Topology and Groupoids Exercises

Chapter 2, Section 2

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Problem 1. What are the open sets of X hen X is discrete, that is, has the discrete topology?, is indiscrete, that is, has the indescrete topology? What is the closure of $\{x\}, x \in X$, in these cases?

Solution. Recall that under the discrete topology, a set $N \subseteq X$ is a neighborhood of a point $x \in X$ if and only if $x \in N$; that is, $\operatorname{Int} N = N$. Hence, under the discrete topology, every set is open.

On the other hand, under the indescrete toplogy, a set $N \subseteq X$ is a neighborhood of a point $x \in X$ if and only if $N = X$ and $x \in N$. Here, the only open sets are and X itself. ■

Problem 2. Let X be a topological space and let $A \subseteq X$. Prove that $\operatorname{Int} A$ is the union of all open sets U such that $U \subseteq A$ and $\overline{A}$ is the intersection of all closed sets C such that $A \subseteq C$.

Proof. First, note that $x \in \operatorname{Int} A$ if and only if there exists some $U \subseteq A$ such that $x \in U$, if and only if $x \in \mathcal{U}$, where $\mathcal{U}$ is the family of all open sets containing A.

For closed sets, first suppose $x \in \overline{A}$, and so every neighborhood of x meets A. If C is a closed subset of X containing A, then every neighborhood of x must also meet C, implying $x \in \overline{C} = C$ since C is closed. Hence x is in the intersection of all closed sets containing A, completing the inclusion in one direction.

Conversely, if $x \notin \overline{A}$, then $\overline{A}$ itself is a closed set containing A that does not contain x, so x is certainly not in the intersection of all closed sets containing A. Thus, $\overline{A}$ is the intersection of all closed sets in X that contain A as a subset. □

The previous result essentially means that the interior of A is the largest open set within A, and the closure of A is the smallest closed set containing A.

Problem 3. Let X be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A if each neighborhood of x contains points of A other than x. The set of limit points of A is written $\widehat{A}$. Prove that $\overline{A} = A \cup \widehat{A}$, and that A is closed iff $\widehat{A} \subseteq A$. Give examples of non-empty subsets A of $\mathbb{R}$ such that:

(i) $\widehat{A} = \varnothing$

(ii) $\widehat{A} \neq \varnothing$ and $\widehat{A} \subseteq A$

(iii) A is a proper subset of $\widehat{A}$

(iv) $\widehat{A} \neq \varnothing$ but $A \cap \widehat{A} = \varnothing$

Solution. First we will prove that $\overline{A} = A \cup \widehat{A}$.

Proof. First suppose that $x \in \overline{A}$. Then, by definition, every neighborhood of x meets A. If x is not in A, then that every neighborhood of x meets A means that every neighborhood of x contains points of A that aren't x, meaning $x \in \widehat{A}$. Hence $\overline{A} \subseteq A \cup \widehat{A}$.

Conversely, suppose $x \in A \cup \widehat{A}$. If $x \in A$, then every neighborhood of x contains $x \in A$. If $x \in \widehat{A}$, then every neighborhood of x contains a point in A. Hence $A \cup \widehat{A} \subseteq \overline{A}$, and so $\overline{A} = A \cup \widehat{A}$. □

Next, we will prove that A is closed iff $\widehat{A} \subseteq A$.

Proof. A is closed iff $A = \overline{A}$, meaning $A = A \cup \widehat{A}$, whence $\widehat{A} \subseteq A$. □

Now, we produce each of the examples requested:

(i) Let A be the singleton set $\{0\}$. Obviously every neighborhood of 0 contains 0, so 0 is not a limit point of A. If $x \neq 0$, then the open interval $(x - |x|, x + |x|)$ does not contain 0, so x is not a limit point of A.

(ii) Let $A = [0, 1] \cup \{2\}$. Then $\widehat{A} = [0, 1] \subseteq A$.

(iii) Let $A = (0, 1)$. Then $\widehat{A} = [0, 1] \supset A$.

(iv) Let $A = \{1/n \mid n \in \mathbb{N}\}$. Then for any $1/n \in A$, the interval $\left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right)$ with $\delta = \frac{1}{n} - \frac{1}{n+1}$ contains only $1/n \in A$, and so $A \cap \widehat{A} = \varnothing$. However, by the Archimedean property of the real numbers, there exists for every $\varepsilon > 0$ an $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Hence every open interval containing 0 also contains a point in A, and so $0 \in \widehat{A}$.

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Problem 4. Let X be a topological space and let $A \subseteq B \subseteq X$. We say that A is *dense* in B if $B \subseteq \overline{A}$, and A is *dense* if $\overline{A} = X$. Prove that if A is dense in X and U is open then

$$U \subseteq \overline{A \cap U}.$$

Proof. Recall from the previous exercise that $\overline{A} = A \cup \widehat{A}$. Since A is dense in X, we then know from this that $X = A \cup \widehat{A}$. With this in mind, we proceed to consider the two cases for any $x \in U \subseteq X$:

If $x \in A$, then of course $x \in U \cap A \subseteq \overline{U \cap A}$.

Otherwise, suppose $x \in \widehat{A}$, and let N be any neighborhood of x. Since U is open, we know that $N \cap U$ is also a neighborhood of x, and because $x \in \widehat{A}$, we know that $N \cap U$ meets A; that is, there exists an $a \in N \cap U \cap A$ with $a \neq x$. Clearly, then, this a is also an element of $U \cap A$, allowing us to conclude that every neighborhood of x meets $U \cap A$, and so $x \in \overline{U \cap A}$, as desired. □

Problem 5. Let $\mathbb{I} = [0,1]$. Define an order relation $\leq$ on $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ by

$$(x,y) \leq (x',y') \Leftrightarrow y < y' \text{ or } (y = y' \text{ and } x \leq x').$$

The *television topology* on $\mathbb{I}^2$ is the order topology with respect to $\leq$. Let A be the set of points $(1/2, 1-n^{-1})$ for positive integral n. Prove that in the television topology on $\mathbb{I}^2$,

$$\overline{A} = A \cup \{(0,1)\}.$$

Solution. (Sketch) Note that an interval of a point $p = (x,y) \in \mathbb{I}^2$ (with respect to the television order) which also contains the point $(x',y') \in \mathbb{I}^2$ for $y' < y$ will contain the "vertical interval" $\mathbb{I}\times]y',y[$, where $]y',y[$ is an open interval with respect to the usual order topology on $\mathbb{R}$.

Since $p = (x,y) \leq (0,1)$ if and only if $y < 1$, we then know that any neighborhood of p contains the open $\mathbb{I}^2$-interval $\mathbb{I}^2\times]r,1[$ for some real number $0 \leq r < 1$. By the Archimedean propoerty of the reals, there exists some positive integral n such that $n^{-1} < 1-r$, whence $1-n^{-1} > r$, and so the point $(1/2, 1-n^{-1})$ is contained in the neighborhood in question. Hence the point $(0,1)$ is in $\overline{A}$.

Points (x,y) with $y < 1$ are not contained in $\overline{A}$ since A will contain some points above $\mathbb{I}^2$-neighborhoods about (x,y) that don't stretch to the top of $\mathbb{I}^2$. Neither will points with $x > 0$, since there exist $\mathbb{I}^2$-neighborhoods about this point which only contain points of one y-value. ■

Problem 7. Prove that if A is the closure of an open set, then $A = \overline{\operatorname{Int} A}$.

Proof. First, suppose that $x \in A = \overline{U}$, where $U \subset X$ is open. By definition, this means that every neighborhood of x meets $U = \operatorname{Int} U$. Since the interior operator preserves inclusions, we know that $U \subseteq \overline{U}$ implies $\operatorname{Int} U \subseteq \operatorname{Int} \overline{U}$, and so, continuing our previous line of reasoning, every neighborhood of x meets $\operatorname{Int} \overline{U}$, meaning $x \in \overline{\operatorname{Int} \overline{U}} = \overline{\operatorname{Int} A}$, by the definition of closure. Therefore, $A \subseteq \overline{\operatorname{Int} A}$.

Conversely, suppose $x \in \overline{\operatorname{Int} A}$. Then, by the definition of closure, every neighborhood of x meets $\operatorname{Int} A$, and so every neighborhood of x meets A, whence $x \in \overline{A}$, which equals A since A is closed. Thus $\overline{\operatorname{Int} \overline{A}} \subseteq A$, and so $\overline{\operatorname{Int} \overline{A}} = A$, completing the proof. □

Problem 9. A topological space H (the *half-open topology*) is defined as follows. The underlying set of H is $\mathbb{R}$, and for each $x \in H$ and $N \subseteq H$, N is a neighborhood of x iff there are real numbers a and b such that

$$x \in [a,b[\subseteq N.$$

Prove hat H is a topological space and that:

1. Each interval $[a,b[$ is both open and closed
2. H is separable
3. If $A \subseteq H$, then $A \setminus \widehat{A}$ is countable

Solution. To begin, we verify that H is a topological space using the neighborhood topology axioms.

First, if N is a neighborhood of x, then there exist a, b in $\mathbb{R}$ such that $x \in [a, b[\subseteq N$, meaning $x \in N$.

Next, if $N \subseteq \mathbb{R}$ contains a neighborhood M of x, then there exists an interval $[a, b[$ with $x \in [a, b[\subseteq M \subseteq N$, whence M is a neighborhood of x.

Next, suppose N_1 and N_2 are neighborhoods of x so that $x \in [a_1, b_1[\subseteq N_1$, and $x \in [a_2, b_2[\subseteq N_2$. We then have $x \in [\max(a_1, a_2), \min(b_1, b_2)[\subseteq N_1 \cap N_2$, and so $N_1 \cap N_2$ is a neighborhood of x.

Finally, let N be a neighborhood of x, and let $I = [a, b[\subseteq N$ be an interval containing x. Then I itself is a subset of N containing x such that N is a neighborhood of every point of I. Now we prove each of the additional statements:

1. Each interval $[a, b[$ is both open and closed.

   Proof. Note that if $x \in [a, b[$, then $[a, b[$ is itself a half-open interval within $[a, b[$ containing x, and so $\operatorname{Int}[a, b[= [a, b[$. Hence $[a, b[$ is open. Additionally, $H \setminus [a, b[=]-\infty, a[\cup [b, \infty[$, which is the union of two open sets, and hence is open. Therefore $[a, b[$ is also closed. □

2. H is separable.

   Proof. Consider the rationals $\mathbb{Q} \subseteq H$. Since half open intervals $[a, b[$ cannot contain a single element, every neighborhood of an irrational number contains a rational number, and so $\overline{\mathbb{Q}} = H$. Hence H is separable. □

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