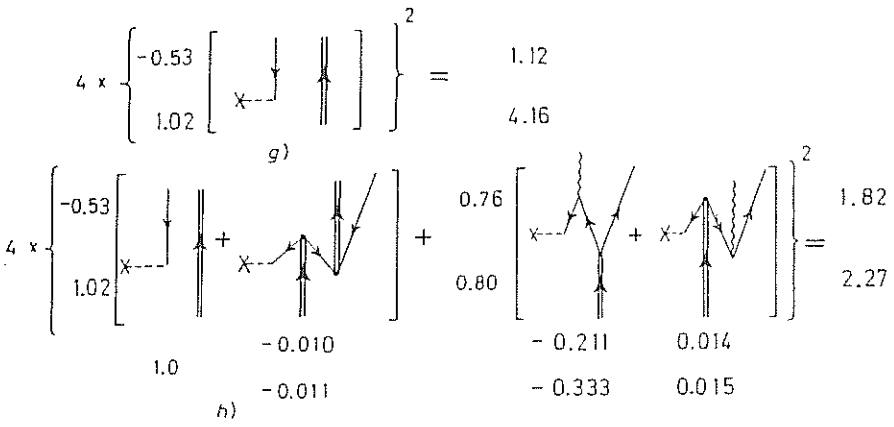
Because of the energy dependence of h_{eff} there is a different matrix element for each final state. The diagonalization of the matrices was carried out self-consistently, *i.e.* the energy denominators of the different graphs are to be calculated by utilizing the exact energies (for more details, cf. ref. [61]).

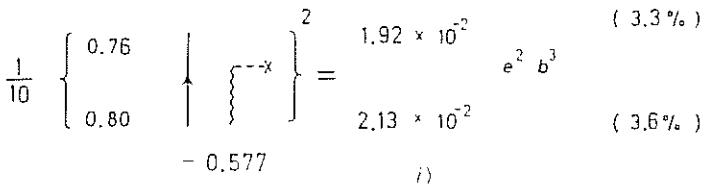
The corresponding graphical contributions to the spectroscopic factor and inelastic cross-sections are also collected in fig. 24. To be noted


a)
b)
M (keV)
I
II
319
503
-93
-103
226
400
c)
$h_{9/2}$
$d_{3/2}$
g s (^{210}Po)
3$^-$ (^{208}Pb)
d)


E (MeV)
3.0
2.8
2.6
$|\tilde{II}\rangle$ E = 3.125
R = 1.5; S = 2.25
experiment
E = 2.73
R = 0; S = 4
E = 2.95
R = 1.1; S = 2.20
$|\tilde{I}\rangle = -0.53\,|\,\rangle + 0.76\,|\,\rangle$
$|\tilde{II}\rangle = 1.02\,|\,\rangle + 0.80\,|\,\rangle$
E = 2.615
R = 1; S = 0
e)
$|\tilde{I}\rangle$ E = 2.48
R = 3.7; S = 1.83
E = 2.494
R = 4.2; S = 1.80


-0.53
1.02
-0.103
-0.103
f)
0.76
0.80
0.135
0.135
2 × 10⁻²
1 × 10⁻⁵


4 ×
-0.53
1.02
g)
1.12
4.16
1.0
-0.010
-0.011
h)
0.76
0.80
-0.211
0.014
-0.333
0.015
1.82
2.27


1/10
0.76
0.80
-0.577
i)
1.92 × 10⁻²
2.13 × 10⁻²
e² b³
(3.3 %)
(3.6 %)

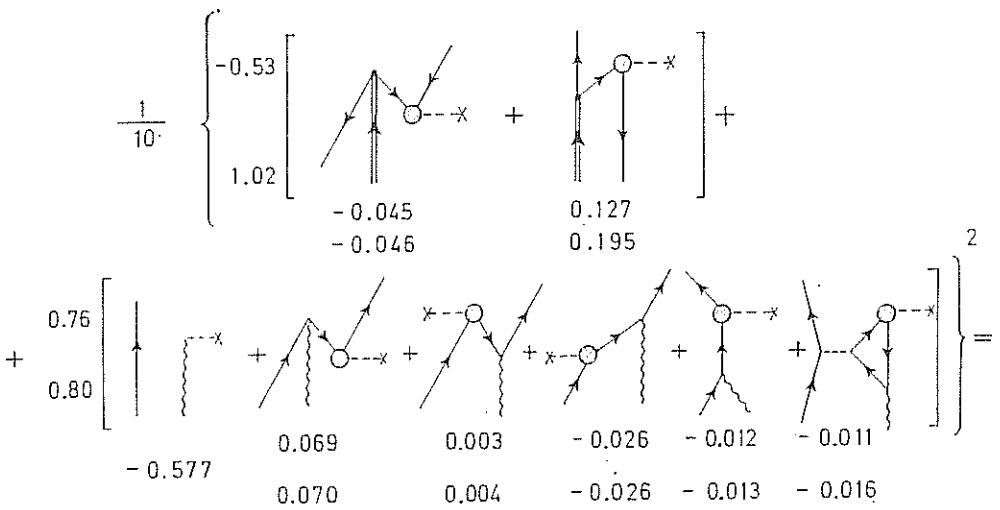


Fig. 24. – In *a*), *b*) and *c*) we give the two contributions to the matrix element $M(E) = \langle d_{3/2}^{-1} \otimes \mathrm{gs}(^{210}\mathrm{Po}) | h_{\mathrm{eff}}(E) | h_{9/2} \otimes 3^-(^{208}\mathrm{Pb}); 3/2 \rangle$ in lowest order in $1/\Omega$. The resulting wave functions $|\tilde{\mathrm{I}}\rangle$ and $|\tilde{\mathrm{II}}\rangle$ are displayed in *e*) normalized according to (7.19). In *c*) we also give the unperturbed, theoretical energies of the levels. The (t, α) spectroscopic factor corresponding to the reaction $^{210}\mathrm{Pi(t,\alpha)}^{209}\mathrm{Bi}$ is denoted by S, while

$$R = \frac{\mathrm{d}\sigma(h_{9/2} \to J)}{\mathrm{d}\sigma(\mathrm{gs}(^{208}\mathrm{Pb}) \to 3^-(^{208}\mathrm{Pb}))}$$

is the ratio of inelastic cross-sections. In *d*) we display the free fields. The zeroth and order $1/\Omega$ contributions to the electromagnetic excitations are collected in *i*) and *j*). The value $0.58\, e^2 b^3$ is the $B(E3; 0 \to 3)$ value associated with the 2.615 MeV state in ^{208}Pb. In *g*) and *k*) we give the zeroth and order $1/\Omega$ contributions to the spectroscopic factor associated with the $^{210}\mathrm{Po(t,\alpha)}^{209}\mathrm{Bi}$ reaction. Finally in *f*) we display the lowest contribution to the spectroscopic factor associated with the $^{208}\mathrm{Pb}(^3\mathrm{He}, \mathrm{d})$ reaction, which gives a measure of the ground-state correlations of ^{208}Pb associated with the existence of an octupole and a pairing vibration.

TABLE VII. – *States strongly excited in one or more of the reactions* ^{208}Pb(^{3}He, d)^{209}Bi, ^{210}Po(t, α)^{209}Bi *and* ^{209}Bi(d, d')^{209}Bi (J^π). In the first column the spin and parity of the states are given. In the second and third columns we give the experimental and theoretical energy difference $\delta E_j = (E_j - 2615)$ keV of the members of the septuplet measured with respect to the energy of the 3^- state in ^{208}Pb. For the second state $3/2^+$ (2.95 MeV) we give instead the absolute excitation energy. In columns 4 and 5 we collect the experimental and theoretical inelastic excitation cross-section normalized to the 3^- (^{208}Pb) cross-section according to

$$\frac{\mathrm{d}\sigma(h_{9/2}(^{209}\mathrm{Bi}) \to J^\pi(^{209}\mathrm{Bi}))}{\mathrm{d}\sigma(\mathrm{gs}(^{208}\mathrm{Pb}) \to 3^-(^{208}\mathrm{Pb}))}.$$

In columns 6 and 7 the experimental and theoretical values of the (t, α) spectroscopic factor are displayed. In the two final columns we give the spectroscopic factors associated with the (^{3}He, d) reaction.

J	δE_j (keV)		$\frac{\mathrm{d}\sigma(h_{9/2}(^{209}\mathrm{Bi}) \to J^\pi(^{209}\mathrm{Bi}))}{\mathrm{d}\sigma(\mathrm{gs}(^{208}\mathrm{Pb}) \to 3^-(^{208}\mathrm{Pb}))}$ (%)		S(t, α)		$S(^3\mathrm{He}, \mathrm{d})$	
	experimental [58]	theoretical	experimental [58]	theoretical	experimental [60]	theoretical	experimental [63]	theoretical
$3/2^+$	-121	-136	4.2 ± 0.3	3.7	1.8 ± 0.3	1.81	< 0.01	0.02
$5/2^+$	3	-46	9.1 ± 0.5	8.6				—
$7/2^+$	-30	30	12.3 ± 0.5	11				—
$9/2^+$	-49	-90	13.8 ± 0.6	15.8		—		—
$11/2^+$	-15	-20	$37.4 \pm (0.7)$ [a]	18.5 39.2		—	0.06	0.03
$13/2^+$		-80		20.7		—		0.01
$15/2^+$	129	190	23.7 ± 0.7	20	< 0.2		< 0.02	0.01
$3/2^+$	2.95 MeV	3.075 MeV	1.1 ± 0.2 [a]	1.3	2.2 ± 0.3	1.96	< 0.01	

(a) Error estimated (not given in original paper).

is the very different ratio of the (d, d′) and (t, α) cross-sections. While $R_1 = B(E3; (\frac{3}{2})_1)/B(E3; (\frac{3}{2})_2)$ is approximately equal to 1, the ratio $R_2 = \sigma((t, \alpha); (\frac{3}{2})_2)/\sigma((t, \alpha); (\frac{3}{2})_1)$ is close to one. Because the component $|2\rangle$ carries the inelastic-scattering strength, while the (t, α) reaction proceeds mainly through the component of type $|1\rangle$, the difference between R_1 and R_2 can be traced back to the over-completeness of the basis which give rise to rather different normalizations of the two physical states (cf. sect. 7).

b) The multiplet.

By utilizing all the states of the basis, and the same techniques discussed above, the states that are strongly excited in at least one of three reactions (d, d′), (t, α) and (^{3}He, d) were calculated. The resulting spectroscopic amplitudes and relative cross-sections are collected in table VII.

The picture of the nuclear states achieved in terms of the elementary modes of excitation displays in a transparent manner the correlation aspects of the nuclear dynamics.

Résumé.

We have shown that the nuclear-field description of the nuclear structure is mathematically correct (cf. sect. **6** and **7**) and that it smoothly joins the ranks of quantum electrodynamics and many-body field theories. We have also shown that its predictions are borne out by the experiment.

One thus achieves a simple picture of the nuclear structure. Each wave function, which is an exact solution of the many-body problem, contains only few components. Each of these components carry a label which reads: « Only to be excited through inelastic scattering » or « Only to be excited in one-particle pickup reactions », *i.e.* each component corresponds to an observable, associated with a concrete experiment one knows how to carry out with the help of accelerators, detectors and targets.

* * *

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