

Algebra: Chapter 0 Exercises

Chapter 2, Section 7

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Problem 7.4. Prove that the relation defined in Exercise 5.10 on a free abelian group $F = F^{ab}(A)$ by

$$f \sim f' \Leftrightarrow (\exists g \in G) : f' - f = 2g$$

is compatible with the group structure. Determine the quotient F/\sim as a better known group.

Solution. The relation \sim is compatible with the group structure on F if and only if, considering Proposition 7.3,

$$(\forall f, f', a \in F) : f \sim f' \implies a + f \sim a + f'$$

(since F is abelian). Suppose then, we have $f \sim f'$, that is $f' - f = 2g$. We then have $(f' + a) - (f + a) = 2g$, and hence $a + f' \sim a + f$.

To determine F/\sim as a better known group, we will establish an isomorphism between the group F and the abelian group $G := (\mathbb{Z}/2\mathbb{Z})^A$.

Define the set function $\kappa : A \rightarrow G$ as follows:

$$(\forall a, a' \in A) : \kappa(a)(a') = [\delta_{a'a}]_2,$$

where δ is the kronecker delta function. We then use this function and the universal property for free (abelian) groups to induce a group homomorphism $\varphi : F \rightarrow G$ that makes the following diagram commute:

$$\begin{array}{ccc} F^{ab}(A) & \overset{\varphi}{\dashrightarrow} & (\mathbb{Z}/2\mathbb{Z})^A \\ j \uparrow & \nearrow \kappa & \\ A & & \end{array}$$

where $j : A \rightarrow F$ is the usual inclusion. We will now prove two lemmas essential to constructing the desired isomorphism.

Lemma 1. *The homomorphism φ is surjective.*

Proof. Suppose $f \in (\mathbb{Z}/2\mathbb{Z})^A$, let $\bar{a} = j(a) \in F$, and allow $f(x)y$ to be "multiplication" by a coset representative of $f(x)$, i.e. $[0]_2y = 0y$ and $[1]_2y = 1y$. We then have, with the symbol at the top of each sigma representing for clarity the abelian group in which the sum is taking place,

$$\begin{aligned} (\forall a, a' \in A) \varphi \left(\sum_{a \in A}^F f(a) \bar{a} \right) (a') &= \sum_{a \in A}^G (\varphi(f(a) \bar{a})) (a') \\ &= \sum_{a \in A}^G (f(a) (\varphi(\bar{a})(a'))) \\ &= \sum_{a \in A}^{\mathbb{Z}/2\mathbb{Z}} (f(a) ([\delta_{a'a}]_2)) \\ &= f(a') \end{aligned}$$

Hence every $f \in G$ is in the image of φ . □

Lemma 2. *The homomorphism φ agrees with the relation \sim ; that is,*

$$f \sim f' \implies \varphi(f) = \varphi(f')$$

Proof. Suppose $f, f' \in F$ and $f \sim f'$. We then have, for all $a \in A$ and for some $g \in (\mathbb{Z}/2\mathbb{Z})^A$:

$$\begin{aligned} (\varphi(f') - \varphi(f))(a) &= \varphi(f' - f)(a) \\ &= (2g)(a) \\ &= 2(g(a)) \\ &= [0]_2 \end{aligned}$$

(as $\mathbb{Z}/2\mathbb{Z}$ has order 2); hence $\varphi(f') = \varphi(f)$, as desired. □

Lemma 3. *If we define $H = [e_F]_\sim$ then we have $F/H = F/\sim$, and $\ker(\varphi) = H$.*

Proof. The first statement follows from Proposition 7.4, Proposition 7.7, and the definition of quotient by a normal subgroup.

For the second statement, first suppose that $h \in H$. It then follows, since $h \sim e_F$, that $\varphi(h) = \varphi(e_F) = e_G$, so $h \in \ker(\varphi)$, and hence $H \subseteq \ker(\varphi)$. For the other direction, suppose $f \in \ker(\varphi)$, and

$$f = \sum_{a \in A} n_a \bar{a}$$

We then have, for all $a' \in A$,

$$\begin{aligned}
\varphi(f)(a') &= \varphi\left(\sum_{a \in A} n_a \bar{a}\right)(a') \\
&= \sum_{a \in A} (n_a \varphi(\bar{a})(a')) \\
&= \sum_{a \in A} (n_a \kappa(a)(a')) \\
&= \sum_{a \in A} (n_a [\delta_{a'a}]_2) \\
&= [n_{a'}]_2
\end{aligned}$$

Since $\varphi(f) = e_G$, we know that each $n_{a'}$ is congruent to 0 modulo two; that is, even. Hence, we have:

$$\begin{aligned}
f - e_F &= f \\
&= 2 \sum_{a \in A} \frac{n_a}{2} \bar{a}
\end{aligned}$$

This shows that $f \sim e_f$ and thus $f \in H$, giving us $\ker(\varphi) \subseteq H$. Therefore $\ker(\varphi) = H$, as desired. \square

With this all established, we can finally find our isomorphism. We now construct a homomorphism $\tilde{\varphi} : F/H \rightarrow G$ using the universal property for quotient by an equivalence relation:

$$\begin{array}{ccc}
F & \xrightarrow{\varphi} & G \\
\pi \downarrow & \nearrow \tilde{\varphi} & \\
F/H & &
\end{array}$$

Here, π is the quotient map $\pi(f) = [f]_{\sim}$, and $\tilde{\varphi}$ is the unique homomorphism making the diagram commute. The first isomorphism theorem shows that $\tilde{\varphi}$ is an isomorphism, and so $F/\sim \cong (\mathbb{Z}/2\mathbb{Z})^A$. \blacksquare