

Algebra: Chapter 0 Exercises

Chapter 1, Section 5

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Problem 5.1. A final object in a category \mathbf{C} is initial in the opposite category \mathbf{C}^{op} .

Proof. Let $T \in \text{Obj}(\mathbf{C})$ be initial and $Z \in \text{Obj}(\mathbf{C})$ be any object. Since $\text{Hom}_{\mathbf{C}}(T, Z)$ is a singleton, $\text{Hom}_{\mathbf{C}^{op}}(Z, T) = \text{Hom}_{\mathbf{C}}(T, Z)$ is also a singleton, making T final in \mathbf{C}^{op} \square

Problem 5.2. The empty set \emptyset is the *unique* initial object in \mathbf{Set} .

Proof. Recall that the number of morphisms between any two sets A and B is given by B^A . Thus, the problem of finding the size of an initial object in \mathbf{Set} boils down to solving $|Z|^{|I|} = 1$ for $|I|$, where Z is an arbitrary set. The only solution to this is $|I| = 0$, which is only satisfied by the the null set, given that $\emptyset^\emptyset = 0$. \square

Problem 5.3. Final objects are unique up to isomorphism.

Proof. Let \mathbf{C} be a category and $T_1, T_2 \in \text{Obj}(\mathbf{C})$ be final. Because T_1 and T_2 are final, we have, for all $Z \in \text{Obj}(\mathbf{C})$:

$$\begin{aligned}\text{End}(T_1) &= \{\text{id}_{T_1}\} \\ \text{End}(T_2) &= \{\text{id}_{T_2}\} \\ \text{Hom}(T_1, T_2) &= \{f\} \text{ for some } f \\ \text{Hom}(T_2, T_1) &= \{g\} \text{ for some } g\end{aligned}$$

We then have $fg = \text{id}_{T_2}$ and $gf = \text{id}_{T_1}$, giving us $T_1 \cong T_2$. \square

Problem 5.4. What are the initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

Solution. For the sake of completeness, a description of this category \mathbf{Set}^* will be included.

Definition. Define \mathbf{Set}^* as follows:

Objects in \mathbf{Set}^* are morphisms $f : \{*\} \rightarrow A$ where A is any set, denoted (f, A) .

Morphisms $\sigma \in \text{Hom}_{\mathbf{Set}^*}((f_A, A), (f_B, B))$ are commutative diagrams:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ f_A \swarrow & & \searrow f_B \\ & \{*\} & \end{array}$$

Now, we will answer the question.

Proposition. Initial and final objects in \mathbf{Set}^* are objects (i, T) , where T is a singleton and i is the unique function that maps from $\{*\}$ to T .

Proof. We will prove that the objects described are both initial and final in \mathbf{Set}^* .

1. Initial:

Consider the object described above. Then, a morphism from this object to another object $(f, B) \in \text{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\sigma} & B \\ i \swarrow & & \searrow f \\ & \{*\} & \end{array}$$

The only choice of σ that makes this diagram commute is defined by $\sigma(i(*)) = f(*)$ (as T is a singleton), so T is initial.

2. Final: Consider the object described above. Then, a morphism to this object from another object $(f, B) \in \text{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\tau} & T \\ f \swarrow & & \searrow i \\ & \{*\} & \end{array}$$

The only choice of τ that makes this diagram commute is defined by $\tau(f(*)) = i(*)$ (as T is a singleton), so T is final.

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Problem 5.5. What are the final objects in the category considered in 5.3?

Solution. Final objects in this category (let's denote it \mathbf{C}) are singletons.

Proof. Let $\{*\}$ be a singleton. Then morphisms τ from $(Z, \varphi) \in \text{Obj}(\mathbf{C})$ to $\{*\}$ are commutative diagrams:

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & \{*\} \\ & \swarrow \varphi \quad \searrow i & \\ & A & \end{array}$$

Since $\{*\}$ is a singleton, the only τ that satisfies $\tau(\varphi(a)) = \iota(a)$ is defined by $\tau(z) = *$ for all $z \in Z$. Thus $(\{*\}, \tau)$ is final in \mathbf{C} . \square

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Problem 5.6. Consider the category corresponding to endowing (as in Example 3.3) the set \mathbb{Z}^+ with the divisibility relation. Thus there is exactly one morphism $d \rightarrow m$ in this category iff d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names?

Solution. Let \mathbf{Div} denote this category and $a, b \in \text{Obj}(\mathbf{Div})$ be objects. That \mathbf{Div} has products means the following:

For every $a, b \in \text{Obj}(\mathbf{Div})$, there exists an object $a \times b \in \text{Obj}(\mathbf{Div})$ such that every $(x, f_a, f_b) \in \text{Obj}(\mathbf{Div}^{a,b})$ admits a unique morphism $\sigma \in \text{Hom}_{\mathbf{Div}}(x, a \times b)$ that makes the following diagram commute:

$$\begin{array}{ccccc} & & f_a & \xrightarrow{\quad} & a \\ & \nearrow & & \nearrow \pi_a & \\ x & \xrightarrow{\sigma} & a \times b & & \\ & \searrow & & \searrow \pi_b & \\ & & f_b & \xrightarrow{\quad} & b \end{array}$$

Meaning, for every pair of integers (a, b) , there exists a unique factor of a and b , $a \times b$, such that every common factor of a and b divides $a \times b$. This is necessarily the greatest common factor of a and b , since it must be a multiple of every common factor of a and b , and $\text{gcf}(a, b)$ is the product of the common factors of a and b . \blacksquare