

## 0.1. VIEWS OF THE NUCLEUS

### 0.1 Views of the nucleus

0.1.1 the liquid drop and the shell model

In the atom, the nucleus provides the Coulomb field in which negatively charged electrons ( $-1 \times e$ ) move independently of each other in single-particle orbitals. The filling of these orbitals explains Mendeleev's periodic table. Thus the valence of the chemical elements as well as the particular stability of the noble gases (He, N, Ar, Kr, Xe and Ra) associated with the closing of shells (Fig. 0.1.1). The dimension of the atom is measured in angstroms ( $\text{\AA} = 10^{-8} \text{ cm}$ ), and typical energies in eV, the electron mass being  $m_e \approx 0.5 \text{ MeV}$  ( $\text{MeV} = 10^6 \text{ eV}$ ).

The atomic nucleus is made out of positively charged protons ( $1 \times e$ ) and of (uncharged) neutrons, nucleons, of mass  $\approx 10^3 \text{ MeV}$  ( $m_p = 938.3 \text{ MeV}$ ,  $m_n = 939.6 \text{ MeV}$ ). Nuclear dimensions are of the order of few fermis ( $\text{fm} = 10^{-13} \text{ cm}$ ). While the stability of the atom is provided by a source external to the electrons, namely the atomic nucleus, this system is self-bound as a result of the strong interaction of range  $a_0 \approx 0.9 \text{ fm}$  and strength  $v_0 \approx -100 \text{ MeV}$  acting among nucleons. Carrying with the parallel, while most of the atom is empty space, the density of the atomic nucleus is conspicuous ( $\rho = 0.17 \text{ nucleon/fm}^3$ ). The "closed packed" nature of this system implies a short mean free path as compared to nuclear dimensions. This can be estimated from classical kinetic theory  $\lambda \approx (\rho \sigma)^{-1} \approx 1 \text{ fm}$ , where  $\sigma \approx 2\pi a_0^2$  is the nucleon-nucleon cross section. It seems then natural to liken the atomic nucleus to a liquid drop (Bohr and Kalckar). This picture of the nucleus provided the framework to describe the basic features of the fission process (Meitner and Frisch (1939); Bohr and Wheeler (1939)). (1937)

The leptodermic properties of the atomic nucleus are closely connected with the semi-empirical mass formula (Weizsäcker (1935))

$$m(N, Z) = (Nm_n + Zm_p) - \frac{1}{c^2} B(N, Z), \quad (0.1.1)$$

the binding energy being

$$B(N, Z) = \left( b_{vol}A - b_{surf}A^{2/3} - \frac{1}{2}b_{sym}\frac{(N-Z)^2}{A} - \frac{3}{5}\frac{Z^2e^2}{R_C} \right). \quad (0.1.2)$$

The second term in (0.1.2) represents the surface energy, while

$$b_{surf} = 4\pi r_0^2 \gamma. \quad (0.1.3)$$

The nuclear radius is written as  $R = r_0 A^{1/3}$ , with  $r_0 = 1.2 \text{ fm}$ , the surface tension energy being  $\gamma \approx 0.95 \text{ MeV/fm}^2$ .

When, in a heavy-ion reaction, the two nuclei come within the range of the nuclear forces, the trajectory of relative motion will be changed by the attraction which will act between the nuclear surfaces. This surface interaction is a fundamental quantity in all heavy ion reactions. Assuming two spherical nuclei at a relative distance  $r_{aA} = R_a + R_A$ , where  $R_a$  and  $R_A$  are the corresponding half-density

N.Bohr and F.Kalckar, On the transmutation of atomic nuclei by impact of material particles, Mat. Fys. Medd. Dans. Vid. Selsk. 14 no. 10 (1937)

radii, the force acting between the two surfaces is

$$\left( \frac{\partial U_{AA}^N}{\partial r} \right)_{r_{AA}} = 4\pi\gamma \frac{R_a R_A}{R_a + R_A} \quad (0.1.4)$$

This result allows for the calculation of the ion-ion (proximity) potential which, supplemented with a position dependent absorption, can be used to accurately describe heavy ion reactions. (Broglia and Wuther (2004) and refs. therein)

In such reactions, not only elastic processes are observed, but also anelastic ones in which one, or both of the nuclear surfaces is set into vibration (Fig. 0.1.2). The restoring force parameter associated with oscillations of multipolarity  $\lambda$  is

$$C_\lambda = (\lambda - 1)(\lambda + 2)R^2\gamma - \frac{3}{2\pi} \frac{\lambda - 1}{2\lambda + 1} \frac{Z^2 e^2}{R} \quad (0.1.5)$$

where the second term corresponds to the contribution of the Coulomb energy to  $C_\lambda$ . Assuming the flow associated with surface vibration to be irrotational, the associated inertia for small amplitude oscillations is,

$$D_\lambda = \frac{3}{4\pi} \frac{1}{\lambda} A M R^2, \quad (0.1.6)$$

the energy of the corresponding mode being

$$\hbar\omega_\lambda = \hbar \sqrt{\frac{C_\lambda}{D_\lambda}}. \quad (0.1.7)$$

Experimental information associated with low-energy quadrupole vibrations, namely  $\hbar\omega_2$  and the electromagnetic transition probabilities  $B(E2)$ , allow to determine  $C_2$  and  $D_2$ . The resulting  $C_2$  values exhibit variations by more than a factor of 10 both above and below the liquid-drop estimate. The observed values of  $D_2$  are large as compared with the mass parameter for irrotational flow.

A picture apparently antithetic to that of the liquid drop, the shell model, emerged from the study of experimental data, plotting them against either the number of protons (atomic number), or the number of neutrons in the nuclei, rather than against the mass number. One of the main nuclear features which led to the development of the shell model was the study of the stability and abundance of nuclear species and the discovery of what are usually called magic numbers (Elsasser (1933); Mayer (1948); Haxel et al. (1949)). What makes a number magic is that a configuration of a magic number of neutrons, or of protons, is unusually stable whatever the associated number of other nucleons (Mayer (1949); Mayer and Teller (1949)).

The strong binding of a magic number of nucleons and weak binding for one more reminds, only relatively much weaker, the results displayed in Fig. 0.1.1 concerning the atomic stability of rare gases. In the nuclear case, at variance with the atomic case, the spin-orbit coupling play an important role, as can be seen from

The label  $\lambda$  stand for the angular momentum of the vibrational mode, including its third component (see Eq. (0.1.18)). Aside from  $\lambda$ , surface vibrations can also be characterized by an integer  $n$  ( $= 1, 2, \dots$ ), an ordering number indicating increasing energy. For simplicity, a common label  $\alpha$  will be also used.

the level scheme shown in Fig. 0.1.3, obtained by assuming that nucleons move independently of each other in an average potential of spherical symmetry.

A closed shell, or a filled level, has angular momentum zero. Thus, nuclei with one nucleon outside (missing from) closed, should have the spin and parity of the orbital associated with the odd nucleon (-hole), a prediction confirmed by the data (available at that time) throughout the mass table. Such a picture implies that the nucleon mean free path is large compared to nuclear dimensions.

The systematic studies of the binding energies leading to the shell model found also that formula (0.1.2) has to be supplemented to take into account the fact that nuclei with both odd number of protons and of neutrons are energetically unfavored compared with even-even ones (inset Fig. 0.1.1) by a quantity of the order of  $\delta \approx 33 \text{ MeV}/A^{3/4}$  called the pairing energy<sup>1</sup>.

The low-lying state of closed shell nuclei can be interpreted as harmonic quadrupole or octupole collective vibrations (Fig. 0.1.4) described by the Hamiltonian

$$\text{excited} \quad H_{coll} = \sum_{\lambda\mu} \left( \frac{1}{2D_\lambda} |\Pi_{\lambda\mu}|^2 + \frac{C_\lambda}{2} |\alpha_{\lambda\mu}|^2 \right) \quad (0.1.8)$$

Following Dirac (1930) one can describe the oscillatory motion introducing boson creation (annihilation) operator  $\Gamma_{\lambda\mu}^\dagger$  ( $\Gamma_{\lambda\mu}$ ) obeying

$$[\Gamma_\alpha, \Gamma_{\alpha'}^\dagger] = \delta(\alpha, \alpha'), \quad (0.1.9)$$

leading to

$$\hat{\alpha}_{\lambda\mu} = \sqrt{\frac{\hbar\omega_\lambda}{2C_\lambda}} (\Gamma_{\lambda\mu}^\dagger + (-1)^\mu \Gamma_{\lambda-\mu}), \quad (0.1.10)$$

and a similar expression for the conjugate momentum variable  $\hat{\Pi}_{\lambda\mu}$ , resulting in

$$\omega_\lambda \quad \hat{H}_{coll} = \sum_\lambda \hbar\omega_\lambda ((-1)^\mu \Gamma_{\lambda\mu}^\dagger \Gamma_{\lambda-\mu} + 1/2). \quad (0.1.11)$$

The frequency is  $\omega_\lambda = (C_\lambda/D_\lambda)^{1/2}$ , while  $(\hbar\omega_\lambda/2C_\lambda)^{1/2}$  is the amplitude of the zero-point fluctuation of the vacuum state  $|0\rangle_B$ ,  $\Gamma_{\lambda\mu}^\dagger |0\rangle_B$  being the one-phonon state. To simplify the notation, in many cases one writes  $|n_{\lambda\mu}=1\rangle$

The ground and low-lying states of nuclei with one nucleon outside closed shell can be described by the Hamiltonian

$$H_{sp} = \sum_\nu \epsilon_\nu a_\nu^\dagger a_\nu, \quad (0.1.12)$$

<sup>1</sup> Connecting with further developments associated with the BCS theory of superconductivity (Bardeen et al. (1957a,b)) and its extension to the atomic nucleus (Bohr et al. (1958)), the quantity  $\delta$  is identified with the pairing gap  $\Delta$  parametrized according to  $\Delta = 12 \text{ MeV}/\sqrt{A}$  (Bohr and Mottelson (1969)). It is of notice that for typical superfluid nuclei like  $^{120}\text{Sn}$ , the expression of  $\delta$  leads to  $\delta \approx 10 \text{ MeV}/\sqrt{A}$ .

a numerical value which can be parameterized as

where  $a_\nu^\dagger(a_\nu)$  is the single-particle creation (annihilation) operator,

$$|\nu\rangle = a_\nu^\dagger |0\rangle_F, \quad (0.1.13)$$

being the single-particle state of quantum numbers  $\nu (\equiv nljm)$  and energy  $\epsilon_\nu$ ,  $|0\rangle_F$  being the Fermion vacuum.<sup>12</sup>

(a) —————

from  
P. (14)  
handwritten

Both the existence of drops of nuclear matter displaying collective surface vibrations, and of independent-particle motion in a self-confining mean field are emergent properties not contained in the particles forming the system, neither in the  $NN$ -force, but on the fact that these particles behave according to the rules of quantum mechanics, move in a confined volume and that there are many of them.

Generalized rigidity as measured by the inertia parameter  $D_\lambda$ , as well as surface tension closely connected to the restoring force  $C_\lambda$ , implies that acting on the system with an external time-dependent (nuclear and/or Coulomb) field, the system reacts as a whole. This behavior is to be found nowhere in the properties of the nucleons, nor in the nucleon-nucleon scattering phase shifts at the basis of Yukawa prediction of the existence of a  $\pi$ -meson as the carrier of the strong force acting among nucleons.

Similarly, the fact that nuclei probed through fields which change in one unit particle number (e.g.  $(d, p)$  and  $(p, d)$  reactions) react in term of independent particle motion, feeling the pushings and pullings of the other nucleons only when trying to leave the nucleus, is not apparent in the detailed properties of the  $NN$ -forces, not even in those carrying the quark-gluon input. Within this context, independent particle motion can be considered a *bona fide* emergent property.

Collective surface vibrations and independent particle motion are examples of what are called elementary modes of excitation in many-body physics, and collective variables in soft-matter physics.

The oscillation of the nucleus under the influence of surface tension implies that the potential  $U(R, r)$  in which nucleons move independently of each other change with time. For low-energy collective vibrations this change is slow as compared with single-particle motion. Within this scenario the nuclear radius can be written as

$$R = R_0 \left( 1 + \sum_{LM} \alpha_{LM} Y_{LM}^* \right)^{(14)} \quad (0.1.14)$$

Assuming small amplitude motion,

$$U(r, R) = U(r, R_0) + \delta U(r), \quad (0.1.15)$$

where

$$\delta U = -\kappa \hat{F}, \quad (0.1.16)$$

and

$$\hat{F} = \sum_{\nu_1 \nu_2} \langle \nu_1 | F | \nu_2 \rangle a_{\nu_1}^\dagger a_{\nu_2}, \quad (0.1.17)$$

\*\*) It is of notice that  $[H_{\text{core}}, \Gamma_{\mu}^{\pm}] = \hbar \omega_{\mu} \Gamma_{\mu}^{\pm}$  and  $[H_{\text{sp}}, a_{\nu}^{\pm}] = E_{\nu} a_{\nu}^{\pm}$ . This is an obvious result resulting from the bosonic ( $[\Gamma_{\mu}, \Gamma_{\nu}^{\pm}] = \delta(\mu, \nu) \Gamma_{\nu}^{\pm}$ ) and fermionic ( $[a_{\nu}, a_{\nu'}^{\pm}] = \delta(\nu, \nu') \Gamma_{\nu}^{\pm}$ ) commutation rules ( $[A, B] = AB - BA$  (commutator),  $\{A, B\} = AB + BA$ , anticommutator).

(a) to p. 4 manuscript

It is of notice that

$$[H_{\text{coll}}, \Gamma_{\alpha' \mu'}^+] = \hbar \omega_{\alpha'} \Gamma_{\alpha' \mu'}^+ \quad (0.1.14)$$

and

$$[H_{\text{sp}}, a_{\nu}^+] = \varepsilon_{\nu} a_{\nu}^+ \quad (0.1.15)$$

This is an obvious outcome resulting from the bosonic

$$[\Gamma_{\alpha}, \Gamma_{\alpha'}^+] = \delta(\alpha, \alpha') \quad (0.1.16)$$

and fermionic

$$\{a_{\nu}, a_{\nu'}^+\} = \delta(\nu, \nu') \quad (0.1.17)$$

commutation relations. ② p. 4

to p. 6 manuscript

(b) An alternative procedure to the diagrammatic one to obtain the HF and RPA solutions associated with the bare NN-interaction  $\nu$  is provided by the relations (0.1.15) and (0.1.14) respectively, replacing the Hamiltonians by  $(T + \nu)$ , where  $T$  is the kinetic energy operator. The phonon operator associated with surface vibrations is defined through

$$\Gamma_{\alpha}^+ = \sum_{ki} X_{ki}^{\alpha} \Gamma_{ki}^+ + Y_{ki}^{\alpha} \Gamma_{ki}^- \quad (0.1.26)$$

and the normalization condition

The label  $\alpha$  stands for the quantum numbers characterizing the vibrational mode

$$[\Gamma_{\alpha}, \Gamma_{\alpha}^+] = \sum_{hi} (X_{hi}^{\alpha 2} - Y_{hi}^{\alpha 2}) = 1 \quad (0.1.27)$$

The operator  $\Gamma_{ki}^+ = a_{ki}^+$  creates a particle-hole excitation (acting on the HF vacuum state  $|0\rangle_F$ ). It is assumed that

$$[\Gamma_{hi}, \Gamma_{h'i'}^+] = \delta(h, h') \delta(i, i') \quad (0.1.28)$$

Within this context, RPA is a harmonic, quasi-boson approximation (b)

Concerning the rules of NFT, they codify the way in which  $H_c$  and  $v$  are to be treated to all orders of perturbation theory. Also which processes (diagrams) are not allowed because they will imply overcounting of correlations already included in the basis states\*) - single-particle HF, collective vibrations, RPA.

### 0.1. VIEWS OF THE NUCLEUS

5

while

$$F = \frac{R_0}{\kappa} \frac{\partial U}{\partial r} Y_{\lambda\mu}^*(\hat{r}). \quad (0.1.18)$$

The coupling between surface oscillation and single-particle motion, namely the particle vibration coupling (PVC) Hamiltonian  $\delta U$  (Fig. 0.1.5) is a consequence of the overcompleteness of the basis. Diagonalizing  $\delta U$  making use of the graphical (Feynman) rules of Nuclear Field Theory (NFT) to be discussed below, one obtains structure results which can be used in the calculation of transition probabilities and reaction cross sections which can be compared with experimental findings.

In fact, within the framework of NFT, single-particle are to be calculated as the Hartree-Fock solution of the  $NN$ -interaction  $v(|\mathbf{r} - \mathbf{r}'|)$  (Fig. 0.1.6), in particular

$$U(r) = \int d\mathbf{r}' \rho(\mathbf{r}') v(|\mathbf{r} - \mathbf{r}'|) \quad (0.1.19)$$

being the Hartree field<sup>2</sup> expressing the selfconsistency between density  $\rho$  and potential  $U$  (Fig. 0.1.6 (b) (1) and (3)), while vibrations are to be calculated in the Random Phase Approximation (RPA) making use of the same interaction<sup>3</sup> (Fig. 0.1.7), extending the selfconsistency to fluctuations  $\delta\rho$  of the density and  $\delta U$  of the mean field, that is,

$$\delta U(r) = \int d\mathbf{r}' \delta\rho(\mathbf{r}') v(|\mathbf{r} - \mathbf{r}'|). \quad (0.1.20)$$

Making use of the selfconsistent solution of the relation (0.1.20), one obtains the transition density  $\delta\rho$ . The matrix elements  $\langle v_i | \delta\rho | v_k \rangle$  provide the particle-vibration coupling strength to work out the variety of coupling processes between single-particle and collective motion (Fig. 0.1.5). That is, the matrix element of the PVC Hamiltonian  $H_c$ . Diagonalizing

$$H = H_{HF} + H_{RPA} + H_c + v, \quad (0.1.21)$$

making use of the rules of NFT to be discussed below, in the basis of single-particle and collective modes, that is solutions of  $H_{HF}$  and of  $H_{RPA}$  respectively, one obtains a solution of the total Hamiltonian. Because of quantal zero point fluctuations, a nucleon propagating in the nuclear medium moves through clouds of bosonic and fermionic virtual excitations to which it couple ( $H_c + v$ ), becoming dressed and acquiring an effective mass, charge, etc. (Fig. 0.1.8). Vice versa,

<sup>3</sup> To this potential one has to add the Fock potential resulting from the fact that nucleons are fermions. This exchange potential (Fig. 0.1.6 (2 and 4)) is essential in the determination of single-particle energies and wavefunctions. Among other things, it takes care of eliminating the nucleon self interaction from the Hartree field.

<sup>4</sup> The sum of the so called ladder diagrams (see Fig. 0.1.7) are taken into account to infinite order in RPA. This is the reason why bubble contributions in the diagonalization of Eq. (0.1.21) are not allowed in NFT, being already contained in the basis states.

\*<sup>5</sup>) a simpler example is provided by Eq (ZA-31) of Bohr and Mottelson (1969) i.e.

$$G = \frac{1}{4} \sum_{v_1, v_2, v_3, v_4} \langle v_3 v_4 | G | v_1 v_2 \rangle a^\dagger(v_4) a^\dagger(v_3) a(v_1) a(v_2)$$

$$= \frac{1}{2} \sum_{v_1, v_2, v_3, v_4} \langle v_3 v_4 | G | v_1 v_2 \rangle a^\dagger(v_4) a^\dagger(v_3) a(v_1) a(v_2)$$

where  $\langle \gamma_a \rangle$  is the antisymmetric matrix element.

such a process  
leads to a  
renormalization of  
vibrations which  
produce the

and refs. therein

vibrational modes can become renormalized through the coupling to dressed nucleons which, in intermediate virtual states, can exchange the vibrational clothing with the second fermion (hole state), and renormalize the PVC vertex (Fig. 0.1.9) (Barranco et al. (2004)), as well as the bare  $NN$ -interaction.

From being antithetic views of the nuclear structure, a proper analysis of the experimental data testifies to the fact that the collective and the independent particle picture of the nuclear structure require and support each other (Bohr, A. and Motelson (1975)). To obtain a quantitative description of nucleon motion and nuclear phonons (vibrations), one needs a proper description of the  $k$ - and  $\omega$ -dependent "dielectric" function of the nuclear medium, in a similar way in which a proper description of the reaction processes used as probes of the nuclear structure requires the use of the optical potential (continuum "dielectric" function). The NFT solution of (0.1.21) provide all the elements to calculate the nuclear structure properties of nuclei, and also the optical potential needed to describe nucleon-nucleus scattering. It furthermore shows that both single-particle and vibrational elementary modes of excitation emerge from the same properties of the  $NN$ -interaction.

The development of experimental techniques and associated hardware has allowed for the identification of a rich variety of elementary modes of excitation aside from collective surface vibrations and of independent particle motion: quadrupole and octupole rotational bands, giant resonance of varied multipolarity and isospin, as well as pairing vibrations and rotation, together with giant pairing vibrations of transfer quantum numbers  $\pm 2$ . Modes which can be specifically excited in inelastic and Coulomb excitation processes, charge exchange, and one- and two-particle transfer reactions.

$\beta =$

$\beta = \pm 2$

One can choose to privilege one among this variety of elementary modes of excitation, for example, independent particle motion. Making use of the shell model eventually the so called no core shell model, understood within this context as a full diagonalization of the  $NN$ -interaction in the single-particle basis, attempt at describing the whole of structure and reactions. Another possibility is to use the elementary modes of excitation basis states to describe both structure and reactions and nuclear field theory to deal with the overcompleteness and Pauli principle violations of the basis states.

From a systematic collaboration between the two approaches and of strong experimental input, it is likely that shell model calculations can help at individuating interaction

(1) A schematic separable force leading to surface vibrations can be written as  $-k_F F$ , where  $F$  is defined in Eq. (0.1.17). The resulting collective modes can thus be viewed as correlated particle-hole ( $p-h$ ) excitations ( $a^\dagger a$ ), a process in which the number of nucleons (fermions) does not change. One speaks in this case of a mode with transfer quantum number  $\beta = 0$ . In connection with the pairing energy mentioned in relation with the inset to Fig. 0.1.1 and its connection to the theory of superconductivity, it is of notice that this theory is based on the concept of Cooper pairs, that is pairs of fermions moving in time reversal states which interact through  $\hat{H}_p = -G \hat{P}^\dagger \hat{P}$ , where  $\hat{P}^\dagger = \sum_{\nu>0} a_\nu^\dagger a_\nu$ . Consequently, in this case the concept of independent particle field  $\hat{F}$  has to be generalized to include  $\hat{P}^\dagger$  and  $\hat{P}$ . The resulting collective modes, pair addition and pair subtraction modes (pairing vibrations), can be viewed as correlated ( $p-p$ ) and ( $h-h$ ) modes, changing the number of nucleons in  $\pm 2$ . One speaks of vibrations with transfer quantum number  $\beta = \pm 2$ .

pairing vibrations,

with but  
with  $F$  in  
Eq. (0.1.22)  
replaced by  
 $Q = \int r^2 Y_{20}(r)$

at



①

to p.6 manuscript  
 ② ③ Pairing vibrations

Let us introduce this new type of elementary mode of excitation by making a parallel with quadrupole surface vibrations, within the framework of the RPA, namely

$$[(H_{sp} + H_i), \Gamma_d^+] = \hbar \omega_d \Gamma_d^+, \quad (0.1.29)$$

where for simplicity we use, instead of  $\nu$ , a quadrupole-quadrupole separable interaction ( $i=QQ$ ) defined as

$$H_{QQ} = -K \hat{Q}^+ \hat{Q} \quad (0.1.30)$$

with

$$\hat{Q}^+ = \sum_{R,C} \langle k | r^2 Y_{2\mu} | i \rangle a_R^+ a_C^i, \quad (0.1.31)$$

while  $H_{sp}$  and  $\Gamma_d^+$  are defined in (0.1.12) and (0.1.26) supplemented by (0.1.27).

In connection with the pairing energy mentioned in relation with the inset of Fig. 0.1.1, it is a consequence of correlation of pairs of like nucleons moving in time reversal states. A similar phenomenon found in metals at low temperatures and giving rise to superconductivity. The pairing interaction ( $i=p$ ) can be written, within the approximation (0.1.30) used in the case of the quadrupole-quadrupole force as

$$H_p = -\hat{P}^+ \hat{P}, \quad (0.1.32)$$

where

$$\hat{P}^+ = \sum_{\nu \neq 0} a_\nu^+ a_\nu^-. \quad (0.1.33)$$

Consequently, in this case the concept of <sup>(2)</sup> independent-particle field  $\hat{Q}$  (see also (0.1.21)) associated with particle-hole excitations and carrying transfer quantum number  $\beta=0$ , has to be generalized to include fields describing inelastic pair motion, in which case  $\alpha \in (\beta=+2, \pi=0^+)$

$$\Gamma_{\alpha}^+ = \sum_k X_{hh}^{\alpha} \Gamma_k^+ + Y_{ii}^{\alpha} \Gamma_i^+ \quad (0.1.30)$$

with

$$\Gamma_{hh}^+ = a_k^+ a_h^+ \quad (\varepsilon_k > \varepsilon_F), \quad \Gamma_i^+ = a_i^+ a_i^+ \quad (\varepsilon_i < \varepsilon_F) \quad (0.1.31)$$

and

$$\sum_k X_{hh}^{\alpha 2} - \sum_i Y_{ii}^{\alpha 2} = 1 \quad (0.1.32)$$

for the pair addition ((pp),  $\beta=+2$ ) mode, and a similar expression for the pair removal (hh),  $\beta=-2$  mode. In Fig. 0.1.10

the NFT graphical representation of the RPA equations for the pair addition

mode is given. The state  $\Gamma_{\alpha}^+(\beta=+2) |0\rangle$ , where  $|0\rangle$  is the correlated ground state of a closed shell nucleus, can be viewed as the nuclear embodiment of

a Cooper pair found at the basis of the microscopic theory of superconductivity.

While surface vibrations are associated with the normal ( $\beta=0$ ) nuclear density, pairing vibrations are connected with the so called abnormal ( $\beta=\pm 2$ ) nuclear density (density of Cooper pairs).

Fig. 0.1.  
nuclear  
pair  
vibration

Similarly to the quadrupole and octupole vibrational bands built out of  $n_\alpha$  phonons of quantum numbers  $\alpha = (\beta=0, \lambda^\pi=2^+, 3^-)$  (3) schematically shown in Fig. 0.1.4 and experimentally observed in inelastic and Coulomb excitation and associated  $\gamma$ -decay processes, pairing vibrational bands build of  $n_\alpha$  phonons of quantum numbers  $\alpha = (\beta=\pm 2, \lambda^\pi=0^+, 2^+)$  have been identified around closed shells in terms of two-nucleon transfer reactions throughout the mass table (Fig. 0.1.11). Fig 16 p. 324  
adv. in  
NP or  
threephonon  
Flynn.

#### 0.4 Spontaneous broken symmetry

Because empty space is homogeneous and isotropic, the nuclear Hamiltonian is translational and rotational invariant. It also conserves particle number and is thus gauge invariant. According to quantum mechanics, the corresponding wavefunctions transform in an irreducible way under the corresponding group of transformations.

When the solution of the Hamiltonian does not have some of these symmetries, for example defines a privileged direction in space violating rotational invariance, one is confronted with the phenomenon of spontaneous broken symmetry. Strictly speaking, this can take place only

for idealized systems that are infinitely large. (4)

#### 0.4.1 Quadrupole deformations in 3D-space

An nuclear embodiment of the spontaneous symmetry breaking phenomenon is provided by a mean field. A situation one is confronted with when the value of the lowest quadrupole frequency  $\omega_2$  of the RPA solution (0.1.14) (see also (0.1.26) and (0.1.27)) tends to zero ( $C_2 \rightarrow 0, D_2$  finite). A phenomenon resulting from the interplay of the interaction  $v$  ( $H_{QQ}$  in (0.1.29)), and of the

nucleons outside closed shell, leading to tidal-like polarization of the spherical core.

Coordinate and linear momentum ( $(x, p_x)$  single-particle motion) as well as Euler angles and angular momentum ( $(\phi, I_z)$  rotation in two-dimensional (2D)-space) are conjugate variables. Similarly, the gauge angle and the number of particles ( $(\phi, N)$  rotation in gauge space), fulfill  $[\phi, N] = i$ . The operators  $e^{-ip_xx}$ ,  $e^{-i\phi I_z}$  and  $e^{-iN\phi}$  induce Galilean transformation and rotations in 2D- and in gauge space respectively.

Making again use, for didactical purposes, of  $H_{QQ}$  instead of  $V$ , and calling  $|N\rangle$  the eventual mean field solution of the Hamiltonian  $T+H_{QQ}$ , one expects

$$\langle N | \hat{Q} | N \rangle = Q_0 , \quad (0.1.37)$$

where, for simplicity, we assumed axial symmetry ( $\lambda=2, \mu=0$ ). That is, the emergence of a static quadrupole deformation.

Rewriting  $H_{QQ}$  in terms of  $(\hat{Q}^+ - Q_0 + Q_0)$  and its Hermitian conjugate, one obtains

$$H = H_{SP} + H_{QQ} = H_{MF} + H_{fluct}, \quad (0.1.38)$$

where

$$H_{MF} = H_{SP} - K(\hat{Q}^+ + Q) , \quad (0.1.39)$$

is the mean field, and

$$H_{fluct} = -K(\hat{Q}^+ - Q_0)(\hat{Q} - Q_0) \quad (0.1.40)$$

the residual interaction inducing fluctuations around  $Q_0$ .

Assuming  $Q_0 \gg (\hat{Q}^+ - Q_0)(\hat{Q} - Q_0)$ , we concentrate on  $H_{MF}$ . The original realization of it is known as the Nilsson Hamiltonian (Nilsson (1955)). It describes the motion of nucleons in a single-particle potential of radius  $R = R_0(1 + \beta_2 Y_{20}(\hat{r}))$ , with  $\beta_2$  proportional to the intrinsic quadrupole moment  $Q_0$ .

The reflection invariance and axial symmetry of the Nilsson Hamiltonian implies, that parity  $\pi$  and projection  $\Omega$  of the total angular momentum along the symmetry axis are constants of motion for the one-particle Nilsson states. These states are two-fold degenerate, since two orbits that differ only in the sign of  $\Omega$  represent the same motion, apart from the clockwise and anticlockwise sense of revolution around the symmetry axis. One can thus write the Nilsson creation operators in terms of a linear combination of creation operators carrying good total angular momentum  $J$ ,

$$\gamma_{a\Omega}^+ = \sum_J A_J^a a_{aj\Omega}^+, \quad (A.1.41)$$

where the label  $a$  stands for all the quantum numbers aside from  $\Omega$ , which specify the orbital.

Expressed in the intrinsic, body-fixed, system of coordinates  $K'$  where the  $z(z')$  axis lies along the symmetry axis and the  $1$  and  $2(x', y')$  axis lie in a plane perpendicular to it, namely

$$\gamma_{a\Omega}^{+t} = \sum_J A_J^a \sum_{\Omega'} D_{\Omega'\Omega}^2(w) a_{aj\Omega'}^+ \quad (A.1.42)$$

one can write the Nilsson state as

$$|N(\omega)\rangle_{J\ell^1} = \prod_{\alpha>0} \gamma_1^{\alpha+} \gamma_2^{\alpha+} |0\rangle, \quad (0.1.43)$$

$\omega$  represent the Euler angle,

where  $|0\rangle$  is the particle vacuum, and  $|\alpha\beta\rangle = \gamma_1^{\alpha+} |0\rangle$  is the state time-reversed to  $|\alpha\beta\rangle$ . For well deformed nuclei, a convenient description of the one-particle motion is based on the similarity of the nuclear potential to that of an anisotropic nuclear potential.

$$\begin{aligned} V &= \frac{1}{2} M (\omega_3^2 x_3^2 + \omega_{\perp}^2 (x_1^2 + x_2^2)) \\ &= \frac{1}{2} M \omega_0 r^2 \left( 1 - \frac{4}{3} \delta P_2(\cos\theta) \right), \end{aligned} \quad (0.1.44)$$

with  $\omega_3 \omega_{\perp}^2 = \omega_0^3$ . That is a volume which is independent of the deformation  $\delta \approx 0.95 \rho_2$ .

The corresponding single-particle states have energy

$$E(n_3 n_{\perp}) = (n_3 + \frac{1}{2}) \hbar \omega_3 + (n_{\perp} + \frac{1}{2}) \hbar \omega_{\perp}, \quad (0.1.44)$$

where  $n_3$  and  $n_{\perp} = n_1 + n_2$  are the number of quanta along and perpendicular to the symmetry axis. The degenerate states with the same value of  $n_{\perp}$  can be specified by the component  $\lambda$  of the orbital angular momentum along the

(8)

symmetry axis,

$$\Lambda = \pm n_{\perp}, \pm(n_{\perp}-2), \dots, \pm 1 \text{ or } 0 \quad (0.1.45)$$

One can then label the Nilsson levels in terms of the asymptotic quantum numbers  $[N n_3 \Lambda \Sigma]$ , where  $N = n_3 + n_{\perp}$ , is the total oscillator quantum number.

The complete expression of the Nilsson potential includes, aside from the central term discussed above, a spin-orbit and a term proportional to the orbital angular momentum quantity squared, so as to make the shape of the oscillator to resemble more that of a Saxon-Woods potential. The resulting levels provide an overall account of the experimental findings, providing detailed evidence in terms of individual states of the interplay between the single-particle and the collective aspects of nuclear structure. An example of relevance for light nuclei ( $N$  and  $Z < 20$ ) is given in Fig. 0.1.12

FIG. 5-1  
B+MP, 221  
Vol II

The Nilsson intrinsic state (0.1.43) does not have a definite angular momentum, but is rather a superposition of such states,

$$|N(\omega)\rangle_{\mathbf{k}'} = \sum_I c_I |I\rangle, \quad (0.1.46)$$

The symmetry axis 3 defines a privileged direction in space, thus breaking rotational invariance.

(9)

Because there is no restoring force associated with different orientations of  $|N(\omega)\rangle_x$ , fluctuations in the Euler angles diverge in just the right way to restore rotational invariance, leading to a rotational band whose members are

$$|IKM\rangle \sim \int d\omega D_{MK}^I(\omega) |N(\omega)\rangle_{x'}, \quad (0.1.47)$$

with energy

$$E_I = \frac{\hbar^2}{2J} I(I+1). \quad (0.1.48)$$

The quantum numbers  $I, M, K$  are the total angular momentum  $I$ , and its third component  $M$  and  $K$  along the laboratory ( $z$ ) and intrinsic ( $z'$ ) frame of reference respectively.

Rotational bands have been observed up to rather high angular momenta in terms of individual transitions. An example extending up to  $I=60\hbar$  is given in Fig. 0.1.13

#### 0.4.2 Deformation in gauge space

Let us now turn to the pairing Hamiltonian. In the case in which  $\hbar\omega_{\beta=+2} \approx \hbar\omega_{\beta=-2} \approx 0$ , the system deforms, this time in gauge space. Calling  $|BCS\rangle$  the eventual mean field solution, leads to a finite value

$$\alpha_0 = \langle BCS | P^+ | BCS \rangle, \quad (0.1.49)$$

of the pair operator  $P^+$  which can be  
viewed as the order parameter of the  
new (gauge deformed) phase of the system. (10)

The total Hamiltonian can be written as

Let us now turn to the pairing Hamiltonian. In the case in which  $\epsilon_{\text{two}} \approx \epsilon_{\text{two}}^0 \approx 0$ , the system deforms, this time in gauge space. Calling  the eventual mean field solution, leads to a finite value

$$\alpha_0 = \langle \text{BCS} | P^+ | \text{BCS} \rangle \quad (0.1.39)$$

of the pair operator  $P^+$ , which can be viewed as the order parameter of this new (deformed) phase of the system. The total Hamiltonian can be

written as

$$H = H_{\text{MF}} + H_{\text{fluct}}, \quad (0.1.40) \quad (0.1.50)$$

where

$$H_{\text{MF}} = H_{\text{sp}} - \Delta (P^+ + P) + \frac{\Delta^2}{G} \quad (0.1.51)$$

and

$$H_{\text{fluct}} = -G (P^- \alpha_0) (P^+ \alpha_0) \quad (0.1.42) \quad (0.1.52)$$

The quantity

$$\Delta = G \alpha_0 \quad (0.1.43) \quad (0.1.53)$$

is the so called pairing gap, which measures the binding energy of Cooper pairs,  $\alpha_0$  counting the number of Cooper pairs.

$[H, F], F$  sum rules

$$\text{two} = \frac{41}{A'^3} \text{nev} \quad \text{GR.} \quad \underline{\text{GPV}}$$

København 11/08/18  
The mean-field pairing Hamiltonian (11) (17)

$$H_{MF} = \sum_{\nu>0} (\varepsilon_\nu - \lambda) (a_\nu^\dagger a_\nu + a_{\bar{\nu}}^\dagger a_{\bar{\nu}}) - \Delta \sum_{\nu>0} (a_\nu^\dagger a_{\bar{\nu}}^\dagger + a_{\bar{\nu}}^\dagger a_\nu) + \frac{\Delta^2}{4} \quad (0.1.54)$$

is a bilinear expression in the creation and annihilation operator,  $\nu$  labeling the quantum numbers of the single-particle orbitals, where nucleon, are allowed to correlate (e.g.  $(n\ell m)$ ), while  $\bar{\nu}$  denotes the time reversal state which in this case is degenerate with  $\nu$  and has quantum numbers  $(n\bar{\ell} - m)$ ,  $\nu > 0$  implying that one sums over  $m > 0$ . It is of notice that

$$\hat{N} = \sum_{\nu>0} (a_\nu^\dagger a_\nu + a_{\bar{\nu}}^\dagger a_{\bar{\nu}}), \quad (0.1.55)$$

is the number operator, and  $\hat{2}\hat{N}$  in Eq (0.1.54) acts as the Coriolis force in the body-fixed frame of reference in gauge space.

One can diagonalize  $H_{MF}$  by a rotation in the  $(a_\nu^\dagger, a_\nu)$ -space. This can be accomplished through the Bogoliubov-Valatin transformation

$$(0.1.56)$$

$$d_\nu^\dagger = U_\nu a_\nu^\dagger - V_\nu a_\nu. \quad (0.1.45)$$

The BCS solution does not change the energies  $\varepsilon_\nu$  (measured in (0.1.44)) from the Fermi energy  $\Delta$  of the single-particle levels, or associated wave functions  $\psi_\nu(\vec{r})$ , but the occupation probabilities for levels around the Fermi energy within an energy range  $2\Delta$  ( $2\Delta/\gamma \approx 2Mev/36meV \approx 0.06$ ). The quasiparticle operator  $d_\nu^\dagger$  creates a particle in the single-particle state  $\nu$  with probability  $U_\nu^2$ , while it creates a hole (annihilates a particle) with probability  $V_\nu^2$ . To be able to create a particle, the state  $\nu$  should be empty, while to annihilate a particle it has to be filled, so  $U_\nu^2$  and  $V_\nu^2$  are the probabilities that the state  $\nu$  is empty and is occupied respectively. Within this context, the one quasiparticle state

$$(0.1.57)$$

$$(0.1.46)$$

$$| \nu \rangle = d_\nu^\dagger | BCS \rangle$$

(12)

are orthonormal. In particular

$$\langle v|v\rangle = 1 = \langle BCS|\alpha_v, \alpha_v^+ | BCS \rangle \quad (O.1.58)$$

$$= \langle BCS|\{\alpha_v, \alpha_v^+\} | BCS \rangle = U_v^2 + V_v^2, \quad (O.1.47)$$

where the relations

$$\{\alpha_v, \alpha_{v'}^+\} = \delta(v, v'). \quad (O.1.48)$$

and

$$\{\alpha_v, \alpha_v\} = \{\alpha_v^+, \alpha_{v'}^+\} = 0 \quad (O.1.49)$$

have been used.

Note that the  $|BCS\rangle$  state is the quasiparticle vacuum

$$\alpha_v |BCS\rangle = 0, \quad (O.1.50)$$

in a similar way in which  $|0\rangle_F$  is the particle vacuum.

Inverting the quasiparticle transformation (O.1.45) and its complex conjugate, i.e. expressing  $\alpha_v^+$  and  $\alpha_v$  (and time reversals ( $t^\dagger$ )) in terms of  $\alpha_v^+$  and  $\alpha_v$  (and  $t^\dagger$ ), one can rewrite (O.1.44) in terms of quasiparticles. (O.1.54)

Minimizing the  $E_0 = \langle BCS|H|BCS \rangle$  in terms of  $V_v$

$$\frac{\partial E_0}{\partial V_v} = 0 \quad (O.1.62)$$

$$(O.1.51)$$

and making use of the expression for the average number of particles (O.1.63)

$$N = \langle BCS|\hat{N}|BCS \rangle = 2 \sum_{v>0} V_v^2, \quad (O.1.52)$$

$$(O.1.64)$$

and of the number of Cooper pairs (O.1.53)

$$N_c = \langle BCS|P^+|BCS \rangle = \sum_{v>0} U_v V_v \quad (O.1.53)$$

$$(O.1.65)$$

and thus of the pairing gap, (O.1.54)

$$\Delta = G \sum_{v>0} U_v V_v, \quad (O.1.54)$$

one obtains,

$$H_{MF} = H_{11} + U$$

(0.1.68)  
(0.1.55)

9  
13

where

$$H_{11} = \sum_v E_v \alpha_v^+ \alpha_v$$

(0.1.67)  
(0.1.56)

(0.1.68)

and

$$U = 2 \sum_{v>0} (\epsilon_v - \lambda) V_v^2 - \frac{\Delta^2}{G}. \quad (0.1.57)$$

(0.1.69)

(0.1.58)

The quantity

$$E_v = \sqrt{(\epsilon_v - \lambda)^2 + \Delta^2}$$

is the quasiparticle energy, while the probability amplitudes are

$$V_v = \frac{1}{\sqrt{2}} \left( 1 - \frac{\epsilon_v - \lambda}{E_v} \right)^{1/2} \quad (0.1.59)$$

(0.1.71)

and

$$U_v = \frac{1}{\sqrt{2}} \left( 1 + \frac{\epsilon_v - \lambda}{E_v} \right)^{1/2} \quad (0.1.60)$$

From the relations (0.1.52) and (0.1.54)

one obtains

$$N_0 = 2 \sum_{v>0} V_v^2, \quad (\text{number equation})$$

(0.1.73) (0.1.56)

and

$$\frac{1}{G} = \sum_{v>0} \frac{1}{2E_v}, \quad (\text{gap equation})$$

from which one can determine the parameters  $\lambda$  and  $\Delta$ , and thus the occupation amplitudes.

These equations allow us to calculate the parameters  $\lambda$  and  $\Delta$  from the knowledge of  $G$  and  $E_v$ , parameters which completely determine  ~~$\epsilon_v$~~   $V_v$  and  $U_v$  and thus the BCS mean field solution (Fig. 0.1.14).

Fig 1.9  
Brueck  
+ Bruecklin

The validity of the BCS description (10) (14)  
of superfluid open shell nuclei have been  
confirmed throughout the mass table. We  
provide below recent examples.

The relation (0.1.50), implies that  
(0.1.61)

$$|BCS\rangle = \frac{1}{N_{\text{even}}} \prod_{\nu>0} \alpha_\nu \alpha_{\bar{\nu}} |10\rangle_F = \prod_{\nu>0} (U_\nu + V_\nu P_\nu^+) |10\rangle_F, \quad (0.1.57)$$

$$= (\prod_{\nu>0} U_\nu) \sum_{N_{\text{even}}} \frac{(\sum_{\nu>0} C_\nu P_\nu^+)^{N/2}}{(N/2)!} |10\rangle_F, \quad (0.1.74)$$

where

$$P_\nu^+ = a_\nu^+ a_{\bar{\nu}}^+ \quad (P^+ = \sum_{\nu>0} P_\nu^+), \quad C_\nu = V_\nu / U_\nu. \quad (0.1.75) \quad (0.1.58)$$

In the above discussion of BCS we have treated in a rather cavalier fashion the fact that the amplitudes  $U_\nu$  and  $V_\nu$  are in fact complex quantities. A possible choice of phasing is\*)

$$U_\nu = U'_\nu ; \quad V_\nu = V'_\nu e^{-2i\phi} \quad (0.1.76) \quad (0.1.59)$$

$U'_\nu$  and  $V'_\nu$  being real quantities, while  $\phi$  is the gauge angle, conjugate variable (0.1.55) to the number of particles operator (0.1.44).

Then  $\hat{\phi} = i \frac{\partial}{\partial N}, \quad N$

$$(0.1.76) \quad (0.1.60)$$

and (Appendix 0.A)

$$[\hat{\phi}, N] = i \quad (0.1.55) \quad (0.1.77) \quad (0.1.61)$$

where  $N \equiv \hat{N}$  (Eq. 0.1.44), gauge transfor-

\*) The same results as those which will be derived are obtained with the alternative choice  $U_\nu = U'_\nu e^{i\phi}, V_\nu = V'_\nu e^{-i\phi}$ .

11/08/18

mations being induced by the operator

$$G(\phi) = e^{-i\phi} \quad (0.1.78)$$

$$. \quad (0.1.62) \quad (0.1.76)$$

Let us introduce the amplitudes, (0.1.59)  
in (0.1.58),

$$|BCS\rangle = (\prod_{v>0} U'_v) \sum_{N \text{ even}} e^{-iN\phi} |\Phi_N\rangle = (\prod_{v>0} U'_v) \sum_{N \text{ even}} e^{-iN\phi} |\Phi'_N\rangle$$

where

$$|\Phi'_N\rangle = \frac{(\sum_{v>0} C'_v P'_v)^{N/2}}{(N/2)!} |0\rangle_F \quad (0.1.80) \quad (0.1.79)$$

with  $C'_v = V'_v / U'_v$ . It is of notice that

$$\sum_{v>0} C'_v P'_v |0\rangle_F \quad (0.1.81)$$

is the single cooper pair state. As already emerged from (0.1.39) and is explicitly confirmed by the above expression, the state  $|BCS\rangle$  does not have a definite number of particles but only in average (0.1.52), being a wavepacket in  $N$ .  
(0.1.79) (0.1.63)

In fact, (0.1.63) defines a privileged direction in gauge space, being an eigenstate of  $\hat{\phi}$

$$\begin{aligned} \hat{\phi} |BCS\rangle &= i \frac{\partial}{\partial \phi} (\prod_{v>0} U'_v) \sum_{N \text{ even}} e^{-iN\phi} |\Phi'_N\rangle \\ &= \phi |BCS\rangle \end{aligned} \quad (0.1.82) \quad (0.1.65)$$

Expressing it differently (0.1.63)

can be viewed as an axially symmetric deformed system whose symmetry axis coincides with the  $z'$  component of the body-fixed frame of reference  $R'$ , which makes an angle  $\phi$  with the laboratory  $z$ -axis.  
(Fig 0.1.4)

Returning to the original, first line

0.1.15

expression of (0.1.57) one can write,

$$|BCS(\phi=0)\rangle_{K'} = \prod_{\nu>0} (U'_\nu + V'_\nu P_\nu^{'+}) |0\rangle_F, \quad (0.1.66)$$

where use was made of the relations (0.1.84)

$$g_j(\phi) a_\nu^+ g_j^{-1}(\phi) = e^{-i\phi} a_\nu^+ = \tilde{a}_\nu^+, \quad (0.1.67)$$

and

$$g_j(\phi) P_\nu^+ g_j^{-1}(\phi) = e^{-2i\phi} P_\nu^+ = \tilde{P}_\nu^+, \quad \begin{matrix} (0.1.68) \\ \text{anti-commutes} \end{matrix}$$

It is to be noted that  $g_j$  induces an counter clockwise rotation (App.0.A),  $g_j(x) \hat{\phi} g_j^{-1}(x) = \hat{\phi} - x$ . (0.1.85)

As a consequence, to rotate  $|BCS(\phi=0)\rangle_{K'}$  back into the laboratory system, use has to be made of the clockwise rotation of angle  $\phi$  induced by  $g_j^{-1}(\phi)$ ,

$$\begin{aligned} g_j^{-1}(\phi) |BCS(\phi=0)\rangle_{K'} &= \prod_{\nu>0} (U'_\nu + V'_\nu g_j^{-1}(\phi) P_\nu^+) |0\rangle_F \\ &= \prod_{\nu>0} (U_\nu + e^{2i\phi} V_\nu P_\nu^+) |0\rangle_F \quad (0.1.68) \\ &= |BCS(\phi)\rangle_{K'} \quad (0.1.69) \end{aligned}$$

where use was made of (0.1.68)

$$g_j^{-1}(\phi) (g_j(\phi) P_\nu^+ g_j^{-1}(\phi)) g_j(\phi) = g_j^{-1}(\phi) P_\nu^+ g_j(\phi) \quad (0.1.70)$$

We note furthermore

$$\begin{aligned} |BCS(\phi)\rangle_{K'} &= \prod_{\nu>0} (U'_\nu + V'_\nu P_\nu^{'+}) |0\rangle_F \quad (0.1.71) \\ &= \prod_{\nu>0} (U_\nu + V_\nu P_\nu^+) |0\rangle_F \quad (0.1.88) \end{aligned}$$