Chapter 4

Reaction cross section

4.1 Direct reaction

Oversimplifying the situation we can define direct reactions as those scattering processes which can be described in terms of coordinates of the entrance and exit channels, and any few internal coordinates which must be considered because the exit and entrance channels are different. These processes do not go through compound nucleus. The differential cross sections are in general peaked at small angles and often oscillate strongly with angle. But variation with energy is smooth. In future lectures we are going to discuss in more detail the separation between compound and direct reactions. For the time being we assume such separation is possible.

4.2 The reaction cross section

As discussed in the previous lecture, $f_{\alpha\beta}(\vec{k}_\alpha, \vec{r}_\beta)$ determines completely the reaction cross section, consequently, we only need to calculate $\Psi^{(+)}(\vec{k}_\alpha)$ in the asymptotic region to obtain the transition matrix. That is we have now to solve the problem

$$H\Psi^{(+)} = \left[-\frac{\hbar^2}{2\mu\beta}\nabla^2_{r_\beta} + H(\xi_\beta) - E\right]\Psi^{(+)} = -V_\beta\Psi^{(+)} \tag{4.1}$$

$$\xi_B + \xi_b = \xi_\beta \tag{4.2}$$

$$\Psi_\beta = \Psi_a(\xi_a)\Psi_B(\xi_B) \tag{4.3a}$$

$$V_\beta = \sum_{\substack{i\in b\\ j\in B}} V(|\vec{r}_i - \vec{r}_j|) \tag{4.3b}$$

$$H_\beta(\xi_\beta)\Psi_\beta(\xi_\beta) = \epsilon_\beta\Psi_\beta(\xi_\beta) \tag{4.4}$$

$$\mu_\beta = \frac{M_b M_B}{M_b + M_B} \tag{4.5}$$

ϵ_β : intrinsic energy of system (b, B).
E : total energy.
$E - \epsilon_\beta$ = kinetic energy in channel β. Then we can define

$$E - \epsilon_\beta = \frac{\hbar^2 k_\beta^2}{2\mu_\beta}. \tag{4.6}$$

Let us introduce an arbitrary potential $U_\beta(r_\beta)$, that is a function of only the magnitude of the channel radius variable $|\vec{r}_\beta|$. This potential is chosen, in general, in such a way that it reproduces the average effect of the interaction V_β, i.e. $U_\beta(r_\beta) \approx \int d\xi_\beta \Psi^*_\beta(\xi_\beta) V_\beta \Psi_\beta(\xi_\beta)$ (This point is taken up again in more detail later).

With the aid of eqs. (4.4) and (4.6) we can rewrite (4.1) as

$$\left(-\frac{\hbar^2}{2\mu_\beta}\nabla^2_{r_\beta} + U(r_\beta) - \frac{\hbar^2 k_\beta^2}{2\mu_\beta}\right)\Psi^{(+)} = -V'_\beta \Psi^{(+)} \tag{4.7}$$

$$\frac{\hbar^2}{2\mu_\beta} U(r_\beta) = \bar{U}(r_\beta) \tag{4.8a}$$

$$\left(V_\beta - U(r_\beta)\right) = V'_\beta(r_\beta) \tag{4.8b}$$

$$\left(-\nabla^2_{r_\beta} + \bar{U}(r_\beta) - k_\beta^2\right)\left\langle \psi(\xi_\beta), \Psi^{(+)}\right\rangle = -\frac{2\mu_\beta}{\hbar^2}\left\langle \psi(\xi_\beta), V'_\beta \Psi^{(+)}\right\rangle \tag{4.8c}$$

$$\varphi(\vec{r}_\beta) = \left\langle \psi(\xi_\beta), \Psi^{(+)}\right\rangle \tag{4.9}$$

$$\left(-\nabla^2_{r_\beta} + \bar{U}(r_\beta) - k_\beta^2\right)\varphi(\vec{r}_\beta) = -\frac{2\mu_\beta}{\hbar^2}\left\langle \psi(r_\beta), V'_\beta \Psi^{(+)}\right\rangle \tag{4.10}$$

The asymptotic form of $\varphi(\vec{r}_\beta)$ will determine the cross section.

Let $\chi^{(+)}_\beta(\vec{k}_\beta, \vec{r}_\beta)$ be the scattering wavefunction governed by the potential $\bar{U}(r_\beta)$, so that

$$\left(-\nabla^2_{r_\beta} + \bar{U}(r_\beta) - k_\beta^2\right)\chi^{(+)}_\beta(\vec{k}_\beta, \vec{r}_\beta) = 0 \tag{4.11}$$

The magnitude of k_β is given by eq. (4.6). Let us expand $\chi^{(+)}_\beta(\vec{k}_\beta, \vec{r}_\beta)$ in partial waves, i.e.

$$\chi^{(+)}_\beta(k_\beta, r_\beta) = \frac{4\pi}{\vec{k}_\beta \vec{r}_\beta} \sum_{l,m} i^l f_{\beta l}(k_\beta, r_\beta) Y^l_m(\hat{r}_\beta) Y^{l*}_m(\hat{k}_\beta) \tag{4.12}$$

(we assume we are dealing with neutral particles i.e. the Coulomb field is equal to zero. Otherwise one has to introduce the Coulomb phase $\sigma_{\beta l} = \arg\Gamma(l + 1 + i\eta_\beta)$). Replacing the function (4.12) in eq. (4.11) we obtain

$$\left\{-\frac{d^2}{dr_\beta^2} + \frac{l(l+1)}{r_\beta^2} + \bar{U}(r_\beta) - k_\beta^2\right\} f_{\beta l}(k_\beta, r_\beta) = 0 \tag{4.13}$$

To solve (4.13) we impose the following boundary conditions

$$\lim_{r_\beta \to \infty} \chi^{(+)}_\beta(\vec{k}_\beta, \vec{r}_\beta) \longrightarrow e^{i\vec{k}_\beta \vec{r}_\beta} + \text{outgoing scattered waves} \tag{4.14a}$$

$$f_{\beta l}(k_\beta, r_\beta = 0) = 0 \tag{4.14b}$$

We can easily see that the boundary condition (4.14a) , (4.14b) can be recasted, with the help of eq. (4.12) into the condition

$$\lim_{r_\beta \to \infty} f_{\beta l}(k_\beta, r_\beta) \longrightarrow \frac{1}{2}\left[e^{-i(k_\beta r_\beta - \frac{l\pi}{2})} - \eta_{\beta l} e^{i(k_\beta r_\beta - \frac{l\pi}{2})}\right] \tag{4.15}$$

(see Appendix) where $\eta_{\beta l}$ is the reflection coefficient for the l^{th} partial wave. From the condition (4.14b), we see that f_l is a regular solution.

The corresponding irregular solution $h_{\beta l}(k_\beta, r_\beta)$ to eq. (4.13) has the following asymptotic properties

$$\lim_{r_\beta \to 0} h_{\beta l}(k_\beta, r_\beta) \longrightarrow \frac{1}{(k_\beta r_\beta)^{l+2}} \tag{4.16}$$

$$\lim_{r_\beta \to \infty} h_{\beta l}(k_\beta, r_\beta) \longrightarrow e^{i(k_\beta r_\beta - \frac{l\pi}{2})} \tag{4.17}$$

In the appendix it is proved that

$$\hat{O} G_{\beta l}(r_\beta, r'_\beta) = \delta(r_\beta - r'_\beta) \tag{4.18}$$

where

$$\hat{O} = -\frac{d^2}{dr_\beta^2} + \frac{l(l+1)}{r_\beta^2} + \bar{U}(r_\beta) - k_\beta^2 \tag{4.19}$$

and

$$G_{\beta l}(r_\beta, r'_\beta) \equiv \begin{cases} -\frac{1}{ik} f_{\beta l}(k_\beta, r_\beta) h_{\beta l}(k_\beta, r'_\beta) & (r_\beta < r'_\beta) \\ -\frac{1}{ik} h_{\beta l}(k_\beta, r_\beta) f_{\beta l}(k_\beta, r'_\beta) & (r_\beta > r'_\beta) \end{cases} \tag{4.20}$$

Then

$$z(r_\beta) \equiv \int_0^\infty G_{\beta l}(r_\beta, r'_\beta) y(r'_\beta) dr'_\beta \tag{4.21}$$

satisfies

$$\left(-\frac{d^2}{dr_\beta^2} + \frac{l(l+1)}{r_\beta^2} + \bar{U}(r_\beta) - k_\beta^2\right) z(r_\beta) = y(r_\beta) \tag{4.22}$$

Using these results we can prove that, for an arbitrary function $y(\vec{r}_\beta)$ (note now the angular dependence $\vec{r}_\beta \equiv (r_\beta, \hat{r}_\beta)$), the function

$$z(\vec{r}_\beta) \equiv \int_0^\infty G_{\beta l}(\vec{r}_\beta, \vec{r'}_\beta) y(\vec{r'}_\beta) d\vec{r'}_\beta \tag{4.23}$$

satisfies

$$\left(-\frac{d^2}{dr_\beta^2} + \bar{U}(r_\beta) - k_\beta^2\right) z(\vec{r}_\beta) = y(\vec{r}_\beta) \tag{4.24}$$

where now

$$G_{\beta l}(\vec{r}_\beta, \vec{r'}_\beta) \equiv \sum_{l,m} \frac{Y^l_m(\hat{r}_\beta) Y^{*l}_m(\hat{r}'_\beta)}{-i\,k\,r_\beta\,r'_\beta} \begin{cases} f_{\beta l}(k_\beta, r_\beta) h_{\beta l}(k_\beta, r'_\beta) & (r_\beta < r'_\beta) \\ h_{\beta l}(k_\beta, r_\beta) f_{\beta l}(k_\beta, r'_\beta) & (r_\beta > r'_\beta) \end{cases} \tag{4.25}$$

By comparing eqs. (4.10) and (4.23), (4.24) and (4.25) we obtain

$$\varphi(\vec{r}_\beta) = -\frac{2\mu_\beta}{\hbar^2} \int G(\vec{r}_\beta, \vec{r'}_\beta) \left\langle \psi_\beta(\xi_\beta), V'_\beta \Psi^{(+)} \right\rangle d\vec{r'}_\beta \tag{4.26}$$

We are interested in the asymptotic form of this wave function, namely for $r_\beta \longrightarrow \infty \quad (r_\beta \gg r'_\beta)$. Physically r'_β has the dimensions of the nuclear system (as both $U(r'_\beta)$ and V'_β go to zero for $r'_\beta \gg R_0$, where R_0 is the nuclear radius), and r_β stands for the distance of the detector from the target. Then

$$\begin{aligned} \varphi(\vec{r}_\beta) \xrightarrow[r_\beta \to \infty]{} -\frac{2\mu_\beta}{\hbar^2} \int_{r'_\beta = 0} \sum_{l,m} \frac{Y^l_m(\hat{r}_\beta) Y^{*l}_m(\hat{r}'_\beta)}{-i\,k_\beta\,r_\beta\,r'_\beta} e^{i(k_\beta r_\beta - l\pi/2)} f_{\beta l}(k_\beta, r'_\beta) \\ \times \left\langle \psi_\beta(\xi_\beta), V'_\beta \Psi^{(+)} \right\rangle d^3 r'_\beta, \quad e^{il\pi/2} = i^l \end{aligned} \tag{4.27}$$

$$\begin{aligned} \varphi(\vec{r}_\beta) \xrightarrow[r_\beta \to \infty]{} -\frac{2i\mu_\beta}{\hbar^2} \frac{e^{ik_\beta r_\beta}}{r_\beta} \sum_{l,m} i^{-l} Y^l_m(\hat{r}_\beta) \\ \times \int_{r'_\beta = 0} \frac{Y^{*l}_m(\hat{r}'_\beta)}{k_\beta r'_\beta} f_{\beta l}(k_\beta, r'_\beta) \left\langle \psi_\beta(\xi_\beta), V'_\beta \Psi^{(+)} \right\rangle d^3 r'_\beta \end{aligned} \tag{4.28}$$

Assuming $\hat{r}_\beta = \hat{k}_\beta$ it is easy to see that

$$\begin{aligned} \chi^{*(-)}(\vec{k}_\beta, \vec{r'}_\beta) &= \chi^{(+)}(-\vec{k}_\beta, \vec{r'}_\beta) \\ &= \frac{4\pi}{k_\beta r'_\beta} \sum_{l,m} i^{-l} Y^{*l}_m(\hat{r}'_\beta) Y^l_m(\hat{k}_\beta) f_{\beta l}(k_\beta, r'_\beta) \end{aligned} \tag{4.29}$$

Equation (4.28) can then be rewritten as

$$\varphi(\vec{r}_\beta) \xrightarrow[r_\beta \to \infty]{} -\frac{2i\mu_\beta}{4\pi\hbar^2} \frac{e^{ik_\beta r_\beta}}{r_\beta} \left\langle \psi_\beta(\xi_\beta) \chi^{(-)}(\vec{k}_\beta, \vec{r'}_\beta), V'_\beta \Psi^{(+)} \right\rangle \tag{4.30}$$

Then from eqs. (**??**) and (**??**) (second lecture) we can write

$$\frac{d\sigma}{d\Omega} = \frac{k_\beta}{k_\alpha} \frac{\mu_\alpha \mu_\beta}{4\pi^2 \hbar^4} \left| \left\langle \psi_\beta(\xi_\beta) \chi^{(-)}(\vec{k}_\beta, \vec{r'}_\beta), V'_\beta \Psi^{(+)} \right\rangle \right|^2 \tag{4.31}$$

So far the result is exact, if the exact wave function $\Psi^{(+)}$ is used. If reaction processes (other than those represented by the optical potentials) are weak compared to elastic scattering, the scattering function $\Psi^{(+)}$ is well represented by the elastic channel alone, i.e.

$$\Psi^{(+)} = \psi(\xi_\alpha) \chi^{(+)}_\alpha(k_\alpha, r_\alpha) \tag{4.32}$$

This approximation is known as the DWBA. The transition matrix element is then equal to

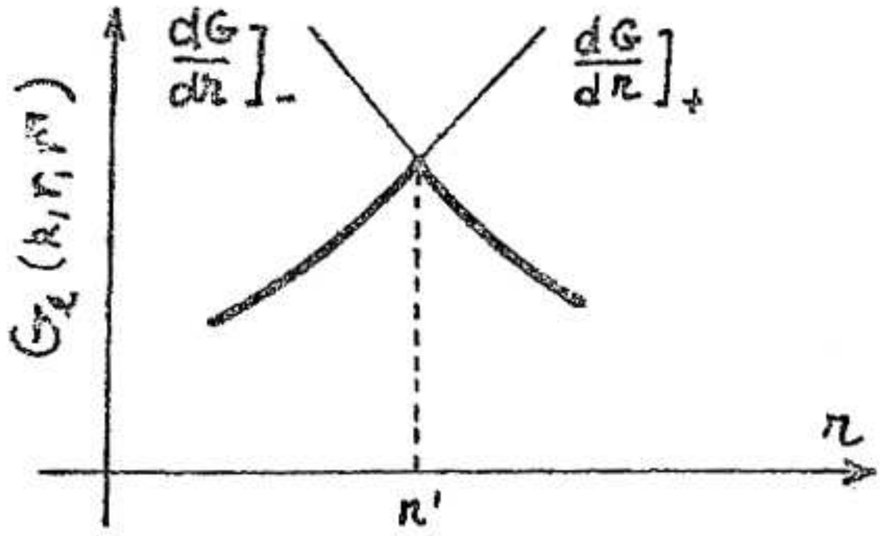

Figure 4.1:

$$
\begin{aligned}
T_{\alpha\beta}^{DWBA} &= \left\langle \psi_\beta(\xi_\beta)\chi_\beta^{(-)}(\vec{k}_\beta, \vec{r}_\beta), V'_\beta \psi_\alpha(\xi_\alpha)\chi_\alpha^{(+)}(\vec{k}_\alpha, \vec{r}_\alpha) \right\rangle \\
&= \int d\vec{r}_\alpha d\vec{r}_\beta \chi_\beta^{*(-)}(\vec{k}_\beta, \vec{r}_\beta) \left\langle \psi_\beta, V'_\beta \psi_\alpha \right\rangle \chi_\alpha^{(+)}(\vec{k}_\alpha, \vec{r}_\alpha)
\end{aligned}
\tag{4.33}
$$

The magnitude $V_{eff}(\vec{r}_\alpha, \vec{r}_\beta) = \langle \psi_\beta, V'_\beta \psi_\alpha \rangle$ can be considered as an effective interaction (formfactor) connecting the entrance and exit channel scattering states. Note that the integral in eq. (4.33) is six–dimensional, and that the relative motion is connected to the intrinsic motion. If one deals with inelastic scattering or one neglects the effects of recoil, the integral becomes three dimensional. In fact, one dimensional in the relative motion variable, as the angular integration gives rise to finite summations.

4.3 Appendix. Green's function

Let us assume that $f_l(k,r)$ and $g_l(k,r)$ are the regular and irregular solutions of the differential equation (4.13), i.e.

$$
\left(-\frac{d^2}{dr_\beta^2} + \frac{l(l+1)}{r_\beta^2} + \bar{U}(r_\beta) - k_\beta^2 \right) \begin{cases} f_l(k,r) = 0 \\ g_l(k,r) = 0 \end{cases}
\tag{4.34}
$$

We want to prove that the function

$$
G_l(k,r,r') = f_l(k,r_<) g_l(k,r_>)
\tag{4.35}
$$

fulfills

$$
\left(-\frac{d^2}{dr_\beta^2} + \frac{l(l+1)}{r_\beta^2} + \bar{U}(r_\beta) - k_\beta^2 \right) G_l(k,r,r') = \delta(r-r') \times \text{constant}
\tag{4.36}
$$

In eq.(4.35), $r_<$ and $r_>$ denote the smallest and largest of the quantities r and r', i.e. we can write (4.35) as

$$G_l(k,r,r') = \begin{cases} g_l(k,r)f_l(k,r') & (r > r') \\ g_l(k,r')f_l(k,r) & (r < r') \end{cases} \tag{4.37}$$

multiplying the first equation by $g_l(r)$ and the second by $f_l(r)$ and subtracting, we obtain

$$\begin{aligned} g_l(k,r)\frac{d^2 f_l(k,r)}{dr^2} - f_l(k,r)\frac{d^2 g_l(k,r)}{dr^2} &= 0 \\ = \frac{d}{dr}(f_l'(k,r)g_l(k,r) - f_l(k,r)g_l'(k,r)) \end{aligned} \tag{4.38}$$

Then

$$f_l'(k,r)g_l(k,r) - f_l(k,r)g_l'(k,r) = \text{constant} \tag{4.39}$$

But

$$r < r' = r + \epsilon \qquad \frac{dG_l(k,r,r+\epsilon)}{dr} = f_l'(k,r)g_l(k,r+\epsilon) \tag{4.40a}$$

$$r > r' = r - \epsilon \qquad \frac{dG_l(k,r,r-\epsilon)}{dr} = f_l(k,r-\epsilon)g_l'(k,r) \tag{4.40b}$$

where $\epsilon > 0$. Let us take the differences between (4.40a) and (4.40b) and make $\epsilon \to 0$, i.e.

$$\begin{aligned} &\lim_{\epsilon\to 0}\left(\frac{dG_l(k,r,r+\epsilon)}{dr} - \frac{dG_l(k,r,r-\epsilon)}{dr}\right) \\ &\quad = \lim_{\epsilon\to 0} f_l'(k,r)g_l(k,r+\epsilon) - f_l(k,r-\epsilon)g_l'(k,r) \\ &\quad = f_l'(k,r)g_l(k,r) - f_l(k,r)g_l'(k,r) = \text{constant} \end{aligned} \tag{4.41}$$

where we have made use of eq. (4.39). Eq. (4.41) says that the function $G_l(k,r,r')$ has a finite discontinuity at $r = r'$.

Note that $G_l(k,r,r')$ is a continuous function as $\lim_{\epsilon\to 0} G_l(k,r,r+\epsilon) = \lim_{\epsilon\to 0} G_l(k,r+\epsilon,r) = f_l(k,r)g_l(k,r)$.

Let us now assume that $\varphi(r)$ is a continuous function, which has also continuous derivatives to all orders.

We calculate now the integral

$$\begin{aligned} I &= \int_a^b \left\{-\frac{d^2 G_l(k,r,r')}{dr^2} + \left(\frac{l(l+1)}{r^2} + \bar{U}(r) - k^2\right)G_l(k,r,r')\right\}\varphi(r)dr \\ &= \underbrace{\int_a^{r'-\epsilon}}_{1} + \underbrace{\int_{r'-\epsilon}^{r'+\epsilon}}_{2} + \underbrace{\int_{r'+\epsilon}^{b}}_{3} \end{aligned} \tag{4.42}$$

We have assumed that the point with coordinate r' is contained in the interval (a,b) In the integral (1) $a \le r \le r' - \epsilon \therefore$

$$G_l(k,r,r') = g_l(k,r')f_l(k,r) \quad (r < r')$$

Then

$$(1) = \int_a^{r'-\epsilon} dr\,\varphi(r)g_l(k,r')\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \bar{U}(r) - k^2\right)f_l(k,r) = 0 \tag{4.43}$$

because of eq. (4.34).

The sama can be said of the integral (3), as $g_l(k,r)$ is also solution of the differential equation.

We have to consider only the integral (2), i.e.

$$\begin{aligned}(2) = &-\int_{r'-\epsilon}^{r'+\epsilon} \frac{d^2}{dr^2}G_l(k,r,r')\varphi(r)\,dr \\ &+\int_{r'-\epsilon}^{r'+\epsilon}\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \bar{U}(r) - k^2\right)G_l(k,r,r')\varphi(r)\,dr\end{aligned} \tag{4.44}$$

We now take the limit of this integral for $\epsilon \to 0$. In this case, the second term is equal to zero, as we are integrating a continuous function $G_l(k,r,r')\varphi(r)$ over an interval of zero measure. Then

$$\begin{aligned}I = &-\lim_{\epsilon\to 0}\int_{r'-\epsilon}^{r'+\epsilon} \frac{d^2}{dr^2}G_l(k,r,r')\varphi(r)\,dr \\ = &-\lim_{\epsilon\to 0}\left[\frac{d}{dr}G_l(k,r,r')\varphi(r)\right]_{r'-\epsilon}^{r'+\epsilon} \\ &+\lim_{\epsilon\to 0}\int_{r'-\epsilon}^{r'+\epsilon}\frac{d}{dr}G_l(k,r,r')\frac{d\varphi(r)}{dr}\,dr\end{aligned} \tag{4.45}$$

The first term may be different from zero, as $\frac{d}{dr}G_l(k,r,r')$ has a finite discontinuity and consequently $\frac{d^2}{dr^2}G_l(k,r,r')$ can have an infinite (but measurable) discontinuity. The second term in (4.45) is zero as is the integral of the product of a continuous function ($\varphi(r)$) and a function with a finite discontinuity ($\frac{d}{dr}G_l(k,r,r')$) over an interval of zero measure. Then

$$\begin{aligned}I = &-\lim_{\epsilon\to 0}\left[\frac{d}{dr}G_l(k,r,r')\varphi(r)\right]_{r'-\epsilon}^{r'+\epsilon} \\ = &\lim_{\epsilon\to 0}\left\{\frac{d}{dr}G_l(k,r'+\epsilon,r')\varphi(r'+\epsilon)\right. \\ &\left.-\frac{d}{dr}G_l(k,r'-\epsilon,r')\varphi(r'-\epsilon)\right\} = \text{constant}\cdot\varphi(r')\end{aligned} \tag{4.46}$$

where use has been made of eq. (4.41).

From eqs. (4.42) and (4.46) we obtain

$$\int_a^b\left\{-\frac{d^2G_l(k,r,r')}{dr^2} + \left(\frac{l(l+1)}{r^2} + \bar{U}(r) - k^2\right)G_l(k,r,r')\right\}\varphi(r)dr = \text{constant}\cdot\varphi(r')$$

which proves eq.(4.36).

(check)

Appendix X W: Elastic scattering cross section in DWBA

Let us consider the scattering by a potential $U(r)$. The scattering wavefunction (DWBA) can be written as

$$X^{(+)}(\vec{k},\vec{r}) = \frac{4\pi}{r}\sum_{l,m} i^l f_l(k,r) Y_m^{l*}(\hat{k}) Y_m^l(\hat{r}) \tag{X.1}$$

where $f_l(k,r)$ is solution of the equation

$$\left\{\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \bar{U} - k^2\right\} f_l(k,r) = 0 \tag{X.2}$$

with the boundary condition

$$f_l(k,0) = 0 \ . \tag{X.3}$$

Starting at $r = 0$ integrate out to $r \to \infty$, imposing the asymptotic condition

$$\lim_{r\to\infty} f_l(k,r) \to \frac{1}{2ikr}\left[\mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)} - \eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}\right] \ . \tag{X.4}$$

Of notice that

$$\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} = \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}\mathrm{e}^{\ln\eta_l} = \mathrm{e}^{i\left(kr-\frac{l\pi}{2}-i\ln\eta_l\right)} \ . \tag{X.5}$$

In other words, η_l is associated with the phase shift included by the potential $U(r)$.

Replacing in (X.12) one obtains,

$$\begin{aligned} X^{(+)}(\vec{k},\vec{r}) &\to -4\pi\sum_l i^l \frac{\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} - \mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)}}{2ikr}\sum_m Y_m^{l*}(\hat{k})Y_m^l(\hat{r}) \ , \\ &= -4\pi\sum_l i^l \frac{\eta_l \mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)} - \mathrm{e}^{-i\left(kr-\frac{l\pi}{2}\right)} + 2i\sin\left(kr-\frac{l\pi}{2}\right)}{2ikr}\frac{2l+1}{4\pi}P_l(\hat{k},\hat{r}) \ , \\ &= -\sum_l i^l(2l+1)P_l(\hat{k},\hat{r})\frac{\sin\left(kr-\frac{l\pi}{2}\right)}{kr} \ , \\ &= -\sum_l i^l \frac{(\eta_l - 1)}{2ikr}\mathrm{e}^{i\left(kr-\frac{l\pi}{2}\right)}(2l+1)P_l(\hat{k},\hat{r}) \ . \end{aligned} \tag{X.6}$$

Making use of the plane wave expansion

$$\mathrm{e}^{i\vec{k}\cdot\vec{r}} = \sum_l i^l(2l+1)P_l(\cos\theta')\mathcal{I}_l(kr) \ , \tag{X.7}$$

and of the asymptotic expression of the Bessel function

$$\mathcal{I}_l(kr) \to_{r\to\infty} \frac{\sin\left(kr-\frac{l\pi}{2}\right)}{kr} \ , \tag{X.8}$$

one obtains,

$$\lim_{r\to\infty} \mathrm{e}^{i\vec{k}\cdot\vec{r}} = \sum_l i^l(2l+1)P_l(\cos\theta')\frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} \ . \tag{X.9}$$

Making also use of the relation

$$\mathrm{e}^{-i\frac{\pi l}{2}} = \cos\frac{l\pi}{2} - i\sin\frac{l\pi}{2} = \left\{\begin{array}{ll} -i & l=1 \\ -1 & l=2 \\ +i & l=3 \\ +1 & l=4 \end{array}\right\} = (-1)^l i^l = i^{-l}$$

on can write

$$\lim_{r\to\infty} X^{(+)}(\vec{k},\vec{r}) \to -\mathrm{e}^{i\vec{k}\cdot\vec{r}} - \frac{\mathrm{e}^{ikr}}{r}\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r}) \ . \tag{X.10}$$

From the relation

$$\frac{\mathrm{d}\sigma_{el}(\theta)}{\mathrm{d}\Omega} = |f_{\alpha\alpha'}(E,\theta)|$$

where

$$\Psi_{\text{elast scatt}} \to \mathrm{e}^{i\vec{k}\cdot\vec{r}} + \frac{\mathrm{e}^{ikr}}{r}f_{\alpha\alpha'}(E,\theta)$$

one obtains

$$\left(\frac{\mathrm{d}\sigma_{el}(\theta)}{\mathrm{d}\Omega}\right)_{DWBA} = \left|\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r})\right|^2 \ . \tag{X.11}$$

1. Order of magnitude estimate

Elastic cross section ^{16}O+^{208}Pb at $E_{lab} \approx 85$ MeV

$$E_B = \frac{Z_A Z_A e^2}{r_b}\left(1 - \frac{a}{r_B}\right) \qquad \left.\frac{\mathrm{d}U}{\mathrm{d}r}\right|_{r_B} = 0$$

$$r_B = \left[1.07\left(A_a^{1/3} + A_A^{1/3}\right) + 2.72\right] \ \text{fm} = [(2.7 + 6.3) + 2.72] \ \text{fm} \approx 11.7 \ \text{fm}$$

$$a = 0.65 \ \text{fm}$$

$$e^2 = 1444 \ \text{fm MeV}$$

$$E_B = \frac{8\times 82\times 1.44 \ \text{MeV fm}}{11.7 \ \text{fm}}\left(1 - \frac{0.65 \ \text{fm}}{11.7 \ \text{fm}}\right) \approx 80\times 0.9 \approx 72 \ \text{MeV}$$

Grazing angular momentum

$$L_g = (r_g)_{fm}\left(\frac{1}{20}\frac{A_a A_A}{A_a + A_A}(E - E_B)_{\mathrm{MeV}}(1+c)\right)^{1/2}\hbar$$

$$r_g = r_B - \delta = 11.7\ \mathrm{fm} - 0.13\ \mathrm{fm} \approx 11.6\ \mathrm{fm}$$

$$\delta = a\ln\left(1 + \frac{2(E - E_B)}{E_B}\right) = 0.13\ \mathrm{fm}$$

$$c \approx \frac{2a}{r_B}\frac{E - E_B}{E_B} = 0.10 \times 0.11 = 0.01$$

$$L_g = 11.6(0.05 \times 14.9 \times 8.6 \times 1.01)^{1/2}\hbar \approx 30\hbar$$

$$E_{CM} = \frac{M_{\text{target}}}{M_{\text{target}} + M_{\text{project}}}E_{\text{lab}} = \frac{208}{208+16}85\ \mathrm{MeV} = 0.93 \times 85\ \mathrm{MeV} \approx 79\ \mathrm{MeV}$$

$$E = \frac{\hbar^2 k^2}{2\mu}; \qquad k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

$$\mu = \frac{16 \times 208}{224}M_n = 14.9M_n; \qquad \frac{2\mu}{\hbar^2} \approx 30\frac{M_c^2}{(\hbar c)^2} = \frac{30}{40\ \mathrm{fm}^2\ \mathrm{MeV}} = 0.75\ \mathrm{fm}^{-2}\ \mathrm{MeV}^{-1}$$

$$k = \sqrt{\frac{0.75}{\mathrm{MeV\ fm}^2} \times 79\ \mathrm{MeV}} \approx 7.7\ \mathrm{fm}^{-1}$$

$$\hbar k(R_a + R_A) \approx 8\ \mathrm{fm}^{-1} \times (3\ \mathrm{fm} + 7\ \mathrm{fm})\hbar \approx 80\hbar$$

which is completely out of mark (see Fig.).

To get the grazing angular momentum we have to use, instead of E the quantity $E_{CM} - E_B \approx 7$ MeV. Thus

$$k = \sqrt{\frac{2\mu}{\hbar^2}(E_{CM} - E_B)} = \sqrt{\frac{0.75}{\mathrm{fm}^2\ \mathrm{MeV}} \times 7\ \mathrm{MeV}} = 2.3\ \mathrm{fm}^{-1}$$

$$\hbar k(R_a + R_A) \approx 2.3\ \mathrm{fm}^{-1} \times 10.13\ \mathrm{fm}\hbar \approx 23\hbar$$

which is not inconsistent with L_g.

$$L_R \approx \eta_{cl}\cot\frac{\Theta_R}{2}$$

$$\Theta_R = 2\cot^{-1}\left(\frac{L_R}{\eta_{cl}}\right) = 2\cot^{-1}\left(\frac{30}{139}\right)$$

$$\cot x = \frac{1}{\tan x}$$

$$\cot\frac{\Theta_R}{2} = \frac{L_R}{\eta_{cl}}$$

$$\frac{1}{\tan\frac{\Theta_R}{2}} = \frac{L_R}{\eta_{cl}}$$

$$\frac{\eta_{cl}}{L_R} = \tan\frac{\Theta_R}{2}$$

$$\Theta_R = 2\arctan\frac{\eta_{cl}}{L_R} = 111°$$

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$$\eta_{cl} = \frac{Z_a Z_A e^2}{\hbar v} = \frac{8 \times 82 \times 1.44 \text{ MeV fm}}{\hbar v} = \frac{945 \text{ MeV fm}}{\hbar v}$$

$$\hbar k = \mu v \;; \qquad v = \frac{\hbar k}{\mu}$$

To calculate the parameter η_{cl} (how classical the trajectory is) we need to use E_{lab} (asymptotic condition). Thus

$$\hbar v = \frac{\hbar^2 k}{\mu} = \frac{(\hbar c)^2}{Mc^2}\frac{1}{15} \times 8 = \frac{40 \times 8}{15} \text{ fm MeV} \approx 21.3$$

$$\eta_{cl} = \frac{945}{21.3} \approx 44$$

$$\eta_l \approx 0.5 \;; \qquad \eta_l = 1 - 0.5 = 0.5 \;; \qquad P_l = \frac{1}{2}$$

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} &= \left|\sum_l \frac{(\eta_l - 1)}{2ik}(2l+1)P_l(\hat{k}\cdot\hat{r})\right|^2 \\
&\sim \frac{(0.5)^2}{4k^2}\frac{(2L_g+1)^2}{4} \sim \frac{(61)^2}{64 \times (2.5 \text{ fm}^{-1})^2}\frac{1}{\text{sr}} \\
&\sim 9 \text{ fm}^{-2} = 0.09 \times 10^2 \times 10^{-26}\frac{\text{cm}^2}{\text{sr}} \\
&\approx 0.1 \text{ b} = \frac{100 \text{ mb}}{\text{sr}} \qquad (\Theta = \Theta_g)
\end{aligned}$$

$$\frac{\mathrm{d}\sigma(\Theta \approx 110°)}{\mathrm{d}\omega} \sim \frac{100 \text{ mb}}{\text{sr}}$$

appendix δ

~~2.~~ Elastic scattering

The scattered wave (asymptotic region) must be the solution of the *free field equation*

$$H_\alpha \psi_{\text{scatt}} = E\psi_{\text{scatt}} \ , \tag{X.12}$$

with

$$E = \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha} \ , \tag{X.13}$$

where

$$H_\alpha = T_\alpha \ , \tag{X.14}$$

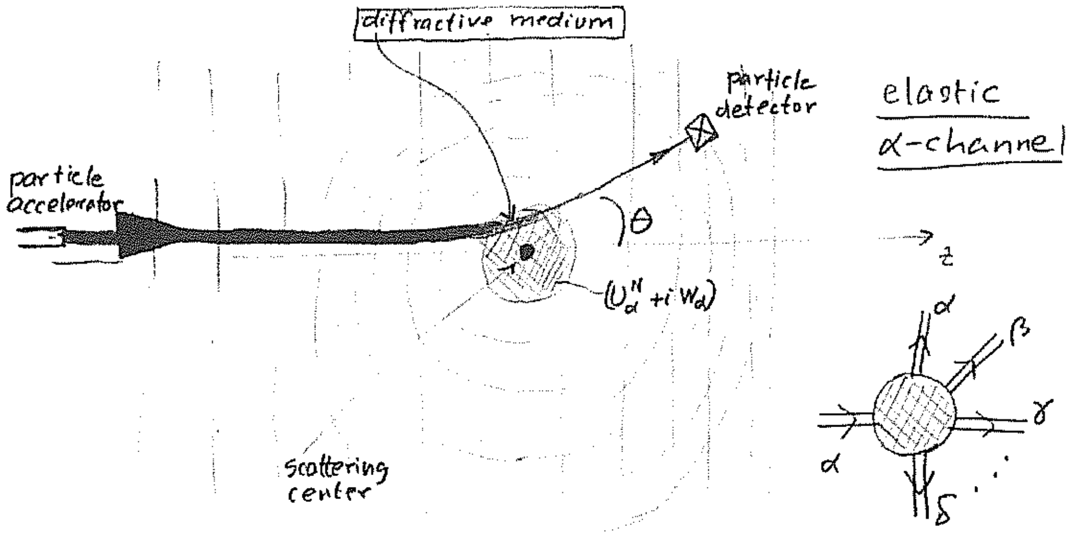
with

$$T_\alpha = \frac{p^2}{2\mu_\alpha} = -\frac{\hbar^2}{2\mu_\alpha}\nabla^2 = -\frac{\hbar^2}{2\mu_\alpha}\left\{\frac{1}{r_\alpha^2}\frac{\partial}{\partial r_\alpha}\left(r_\alpha^2\frac{\partial}{\partial r_\alpha}\right) + \frac{\hat{L}_2}{r\alpha^2}\right\} \ , \tag{X.15}$$

and

$$\mu_\alpha = \frac{M_a M_A}{M_a + M_A} \ , \tag{X.16}$$

is the reduced mass in the entrance channel $\alpha \equiv (a, A)$, while k_α and r_α are the relative momentum and coordinate associated with the elastic process $a + A \to a + A$.



Asymptotically $r \gg R_a + R_A, H_\alpha = T_\alpha$ and the centrifugal barrier term drops out as $1/r^2$, and can be neglected. Consequently, the asymptotic solution is

$$\psi_{\text{scatt}} = \frac{e^{ik_\alpha r_\alpha}}{r_\alpha} f_{\alpha\alpha}(E, \theta, \phi)\psi_\alpha(\xi_\alpha) \ . \tag{X.17}$$

Making use of the fact that the incident scattering wavefunction is

$$\psi_{\rm inc} = e^{ik_\alpha z_\alpha} , \tag{X.18}$$

one can calculate the differential elastic scattering cross section, as the ratio of the scattered flux going through an asymptotic differential surface perpendicular to the (asymptotic) scattering beam and the incident intensity. One obtains (see appendix Y)

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\omega} = |f_{\alpha\alpha}(E,\theta)|^2 , \tag{X.19}$$

making the assumption of the spherical symmetry of the process (independent on ϕ, $\frac{1}{2}\int_0^\pi \mathrm{d}\phi \sin\phi = \frac{1}{2}\int_0^\pi(-\mathrm{d}\cos\theta) = \frac{1}{2}(\cos\theta)_0^\pi = 1$).

The next step consists in the calculation of $f_{\alpha\alpha}(E,\theta,\phi)$. To do this we need to find the scattering wavefunction $\Psi^{(+)}(\vec{k_\alpha},\vec{r_\alpha},\xi_\alpha)$, that is the solution of the full Hamiltonian

$$\begin{aligned} H &= T_a + T_A + H_a + H_A + V_{aA} \\ &= T_{CM} + T_\alpha + H_\alpha + V_\alpha \\ (&= T_{CM} + T_\beta + H_\beta + V_\beta = \ldots) \end{aligned} \tag{X.20}$$

in the center of mass system. In other words, one has to solve the equation

$$\left[-\frac{\hbar^2}{2\mu_\alpha}\nabla^2_{r_\alpha} + H_\alpha(\xi_\alpha) + V_\alpha(\xi_\alpha;\vec{r_\alpha})\right]\Psi_\alpha^{(+)}(\vec{k_\alpha},\vec{r_\alpha}) = E_\alpha\Psi_\alpha^{(+)}(\vec{k_\alpha},\vec{r_\alpha}) , \tag{X.21}$$

where

$$\mu_\alpha = \frac{M_a M_A}{M_a + M_A} , \tag{X.22}$$

$$\xi_\alpha = \xi_a + \xi_A , \tag{X.23}$$

$$\phi_\alpha(\xi_\alpha) = \phi_a(\xi_a)\phi_A(\xi_A) , \tag{X.24}$$

and

$$H_\alpha\phi_\alpha(\xi_\alpha) = \varepsilon_\alpha\phi_\alpha(\xi_\alpha) . \tag{X.25}$$

Of notice that the relative, intrinsic variables ξ_α refer to the structure information contained int he "exact" scattering wavefunction $\Psi_\alpha^{(+)}(\vec{k_\alpha},\vec{r_\alpha};\xi_\alpha)$. In particular

$$H_\alpha(\xi_\alpha)\Psi_\alpha^{(+)} = \varepsilon_\alpha\Psi_\alpha^{(+)} , \tag{X.26}$$

where $H_\alpha(\xi_\alpha) = H_a(\xi_a) + H_A(\xi_A)$ is the total intrinsic Hamiltonian describing the structure of nuclei a and A by themselves, that is, far away from the range in which the interaction

$$V_\alpha(\vec{r_\alpha};\xi_\alpha) = \sum_{\substack{i\in a, j\in A \\ i<j}} V_\alpha(|\vec{r_i} - \vec{r_j}|) , \tag{X.27}$$

is effective.

As seen from the above relations, the structure information associated with the intrinsic degrees of freedom related to the variables ξ_α are melted together the scattering degrees of freedom associated with the variables $(\vec{k_\alpha}, \vec{r_\alpha})$.

Introducing the mean field potential

$$U_\alpha(r_\alpha) = \int d\xi_\alpha |\psi_\alpha(\xi_\alpha)|^2 V_\alpha(\vec{r}, \xi_\alpha) , \tag{X.28}$$

one can write

$$\left(-\frac{\hbar^2}{2\mu_\alpha}\nabla_r^2 + U_\alpha - \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha}\right)\Psi^{(+)} = -V'_\alpha \Psi^{(+)} , \tag{X.29}$$

in keeping with the fact that the difference $E_\alpha - \varepsilon_\alpha$ is equal to the kinetic energy of relative motion in channel α, that is,

$$E_\alpha - \varepsilon_\alpha = \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha} . \tag{X.30}$$

Multiplying the above Schrödinger equation from the left with the intrinsic wavefunction $\psi_\alpha(\xi_\alpha)$, integrating over ξ_α and introducing the definitions

$$V'_\alpha = V_\alpha - U_\alpha , \tag{X.31}$$

$$\bar{U}_\alpha = \frac{2\mu_\alpha}{\hbar^2} U_\alpha(r_\alpha) , \tag{X.32}$$

and

$$\varphi_\alpha(\vec{k_\alpha}, \vec{r_\alpha}) = \int d\xi_\alpha, \psi^*(\xi_\alpha)\Psi^{(+)} = \langle \psi_\alpha, \Psi_\alpha^{(+)} \rangle , \tag{X.33}$$

one can write

$$\left(-\nabla_{r_\alpha}^2 + \bar{U}(r_\alpha) - k_\beta^2\right)\varphi_\alpha^{(+)}(\vec{k_\alpha}, \vec{r_\alpha}) = -\frac{2\mu_\alpha}{\hbar^2}\langle \psi_\alpha(\xi_\alpha), V'_\alpha \psi_\alpha^{(+)} \rangle . \tag{X.34}$$

The asymptotic form of $\varphi_\alpha^{(+)}(\vec{k_\alpha}, \vec{r_\alpha})$ will determine the elastic differential cross section[17].

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[17] Because of the absence of potential (interaction terms) in the free field Eqs (X.12)–(?) one obtains the same wavefunction by first solving the full Schrödinger equation and then taking the asymptotic limit ($r \to \infty$) or vice versa. In connection with equation (X.34) (see also (X.35)), only the first procedure is correct, in keeping with the presence of short range interactions in the Hamiltonian.

Of notice that the rhs term gives the coupling of the entrance channel with all other channels. These couplings lead to both depopulation and distortion of entrance channel outgoing spherical wave. To the extent that these couplings are weak, the rhs term can be just viewed as a (source – sink)–like term, that is, as an imaginary potential depopulating (populating) the entrance channel. In other words, the above equation can be rewritten as

$$\left(-\nabla^2_{r_\alpha} + \bar{U}(r_\alpha) + i\bar{W}(r_\beta) - k^2_\alpha\right)\chi^{(+)}_\alpha(\vec{k_\alpha}, \vec{r_\alpha}) = 0 \tag{X.35}$$

where $\chi^{(+)}_\alpha$ is the standard notation for $\varphi^{(+)}_\alpha$ when the coupling term $\langle\psi_\alpha(\xi_\alpha), V'_\alpha\psi^{(+)}_\alpha\rangle$ is not considered. This is known as the Distorted Wave Born Approximation (DWBA).

Appendix E ~~Y~~: Elastic scattering solution

$$\lim \psi_{\text{scatt}} =_{r\to\infty} \frac{e^{ikr}}{r} f(E,\theta,\phi)\psi_\alpha(\xi_\alpha)$$

$$\frac{\partial}{\partial r}\frac{e^{ikr}}{r} = -\frac{1}{r^2}e^{ikr} + \frac{ike^{ikr}}{r}$$

$$r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r} = -e^{ikr} + rike^{ikr}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) = -ike^{ikr} + ike^{ikr} + r(ik)^2e^{ikr} = -rk^2e^{ikr}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) = -\frac{k^2e^{ikr}}{r}$$

$$\lim_{r\to\infty} H = \lim_{r\to\infty}(T + U + iW) = \lim_{r\to\infty} T = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)$$

Thus

$$\begin{aligned}
\lim_{r\to\infty} H\psi_{\text{scatt}} &= -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)\frac{e^{ikr}}{r}f\psi \\
&= -\frac{\hbar^2}{2\mu}\left(-k^2\frac{e^{ikr}}{r}\right)f\psi \\
&= \frac{\hbar^2k^2}{2\mu}\frac{e^{ikr}}{r}f\psi = E\lim_{r\to\infty}\psi_{\text{scatt}}
\end{aligned}$$

Current

$$\vec{I} = \frac{\hbar}{\mu}\mathfrak{Im}(\psi^\star\vec{\nabla}\psi)$$

$$\psi = \int \mathrm{d}\xi \psi_i \psi(\xi) = \begin{cases} e^{ikz} \\ \frac{e^{ikr}}{r} f \end{cases}$$

where

$$\psi_i = \begin{cases} e^{ikz}\psi(\xi)\hat{z} & \text{(incident)} \\ \frac{e^{ikz}}{r} f(E,\theta,\phi)\psi(\xi) & \text{(scattered)} \end{cases}$$

$$\vec{\nabla} = \begin{cases} \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} & \text{(cartesian coordinates)} \\ \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial \psi}{\partial \phi} & \text{(spehrical coordinates)} \end{cases}$$

Incident flux

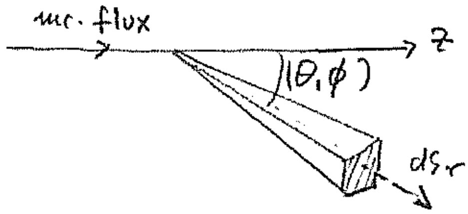
$$\vec{\nabla}\psi_{\text{inc}} = \hat{z}\frac{\partial}{\partial z} e^{ikz} = ike^{ikz}$$

$$\lim_{r\to\infty} \vec{\nabla}\psi_{\text{scatt}} = \lim_{r\to\infty} \hat{r}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r} f\right) = \lim_{r\to\infty}\left(-\frac{e^{ikr}}{r^2} f + ik\frac{e^{ikr}}{r} f\right)\hat{r} \approx ik\frac{e^{ikr}}{r} f\hat{r}$$

$$I_{\text{inc}} = \frac{\hbar}{\mu}\mathfrak{Im}(e^{-ikz} ike^{ikz})\hat{z} = \frac{\hbar k}{\mu}\hat{z} = v_\infty \hat{z} \ ,$$

$$\begin{aligned} \vec{I_{\text{scatt}}} &= \frac{\hbar}{\mu}\mathfrak{Im}\left(\frac{e^{-ikr}}{r} f^\star ik \frac{e^{ikr}}{r} f\right)\hat{r} \\ &= \frac{\hbar k}{\mu}\frac{|f|^2}{r^2}\hat{r} = v_\infty \frac{|f|^2}{r^2}\hat{r} \end{aligned}$$

$$\mathrm{d}\vec{s} = r^2 \mathrm{d}\Omega \hat{r}$$



$$\vec{I_{\text{scatt}}} \cdot \mathrm{d}\vec{s} = |f|^2 v_\infty \mathrm{d}\Omega$$

$$\mathrm{d}\sigma_\alpha(\theta,\phi) = \frac{\vec{I_{\text{scatt}}} \cdot \mathrm{d}\vec{s}}{\vec{I_{\text{inc}}} \cdot \hat{z}} = |f_{\alpha\alpha}(E,\theta,\phi)|^2 \,\mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma_\alpha(\theta,\phi)}{\mathrm{d}\Omega} = |f_{\alpha\alpha}(E,\theta,\phi)|^2$$

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