Algebra: Chapter 0 Exercises

Chapter 2, Section 3

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Problem 3.1. Let $\varphi : G \to H$ be a morphism in a category $\mathbf{C}$ with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \to H \times H$$

compatible in the evident way with the natural projections.

Solution. I'm going to assume the author means the following:
Since $H \times H$ is a product in $\mathbf{C}$, we have that $G \times G$ along with two "copies" of $\varphi : G \to H$ admits a unique morphism $(\varphi \times \varphi) : G \times G \to H \times H$ that makes the following diagram commute:

$$\begin{tikzcd}
 & & H \\
G \times G \arrow[r, "\varphi\times\varphi"] \arrow[urr, bend left, "\varphi\pi_{G_1}"] \arrow[drr, bend right, "\varphi\pi_{G_2}"'] & H \times H \arrow[ur, "\pi_{H_1}"] \arrow[dr, "\pi_{H_2}"'] & \\
 & & H
\end{tikzcd}$$

where π_{G_1} and π_{G_2} are the canonical projections of $G \times G$ onto G. ■

Problem 3.2. Let $\varphi : G \to H$ and $\psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G$, $H \times H$, and $K \times K$ as in Exercise 3.1. Prove that

$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$$

Solution. To demonstrate this result, first stare at this diagram, mapping $G \times G$ to the product $H \times H$ and $H \times H$ to the product $K \times H$.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \arrow[r, "\psi"] & K \\
G \times G \arrow[u] \arrow[d] \arrow[r, "\varphi\times\varphi"] & H \times H \arrow[u] \arrow[d] \arrow[r, "\psi\times\psi"] & K \times K \arrow[u] \arrow[d] \\
G \arrow[r, "\varphi"'] & H \arrow[r, "\psi"'] & K
\end{tikzcd}$$

Compare it to the following diagram, mapping $G \times G$ straight to $K \times K$:

$$\begin{tikzcd}
G \arrow[r, "\psi\varphi"] & K \\
G \times G \arrow[u] \arrow[d] \arrow[r, "(\psi\varphi)\times(\psi\varphi)"] & K \times K \arrow[u] \arrow[d] \\
G \arrow[r, "\psi\varphi"'] & K
\end{tikzcd}$$

Notice that we have morphisms $(\psi\varphi) \times (\psi\varphi)$ and $(\psi \times \psi)(\varphi \times \varphi)$ from $G \times G$ to $K \times K$, both determined by the universal property for products with regards to $K \times K$. These must be equal per Problem 3.1. ■

Problem 3.3. Show that if G and H are both *abelian* groups, then $G \times H$ satisfies the universal property for coproducts in **Ab**.

Solution. That $G \times H$ satisfies the universal property for coproducts in **Ab** means the following:
Every triple (X, δ_G, δ_H) (as in the diagram) admits a unique morphism σ such that the following diagram commutes:

$$\begin{tikzcd}
 & G \arrow[dl, "f_G"'] \arrow[d, "\delta_G"] \\
X & G \times H \arrow[l, "\sigma"'] \\
 & H \arrow[ul, "f_H"] \arrow[u, "\delta_G"']
\end{tikzcd}$$

That is,

$$\sigma\delta_G = f_G$$

$$\sigma\delta_H = f_H$$

Define $\sigma : G \times H \to X$ by

$$\sigma(g, h) = f_G(g) \cdot f_H(h)$$

Also define $\delta_G : G \to G \times H$ $\delta_H : H \to G \times H$ by

$$\delta_G(g) = (g, e_H)$$
$$\delta_G(h) = (e_G, h)$$

The proof that these are homomorphisms is trivial. To show that σ makes the diagram commute, note that

$$\begin{aligned}
\sigma(\delta_G(g)) &= \sigma(g, e_H) \\
&= f_G(g) \cdot f_H(e_H) \\
&= f_G(h)
\end{aligned}$$

The same proof applies to H.
This is where our groups being abelian comes in. We will show that σ is a homomorphism:

$$\begin{aligned}
\sigma\left((g_1, h_1)(g_2, h_2)\right) &= \sigma(g_1 g_2, h_1 h_2) \\
&= f_G(g_1 g_2) \cdot f_H(h_1 h_2) \\
&= f_G(g_1) g_G(g_2) f_H(h_1) f_H(h_2) \\
&= \left(f_G(g_1) f_H(h_1)\right)\left(f_G(g_2) f_h(h_2)\right) \\
&= \sigma(g_1, h_1) \cdot \sigma(g_2, h_2)
\end{aligned}$$

done ■