and

$$
\begin{aligned}
s_{P,l_a,l_c}(r_{Cc}) = & \int r'^2_{Cc}\, dr'_{Cc}\, r'^2_{A2}\, dr'_{A2}\, \sin\theta'\, d\theta'\, v(r'_{c2}) u_{l_f}(r'_{A2}) u_{l_i}(r'_{c2}) \\
& \times \frac{F_{l_a}(r'_{Aa})}{r'_{Aa}} \frac{f_{l_c}(k_{Cc}, r_<) P_{l_c}(k_{Cc}, r_>)}{r'_{Cc}} \\
& \times \sum_M \langle l_c\ 0\ l_a\ M | P\ M \rangle \left[Y^{l_f}(\hat{r}'_{A2}) Y^{l_a}(\hat{r}'_{c2}) \right]^P_M Y^{l_a}_{-M}(\hat{r}'_{Aa}).
\end{aligned}
\tag{7.2.178}
$$

We have evaluated the transition matrix element for a particular projection μ of the initial angular momentum of the two transferred nucleons. If they are coupled to a core of angular momentum J_f to total angular momentum J_i, M_i, the fraction of the initial wavefunction with projection μ is $\langle \Lambda\ \mu\ J_f\ M_i - \mu | J_i\ M_i \rangle$, and the cross section will be

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \left| \sum_\mu \langle \Lambda\ \mu\ J_f\ M_i - \mu | J_i\ M_i \rangle T_\mu \right|^2 . \tag{7.2.179}
$$

For a non polarized incident beam,

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2J_i + 1} \sum_{M_i} \left| \sum_\mu \langle \Lambda\ \mu\ J_f\ M_i - \mu | J_i\ M_i \rangle T_\mu \right|^2 . \tag{7.2.180}
$$

This would be the differential cross section for a transition to a definite final state M_f. If we do not measure M_f we have to sum for all M_f,

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2J_i + 1} \sum_\mu |T_\mu|^2 \sum_{M_i, M_f} \left| \langle \Lambda\ \mu\ J_f\ M_f | J_i\ M_i \rangle \right|^2 . \tag{7.2.181}
$$

The sum over M_i, M_f of the Clebsh–Gordan coefficients gives $(2J_i + 1)/(2\Lambda + 1)$ (see 7.K.26). One then gets,

(see (7.K.26))

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{(2\Lambda + 1)} \sum_\mu |T_\mu|^2, \tag{7.2.182}
$$

where one can write

$$
\begin{aligned}
T_\mu &= \sum_{l_a, l_b} C_{l_a, l_b} \left[Y^{l_a}(\hat{\mathbf{k}}_{aA}) Y^{l_b}(\hat{\mathbf{k}}_{bB}) \right]^\Lambda_\mu \\
&= \sum_{l_a, l_b} C_{l_a, l_b}\, i^{l_a} \sqrt{\frac{2l_a + 1}{4\pi}} \langle l_a\ l_b\ 0\ \mu | \Lambda\ \mu \rangle Y^{l_b}_\mu(\hat{\mathbf{k}}_{bB}).
\end{aligned}
\tag{7.2.183}
$$

Gregory

Note that (7.2.182) takes into account only the spins of the heavy nucleus. In a (t, p) or (p, t) reaction, we have to sum over the spins of the proton and of the triton and divide by 2. If a spin orbit term is present in the optical potential, the sum yields the combination of terms shown in section 22:

$$
\frac{d\sigma}{d\Omega}(\hat{\mathbf{k}}_{bB}) = \frac{k_{bB}}{k_{aA}} \frac{\mu_{aA}\mu_{bB}}{(2\pi\hbar^2)^2} \frac{1}{2(2\Lambda + 1)} \sum_\mu |A_\mu|^2 + |B_\mu|^2 . \tag{7.2.184}
$$

Lates 21/10/13

Appendix 7.A ZPF and Pauli principle at the basis of medium polarization effects: self–energy, vertex corrections and induced interaction

(1928) (1927)

In keeping with a central objective of the formulation of quantum mechanics, namely that the basic concepts on which it is based relate directly to experiment[2] (Heisenberg (1925)), elementary modes of nuclear excitation (single–particle, collective vibrations and rotations), are solidly anchored on observation (inelastic and Coulomb excitation, one– and two–particle transfer reactions). Of all quantal phenomena, zero point fluctuations (ZPF), closely connected with virtual states, are likely to be most representative of the essential difference existing between quantum and classical mechanics. In fact, ZPF are intimately connected with the complementary principle (Bohr), and thus with the indeterminacy (Heisenberg) and non–commutative (Born Jordan) relations, and with the probabilistic interpretation (Born) of the (modulus squared) of the wavefunctions, solution of Schrödinger's or Dirac's equations (Schrödinger (1925), Dirac (1926))

and

((1925), Born, Heisenberg and Jordan (1925))

Pauli principle brings about essential modifications of the virtual fluctuations of the many–body system, modifications which are instrumental in the dressing and interweaving of the elementary modes of excitation (see Figs. 7.A.1 and 7.A.2); within the present context, see also Schrieffer (1964).

(Pauli (1925))

Figs here from p. 115 and 116

Appendix 7.B Coherence and effective formfactors

In what follows we shall work out a simplified derivation of the simultaneous two–nucleon transfer amplitude, within the framework of first order DWBA specially suited to discuss correlation aspects of pair transfer in general, and of the associated effective formfactors in particular.

We will concentrate on (t, p) reaction, namely reactions of the type $A(\alpha,\beta)B$ where $\alpha = \beta + 2$ and $B = A + 2$.

[2]The abstract of this reference reads: "In this paper it will be atempted to secure foundations for a quantum theoretical mechanics which is exclusively based on relations between quantities which in principle are observables". Within the present context, namely that of probing the nuclear structure (e.g. pairing correlations) with direct nuclear reactions, in particular Cooper pair transfer, one can hardly think of a better *incipit* for the introduction of elementary modes of excitation, modes which carry within them most of the correlations ~~and~~ thus requiring an effective field theory, like e.g. NFT to take properly into account the essential overcompleteness of the basis (non–orthogonality) as well as Pauli violating processes.

of

for their theoretical treatment

~~The~~ i.e. (Born and Jordan (1925), Born, Heisenberg and Jordan (1925))

The intrinsic wave functions are in this case

$$\begin{aligned}\psi_\alpha &= \psi^{J_i}_{M_i}(\xi_A) \sum_{ss'_f} \left[\chi^{s}(\sigma_\alpha)\chi^{s'_f}(\sigma_\beta)\right]^{s_i}_{M_{s_i}} \phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|) \\ &= \psi^{J_i}_{M_i}(\xi_A) \sum_{M_s M'_{s_f}} (sM'_s s'_f M'_{s_f} | s_i M_{s_i}) \chi^{s}_{M_s}(\sigma_\alpha)\chi^{s'_f}_{M'_{s_f}}(\sigma_\beta) \\ &\times \phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|)\end{aligned} \tag{7.B.1}$$

M_s'

while

$$\begin{aligned}\psi_\beta &= \psi^{J_f}_{M_f}(\xi_{A+2})\chi^{s_f}_{M_{s_f}}(\sigma_\beta) \\ &= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} B(n_1 l_1 j_1, n_2 l_2 j_2); J J'_i J_f) \left[\phi^J(j_1 j_2)\phi^{J'_i}(\xi_A)\right]^{J_f}_{M_f} \\ &\times \chi^{s_f}_{M_{s_f}}(\sigma_\beta)\end{aligned} \tag{7.B.2}$$

Making use of the above equation one can define the spectroscopic amplitude B as

$$\begin{aligned}&B(n_1 l_1 j_1, n_2 l_2 j_2); J J'_i J_f) \\ &= \left\langle \psi^{J_f}(\xi_{A+2}) \middle\| \left[\phi^J(j_1 j_2)\phi^{J_i}(\xi_A)\right]^{J_f} \right\rangle,\end{aligned} \tag{7.B.3}$$

where

$$\phi^J(j_1 j_2) = \frac{\left[\phi_{j_1}(\vec{r}_1)\phi_{j_2}(\vec{r}_2)\right]^J - \left[\phi_{j_1}(\vec{r}_2)\phi_{j_2}(\vec{r}_1)\right]^J}{\sqrt{1 + \delta(j_1, j_2)}}, \tag{7.B.4}$$

is an antisymetrized, normalized wave function of the two transferred particles. The function $\chi^{s}_{M_s}(\sigma_\beta)$ appearing both in eq. (7.B.1) and (7.B.2) is the spin wave function of the proton while

$$\chi^{s}(\sigma_\alpha) = [\chi^{s_1}(\sigma_{n_1})\chi^{s_2}(\sigma_{n_2})]^{s}, \tag{7.B.5}$$

is the spin function of the two–neutron system.

A convenient description of the intrinsic degrees of freedom of the triton is obtained by using a wavefunction symmetric in the coordinates of all particles, i.e.

$$\begin{aligned}\phi_t^{L=0}(\sum_{i<j} |\vec{r}_i - \vec{r}_j|) &= N_t\, e^{[(r_1 - r_2)^2 + (r_1 - r_p)^2 + (r_2 - r_p)^2]} \\ &= \phi_{000}(\vec{r})\phi_{000}(\vec{\rho}),\end{aligned} \tag{7.B.6}$$

where

$$\phi_{000}(\vec{r}) = R_{nl}(\nu^{1/2} r) Y_{lm}(\hat{r}).$$

Gregory

The coordinate $\vec{\rho}$ is the radius vector which measures the distance between the center of mass of the dineutron and the proton, while the vector $\vec{r}$ is the dineutron relative coordinate (cf. Fig. 7.B.1).

Bring here Fig. from p. 117

To obtain the DWBA cross section we have to calculate the integral

$$T = \int d\xi_A \, d\vec{r}_1 \, d\vec{r}_2 \, d\vec{r}_p \chi_p^{(-)}(\vec{R}_2)\psi_\beta^*(\xi_{A+2}, \sigma_\beta) V_\beta \psi_\alpha(\xi_A, \sigma_\alpha, \sigma_\beta)\psi_t^{(+)}(\vec{R}_1) \qquad (7.B.7)$$

where the final state effective interaction $V_\beta(\rho)$ is assumed to depend only on the distance ρ between the center of mass of the di–neutron and of the proton. Instead of integrating over $\xi_A, \vec{r}_1, \vec{r}_2$ and $\vec{r}_p$ we would integrate over $\xi_A, \vec{r}, \vec{r}$ and $\vec{r}_p$. The Jacobian of the transformation is equal to 1, i.e. $\partial(\vec{r}_1, \vec{r}_2)/\partial(\vec{r}, \vec{r}) = 1$.

To carry out the integral (7.B.7) we transform the wave function (7.B.4) into center of mass and relative coordinates. If we assume that both $\phi_{j_1}(\vec{r}_1)$ and $\phi_{j_2}(\vec{r}_2)$ are harmonic oscillator wave functions (used as a basis to expand the Saxon–Woods single–particle wavefunctions), this transformation can be carried with the aid of tha Moshinsky brackets. If $|n_1 l_1, n_2 l_2; \lambda\mu\rangle$ is a complete system of wave functions in the harmonic oscillator basis, depending on $\vec{r}_1$ and $\vec{r}_2$ and $|nl, NL; \lambda\mu\rangle$ is the corresponding one depending on $\vec{r}$ and $\vec{R}$, we can write

$$\begin{aligned} |n_1 l_1, n_2 l_2; \lambda\mu\rangle &= \sum_{nlNL} |nl, NL; \lambda\mu\rangle\langle nl, NL; \lambda\mu|)|n_1 l_1, n_2 l_2; \lambda\mu\rangle \\ &= \sum_{nlNL} |nl, NL; \lambda\mu\rangle\langle nl, NL; \lambda\mu|n_1 l_1, n_2 l_2; \lambda\rangle \end{aligned} \qquad (7.B.8)$$

The labels n, l are the principal and angular momentum quantum numbers of the relative motion, ehile N, L are the corresponding ones corresponding to the center of mass motion of the two–neutron system. Because of energy and parity conservation we have

$$\begin{aligned} &2n_1 + l_1 + 2n_2 + l_2 = 2n + l + 2N + L \\ &(-1)^{l_1+l_2} = (-1)^{l+L}. \end{aligned} \qquad (7.B.9)$$

The coefficients $\langle nl, NL, L|n_1 l_1, n_2 l_2, L\rangle$ are tabulated and were first discussed by (Moshinsky,(1959)).

With the help of eq. (7.B.8) we can write the wave function $\psi_{M_f}^{J_f}(\xi_{A+2})$ as

$$
\begin{aligned}
\psi^{J_f}_{M_f}(\xi_{A+2}) &= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2 \\ J J_i}} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \left[\phi^J(j_1 j_2)\phi^{J_i'}(\xi_A)\right]^{J_f}_{M_f} \\
&= \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} \sum_{J J_i} B(n_1 l_1 j_1, n_2 l_2 j_2; J J_i' J_f) \\
&\times \sum_{M_J M'_{J_i}} \langle J M_J J_i' M_{J_i} | J_f M_{J_f} \rangle \psi^{J_i'}_{M'_{J_i}}(\xi_A) \\
&\times \sum_{LS'} \langle S' L J | j_1 j_2 J \rangle \sum_{M_L M_S'} \langle L M_L S' M_S' | J M_J \rangle \chi^{S'}_{M_S'}(\sigma_\alpha) \\
&\times \sum_{nlN\Lambda} \langle nl, N\Lambda, L | n_1 l_1, n_2 l_2, L \rangle \\
&\times \sum_{m_l M_\Lambda} \langle l m_l \Lambda M_\Lambda | L M_L \rangle \phi_{nlm_l}(\vec{r}) \phi_{N\Lambda M_\Lambda}(\vec{R})
\end{aligned}
\tag{7.B.10}
$$

Integration over $\vec{r}$ gives

$$
\langle \phi_{nlm_l}(\vec{r}) | \phi_{000}(\vec{r}) \rangle = \delta(l,0)\delta(m_l,0)\Omega_n \tag{7.B.11}
$$

where

$$
\Omega_n = \int R_{nl}(\nu_1^{1/2} r) R_{00}(\nu_2^{1/2} r) r^2 \, dr \tag{7.B.12}
$$

Note that there is no selection rule in the principal quantum number n, as the potential in which the two neutrons move in the triton has a frequency ν_2 which is different from the one that the two neutrons are subjected to, when moving in the system A (non–orthogonality effect).

Integration over ξ_A and multiplication of the spin functions gives

$$
\begin{aligned}
(\psi^{J_i}_{M_{J_i}}, V_\beta'(\rho)\psi^{J_i'}_{M'_{J_i}}) &= \delta(J_i, J_i')\delta(M_{J_i}, M_{J_i'})V(\rho), \\
(\chi^S_{M_S}(\sigma_\alpha), \chi^{S'}_{M_{S'}}(\sigma_\alpha)) &= \delta(S, S')\delta(M_S, M_{S'}), \\
(\chi^{S_f}_{M_{S_f}}(\sigma_\beta), \chi^{S_f'}_{M_{S_f'}}(\sigma_\beta)) &= \delta(S_f, S_f')\delta(M_{S_f}, M_{S_f'}).
\end{aligned}
\tag{7.B.13}
$$

The integral (7.B.7) can then be written as

$$
\begin{aligned}
T = & \sum_{\substack{n_1 l_1 j_1 \\ n_2 l_2 j_2}} \sum_{JM_J} \sum_{nN} \sum_{S} B(n_1 l_1 j_1, n_2 l_2 j_2; JJ_i' J_f) \\
& \times \langle JM_J J_i M_{J_i} | J_f M_{J_f} \rangle \langle S\, LJ | j_1 j_2 J \rangle \\
& \times \langle L M_L S M_S | J M_J \rangle \langle n0, NL, L | n_1 l_1, n_2 l_2, L \rangle \\
& \times \langle S M_S S_f M_{S_f} | S_i M_{S_i} \rangle \Omega_n \\
& \times \int d\vec{R}\, d\vec{r}_p \chi_t^{(+)*}(\vec{R}_1) \phi^*_{NLM_L}(\vec{R}) V(\rho) \phi_{000}(\vec{\rho}) \chi_t^{(+)}(\vec{R}_1),
\end{aligned}
\tag{7.B.14}
$$

where we have approximated V'_β by an effective interaction depending on $\rho = |\vec{\rho}|$.
We now define the effective two–nucleon transfer ~~effective~~ form factor as

$$
\begin{aligned}
u^{j_i J_f}_{LSJ}(R) = & \sum_{n_1 l_1 j_1} B(n_1 l_1 j_1, n_2 l_2 j_2; JJ_i J_f) \langle S\, LJ | j_1 j_2 J \rangle \\
& \langle n0, NL, L | n_1 l_1, n_2 l_2; L \rangle \Omega_n R_{nL}(R).
\end{aligned}
\tag{7.B.15}
$$

We can now rewrite eq. (7.B.14) as

$$
\begin{aligned}
T = & \sum_J \sum_L \sum_S \langle JM_J J_i M_{J_i} | J_f M_{J_f} \rangle \langle S M_S S_f M_{S_f} | S_i M_{S_i} \rangle \langle L M_L S M_S | J M_J \rangle \\
& \times \int d\vec{R}\, d\vec{r}_p \chi_p^{*(-)}(\vec{R}_2) u^{j_i J_f}_{LSJ}(R) Y^*_{LM_L} V(\rho) \phi_{000}(\vec{\rho}) \chi_t^{(+)}(\vec{R}_1).
\end{aligned}
\tag{7.B.16}
$$

Because the di–neutron has $S = 0$, we have that

$$
\langle L M_L 00 | J M_J \rangle = \delta(J, L) \delta(M_L, M_J), \tag{7.B.17}
$$

and the summations over S and L disappear from eq. (7.B.16). Let us now make also here, as done in App. 6.F, Eq.(6.F.15) for one–particle transfer reactions, the zero range approximation, that is,

$$
V(\rho) \phi_{000}(\vec{\rho}) = D_0 \delta(\vec{\rho}). \tag{7.B.18}
$$

This means that the proton interacts with the center of mass of the di–neutron, only when they are at the same point in space. Within this approximation (cf. Fig. ~~6.F.1~~)

7.B.1

$$
\begin{aligned}
\vec{R} = & \vec{R}_1 = \vec{r}, \\
\vec{R}_2 = & \frac{A}{A+2} \vec{R},
\end{aligned}
\tag{7.B.19}
$$

Then eq. (7.B.14) can be written as

$$
\begin{aligned}
T = D_0 \sum_L & (LM_L J_i M_{J_i} | J_f M_{J_f}) \\
& \times \int d\vec{R} \chi_p^{*(-)}\left(\frac{A}{A+2}\vec{R}\right) u_L^{J_i J_f}(R) Y^*_{LM_L}(\hat{R}) \chi_t^{(+)}(\vec{R})
\end{aligned}
\tag{7.B.20}
$$

From Eq. (7.B.20) it is seen that the change in parity implied by the reaction is given by $\Delta\pi = (-1)^L$. Consequently, the selection rules for (t, p) and (p, t) reactions in zero–range approximation are,

$$
\begin{aligned}
\Delta S &= 0 \\
\Delta J &= \Delta L = L \\
\Delta\pi &= (-1)^L
\end{aligned}
\tag{7.B.21}
$$

(6.E.16)

i.e. only normal parity states are excited.

The integral appearing in ~~eq.~~ (7.B.20) has the same structure as the DWBA integral appearing in Eq. (??) which was derived for the case of one–nucleon transfer reactions.

(cf. Eq.(6.E.10)) The difference between the two processes manifest(s) itself through the different structure of the two form factors. While $u_l(r)$ ~~in Equation (??)~~ is a single–particle bound state wave function, $u_L^{J_i J_f}$ is a coherent summation over the center of mass states of motion of the two transferred neutrons. In other words, an effective quantity (function) vanishes, when one considers dressed particles resulting from the coupling to collective motion, and leading, among other things, to ω–dependent effective masses. Examples of two–nucleon transfer form factors are given in Figs. 7.B.2, 7.B.3 and 7.B.4 (see Eq. 7.B.15)

← Bring Figs. here

Appendix 7.C Relative importance of successive and simultaneous transfer and non-orthogonality corrections

In what follows we discuss the relative importance of successive and simultaneous two-neutron transfer and of non-orthogonality corrections associated with the reaction

and Broglia (1975)

$$
\alpha \equiv a(= b + 2) + A \rightarrow b + B(= A + 2) \equiv \beta \tag{7.C.1}
$$

in the limits of independent particles and ~~of strongly~~ correlated Cooper pairs, making use for simplicity of ~~the~~ semiclassical approximation (for details cf. Broglia and Winther (2004), and refs. therein), in which case the two-particle transfer differential cross section can be written as

(It is of notice ~~that~~ this difference essentially)

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = P_{\alpha\to\beta}(t = +\infty)\sqrt{\left(\frac{d\sigma_\alpha}{d\Omega}\right)_{el}}\sqrt{\left(\frac{d\sigma_\beta}{d\Omega}\right)_{el}}, \qquad (7.C.2)$$

where P is the absolute value squared of a quantum mechanical transition amplitude. It gives the probability that the system at $t = +\infty$ is found in the final channel. The quantities $(d\sigma/d\Omega)_{el}$ are the classical elastic cross sections in the center of mass system, calculated in terms of the deflection function, namely the functional relating the impact parameter and the scattering angle.

The transfer amplitude can be written as

$$a(t = +\infty) = a^{(1)}(\infty) - a^{(NO)}(\infty) + \tilde{a}^{(2)}(\infty), \qquad (7.C.3)$$

where $\tilde{a}^{(2)}(\infty)$ at $t = +\infty$ labels the successive transfer amplitude expressed in the post-prior representation (see below). The simultaneous transfer amplitude is given by (see Fig. ~~A2~~(I)) 7.C.1 (see Fig. 7.C.1 (I))

$$\begin{aligned} a^{(1)}(\infty) &= \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt(\psi^b\psi^B, (V_{bB} - < V_{bB} >)\psi^a\psi^A) \times \exp[\frac{i}{\hbar}(E^{bB} - E^{aA})t] \qquad (7.C.4) \\ &\approx \frac{2}{i\hbar}\int_{-\infty}^{\infty} dt\left(\phi^{B(A)}(S^B_{(2n)}; \vec{r}_{1A}, \vec{r}_{2A}), U(r_{1b})e^{i(\sigma_1+\sigma_2)}\phi^{a(b)}(S^a_{(2n)}; \vec{r}_{1b}, \vec{r}_{2b})\right) \\ &\times \exp[\frac{i}{\hbar}(E^{bB} - E^{aA})t + \gamma(t)] \end{aligned}$$

7.C.3

where

$$\sigma_1 + \sigma_2 = \frac{1}{\hbar}\frac{m_n}{m_A}(m_{aA}\vec{v}_{aA}(t) + m_{bB}v_{bB}(t)) \cdot (\vec{r}_{1\alpha} + \vec{r}_{2\alpha}), \qquad (7.C.\cancel{4}\,5)$$

in keeping with the fact that $\exp(i(\sigma_1 + \sigma_2))$ takes care of recoil effects (Galilean transformation associated with the mismatch between entrance and exit channels).

The phase $\gamma(t)$ is related with the effective Q–value of the reaction. In the above expression, ϕ indicates an antisymmetrized, correlated two-particle (Cooper pair) wavefunction, $S(2n)$ being the two-neutron separation energy (see Fig. ~~A1~~), $U(r_{1b})$ being the single particle potential generated by nucleus b ($U(r) = \int d^3r'\rho^b(r')v(|r - r'|)$). The contribution arising from non-orthogonality effects can be written as (see Fig. ~~A2~~(II))

$$\begin{aligned} a^{(NO)}(\infty) &= \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt(\psi^b\psi^B, (V_{bB} - < V_{bB} >)\psi^f\psi^F)(\psi^f\psi^F, \psi^a\psi^A)\exp[\frac{i}{\hbar}(E^{bB} - E^{aA})t] \\ &\approx \frac{2}{i\hbar}\int_{-\infty}^{\infty} \phi^{B(F)}(S^B_{(n)}, \vec{r}_{1A}), U(r_{1b})e^{i\sigma_1}(\phi^{f(b)}(S^f(n), \vec{r}_{1b}) \qquad (7.C.6) \\ &\times \phi^{F(A)}(S^F(n), \vec{r}_{2A})e^{i\sigma_2}\phi^{a(f)}(S^a(n), \vec{r}_{2b}))\exp[\frac{i}{\hbar}(E^{bB} - E^{aA})t + \gamma(t)], \end{aligned}$$

7.C.1

the reaction channel $f = (b + 1) + F(= A + 1)$ having been introduced, the quantity $S(n)$ being the one-neutron separation energy (see Fig. ~~A1~~). The summation over

7.C.3

7.C.2

$f(\equiv a'_1, a'_2)$ and $F(\equiv a_1, a_2)$ involves a restricted number of states, namely the valence shells in nuclei B and a.

The successive transfer amplitude $\tilde{a}^{(2)}_{\infty}$ written making use of the post-prior representation is equal to (see Fig. ~~A1~~(III))

$$\tilde{a}^{(2)}(\infty) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt(\psi^b\psi^B, (V_{bB} - < V_{bB} >)e^{i\sigma_1}\psi^f\psi^F) \times \exp[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)]$$
$$\times \frac{1}{i\hbar}\int_{-\infty}^{t} dt'(\psi^f\psi^F, (V_{fF} - < V_{fF} >)e^{i\sigma_2}\psi^a\psi^A > \times \exp[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')] \quad (\cancel{7.C.3})\ (7.C.7)$$

( )

To gain insight into the relative importance of the three terms contributing to Eq. (7.C.3) we discuss two situations, namely, the independent-particle model and the strong-correlation limits.

7.C.1 Independent particle limit

7.C.3

In the independent particle limit, the two transferred particles do not interact among themselves but for antisymmetrization. Thus, the separation energies fulfill the relations (see Fig. ~~A2~~)

$$S^B(2n) = 2S^B(n) = 2S^F(n), \quad (7.C.\cancel{6})\ 8$$

and

$$S^a(2n) = 2S^a(n) = 2S^f(n). \quad (7.C.\cancel{7})\ 9$$

In this case

$$\phi^{B(A)}(S^B(2n), \vec{r}_{1A}, \vec{r}_{2A}) = \sum_{a_1a_2}\phi^{B(F)}_{a_1}(S^B(n), \vec{r}_{1A})\phi^{F(A)}_{a_2}(S^F(n), \vec{r}_{2a}), \quad (7.C.\cancel{8})\ 10$$

and

$$\phi^{a(b)}(S^a(2n), \vec{r}_{1b}, \vec{r}_{2b}) = \sum_{a'_1a'_2}\phi^{a(f)}_{a'_1}(S^a(n), \vec{r}_{2b})\phi^{f(b)}_{a'_2}(S^f(n), \vec{r}_{1b}), \quad (7.C.\cancel{9})\ 11$$

where $(a_1, a_2) \equiv F$ and $(a'_1, a'_2) \equiv f$ span, as mentioned above, shells in nuclei B and a respectively.

Inserting Eqs (7.C.8)−(7.C.11) ~~(A.9) and (A.10)~~ in Eq. (7.C.4) ~~(A4)~~ one can show that

$$a^{(1)}(\infty) = a^{(NO)}(\infty). \quad (7.C.\cancel{10})\ 12$$

It can be further demonstrated that within the present approximation, $Im\ \tilde{a}^{(2)} = 0$, and that

$$\tilde{a}^{(2)}(\infty) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt(\psi^b\psi^B, (V_{bB} - < V_{bB} >)e^{i\sigma_1}\psi^f\psi^F > \times \exp[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)]$$
$$\times \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt'(\psi^f\psi^F, (V_{fF} - < V_{fF} >)e^{i\sigma_2}\psi^a\psi^A) \times \exp[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')] \quad (7.C.\cancel{11})\ 13$$

( ) Ch.6,

6.1.1

The total absolute differential cross section (7.C.2), where $P = |a(\infty)|^2 = |\tilde{a}^{(2)}|^2$, is then equal to the product of two one-particle transfer cross sections (see Fig. [illegible]), associated with the (virtual) reaction channels

$$\alpha \equiv a + A \rightarrow f + F \equiv \gamma, \qquad (7.C.\cancel{12})\ 14$$

and

13

$$\gamma \equiv f + F \rightarrow b + B \equiv \beta. \qquad (7.C.\cancel{13})\ 15$$

In fact, Eq.(7.C.~~11~~) involves no time ordering and consequently the two processes above are completely independent of each other. This result was expected because being $v_{12} = 0$, the transfer of one nucleon cannot influence, aside form selecting the initial state for the second step, the behaviour of the other nucleon.

7.C.2 Strong correlation (cluster) limit 7.C.3

The second limit to be considered is the one ~~in~~ which the correlation betwen the two nucleons is so strong that (see Fig. [illegible])

$$S^a(2n) \approx S^a(n) >> S^f(n), \qquad (7.C.\cancel{14})\ 16$$

and

$$S^B(2n) \approx S^B(n) >> S^F(n). \qquad (7.C.\cancel{15})\ 17$$

That is, the magnitude of the one-nucleon separation energy is strongly modified by the pair breaking.

There is a different , although equivalent way to express (7.C.3) which is ~~the~~ more convenient to discuss the strong coupling limit. In fact, making use of the post-prior representation one can write

$$a^{(2)}(t) = \tilde{a}^{(2)}(t) - a^{(NO)}(t) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt(\psi^b\psi^B, (V_{bB} - < V_{bB} >)e^{i\sigma_1}\psi^f\psi^F)$$

$$\times \exp[\frac{i}{\hbar}(E^{bB} - E^{fF})t + \gamma_1(t)] \times$$

$$\frac{1}{i\hbar}\int_{-\infty}^{t} dt'(\psi^f\psi^F, (V_{aA} - < V_{aA} >)\psi^a\psi^A) \times \exp[\frac{i}{\hbar}(E^{fF} - E^{aA})t' + \gamma_2(t')] \qquad (7.C.\cancel{16})\ 18$$

The relations (7.C.~~14~~) 16, (7.C.~~15~~) 17 imply

$$E^{fF} - E^{aA} = S^a(n) - S^F(n) >> \frac{\hbar}{\tau}, \qquad (7.C.\cancel{17})\ 19$$

where τ is the collision time. Consequently the real part of $a^{(2)}(\infty)$ vanishes exponentially with the Q-value of the intermediate transition, while the imaginary part vanishes inversely proportional to this energy. One can thus write,

$$Re\ a^{(2)}(\infty) \approx 0, \qquad (7.C.\cancel{18})\ 20$$

and

$$a^{(2)}(\infty) \approx \frac{1}{i\hbar}\frac{\tau}{< E^{fF} > -E^{bB}} \sum_{fF} (\psi^b\psi^B, (V_{bB} - < V_{bB} >)\psi^f\psi^F)_{t=0} \times (\psi^f\psi^F, (V_{aA} - < V_{aA})\psi^a\psi^A)_{t=0}, \qquad (7.C.\cancel{19})\ 21$$

where one has utilized the fact that $E^{bB} \approx E^{aA}$. For $v_{12} \to \infty$, $(< E^{fF} > -E^{bB}) \to \infty$ and, consequently,

$$lim_{v_{12}\to\infty} a^{(2)}(\infty) = 0. \qquad (7.C.\cancel{20})\ 22$$

Thus the total two-nucleon transfer amplitude is equal, in the strong coupling limit, to the amplitude $a^{(1)}(\infty)$.

Summing up, only when successive transfer and non-orthogonal corrections are included in the description of the two-nucleon transfer process, does one obtain a consistent description of the process, which correctly converges to the weak and strong correlation limiting values.

Appendix 7.D Simple numerical estimates of successive and simultaneous transfer amplitudes

Let us denote

$$H = T + V, \qquad (7.D.1)$$

the total hamiltonian describing the nuclear system, where V is the nuclear two–body interaction.

The fact that the nuclear quantality parameter has a value of $Q \approx 0.4$ testifies to the validity of independent particle motion in nuclei. This is tantamount to saying that there exist a single–particle potential U, such that

$$\langle\Psi_0|U|\Psi_0\rangle \ll \langle\Psi_0|(V - U)|\Psi_0\rangle, \qquad (7.D.2)$$

where Ψ_0 is the exact ground state wavefunction, that is, $H\Psi_0 = E_0\Psi_0$. One can the write 7.D.1 as

$$H = T + V_{eff}, \qquad (7.D.3)$$

where

$$V_{eff} = U + (V - U). \qquad (7.D.4)$$

Let us now consider a reaction in which two nucleons are transferred between target and projectile, that is,

$$a(= b + 2) + A \to b + B(= A + 2). \qquad (7.D.5)$$

The transfer cross section is proportional to the square of the amplitude

$$\sqrt{\sigma} \sim \langle bB|V_{eff}|aA\rangle = \langle bB|U|aA\rangle + \langle bB|(V - U)|aA\rangle. \qquad (7.D.6)$$

Let us assume that the transferred nucleons are e.g. two neutrons moving in time reversal states lying close t othe Fermi energy (Cooper pair). In this case it is natural to assume that the operative component of $(V-U)$ is the pairing interaction

$$V_p = -GP^\dagger P, \tag{7.D.7}$$

where

$$P^\dagger = \sum_{\nu>0} a^\dagger_\nu a^\dagger_{\bar{\nu}}, \tag{7.D.8}$$

is the pair operator, and

$$G \approx \frac{18}{A}\ \mathrm{MeV}, \tag{7.D.9}$$

is the pairing coupling constant for nucleons moving in an extended (2–3 major shell) configuration.

One can then write Eq. 7.D.6 as

$$\sqrt{\sigma} = \sqrt{\sigma_1} + \sqrt{\sigma_2}, \tag{7.D.10}$$

where

$$\sqrt{\sigma_1} \sim \langle Bb|U|aA\rangle \approx 2\left(\frac{|V_0|}{2}\right)O,\ \text{(SUCC+NO)} \tag{7.D.11}$$

and

$$\begin{aligned}\sqrt{\sigma_2} \sim& \langle Bb|V-U|aA\rangle = \langle Bb|H_p|aA\rangle \\ &\approx GU(b)V(B) \approx \frac{G}{2},\ \text{(PAIRING)}\end{aligned} \tag{7.D.12}$$

In the process described by the transfer amplitude $\langle Bb|U|aA\rangle$, one nucleon is transferred under the effect of the single–particle potential of depth $V_0(\approx -50\ \mathrm{MeV}$) while, simultaneously, the second nucleon moves over from a single–particle orbit centered around b to one centered around A profiting of the non–orthogonality of the corresponding wavefunctions $\varphi^{(b)}(r_{1b})$ and $\varphi^{(A)}(r_{1A})$. Within this context, it is then natural that O stands for the overlap between these two wavefunctions, that is, (see below simple estimate of O),

$$O = \langle \varphi^{(b)}|\varphi^{(A)}\rangle \approx 0.3\times 10^{-2}, \tag{7.D.13}$$

and that (7.D.11) is known as the sum of the simultaneous plus non–orthogonality contributions to the two–nucleon transfer amplitude. Of notice that the prefactor 2 in (7.D.11) is associated with the fact that two nucleons can choose to jump from one system to the other through non–orthogonality while the factor $|V_0|/2$ is associated with the fact that transfer takes mainly place at the surface.

The term (7.D.12) corresponds to the simultaneous (t,p) transfer via the pairing two–body interaction V_p (see Eq. (7.D.7)), $U(A)$ and $V(B)$ being the product of two occupation amplitudes: $U(A)$ measures the availability of free single–particle orbitals around the Fermi energy in the target nucleus, while $V(B)$ reflects the degree of occupancy of levels in the residual system. Close to the Fermi energy $U(b)V(B) \approx (1/\sqrt{2})^2 = 1/2$, leading to the final expression of (7.D.12).

In keeping with the fact that the ratio of transfer amplitudes

$$\begin{aligned}\left(\frac{\sigma_1}{\sigma_2}\right)^{1/2} &\approx 2\frac{|V_0|}{2} \times O\frac{1}{G/2} \approx 2 \times A \times 10^{-2} \\ &\approx 2 (A \approx 100),\end{aligned} \tag{7.D.14}$$

is larger that one, and that the correlation length of nuclear Cooper pairs ($\xi \approx \hbar v_F/2\Delta \approx 30$ fm) is larger than nuclear dimensions, one can expect that the successive transfer of two nucleons under the influence of the single–particle field, can give an important contribution to the total transfer amplitude $\sqrt{\sigma}$. In other words, we expect the process

$$a(= b+2) + A \rightarrow f(= b+1) + F(A+1) \rightarrow b + B(= A+2) \tag{7.D.15}$$

gives a consistent contribution to $\sqrt{\sigma}$. The associated amplitude can be written as

$$\begin{aligned}\sqrt{\sigma_3} &\sim \sum_{fF} \frac{\langle bB|U|fF\rangle\langle fF|U|aA\rangle}{E_{aA} - E_{fF}} \\ &\approx \frac{(V_0/13)(V_0/13)}{\Delta E},\end{aligned} \tag{7.D.16}$$

the factor 1/7 appears in each of the steps (instead of 1/2, see (7.D.11)) in keeping with the fact that many other reactions channels and then, absorptive processes will take place at closer distance in two–step channels (of notice that 1/7 corresponds to $r = R_0 + 2.5a$).

Typical values of the energy denominator in (7.D.16) are $\Delta E = 30$ MeV for medium heavy nuclei lying along the stability valley.

Appendix 7.E Transfer amplitudes

Making use of a simplified expression for the elastic scattering amplitude, that is

$$\sqrt{\sigma_{el}} \sim \langle aA|U|aA\rangle, \tag{7.E.1}$$

one can calculate the transfer probabilities associated with the different processes discussed above, namely

$$P_i = \left(\frac{\sigma_i}{\sigma_{el}}\right) = \begin{cases} \left|\frac{\langle bB|U|aA\rangle}{\langle aA|U|aA\rangle}\right|^2 \approx O^2 \approx 0.9 \times 10^{-5} \quad (i=1), \\ \left|\frac{\langle bB|V_p|aA\rangle}{\langle aA|U|aA\rangle}\right|^2 \approx \left(\frac{G}{2V_0}\right)^2 \approx 1.4 \times 10^{-6} \quad (i=2), \\ \left|\frac{\langle bB|U|fF\rangle\langle fF|U|aA\rangle}{\Delta E\langle aA|U|aA\rangle}\right|^2 \approx \left(\frac{V_0}{170\Delta E}\right)^2 \approx 0.96 \times 10^{-4} \quad (i=3). \end{cases} \tag{7.E.2}$$

Because all these probabilities are small, one can write

$$\sigma_i = P_i\sigma_{el}, \tag{7.E.3}$$

where

$$\begin{aligned}\sigma_{el} &= \left(\frac{\mu_\alpha}{2\pi\hbar^2}\right)^2 |\langle aA|U|aA\rangle|^2\\ &\approx \left(\left(\frac{\mu_\alpha}{2\pi\hbar^2}\right)(V_0)\right)^2 U_0^2\\ &\approx \left(1.8\text{MeV}^{-1}\text{fm}\right)^2 (50\text{MeV})^2\\ &\approx (90\text{fm})^2 = 0.8\text{b}\end{aligned} \tag{7.E.4}$$

where use has been made of the effective volume associated with the reaction process (see)

$$V_{ol} \approx \frac{4\pi}{3}3R^2a \approx 12A^{2/3}\text{fm}^3 \approx 260\text{fm}^3, \tag{7.E.5}$$

as well as of $\frac{\mu_\alpha}{2\pi\hbar^2}$ factor of the typical two–nucleon transfer reaction ^{120}Sn+p→^{118}Sn+t, that is (see),

$$\frac{\sqrt{\mu_\alpha\mu_\beta}}{2\pi\hbar^2} \approx \frac{\sqrt{3}M}{2\pi\hbar^2} \approx \frac{1}{145}\text{MeV}^{-1}\text{fm}^{-2}. \tag{7.E.6}$$

Summing up, one can write

$$\sigma_i = P_i 0.8\text{b}. \tag{7.E.7}$$

Making use of (7.E.2) one obtains

$$\sigma_i = \begin{cases} 0.7\times 10^{-2}\text{mb} & (i=1),\\ 1.1\times 10^{-3}\text{mb} & (i=2),\\ 8\,\text{mb} \quad (i=1), & (i=3). \end{cases} \tag{7.E.8}$$

These numbers, although worked out for A=100 can be rescaled in connection with the reaction $^{11}\text{Li}(p,t)^9\text{Li(gs)}$, in which case, microscopic calculation lead to $d\sigma_1(\theta = 60^\circ)/d\Omega \approx 0.01$mb/sr and $d\sigma_3(\theta = 60^\circ)/d\Omega \approx 5$mb/sr.

Appendix 7.F Inelastic scattering following two–particle transfer: final state interaction

This subject is qualitatively discussed in connection with the $^{11}\text{Li}(p,t)^9\text{Li}(1/2^-;2.69)$, but of course is a general question, also in connection with the validity of considering perturbation theory instead of coupled channels.

In keeping with the fact that the first excited state of ^{9}Li can be viewed as

$$|^9\text{Li}(1/2^-;2.69\text{MeV})\rangle \approx |2^+\left(^8\text{Be}\otimes p_{3/2}(\pi)\right)_{1/2^-}\rangle, \tag{7.F.1}$$

this state can, in principle, be excited in a two–step process, namely

$$\text{gs } (^{11}\text{Li})\text{+t} \longrightarrow \text{ gs } (^{9}\text{Li})\text{+p} \longrightarrow 1/2^- \, (^{9}\text{Li})\text{+p}. \tag{7.F.2}$$

Let us calculate the probability associated with the inelastic scattering of the lowest 2^+ of ^{8}Li. In this case, we are interested in the component of $V - U$ corresponding to $\delta U_C = -KF\alpha = -R_0\frac{\partial U}{\partial r}\beta_L$, namely the field associated with the inelastic excitation of multipole vibrations. Making use of the Saxon–Woos potential one obtains

$$R_0\frac{\partial U}{\partial r} = \frac{R_0}{a}\frac{\exp\left(\frac{(r-R_0)}{a}\right)}{\left(1+\exp\left(\frac{(r-R_0)}{a}\right)\right)^2}. \tag{7.F.3}$$

In keeping with the fact that

$$\left\langle R_0\left.\frac{\partial U}{\partial r}\right|_{r=R_0}\right\rangle \approx \left\langle\frac{R_0U_0}{a}\right\rangle \approx 1.2U_0\text{MeV} \approx -60\text{MeV} \tag{7.F.4}$$

and that the main contributions of surface dominated reactions processes is estimated to arize from distances of the order of $r \approx R_0 + 2.5a$, one obtains

$$\begin{aligned}\left\langle\frac{R_0}{a}\frac{e^{2.5}U_0}{(1+\exp 2.5)^2}\right\rangle &= \left\langle\frac{R_0U_0}{a}\right\rangle\frac{e^{2.5}}{(1+\exp 2.5)^2} \\ &\approx 1.2U_0 \times 0.7\times 10^{-1} = 0.84\times 10^{-1}U_0.\end{aligned} \tag{7.F.5}$$

Thus

$$\langle bB^*|\delta U_C|bB\rangle \approx 0.84\times 10^{-1}U_0\beta_L. \tag{7.F.6}$$

Consequently

$$\begin{aligned}P_{inel} &\approx \left|\frac{\langle bB^*|\delta U_C|aA\rangle}{\langle aA|U|aA\rangle}\right|^2 = \left(0.84\times 10^{-1}\beta_L\right)^2 \\ &\approx 0.7\times 10^{-2}\beta_L^2.\end{aligned} \tag{7.F.7}$$

In keeping with the fact that the β_L associated with the lowest 2^+ vibrational states of the Sn–isotopes and of ^{8}are ≈ 0.1 and ≈ 1 respectively one can write

$$P_{inel} = \begin{cases} 0.7\times 10^{-4} & \text{(Sn–isotopes)}, \\ 0.7\times 10^{-2} & (^{11}\text{Li}). \end{cases} \tag{7.F.8}$$

Making use of the results collected in (7.E.2),

$$\begin{aligned}\sqrt{P(p,t)} &= \sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3} \\ &= \sqrt{0.9\times 10^{-1}} + \sqrt{1.4\times 10^{-6}} + \sqrt{0.96\times 10^{-4}} \\ &\approx 3\times 10^{-3} + 1.2\times 10^{-3} + 0.98\times 10^{-2} \\ &\approx 1.4\times 10^{-2}.\end{aligned} \tag{7.F.9}$$

Thus

$$P((p,t)\otimes P(\text{inel})) = P(p,t)P(\text{inel}) = \begin{cases} 2\times 10^{-4}\times 10^{-4}\approx 10^{-8} & (\text{Sn}),\\ 2\times 10^{-4}\times 10^{-2}\approx 10^{-6} & (^{11}\text{Li}), \end{cases} \tag{7.F.10}$$

in overall agreement with th result of microscopic calculations (for ^{11}Li).

Appendix 7.G Simple estimate O

The nuclear density can be parametrized according to

$$\rho(r) = \frac{\rho_0}{1+\exp\left(\frac{r-R_0}{a}\right)}. \tag{7.G.1}$$

Let us calculate this function for

$$r = R_0 + 3a, \tag{7.G.2}$$

that is

$$\rho(r = R_0 + 3a) = \frac{\rho_0}{1+\exp 3} = 5\times 10^{-2}\rho_0. \tag{7.G.3}$$

In other words, we assume that the main transfer takes place from densities of the order of 5% saturation density

$$O \approx \frac{\rho_A(R_0^A+3a)\rho_a(R_0^a+3a)}{\rho_0^2} = 25\times 10^{-4}\approx 0.3\times 10^{-2}. \tag{7.G.4}$$

another estimate

$$r = R_0 + 2.5a, \tag{7.G.5}$$

for which

$$\rho(r = R_0 + 2.5a) = \frac{\rho_0}{1+\exp 2.5}\approx 0.76\times 10^{-1}\rho_0, \tag{7.G.6}$$

leading to

$$O \approx 0.5\times 10^{-2}. \tag{7.G.7}$$

Appendix 7.H Simple estimate of $\frac{(\mu_\alpha\mu_\beta)^{1/2}}{2\pi\hbar^2}$.

Let us do it for the case of ^{120}Sn+p $\longrightarrow$ ^{118}Sn+t. In this case

$$\begin{aligned}\mu_\alpha &= \frac{120}{121}M\approx M,\\ \mu_\beta &= \frac{118\times 3}{121}\approx 2.9M.\end{aligned} \tag{7.H.1}$$

Thus

$$\begin{aligned}\frac{\sqrt{\mu_\alpha\mu_\beta}}{2\pi\hbar^2} &\approx \frac{\sqrt{3}M}{2\pi\hbar^2} = \frac{\sqrt{3}}{2\pi 40\text{MeV fm}^2}\\ &\approx \frac{1}{145}\times\text{MeV}^{-1}\times\text{fm}^{-2}\end{aligned} \tag{7.H.2}$$

Appendix 7.I Simple estimate of V_{ol}

In keeping with the assumption that transfer processes are expected to take place at the nuclear surface, the effective volume associate with such processes can be estimated to be

$$
\begin{aligned}
V_{ol} &= \frac{4\pi}{3}(R^3 - (R+a)^3) \\
&\approx \frac{4\pi}{3} 3aR^2 \approx \frac{4\pi}{3}(2\mathrm{fm})R^2 \\
&\approx \frac{8\pi}{3}(1.2A^{1/3})^2 \mathrm{fm}^3 \\
&\approx 1.2A^{1/3}\mathrm{fm}^3 \approx 260\mathrm{fm}^3 (A \approx 100)
\end{aligned}
\tag{7.I.1}
$$

Appendix 7.J Example of coherent states

A central feature of heavy ion collisions seems to be the importance of the coherent response of the different degrees of freedom. Thus in the description of the excitation of the surface modes it is not enough to know the population of the vibrational states but also the relative phases which determine the shape of the nuclei also as a function of time. Let us assume a collision in which the ions, which display vibrational modes, interact only through the Coulomb field. Solving the problem quantum mechanically leads to the following total wavefunction (cf. Broglia et al. (2004) and references therein, cf. (Glauber(1959))

$$|\Psi(t)\rangle = e^{-i\frac{H_0 t}{\hbar}} |\phi(t)\rangle \tag{7.J.1}$$

$$= \sum_{\{n_\mu\}} \left(\Pi_\mu \, e^{\frac{|I_\mu|^2}{2}} \frac{(I_\mu(t))^{n_\mu}}{\sqrt{n_\mu!}} \right) |\{n_\mu\}\rangle \tag{7.J.2}$$

where

$$I_\mu(t) = \frac{1}{\hbar} \int^t f_\mu^*(t') e^{i\omega t'} dt' \tag{7.J.3}$$

and where H_0 is the Hamiltonian describing the intrinsic degrees of freedom of each nuclei. The wavefunction $\phi(t)$ is the solution of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = \tilde{V}\phi(t) \tag{7.J.4}$$

where $\tilde{V} = \exp(iH_0 t\hbar) V \exp(-iH_0 t\hbar)$, V being the external field. The integral I_μ is related to the average number of phonons by

$$\langle n_\mu \rangle = |I_\mu(t)|^2 \tag{7.J.5}$$

The state $|Psi(t)\rangle$ is known in quantum mechanics as a coherent state (Glauber 1959). Its name stems from the fact that the associated uncertainty rltions in momentum and coordinate associated with it fulfills the absolute minimum consistent with quantum mechanics, that is,

$$\Delta X_\psi \Delta \Pi_\psi = \frac{\hbar}{2}. \tag{7.J.6}$$

Note that this value is normally associated with the ground state. In general states described by a wavefunction of the type $\exp(\frac{i\hat{O}}{\hbar})\phi(t)$ exhaust the energy weighted sum rule of the associated operator which in the present case is the Hamiltonian.

Heavy ion collisions seem thus specific to study the nuclear spectroscopy of the coherent nuclear state. ~~Note that we have left behind the field of experiments where the system that is probed can be described as if the probe was not present.~~

This ~~The~~ coherent state which pictorially looks so simple, being almost a classical state, arises from the excitation and delicate phase relation of many collective and non–collective states of the individual nuclei. ~~Thus, the full response function is tested in these reactions in a totally novel way. Note that collective vibrations as those e.g example surface modes are also coherent states and arise from the correlated efforts of many particle–hole excitation.~~

~~It is interesting to speculate whether the coherent excitation of the gas of phonons will lead to new super–collectivities displaying different condensation or phases as a function of the continuous excitation energy.~~

Appendix 7.K Spherical harmonics and angular momenta

With Condon–Shortley phases

$$Y^l_m(\hat{z}) = \delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}, \quad Y^{l*}_m(\hat{r}) = (-1)^m Y^l_{-m}(\hat{r}). \tag{7.K.1}$$

Time–reversed phases consist in multiplying Condon–Shortley phases with a factor i^l, so

$$Y^l_m(\hat{z}) = \delta_{m,0} i^l \sqrt{\frac{2l+1}{4\pi}}, \quad Y^{l*}_m(\hat{r}) = (-1)^{l-m} Y^l_{-m}(\hat{r}). \tag{7.K.2}$$

With this phase convention, the relation with the associated Legendre polynomials includes an extra i^l factor with respect to the Condon–Shortley phase,

$$Y^l_m(\theta,\phi) = i^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta) e^{im\phi}. \tag{7.K.3}$$

7.K.1 addition theorem

The addition theorem for the spherical harmonics states that

$$P_l(\cos\theta_{12}) = \frac{4\pi}{2l+1}\sum_m Y^l_m(\mathbf{r}_1) Y^{l*}_m(\mathbf{r}_2), \tag{7.K.4}$$

where θ_{12} is the angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. This result is independent of the phase convention. With *time–reversed phases,*

$$P_l(\cos\theta_{12}) = \frac{4\pi}{\sqrt{2l+1}}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]_0^0. \tag{7.K.5}$$

With *Condon–Shortley phases,*

$$P_l(\cos\theta_{12}) = (-1)^l\frac{4\pi}{\sqrt{2l+1}}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]_0^0. \tag{7.K.6}$$

7.K.2 expansion of the delta function

The Dirac delta function can be expanded in multipoles, yielding

$$\begin{aligned}\delta(\mathbf{r}_2-\mathbf{r}_1) &= \sum_l \delta(r_1-r_2)\frac{2l+1}{4\pi r_1^2}P_l(\cos\theta_{12})\\ &= \sum_l \delta(r_1-r_2)\frac{1}{r_1^2}\sum_m Y^l_m(\mathbf{r}_1)Y^{l*}_m(\mathbf{r}_2).\end{aligned} \tag{7.K.7}$$

This result is independent of the phase convention. With *time–reversed phases,*

$$\delta(\mathbf{r}_2-\mathbf{r}_1) = \sum_l \delta(r_1-r_2)\frac{\sqrt{2l+1}}{r_1^2}\left[Y^l(\hat{\mathbf{r}}_1)Y^l(\hat{\mathbf{r}}_2)\right]_0^0. \tag{7.K.8}$$

7.K.3 coupling and complex conjugation

If $\Psi^{I_1*}_{M_1} = (-1)^{I_1-M_1}\Psi^{I_1}_{-M_1}$ and $\Phi^{I_2*}_{M_2} = (-1)^{I_2-M_2}\Phi^{I_2}_{-M_2}$, as it happens to be the case for spherical harmonics with time-reversed phases, then

$$\begin{aligned}\left[\Psi^{I_1}\Phi^{I_2}\right]^{I*}_M &= \sum_{\substack{M_1M_2\\(M_1+M_2=M)}} \langle I_1\, I_2\, M_1\, M_2|IM\rangle\Psi^{I_1*}_{M_1}\Phi^{I_2*}_{M_2}\\ &= \sum_{\substack{M_1M_2\\(M_1+M_2=M)}} (-1)^{I-M_1-M_2}\langle I_1\, I_2\, -M_1\, -M_2|I-M\rangle\Psi^{I_1}_{-M_1}\Phi^{I_2}_{-M_2}\\ &= (-1)^{I-M}\sum_{\substack{M_1M_2\\(M_1+M_2=M)}} \langle I_1\, I_2\, -M_1\, -M_2|I-M\rangle\Psi^{I_1}_{-M_1}\Phi^{I_2}_{-M_2}\\ &= (-1)^{I-M}\left[\Psi^{I_1}\Phi^{I_2}\right]^I_{-M},\end{aligned} \tag{7.K.9}$$

Check sign

where we have used (7.K.23).

Let us care now about the spinor functions $\chi^{1/2}_m(\sigma)$, which have the form

$$\chi^{1/2}(\sigma=1/2) = \begin{bmatrix}1\\0\end{bmatrix} \quad \chi^{1/2}(\sigma=-1/2) = \begin{bmatrix}0\\1\end{bmatrix}, \tag{7.K.10}$$