Central Upwind Scheme for Hyperbolic Conservation Law

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Conservation of mass in 1D

Consider some gas in a tube where the density and velocity are constant across each cross section of the tube.
Let $\rho(x,t)$ represent the density of the gas and $v(x,t)$ represent the velocity of the gas at some cross section of the tube.

$$\text{mass in } [x_1, x_2] = \int_{x_1}^{x_2} \rho(x,t)dx \tag{1}$$

$$\text{the rate of mass flow} = \rho(x,t) \cdot v(x,t) \tag{2}$$

the rate of change of mass in $[x_1, x_2]$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t)$$

[1]

[1] LeV92.

Conservation of mass in 1D

Another Integral form

$$\int_{x_1}^{x_2} \rho(x, t_2)dx = \int_{x_1}^{x_2} \rho(x, t_1)dx + \int_{t_1}^{t_2} \rho(x_1, t)v(x_1, t)dt - \int_{t_1}^{t_2} \rho(x_2, t)v(x_2, t)dt \tag{3}$$

If $\rho(x, t)$ and $v(x, t)$ are both differentiable, we then obtain

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \left\{ \frac{\partial}{\partial t}\rho(x, t) + \frac{\partial}{\partial x}(\rho(x, t)v(x, t)) \right\} dxdt = 0 \tag{4}$$

and the differential form of the conservation law for the conservation of mass

$$\rho_t + (\rho v)_x = 0 \tag{5}$$

When $v(x, t)$ is a known function of $\rho(x, t)$, the conservation of mass equation can be written as

$$\rho_t + f(\rho)_x = 0 \tag{6}$$

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[2] LeV92.

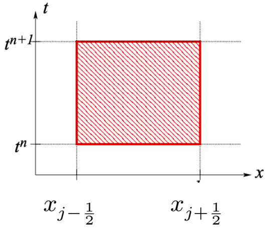
Finite Volume Scheme(BOOK)

Consider the conservation law in 1D,

$$u_t + f(u)_x = 0 \tag{7}$$

Integrate it over $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$ where x_j and t_n come from the discretization grid.

$$\begin{aligned}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_{n+1}) &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_n)dx \\ &- \Big[\int_{t_n}^{t_{n+1}} f(u(x_{j+\frac{1}{2}}, t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-\frac{1}{2}}, t))dt\Big]\end{aligned} \tag{8}$$



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[3] Kur18.

With the definition of cell average at n-th time step and j-th spatial step

$$\bar{u}_j^n = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t_n)dx \tag{9}$$

and the definition of numerical flux

$$F_{j+\frac{1}{2}}^n = F(u_j^n, u_{j+1}^n) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+\frac{1}{2}},t))dt \tag{10}$$

Finite volume method with Lax-Friedrich numerical flux

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}\left[F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right] \tag{11}$$

$$F_{j+\frac{1}{2}}^n = \frac{1}{2}((f(u_j^n) + f(u_j^{n+1})) - \frac{\alpha}{2}(u_j^{n+1} - u_j^n) \tag{12}$$

where $\alpha = max(|f'(u_j^n)|, |f'(u_j^{n+1})|)$ is the local wave speed.[a]

[a] Hes18.

Finite volume method with Lax-Friedrich numerical flux can achieve second-order accuracy.

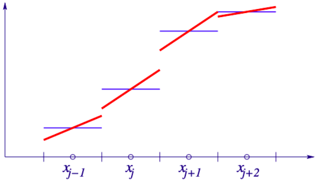
Central Upwind Scheme

Central Upwind Scheme is a finite-volume method so the semi-discretization follows the form below

$$\frac{d}{dt}\bar{u}_j^n = -\frac{1}{\Delta x}\left[H^n_{j+\frac{1}{2}} - H^n_{j-\frac{1}{2}}\right] \tag{13}$$

Construct a piecewise linear polynomial by the cell average $\bar{u}_j^n$ over $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$p_j^n(x) = \bar{u}_j^n + (u_x)_j^n (x - x_j) \tag{14}$$



where $(u_x)_j^n =$

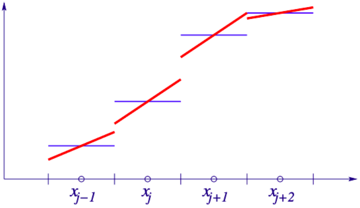
$$minmod\left(\frac{\bar{u}_j^n - \bar{u}_{j-1}^n}{\Delta x}, \frac{\bar{u}_{j+1}^n - \bar{u}_{j-1}^n}{2\Delta x}, \frac{\bar{u}_{j+1}^n - \bar{u}_j^n}{\Delta x}\right)$$

and minmod function is defined as follows

$$minmod(z_1, z_2, ..) = \begin{cases} \min_j\{z_j\} & \text{if } z_j > 0 \\ \max_j\{z_j\} & \text{if } z_j < 0 \\ 0 & \text{otherwise} \end{cases}$$ [4]

[4] Kur18.

Central Upwind Scheme



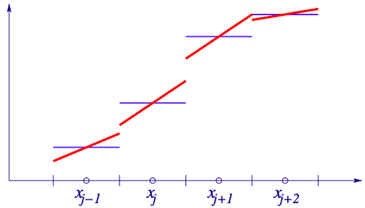
Approximate $u_{j+\frac{1}{2}}$ from u_{j+1} and u_j[5]

$$u^{+}_{j+\frac{1}{2}} = p^{n}_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}^{n}_{j+1} - (u_x)^{n}_{j+1}\frac{\Delta x}{2} \tag{15}$$

$$u^{-}_{j+\frac{1}{2}} = p^{n}_{j}(x_{j+\frac{1}{2}}) = \bar{u}^{n}_{j} + (u_x)^{n}_{j}\frac{\Delta x}{2} \tag{16}$$

[5] Kur18.

Central Upwind Scheme



The propagation speeds from the left and right to $x_{j+\frac{1}{2}}$ are

$$a^+_{j+\frac{1}{2}} = \max\left\{\lambda_N\left(\frac{\partial f}{\partial u}(u^-_{j+\frac{1}{2}})\right), \lambda_N\left(\frac{\partial f}{\partial u}(u^+_{j+\frac{1}{2}})\right), 0\right\} \tag{17}$$

$$a^-_{j+\frac{1}{2}} = \min\left\{\lambda_1\left(\frac{\partial f}{\partial u}(u^-_{j+\frac{1}{2}})\right), \lambda_1\left(\frac{\partial f}{\partial u}(u^+_{j+\frac{1}{2}})\right), 0\right\} \tag{18}$$

where $\lambda_1 < \lambda_2 < ...\lambda_N$ are eigenvalues of the Jacobian $\frac{\partial f}{\partial u}$.[6]

[6] Kur18.

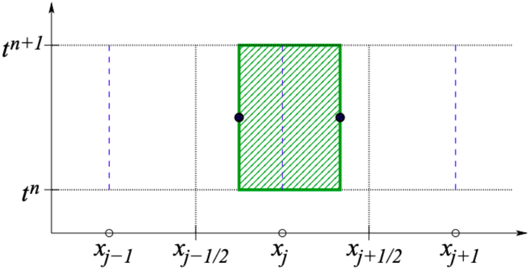
Central Upwind Scheme

Let

$$x_{j+\frac{1}{2},l} = x_{j+\frac{1}{2}} + \Delta t a^-_{j+\frac{1}{2}}, x_{j+\frac{1}{2},r} = x_{j+\frac{1}{2}} + \Delta t a^+_{j+\frac{1}{2}} \tag{19}$$

The cell average over the interval $[x_{j-\frac{1}{2},r}, x_{j+\frac{1}{2},l}] \times [t_n, t_{n+1}]$ is

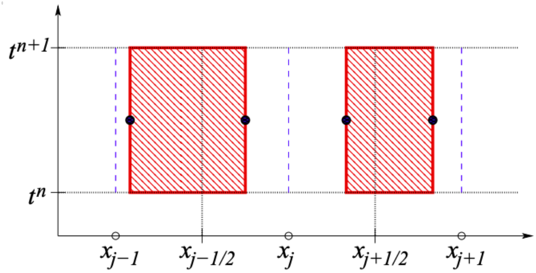
$$\begin{aligned}\bar{w}_j^{n+1} = \frac{1}{x_{j+\frac{1}{2},l} - x_{j-\frac{1}{2},r}} \Big[& \int_{x_{j-\frac{1}{2},r}}^{x_{j+\frac{1}{2},l}} p_j^n(x)dx \\ & - \int_{t_n}^{t_{n+1}} \Big(f(u(x_{j+\frac{1}{2},l}, t))dt - f(u(x_{j-\frac{1}{2},r}, t)) \Big) dt \Big]\end{aligned} \tag{20}$$



Central Upwind Scheme

Similarly, the cell average over the interval $[x_{j-\frac{1}{2},l}, x_{j+\frac{1}{2},r}] \times [t_n, t_{n+1}]$ is

$$\bar{w}^{n+1}_{j+\frac{1}{2}} = \frac{1}{x_{j+\frac{1}{2},r} - x_{j+\frac{1}{2},l}} \Big[\int_{x_{j+\frac{1}{2},l}}^{x_{j+\frac{1}{2}}} p^n_j(x,t)dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2},r}} p^n_{j+1}(x,t)dx - \int_{t_n}^{t_{n+1}} \Big(f(u(x_{j+\frac{1}{2},r},t))dt - f(u(x_{j+\frac{1}{2},l},t)) \Big) dt \Big] \quad (21)$$



Central Upwind Scheme

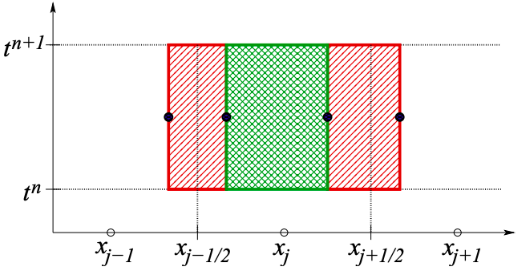
By projecting the above intermediate cell averages onto the original grid over interval $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, a piecewise polynomial interpolant $\tilde{w}$ is constructed.

For $x \in [x_{j\pm\frac{1}{2},l}, x_{j\pm\frac{1}{2},r}]$,

$$\tilde{w}^{n+1}_{j\pm\frac{1}{2}} = \bar{w}^{n+1}_{j\pm\frac{1}{2}} + \mathcal{O}(\Delta t) \tag{22}$$

For $x \in [x_{j-\frac{1}{2},r}, x_{j+\frac{1}{2},l}]$,

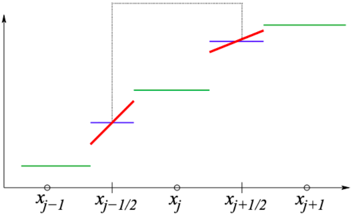
$$\frac{1}{x_{j+\frac{1}{2},l} - x_{j-\frac{1}{2},r}} \int_{x_{j-\frac{1}{2},r}}^{x_{j+\frac{1}{2},l}} \tilde{w}^{n+1}_{j}(x)dx = \bar{w}^{n+1}_{j}(x) \tag{23}$$



Central Upwind Scheme

By projecting the above intermediate cell averages onto the original grid over interval $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, a piecewise polynomial interpolant $\tilde{w}$ is constructed. The cell average

$$\begin{aligned}
\bar{u}_j^{n+1} &= \frac{1}{\Delta x}\Big[\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{w}^{n+1}(x)dx\Big] \\
&= \frac{1}{\Delta x}\Big[\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2},r}} \tilde{w}^{n+1}(x)dx + \int_{x_{j-\frac{1}{2},r}}^{x_{j+\frac{1}{2},l}} \tilde{w}^{n+1}(x)dx + \int_{x_{j+\frac{1}{2},l}}^{x_{j+\frac{1}{2}}} \tilde{w}^{n+1}(x)dx\Big] \\
&= \frac{1}{\Delta x}\Big[(x_{j-\frac{1}{2},r} - x_{j-\frac{1}{2}})\bar{w}^{n+1}(x) + (x_{j+\frac{1}{2},l} - x_{j-\frac{1}{2},r})\bar{w}^{n+1}(x) \\
&\quad + (x_{j+\frac{1}{2}} - x_{j+\frac{1}{2},l})\bar{w}^{n+1}(x)\Big]
\end{aligned} \tag{24}$$



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Central Upwind Scheme

Semidiscrete Central Upwind Scheme

$$\frac{d}{dt}\bar{u}_j^n = -\frac{1}{\Delta x}\left[H^n_{j+\frac{1}{2}} - H^n_{j-\frac{1}{2}}\right] \tag{25}$$

where

$$H_{j+\frac{1}{2}}(t) = \frac{a^+_{j+\frac{1}{2}} f(u^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}} f(u^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}[u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}] \tag{26}$$

Remarks

- Central upwind scheme is a third-order accuracy scheme.
- If one takes $a^+_{j+\frac{1}{2}} = -a^-_{j+\frac{1}{2}} = a_{j+\frac{1}{2}}$, the numerical flux $H_{j+\frac{1}{2}}(t)$ becomes

$$H_{j+\frac{1}{2}}(t) = \frac{f(u^+_{j+\frac{1}{2}}) + f(u^-_{j+\frac{1}{2}})}{2} - \frac{a_{j+\frac{1}{2}}}{2}[u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}] \tag{27}$$

[8]

[8] Kur01.

Numerical Experiements

Apply central upwind scheme and Lax-Friedrichs scheme to the invisicid Burgers' equation with periodic condition.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,\, u(x,0) = sin(x),\, x \in [0, 2\pi] \tag{28}$$

Tables below demonstrate the error and converenge rates with different grid points N.

Central Upwind			Lax-Friedrichs	
N	error	rate	error	rate
40	2.6441e-04	-	6.9891e-04	-
80	3.6891e-05	2.8414	2.0895e-04	1.742
160	4.9314e-06	2.9032	5.6366e-05	1.8903
320	6.448e-07	2.9351	1.5317e-05	1.8797
640	8.2814e-08	2.9609	4.0741e-06	1.9105
1280	1.0531e-08	2.9753	1.0522e-06	1.9531

References

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