Algebra: Chapter 0 Exercises

Chapter 3, Section 5

Modules over a ring

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Problem 5.4. Let R be a ring. A nonzero R-module M is *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M. Let M, N be simple modules, and let $\varphi : M \to N$ be a homomorphism of R-modules. Prove that either $\varphi = 0$ or φ is an isomorphism.

Solution. Let $\varphi : M \to N$ be an R-module homomorphism. Then $\ker\varphi$ is a submodule of M, and hence is either $\{0\}$ or M. If $\ker\varphi = M$, then φ is the zero homomorphism.

Otherwise, we have $\ker\varphi = \{0\}$, telling us that φ is injective, and we turn our attention to im φ. Since $\ker\varphi = 0$, and M, N are nonzero, we know that im φ has at least one nonzero element. Since im φ is a submodule of N, this implies that im $\varphi = N$, as N is simple. Thus φ is surjective, and so it is an isomorphism, as desired. ■

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Problem 5.5. Let R be a commutative ring, viewed as an R-module over itself, and let M be an R-module. Prove that $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R, M) \cong M$ as R-modules.

Solution. Let $\varphi : M \to \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R, M)$ be the function defined by

$$\varphi(m)(r) = rm.$$

Then note that

$$\begin{aligned}
\varphi(m+n)(r) &= r(m+n) \\
&= rm + rn \\
&= \varphi(m)(r) + \varphi(n)(r) \\
&= (\varphi(m) + \varphi(n))(r),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(sm)(r) &= r(sm) \\
&= (rs)m \\
&= (sr)m \\
&= s(rm) \\
&= s\varphi(m)(r) \\
&= (s\varphi(m))(r),
\end{aligned}$$

and so φ is an $R-\mathbf{Mod}$ homomorphism.
Additionally, we have:

$$\begin{aligned}\varphi(m)(r+s) &= (r+s)m \\ &= rm+sm \\ &= \varphi(m)(r)+\varphi(n)(r)\end{aligned}$$

and

$$\begin{aligned}\varphi(m)(rs) &= (rs)m \\ &= r(sm) \\ &= r\varphi(m)(s),\end{aligned}$$

and so $\varphi(m)$ is an $R-\mathbf{Mod}$ homomorphism for all $m \in M$.

Now, to prove that φ is injective, note that if $\varphi(m)=0$, then $m = 1_R m = \varphi(m)(1) = 0$, and so φ is injective. For surjectivity, we need the following insight: For all $m \in M$ and $r \in R$, we have

$$\begin{aligned}\varphi(m)(r) &= \varphi(m)(r \cdot 1_R) \\ &= r\varphi(m)(1_R),\end{aligned}$$

and so if $\psi \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R, M)$, then we have, for all $r \in R$,

$$\begin{aligned}\psi(r) &= r\psi(1_R) \\ &= \varphi(\psi(1_R)(r);\end{aligned}$$

thus, ψ is in the image of φ and φ is surjective. Therefore, ψ is an isomorphism and the modules are isomorphic as desired. ∎

Problem 5.6. Let G be an abelian group. Prove that if G has a structure of $\mathbb{Q}$-vector space, then it has only one such structure. (Hint: First prove that every element of G has necessarily infinite order. Alternative hint: The unique ring homomorphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.)

Solution. Let G be an abelian group. A $\mathbb{Q}$-vector space structure on G is precisely a ring homomorphism $\sigma : G \to \mathrm{Hom}_{\mathbf{Ab}}(G)$. Let σ_1, σ_2, then, be two of these ring homomorphisms. Note that σ_1 and σ_2 agree on the integers, as if we view $\mathbb{Q}$ and $\mathrm{Hom}_{\mathbf{Ab}}(G)$ as $\mathbb{Z}$-modules, we then have, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\varphi_1(n) &= \varphi_1(n \cdot 1) \\ &= n \cdot \varphi_1(1) \\ &= n \cdot \mathrm{id} \\ &= n \cdot \varphi_2(1) \\ &= \varphi_2(n \cdot 1) \\ &= \varphi_2(n).\end{aligned}$$

Thus, if $\iota : \mathbb{Z} \to \mathbb{Q}$ is the unique ring homomorphism $\mathbb{Z} \to \mathbb{Q}$, i.e. the inclusion, we have $\sigma_1\iota = \sigma_2\iota$. Since ι is a ring epimorphism, this then implies that $\sigma_1 = \sigma_2$, and so there is only one $\mathbb{Q}$-vector space structure on $\mathbb{G}$, as desired. ∎

Problem 5.7. Let K be a field, and let $k \subseteq K$ be a subfield of K. Show that K is a vector space over k (and in fact a k-algebra) in a natural way. In this situation, we say that K is an *extension* of k.

Solution. Note that the inclusion $\sigma : k \to \mathrm{Hom}_{\mathbf{Ab}}(K)$ is a ring homomorphism, and thus a natural k-vector space structure on K. This σ also gives us a k-algebra structure on K since the center of K is K itself, and so im $\sigma \subseteq Z(K)$.

More explicitly, the "scalar" multiplication κx for $\kappa \in k$ and $x \in K$ is just multiplication within the field K, and the k-algebra structure on K also consists of multiplication as defined in the field K. ■

Problem 5.8. What is the initial object of the category R-**Alg**?

Solution. Let A be an R-algebra, and let $\varphi : R \to S$ be an R-**Alg** homomorphism, where the R-algebra structure on R is given by the identity map. The conditions on R-algebra homomorphisms then force, for all $r \in R$,

$$\begin{aligned}\varphi(r) &= \varphi(r \cdot 1_R) \\ &= r \cdot \varphi(1_R) \\ &= r \cdot 1_A.\end{aligned}$$

To verify that φ is an R-**Alg** homomorphism, note that:

$$\begin{aligned}\varphi(r_1 + r_2) &= (r_1 + r_2) \cdot 1_A \\ &= r_1 \cdot 1_A + r_2 \cdot 1_A \\ &= \varphi(r_1) + \varphi(r_2),\end{aligned}$$

$$\begin{aligned}\varphi(r_1 r_2) &= (r_1 r_2) \cdot 1_A \\ &= (r_1 r_2) \cdot (1_A 1_A) \\ &= (r_1 \cdot 1_A)(r_2 \cdot 1_A) \\ &= \varphi(r_1)\varphi(r_2),\end{aligned}$$

$$\begin{aligned}\varphi(1_R) &= 1_R \cdot 1_A \\ &= 1_A,\end{aligned}$$

and

$$\begin{aligned}\varphi(s r_1) &= (s r_1) \cdot 1_R \\ &= s \cdot (r_1 \cdot 1_R) \\ &= s \cdot \varphi(r_1),\end{aligned}$$

using R's properties as a ring, R-module, and R-algebra.

Since φ is the unique homomorphism $R \to A$ for all R-algebras A, we then have that R is initial in R-**Alg**. ■

Problem 5.9. Let R be a commutative ring, and let M be an R-module. Prove that the operation of composition on the R-module $\text{End}_{R\text{-}\mathbf{Mod}}(M)$ makes the latter an R-algebra in a natural way.

Prove that $\mathcal{M}_n(R)$ is an R-algebra, in a natural way.

Solution. Let $\alpha : R \to \text{End}_{R\text{-}\mathbf{Mod}}(M)$ be the ring homomorphism defined by

$$\varphi(r)(m) = rm;$$

it is easy to verify that $\varphi(r)$ is an R-module endomorphism for all $r \in R$, and that φ itself is a ring homomorphism.

Note that if φ is an R-module endomorphism of M, then we have, for all $r \in R$,

$$\begin{aligned}(\alpha(r) \circ \varphi)(m) &= \varphi(r)(\varphi(m)) \\ &= r \cdot \varphi(m) \\ &= \varphi(r \cdot m) \\ &= (\varphi \circ \alpha(r))(m),\end{aligned}$$

and so $\varphi(r)$ is in the center of $\text{End}_{R\text{-}\mathbf{Mod}}(M)$ for all $r \in R$.
Because of this, α then gives us an R-module (and indeed an R-algebra) structure on the ring $\text{End}_{R\text{-}\mathbf{Mod}}(M)$, which is precisely the usual R-module structure on $\text{End}_{R\text{-}\mathbf{Mod}}(M)$, as desired.

In the case of the ring $\mathcal{M}_n(R)$, we can endow the ring with an R-algebra structure using the homomorphism $\alpha : R \to \mathcal{M}_n(R)$, defined by

$$\alpha(r)(A) = rA,$$

where the multiplication on the right-hand side is just scalar multiplication. Another way to think of this is the fact that α maps $r \in R$ to the matrix with r's on the diagonal and 0 elsewhere. It's pretty clear that this is a ring homomorphism whose image is contained in the center of $\mathcal{M}_n(A)$ (since diagonal matrices over a commutative ring commute with all other matrices), so I'll stop there. ■

Problem 5.10. Let R be a commutative ring, and let M be a simple R-module. Prove that $\text{End}_{R\text{-}\mathbf{Mod}}(M)$ is a division R-algebra.

Solution. Since M is simple, every R-module endomorphism of M is either zero or an isomorphism, i.e. has an inverse in $\text{End}_{R\text{-}\mathbf{Mod}}(M)$. Hence $\text{End}_{R\text{-}\mathbf{Mod}}(M)$ is a division ring, and thus a division algebra over R by the previous exercise. ■

Problem 5.11. Let R be a commutative ring, and let M be an R-module. Prove that there is a natural bijection between the set of $R[x]$-module structures on M and $\text{End}_{R\text{-}\mathbf{Mod}}(M)$.

Solution. Todo ■

Problem 5.12. Let R be a ring. Let M, N be R-modules, and let $\varphi : M \to N$ be a homomorphism of R-modules. Assume φ is a bijection, so that it has an inverse φ^{-1} as a set-function. Prove that φ^{-1} is a homomorphism of R-modules, and hence that a bijective R-module homomorphism is an isomorphism of R-modules.

Solution. Since φ is a bijective homomorphism of groups, we know already that φ is a isomorphism between the groups M and N, and φ^{-1} is a group homomorphism.

Let $r \in R$ and $m \in M$. Note, then, that

$$\begin{aligned} r\varphi^{-1}(m) &= \varphi^{-1}(\varphi(r\varphi^{-1}(m))) \\ &= \varphi^{-1}(r\varphi(\varphi^{-1}(m))) \\ &= \varphi^{-1}(rm), \end{aligned}$$

and so φ^{-1} is a homomorphism of R-modules. Therefore, φ is an isomorphism of R-modules, as desired. ■

Problem 5.13. Let R be an integral domain, and let I be a nonzero principal ideal of R. Prove that I is isomorphic to R as an R-module.

Solution. Let $a \in R$ be nonzero, and let $I = (a) \subseteq R$ be the principal ideal of R generated by a. Note that if $x \in I$, then there exists some $r \in R$ such that $x = ra$. Since R is an integral domain, we can see that $x = r_1 a = r_2 a$ implies $r_1 = r_2$, and so the following function is well-defined:

$$\begin{aligned} \varphi : I &\to R; \\ ra &\mapsto r \end{aligned}$$

To verify that this is an R-module homomorphism, note that we have, for all $r \in R$,

$$\begin{aligned} \varphi(r_1 a + r_2 a) &= \varphi((r_1 + r_2)a) \\ &= r_1 + r_2 \\ &= \varphi(r_1 a) + \varphi(r_2 a). \end{aligned}$$

Additionally, we have for all $r, s \in R$,

$$\begin{aligned} \varphi(s(ra)) &= \varphi((sr)a) \\ &= sr \\ &= s\varphi(ra); \end{aligned}$$

hence φ is an R-module homomorphism, as desired.

Additionally, note that if $r \in R$, then $r\varphi(ra)$, and so φ is surjective. If $\varphi(ra) = 0$, then $r = 0$ by definition, and so φ is injective. Hence φ is an isomorphism of R-modules between I and R, as desired. ■