Linear Algebra Done Right Exercises

Chapter 4

David Melendez

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Problem 5. Suppose m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \ldots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m+1$.

Proof. Let $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^X)$ where $X = \{z_1, \ldots, z_{m+1}\}$. Define T by

$$T(p)(x_j) = p(x_j)$$

It is easy to show that this transformation is linear.
Now we compute the null space of T. Note that when $Tp = 0$, we have that $p(x) = 0$ for all $x \in X$, and thus p has at least $|X| = m+1$ distinct zeroes. But since the degree of p is at most m, this must mean that p is the zero polynomial. Hence T is injective and $\dim \operatorname{null} T = 0$. We then have

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_m(\mathbb{F}) - \dim \operatorname{null} T \\ &= m + 1 \\ &\overset{1}{=} \dim \mathbb{F}^X. \end{aligned}$$

This shows that range $T = \mathbb{F}$, and therefore that T is an isomorphism between $\mathcal{P}_m(\mathbb{F})$ and $\mathbb{F}^X$. Since T^{-1} assigns a unique polynomial to each function (i.e. set of ordered pairs distinct in the first slot) as stated in the problem, this completes the proof. $\square$

Problem 6. Suppose $p \in \mathcal{P}(\mathbb{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

Proof. We will prove the contrapositive. Suppose z is a zero of both p and p'. Note that

$$\begin{aligned} p(x) &= (x - z)q(x) \\ p'(x) &= q(x) + (x - z)q'(x) \end{aligned}$$

[1] See the following Stack Exchange post: https://math.stackexchange.com/questions/2288812/finding-dimension-of-a-vector-space-v/2289241#2289241

We then have

$$\begin{aligned}0 &= p'(z)\\ &= q(z)\end{aligned}$$

and hence z is a zero of q. This means that $(x-z)^2$ is a factor of p, and therefore it can have at most $m-1$ distinct zeros. $\square$

Problem 7. Every polynomial of odd degree with real coefficients has a real zero.

Proof. According to 4.17, every polynomial with real coefficients can be factored into a product of linear and quadratic factors with real coefficients, with each linear factor corresponding to a real zero and each quadratic factor representing a pair of nonreal zeros. Since the polynomial in question is of odd degree, this factorization must have an odd number of linear factors, and thus at least one real zero. $\square$

Problem 8. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^{\mathbb{R}}$ by

$$Tp = \begin{cases} \dfrac{p - p(3)}{x-3} & \text{if } x \neq 3 \\ p'(3) & \text{if } x = 3 \end{cases}$$

Show that $Tp \in \mathcal{P}(\mathbb{R})$ for every polynomial $p \in \mathcal{P}(\mathbb{R})$ and that T is a linear map.

Solution. Since 3 is a zero of the polynomial $r(x) = p(x) - p(3)$, we have

$$p(x) - p(3) = (x-3)q(x)$$

for some $q \in \mathcal{P}(\mathbb{R})$. Taking the derivative of both sides, we have

$$p'(x) = (x-3)q'(x) + q(x)$$

and hence $p'(3) = q(3)$. Since $q(x) = (Tp)(x)$ for all $x \in \mathbb{R}$, we then have $Tp = q$.

$\blacksquare$

Problem 9. Suppose $p \in \mathcal{P}(\mathbb{C})$. Define $q : \mathbb{C} \to \mathbb{C}$ by

$$q(z) = p(z)\overline{p(\overline{z})}$$

Prove that q is a polynomial with real coefficients.

Solution. We will work with the corresponding factorizations. Let $m = \deg p$ and each ζ_k be a zero of p. We then have

$$\begin{aligned} q(z) &= p(z)\overline{p(\overline{z})} \\ &= \prod_{k=0}^{m}(z-\zeta_k)\overline{\prod_{k=0}^{m}(\overline{z}-\zeta_k)} \\ &= \prod_{k=0}^{m}(z-\zeta_k)\overline{(\overline{z}-\zeta_k)} \\ &= \prod_{k=0}^{m}(z-\zeta_k)(z-\overline{\zeta_k}) \\ &= \prod_{k=0}^{m}(z^2-2(\text{Re }\zeta_k)z+|\zeta_k|^2 z) \end{aligned}$$

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Problem 10. Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbb{C})$ is such that there exist distinct real numbers $x_0, x_1, \ldots, x_m$ such that $p(x_j) \in \mathbb{R}$ for $j = 0, 1, \ldots, m$. Prove that all the coefficients of p are real.

Solution. This follows directly from the uniqueness part of problem 5. ■

Problem 11.