Algebra: Chapter 0 Exercises

Chapter 2, Section 5

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Problem 1. Let $\mathscr{F}^A$ be a category whose objects (f, G) are pairs consisting of a group G along with a function $f : A \to G$, and whose morphisms $(f_1, G_1) \to (f_2, G_2)$ are morphisms $\varphi : G_1 \to G_2$ such that the following diagram commutes:

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \\ A \arrow[u, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

Does this category have final objects?

Solution. Yes. If T is the trivial group and κ_e is the trivial homomorphism (i.e. sends all elements to the identity in the destination group), then (φ_e, T) is final in this group.

Proof. Let $(f, G) \in \mathrm{Obj}(\mathscr{F}^A)$, and suppose φ is such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] & T \\ A \arrow[u, "f"] \arrow[ur, "\kappa_e"'] & \end{tikzcd}$$

Clearly the only morphism $G \to T$ is the trivial morphism (i.e. $\varphi(g) = e$ for all $g \in G$), and this diagram does commute for such a φ since $\varphi(f(a)) = e$. Hence (κ_e, T) is final in $\mathscr{F}^A$. $\square$

$\blacksquare$

Problem 5.2. Explain why (κ_e, T) is not initial in $\mathcal{F}^A$ (unless $a = \emptyset$).

Solution. Let $(f, G) \in \mathrm{Obj}(\mathscr{F}^A)$, and consider the following diagram:

$$\begin{tikzcd} T \arrow[r, dashed, "\varphi"] & G \\ A \arrow[u, "\kappa_e"] \arrow[ur, "f"'] & \end{tikzcd}$$

Since the only group homomorphism $T \to G$ is the trivial one, we must have $f(a) = e_G$ for all $a \in A$ for this diagram to commute. This is the only case for all (f, G) if $A = \emptyset$. $\blacksquare$

Problem 5.3. Use the universal property of free groups to prove that the map $j : f \to F(A)$ is injective, for all sets A.

Solution. Recall that a free group along with inclusion $\iota, F(A)$ is initial in the category $\mathscr{F}^A$. For any $a, b \in A$ with $a \neq b$, consider the following diagram:

$$\begin{tikzcd} F(A) \arrow[r, dashed, "\varphi"] & C_2 \\ A \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

Define f by $f(a) = e$, and $f(x) = g$ if $x \neq a$ (where e and g are the identity and generator in C_2, respectively. We then have $f(a) \neq f(b)$, so $(\varphi \circ \iota)(a) \neq (\varphi \circ \iota)(b)$, and hence $\iota(a) \neq \iota(b)$, making ι injective. ∎

Problem 5.5. Verify explicitly that $H^{\oplus A}$ is a group.

Solution. Let H be a group. If α is a function from A to H, define

$$\ker' \alpha := \{a \in A \mid \alpha(a) \neq e_H\}.$$

We then define $H^{\oplus A}$ by

$$H^{\oplus A} := \{\alpha : A \to H \mid \ker' \alpha \text{ is finite}\}$$

To define the group operation on $H^{\oplus A}$, let $\alpha_1, \alpha_2 \in H^{\oplus A}$. We then define

$$(\alpha_1 \cdot \alpha_2)(a) = \alpha_1(a) \cdot \alpha_2(a)$$

This operation "inherits" associativity and inverses from the group H. Furthermore, this group is closed under inverses, since the inverse of the identity is the identity. To prove closure, we must show that

$$\ker'(\alpha_1 \cdot \alpha_2) \text{ is finite.}$$

Some examination shows us that

$$\ker'(\alpha_1 \cdot \alpha_2) = (\ker{}`\alpha_1 \cup \ker{}`\alpha_2) \setminus \{a \in A \mid \alpha_1(a) = \alpha_2(a)^{-1}\},$$

which is clearly finite; hence $(\alpha_1 \cdot \alpha_2)$ is in $H^{\oplus A}$. ∎

Problem 5.6. Prove that the group $F(\{x, y\})$ is a coproduct $\mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$ by itself in the category **Grp**.

Proof. Let S be the set $\{x, y\}$, let G be a group, and suppose we have group homomorphisms as in the following diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\epsilon_x"] \arrow[dr, "f_x"'] & F(S) \arrow[d, dashed, "\varphi"] & \mathbb{Z} \arrow[l, "\epsilon_y"'] \arrow[dl, "f_y"] \\ & G & \end{tikzcd}$$

where ϵ_x and ϵ_y are the "exponential maps" $\epsilon_g(n) = g^n$. If a φ exists that makes this diagram commute, note that we must have

$$\begin{aligned}\varphi(x) &= \varphi(\epsilon_x(1)) \\ &= f_x(1)\end{aligned}$$

and similarly

$$\begin{aligned}\varphi(y) &= \varphi(\epsilon_y(1)) \\ &= f_y(1)\end{aligned}$$

Keeping this in mind, we turn our attention to the following diagram:

$$\begin{tikzcd} F(S) \arrow[r, dashed, "\varphi"] & G \\ S \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

where $\iota : S \to F(S)$ is the usual inclusion and $f : S \to G$ is a set function mapping $x \mapsto f_1(1)$ and $y \mapsto f_2(1)$.
The group homomorphism φ making this diagram commute exists and is unique by the universal property for free groups. Since this is the only group homomorphism satisfying those properties listed above required to make the coproduct diagram commute, we simply must check that this homomorphism does indeed make that diagram commute. We have

$$\begin{aligned}\varphi(\epsilon_x(n)) &= \varphi(x^n) \\ &= \varphi(x)^n \\ &= f_x(1)^n \\ &= f_x(n),\end{aligned}$$

and similarly for f_y. This completes the proof. $\square$

Problem 5.10. Let $F = F^{ab}(A)$.

1. Define an equivalence relation $\sim$ on F by setting

$$f' \sim f \Leftrightarrow (\exists g \in F) : \quad f - f' = 2g.$$

   Prove that $F/\sim$ is a finite set if and only if A is finite, and in that case $|F/\sim| = 2^{|A|}$.

2. Assume $F^{ab}(B) \cong F^{ab}(A)$. If A is finite, prove that so is B, and $A \cong B$ as sets.

Solution.

1. This follows immediately from the fact that $F \cong (\mathbb{Z}/2\mathbb{Z})^A$ (see exercise 7.4). More specifically, if A is infinite then $(\mathbb{Z}/2\mathbb{Z})^A$ is also infinite, and if it is finite, then $|F| = |(\mathbb{Z}/2\mathbb{Z})^A| = 2^{|A|}$.

2. If A is finite, then

$$\begin{aligned} 2^{|A|} &= |F^{ab}(A)| \\ &= |F^{ab}(B)| \\ &= 2^{|B|} \end{aligned}$$

Hence $|A| = |B|$.

$\blacksquare$