Algebra: Chapter 0 Exercises

Chapter 2, Section 7

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Problem 7.4. Prove that the relation defined in Exercise 5.10 on a free abelian group $F = F^{ab}(A)$ by

$$f \sim f' \Leftrightarrow (\exists g \in G): \; f' - f = 2g$$

is compatible with the group structure. Determine the quotient $F/\sim$ as a better known group.

Solution. The relation $\sim$ is compatible with the group structure on F if and only if, considering Proposition 7.3,

$$(\forall f, f', a \in F): \; f \sim f' \implies a + f \sim a + f'$$

(since F is abelian). Suppose then, we have $f \sim f'$, that is $f' - f = 2g$. We then have $(f' + a) - (f + a) = 2g$, and hence $a + f' \sim a + f$.

To determine $F/\sim$ as a better known group, we will establish an isomorphism between the group F and the abelian group $G := (\mathbb{Z}/2\mathbb{Z})^A$.
Define the set function $\kappa : A \to G$ as follows:

$$(\forall a, a' \in A): \; \kappa(a)(a') = [\delta_{a'a}]_2,$$

where δ is the kronecker delta function. We then use this function and the universal property for free (abelian) groups to induce a group homomorphism $\varphi : F \to G$ that makes the following diagram commute:

$$\begin{tikzcd}
F^{ab}(A) \arrow[r, dashed, "\varphi"] & (\mathbb{Z}/2\mathbb{Z})^A \\
A \arrow[u, "j"] \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

where $j : A \to F$ is the usual inclusion. We will now prove two lemmas essential to constructing the desired isomorphism.

Lemma 1. *The homomorphism φ is surjective.*

Proof. Suppose $f \in (\mathbb{Z}/2\mathbb{Z})^A$, let $\bar{a} = j(a) \in F$, and allow $f(x)y$ to be "multiplication" by a coset representitive of $f(x)$, i.e. $[0]_2 y = 0y$ and $[1]_2 y = 1y$. We then have, with the symbol at the top of each sigma representing for clarity the abelian group in which the sum is taking place,

$$\begin{aligned}(\forall a, a' \in A)\varphi\left(\sum_{a\in A}^{F} f(a)\bar{a}\right)(a') &= \sum_{a\in A}^{G} \left(\varphi(f(a)\bar{a})\right)(a') \\ &= \sum_{a\in A}^{G} \left(f(a)(\varphi(\bar{a})(a'))\right) \\ &= \sum_{a\in A}^{\mathbb{Z}/2\mathbb{Z}} \left(f(a)([\delta_{a'a}]_2)\right) \\ &= f(a')\end{aligned}$$

Hence every $f \in G$ is in the image of φ. $\square$

Lemma 2. *The homomorphism φ agrees with the relation $\sim$; that is,*

$$f \sim f' \implies \varphi(f) = \varphi(f')$$

Proof. Suppose $f, f' \in F$ and $f \sim f'$. We then have, for all $a \in A$ and for some $g \in (\mathbb{Z}/2\mathbb{Z})^A$:

$$\begin{aligned}(\varphi(f') - \varphi(f))(a) &= \varphi(f' - f)(a) \\ &= (2g)(a) \\ &= 2(g(a)) \\ &= [0]_2\end{aligned}$$

(as $\mathbb{Z}/2\mathbb{Z}$ has order 2); hence $\varphi(f') = \varphi(f)$, as desired. $\square$

Lemma 3. *If we define $H = [e_F]_\sim$ then we have $F/H = F/\sim$, and $\ker(\varphi) = H$.*

Proof. The first statement follows from Proposition 7.4, Proposition 7.7, and the definition of quotient by a normal subgroup.

For the second statement, first suppose that $h \in H$. It then follows, since $h \sim e_F$, that $\varphi(h) = \varphi(e_F) = e_G$, so $h \in \ker(\varphi)$, and hence $H \subseteq \ker(\varphi)$. For the other direction, suppose $f \in \ker(\varphi)$, and

$$f = \sum_{a\in A} n_a \bar{a}$$

We then have, for all $a' \in A$,

$$\begin{aligned}
\varphi(f)(a') &= \varphi\left(\sum_{a\in A} n_a \bar{a}\right)(a') \\
&= \sum_{a\in A} (n_a \varphi(\bar{a})(a')) \\
&= \sum_{a\in A} (n_a \kappa(a)(a')) \\
&= \sum_{a\in A} (n_a [\delta_{a'a}]_2) \\
&= [n_{a'}]_2
\end{aligned}$$

Since $\varphi(f) = e_G$, we know that each $n_{a'}$ is congruent to 0 modulo two; that is, even. Hence, we have:

$$\begin{aligned}
f - e_F &= f \\
&= 2\sum_{a\in A} \frac{n_a}{2} \bar{a}
\end{aligned}$$

This shows that $f \sim e_f$ and thus $f \in H$, giving us $\ker(\varphi) \subseteq H$. Hherefore $\ker(\varphi) = H$, as desired. $\square$

With this all established, we can finally find our isomorphism. We now construct a homomorphism $\widetilde{\varphi} : F/H \to G$ using the universal property for quotient by an equivalence relation:

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G \\
F/H \arrow[ur, dashed, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

Here, π is the quotient map $\pi(f) = [f]_\sim$, and $\widetilde{\varphi}$ is the unique homomorphism making the diagram commute. The first isomorphism theorem shows that $\widetilde{\varphi}$ is an isomorphism, and so $F/\sim \cong (\mathbb{Z}/2\mathbb{Z})^A$. ■

Problem 7.6. Let G be a group, and let n be a positive integer. Consider the relation

$$a \sim b \Leftrightarrow (\exists g \in G)\colon ab^{-1} = g^n$$

1. Show that in general $\sim$ is not an equivalence relation.

2. Prove that $\sim$ is an equivalence relation if G is commutative, and determine the corresponding subgroup of G.

Solution.

1. Reflexivity and symmetry are easy to prove, so we will show that $\sim$ is not always transitive.

Example. Consider the following matrices in $\mathrm{GL}_2(\mathbb{R})$, and let $n = 2$ for the relation:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Clearly we have $A \sim I$ and $B \sim I$, so transitivity would imply that $AB \sim I$. Hence we should have

$$\begin{aligned} AB^{-1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Since this matrix has a negative eigenvalue, it has no square roots in $\mathrm{GL}_2(\mathbb{R})$.

2. Suppose $a, b, c \in G$ and:

$$\begin{aligned} ab^{-1} &= g_1^n \\ bc^{-1} &= g_2^n \end{aligned}$$

so $a \sim b$ and $b \sim c$. We then have:

$$\begin{aligned} ac^{-1} &= (ab^{-1})(bc^{-1}) \\ &= g_1^n g_2^n \\ &= (g_1 g_2)^n \end{aligned}$$

Hence $a \sim c$, making $\sim$ transitive. The corresponding subgroup of G is the equivalence class $[e]_\sim$; that is the set of all g^n for $g \in G$.

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Problem 7.7. Let G be a group, n a positive integer, and let $H \subseteq G$ be the subgroup generated by all elements of order n in G. Prove that H is normal.

Solution. Let $N \subseteq G$ be the set of all elements of order n in G, and let $H = \langle N \rangle$. Any $h \in H$ can then be written as a product of powers of generators in N, as follows:

$$h = \prod_{i=1}^{m} h_i^{k_i}$$

Given any $g \in G$, we have

$$\begin{aligned} ghg^{-1} &= g \left(\prod_{i=1}^{m} h_i^{k_i} \right) g^{-1} \\ &= \prod_{i=1}^{m} g h_i^{k_i} g^{-1} \\ &= \prod_{i=1}^{m} (g h_i g^{-1})^{k_i} \\ &\in H \end{aligned}$$

where the ostensibly innocuous second and third equalities follow from the fact that $(gag^{-1})(gbg^{-1}) = g(ab)g^{-1}$. ∎

Problem 7.10. Let G be a group, and $h \subseteq G$ a subgroup. With notation in Exercise 6.7, show that H is normal in G if and only if $\forall \gamma \in \mathrm{Inn}(G), \gamma(H) \subseteq H$.
Conclude that if H is normal in G then there is an interesting homomorphism $\mathrm{Inn}(G) \to \mathrm{Aut}(H)$

Solution. By exercise 7.3:

$$\begin{aligned} H \text{ normal } &\Leftrightarrow (\forall g \in G)\colon gHg^{-1} \subseteq H \\ &\Leftrightarrow (\forall g \in G)\colon \gamma_g(H) \subseteq H \\ &\Leftrightarrow (\forall \gamma \in \mathrm{Inn}(G))\colon \gamma(H) \subseteq H \end{aligned}$$

Every inner automorphism of G is an automorphism of H $(\gamma_g(h) = e \implies ghg^{-1} = e \implies h = e)$, so the inclusion morphism $\iota : \mathrm{Inn}(G) \to \mathrm{Aut}(H)$ is a perfectly fine homomorphism. ∎

Problem 7.11. Let G be a group, and let $[G, G]$ be the subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$.

1. Prove that $[G, G]$ is normal in G.
2. Prove that $G/[G, G]$ is commutative.

Solution.

1. Suppose $\gamma \in \mathrm{Inn}(G)$ and $h \in [G, G]$ so that $h = \prod_{i=1}^n a_i b_i a_i^{-1} b_i^{-1}$. We then have:

$$\begin{aligned} \gamma_g(h) &= \gamma\left(\prod_{i=1}^n a_i b_i a_i^{-1} b_i^{-1}\right) \\ &= \prod_{i=1}^n \gamma(a_i b_i a_i^{-1} b_i^{-1}) \\ &= \prod_{i=1}^n \left(\gamma(a_i)\gamma(b_i)\gamma(a_i)^{-1}\gamma(b_i)^{-1}\right) \\ &\in [G, G] \end{aligned}$$

   Hence $\gamma([G, G]) \subseteq [G, G]$, so $[G, G]$ is normal.

2. Suppose $g_1, g_2 \in [G, G]$. We then have:

$$\begin{aligned} (g_1 g_2)(g_2 g_1)^{-1} &= g_1 g_2 g_1^{-1} g_2^{-1} \\ &\in [G, G] \end{aligned}$$

Hence, with $H := [G, G]$,

$$\begin{aligned}(g_1H)(g_2H) &= (g_1g_2)H \\ &= (g_2g_1)H \\ &= (g_1H)(g_2H)\end{aligned}$$

for all $g_1H, g_2H \in G/[G, G]$, making $G/[G, G]$ commutative.

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Problem 7.12. Let $F = F(A)$ be a free group, and let $f : A \to G$ be a set-function from the set A to a commutative group G. Prove that f induces a unique homomorphism $F/[F, F] \to G$, where $[F, F]$ is the commutator subgroup of F. Conclude that $F/[F, F] \cong F^{ab}(A)$.

Solution. For this, we simply invoke the universal property for free groups and quotients, and "merge" them together. Let A be a set, G a group, $f : A \to G$ be any set function, $j : A \to F(A)$ be the canonical injection, and π be the canonical projection. Invoking the universal properties for free groups and quotients respectively, we find a unique $\sigma : F \to G$ such that $\sigma j = f$. Since G is abelian, we know that $[F, F] \subseteq \ker(\varphi)$, since, for all $a, b \in F(A)$,

$$\begin{aligned}\varphi([a, b]) &= [\varphi(a), \varphi(b)] \\ &= e_G\end{aligned}$$

so we can invoke the universal property for quotients (by Theorem 7.12) to find a unique $\varphi : F/[F, F] \to G$ such that $\varphi\pi = \sigma$. This is all illustrated in the following diagram:

$$\begin{tikzcd}
F/[F,F] \arrow[dr, dashed, "\varphi"] & \\
F(A) \arrow[u, "\pi"] \arrow[r, dashed, "\sigma"] & G \\
A \arrow[u, "j"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Hence, given a set function $f : A \to G$, there is a unique group homomorphism φ such that

$$\begin{aligned}\varphi\pi = \sigma &\implies \varphi\pi j = \sigma j \\ &\implies \varphi(\pi j) = f\end{aligned}$$

Hence the pair $(F/[F, F], \pi j)$ satisfies the universal property for free abelian groups, so $F/[F, F] \cong F^{ab}(A)$. ■

Problem 7.13. Let A, B be sets and $F(A), F(B)$ the corresponding free groups. Assume $F(A) \cong F(B)$. Prove that if A is finite, then so is B, and $A \cong B$.

Solution. Since $F(A) \cong F(B)$, we know there is some morphism $j : A \to F(B)$ such that (A, j) satisfies the universal property for the free group over B. Hence, applying the same argument from the previous exercise (7.12), we can examine the following diagram

$$\begin{tikzcd}
F(B)/[F(B), F(B)] \arrow[dr, dashed, "\varphi"] & \\
F(B) \arrow[u, "\pi"] \arrow[r, dashed, "\sigma"] & G \\
A \arrow[u, "j"] \arrow[ur, "f"'] &
\end{tikzcd}$$

to find that $F(B)/[F(B), F(B)]$ satisfies the universal property for the abelian free group over A. Hence, we have:

$$\begin{aligned} F^{ab}(B) &\cong F(B)/[F(B), F(B)] \\ &\cong F^{ab}(A) \end{aligned}$$

Therefore, by Exercise 5.10, $A \cong B$. ■

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