# Algebra: Chapter 0 Exercises

Chapter 3, Section 5

Modules over a ring

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**Problem 5.5.** Let $R$ be a commutative ring, viewed as an $R$-module over itself, and let $M$ be an $R$-module. Prove that $\mathrm{Hom}_{R-\mathbf{Mod}}(R, M) \cong M$ as $R$-modules.

*Solution.* Let $\varphi : M \to \mathrm{Hom}_{R-\mathbf{Mod}}(R, M)$ be the function defined by

$$\varphi(m)(r) = rm.$$

Then note that

$$\begin{aligned}
\varphi(m+n)(r) &= r(m+n) \\
&= rm + rn \\
&= \varphi(m)(r) + \varphi(n)(r) \\
&= (\varphi(m) + \varphi(n))(r),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(sm)(r) &= r(sm) \\
&= (rs)m \\
&= (sr)m \\
&= s(rm) \\
&= s\varphi(m)(r) \\
&= (s\varphi(m))(r),
\end{aligned}$$

and so $\varphi$ is an $R - \mathbf{Mod}$ homomorphism.
Additionally, we have:

$$\begin{aligned}
\varphi(m)(r+s) &= (r+s)m \\
&= rm + sm \\
&= \varphi(m)(r) + \varphi(n)(r)
\end{aligned}$$

and

$$\begin{aligned}\varphi(m)(rs) &= (rs)m \\ &= r(sm) \\ &= r\varphi(m)(s),\end{aligned}$$

and so $\varphi(m)$ is an $R-\mathbf{Mod}$ homomorphism for all $m \in M$.

Now, to prove that $\varphi$ is injective, note that if $\varphi(m) = 0$, then $m = 1_R m = \varphi(m)(1) = 0$, and so $\varphi$ is injective. For surjectivity, we need the following insight: For all $m \in M$ and $r \in R$, we have

$$\begin{aligned}\varphi(m)(r) &= \varphi(m)(r \cdot 1_R) \\ &= r\varphi(m)(1_R),\end{aligned}$$

and so if $\psi \in \mathrm{Hom}_{R-\mathbf{Mod}}(R, M)$, then we have, for all $r \in R$,

$$\begin{aligned}\psi(r) &= r\psi(1_R) \\ &= \varphi(\psi(1_R)(r);\end{aligned}$$

thus, $\psi$ is in the image of $\varphi$ and $\varphi$ is surjective. Therefore, $\psi$ is an isomorphism and the modules are isomorphic as desired. ■

**Problem 5.6.** Let $G$ be an abelian group. Prove that if $G$ has a structure of $\mathbb{Q}$-vector space, then it has only one such structure. (Hint: First prove that every element of $G$ has necessarily infinite order. Alternative hint: The unique ring homomorphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.)

*Solution.* Let $G$ be an abelian group. A $\mathbb{Q}$-vector space structure on $G$ is precisely a ring homomorphism $\sigma : G \to \mathrm{Hom}_{\mathbf{Ab}}(G)$. Let $\sigma_1, \sigma_2$, then, be two of these ring homomorphisms. Note that $\sigma_1$ and $\sigma_2$ agree on the integers, as if we view $\mathbb{Q}$ and $\mathrm{Hom}_{\mathbf{Ab}}(G)$ as $\mathbb{Z}$-modules, we then have, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\varphi_1(n) &= \varphi_1(n \cdot 1) \\ &= n \cdot \varphi_1(1) \\ &= n \cdot \mathrm{id} \\ &= n \cdot \varphi_2(1) \\ &= \varphi_2(n \cdot 1) \\ &= \varphi_2(n).\end{aligned}$$

Thus, if $\iota : \mathbb{Z} \to \mathbb{Q}$ is the unique ring homomorphism $\mathbb{Z} \to \mathbb{Q}$, i.e. the inclusion, we have $\sigma_1\iota = \sigma_2\iota$. Since $\iota$ is a ring epimorphism, this then implies that $\sigma_1 = \sigma_2$, and so there is only one $\mathbb{Q}$-vector space structure on $\mathbb{G}$, as desired. ■

**Problem 5.7.** Let $K$ be a field, and let $k \subseteq K$ be a subfield of $K$. Show that $K$ is a vector space over $k$ (and in fact a $k$-algebra) in a natural way. In this situation, we say that $K$ is an *extension* of $k$.

*Solution.* Note that the inclusion $\sigma : k \to \mathrm{Hom}_{\mathbf{Ab}}(K)$ is a ring homomorphism, and thus a natural $k$-vector space structure on $K$. This $\sigma$ also gives us a $k$-algebra structure on $K$ since the center of $K$ is $K$ itself, and so $\mathrm{im}\ \sigma \subseteq Z(K)$.

More explicitly, the "scalar" multiplication $\kappa x$ for $\kappa \in k$ and $x \in K$ is just multiplication within the field $K$, and the $k$-algebra structure on $K$ also consists of multiplication as defined in the field $K$. ■

**Problem 5.8.** What is the initial object of the category $R$-**Alg**?

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