Algebra: Chapter 0 Exercises

Chapter 3, Section 6

Products, coproducts, etc. in R-**Mod**

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Problem 6.1. Prove that $R^{\oplus A} \cong F^R(A)$.

Proof. First, define $j : A \to R^{\oplus A}$ by $j(a)(b) = \delta_{ab}$, where δ is the Kronecker delta. We then have, for all $\alpha \in R^{\oplus A}$, that

$$\alpha = \sum_{a \in A} \alpha(a) j(a),$$

since we have for all $x \in A$ that

$$\begin{aligned}\left(\sum_{a \in A} \alpha(a) j(a)\right)(x) &= \sum_{a \in A} (\alpha(a) j(a))(x) \\ &= \sum_{a \in A} \alpha(a)(j(a)(x)) \\ &= \sum_{a \in A} \alpha(a) \delta_{ax} \\ &= \alpha(x).\end{aligned}$$

Of course this representation of α as a linear combination of $j(a)$ for all $a \in A$ is unique, as the coefficients are clearly uniquely determined by the image of each $a \in A$ under α.

Thus, if N is an R-module, $f : A \to N$, and $\varphi : R^{\oplus A} \to N$ is an R-module homomorphism such that $\varphi j = f$, we then have, for all $\alpha \in R^{\oplus A}$,

$$\begin{aligned}\varphi(\alpha) &= \varphi\left(\sum_{a \in A} \alpha(a) j(a)\right) \\ &= \sum_{a \in A} \varphi(\alpha(a) j(a)) \\ &= \sum_{a \in A} \alpha(a) \varphi(j(a)) \\ &= \sum_{a \in A} \alpha(a) f(a);\end{aligned}$$

thus such a homomorphism is unique, if it exists. Of course, this definition indeed defines a homomorphism that satisfies the desired property, as is easy to verify, and so $R^{\oplus A}$ does satisfy the universal property for the free R-module over A. $\square$

Problem 6.2. Prove or disprove that if R is a ring and M is a nonzero R-module, then M is not isomorphic to $M \oplus M$.

Solution. As a counterexample, let R be a ring and consider the R-module $M = R^{\oplus \mathbb{N}}$ (where $\mathbb{N}$ does not include 0), generated by the set $\{e_1, e_2, \dots\}$. Then, $M \oplus M$ is the cartesian product of M with itself. Consider, then, the function $\varphi : M \to M \oplus M$, defined by

$$\varphi\left(\sum_i r_i e_i\right) = \left(\sum_i r_{2i-1} e_i, \sum_i r_{2i} e_i\right).$$

As can be verified, φ is an R-module homomorphism which is injective and surjective. Hence $M \cong M \oplus M$. $\blacksquare$

Problem 6.3. Let R be a ring, M an R-module, and $p : M \to M$ an R-module homomorphism such that $p^2 = p$ (Such a map is called a *projection*). Prove that $M \cong \ker p \oplus \operatorname{im} p$.

Proof. Define the functions $\varphi : M \to \ker p \oplus \operatorname{im} p$ and $\psi : \ker p \oplus \operatorname{im} p$ by

$$\begin{aligned} \varphi(m) &= (m - p(m), p(m)) \\ \psi(u, v) &= u + v. \end{aligned}$$

Note that $p(m) \in \operatorname{im} p$, and if $m \in M$, then

$$\begin{aligned} p(m - p(m)) &= p(m) - p(p(m)) \\ &= p(m) - p(m) \\ &= 0; \end{aligned}$$

hence $m - p(m) \in \ker p$. Thus the definition of φ makes sense. Past this, it is easy to verify that φ and ψ are R-module homomorphisms and that ψ is a left and right inverse for φ; hence, φ is an isomorphism between M and $\ker p \oplus \operatorname{im} p$. $\square$

Problem 6.5. For any ring R and any two sets A_1, A_2, prove that $\left(R^{\oplus A_1}\right)^{\oplus A_2} \cong R^{\oplus (A_1 \times A_2)}$.

Proof. Let $\varphi : R^{\oplus (A_1 \times A_2)} \to \left(R^{\oplus A_1}\right)^{\oplus A_2}$ be a function defined by

$$\Phi(\varphi)(a)(b) = \varphi(a, b).$$

Then Φ is an R-module isomorphism. $\square$

Problem 6.6. Let R be a ring, and let $F = R^{\oplus n}$ be a finitely generated free R-module. Prove that $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, R) \cong F$.

Proof. Let $e_1, \dots, e_n$ be the generators of F, and for $0 \leq i \leq n$, let $\psi_i : F \to R$ be defined by

$$\psi_i\left(\sum_{j=1}^{n} r_j e_j\right) = r_i.$$

Then each ψ_i is well-defined and an R-module homomorphism.

Note, then, that for each $\varphi \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, M)$ and $v = \sum_i r_i e_i$, we have that

$$\begin{aligned}
\varphi(v) &= \varphi\left(\sum_i r_i e_i\right) \\
&= \sum_i \varphi(r_i e_i) \\
&= \sum_i r_i \varphi(e_i) \\
&= \sum_i \psi_i(v)\varphi(e_i) \\
&= \left(\sum_i \varphi(e_i)\psi_i\right)(v);
\end{aligned}$$

thus, if we let $s_i = \varphi(e_i)\psi_i$, then we have that $\varphi = \sum_i s_i \psi_i$, and so $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, R)$ is generated by $(\psi)_i$ Indeed, each ψ_i is in $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, R)$, and so the module generated by them is contained within $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, R)$, as well.

We can then define a function $\Phi : \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, R) \to F$ by

$$\Phi\left(\sum_i r_i \psi_i\right) = \sum_i r_i e_i, \tag{1}$$

It is then easy to show that Φ is an R-module isomorphism. $\square$

Problem 6.7. Let A be any set. For any family $\{M_a\}_{a \in A}$ of modules over a ring R, define the *product* $\prod_{a \in A} M_a$ and coproduct $\bigoplus_{a \in A} M_a$.

Solution. We define the product $P = \prod_{a \in A} M_a$ as follows: We say that P, along with a family of R-module homomorphisms $\{\pi_a : P \to M_a\}_{a \in A}$ is a product of the family $\{M_a\}_{a \in A}$ if for each R-module N and family of morphisms $\{\varphi_a : N \to M_a\}_{a \in A}$, there exists a unique R-module homomorphism $\psi = \prod_{a \in A} \varphi_a : N \to P$ such that for all $a \in A$, we have $\pi_a \psi = \varphi_a$.

In the case where $M_a = R$ for all $a \in A$, we have that the set R^A of functions from A to R, along with the projections $\pi_a(g) = g(a)$ satisfies this universal property. Indeed, if M is an R-module and we have a family of R-module homomorphisms $\{f_a : M \to R\}$, then we have that if $\psi : M \to R^A$ is a function satisfying the condition $\pi_a \psi = f_a$, then

$$\begin{aligned}\psi(m)(a) &= \pi_a(\psi(m)) \\ &= f_a(m);\end{aligned}$$

thus, $\psi(m)$ is the function taking a to $f_a(m)$. It is easy to check that ψ is an R-module homomorphism, and hence that it satisfies the desired universal property.

We define the coproduct $K = \bigoplus_{a \in A} M_a$ as follows: We say that P, along with a family of R-module homomorphisms $\{\iota_a : M_a \to K\}_{a \in A}$ is a coproduct of the family $\{M_a\}_{a \in A}$ if for each R-module N and family of morphisms $\{\varphi_a : M_a \to N\}_{a \in A}$, there exists a unique R-module homomorphism $\psi = \bigoplus_{a \in A} \varphi_a : K \to N$ such that for all $a \in A$, we have $\psi \iota_a = \varphi_a$. ■

Prove that $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$. (Hint: Cardinality.)

Proof. Note that $\mathbb{Z}^{\mathbb{N}}$ is the set of all infinite sequences of integers, which has cardinality equal to that of the reals. By contrast, $\mathbb{Z}^{\oplus \mathbb{N}}$ is countable (proof?). □

Problem 6.8. Let R be a ring. If A is any set, prove that $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R^{\oplus A}, R)$ satisfies the universal property for the *product* of the family $\{R_a\}_{a \in A}$, where $R_a \cong R$ for all a; thus, $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R^{\oplus A}, R) \cong R^A$. Conclude that $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R^{\oplus A}, R)$ is not isomorphic to $R^{\oplus A}$ in general.

Solution. Alternatively, we can just prove directly using our characterization of the infinite product of a module with itself (done above) the desired isomorphism.

Let $\Phi : \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R^{\oplus A}, R)$ be the function defined by

$$\Phi(\rho) = \rho j,$$

where j is the usual inclusion from A into $R^{\oplus A}$. It is easily verified that this is an R-module homomorphism. We can then see that since $R^{\oplus A}$ is the free R-module generated by A, that for every $f \in R^A$, there exists a unique $\rho^{\oplus A} \to R$ such that $\rho j = f$; that is, Φ is a bijection, and hence an isomorphism, as desired.

■

Problem 6.9. Let R be a ring, F a nonzero free R-module, and let $\varphi : M \to N$ be an R-module homomorphism. Prove that φ is onto if and only if for all R-module homomorphisms $\alpha : F \to N$, there exists an R-module homomorphism $\beta : F \to M$ such that $\alpha = \varphi \circ \beta$. (Free modules are *projective*)

Proof. First suppose φ is surjective. Let A be the set of generators of F let $j : A \to F$ be the usual inclusion, and let $f = \alpha j$. Note that for each $n \in \alpha j(A)$, there exists a (not necessarily unique) $m_n \in M$ such that $\varphi(m_n) = n$, since φ is surjective. Define, then, a function $g : A \to M$ by $g(a) = m_{f(a)}$, and extend g to a function $\beta : F \to M$. We then have that

$$\begin{aligned}
\varphi\beta \sum_{a\in A} r_a j(a) &= \varphi \sum_{a\in A} r_a \beta j(a) \\
&= \varphi \sum_{a\in A} r_a g(a) \\
&= \varphi \sum_{a\in A} r_a m_{f(a)} \\
&= \sum_{a\in A} r_a \varphi(m_{f(a)}) \\
&= \sum_{a\in A} r_a f(a) \\
&= \sum_{a\in A} r_a \alpha j(a) \\
&= \alpha \sum_{a\in A} r_a j(a),
\end{aligned}$$

where the sum in the first expression is an arbitrary element of F. Thus $\varphi\beta = \alpha$, as desired.

Conversely, suppose that for all R-module homomorphisms $\alpha : F \to N$, there exists an R-module homomorphism $\beta : F \to M$ such that $\alpha = \varphi\beta$. Then, suppose $n \in N$, and consider the R-module homomorphism $\alpha : F \to N$ extending the constant set-function $a \mapsto n$. We then have that there exists a $\beta : F \to M$ such that $\alpha = \varphi\beta$. In particular, $\varphi\beta(j(a)) = \alpha(j(a)) = f(a) = n$, and so we have that $n \in \text{im}\ \varphi$. Since n was arbitrary, it then follows that φ is surjective, as desired. □

Problem 6.10. Let M, N, Z be R-modules, and let $\mu : M \to Z$ and $\nu : N \to Z$ be homomorphisms of R-modules.

Prove that R-**Mod** has 'fibered products': there exists an R-module $M \times_Z N$ with R-module homomorphisms $\pi_M : M \times_Z N \to M$ and $\pi_N : M \times_Z N \to N$ such that $\mu\pi_M = \nu\pi_N$, and which is universal with respect to this requirement. That is, for every R-module P and R-module homomorphisms $\varphi_M : P \to M$,$\varphi_N : P \to N$ such that $\mu\varphi_M = \nu\varphi_N$, there exists a unique R-module homomorphism $\psi : P \to M \times_Z N$ making the diagram

$$\begin{tikzcd}
 & M \arrow[dr, "\mu"] & \\
P \arrow[ur, "\varphi_M"] \arrow[r, dashed, "\psi"] \arrow[dr, "\varphi_N"'] & M \times_Z N \arrow[u, "\pi_M"'] \arrow[d, "\pi_N"] & Z \\
 & N \arrow[ur, "\nu"'] &
\end{tikzcd}$$

commute.

Solution. Define $M \times_Z N = \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$. That this is an R-module follows immediately from μ, ν being R-module homomorphisms. The usual projections also clearly satisfy $\mu\pi_M = \nu\pi_N$, and the desired unique homomorphism ψ is defined by $\psi(p) = (\varphi_M(p), \varphi_N(p))$, which makes sense since we required that $\mu\varphi_M$ and $\nu\varphi_N$ agree. The case for fibered coproducts is similarly. ■

Problem 6.12. Prove Proposition 6.2: For an R-module homomorphism φ, the following are equivalent:

(a) φ is a monomorphism

(b) $\ker \varphi$ is trivial

(c) φ is injective as a set function.

Additionally, the following are equivalent:

(a) φ is an epimorphism

(b) coker φ is trivial

(c) φ is surjective as a set function.

Solution. For the first, part, let $\varphi : M \to N$ be an R-module homomorphism. If φ is injective as a set-function, then φ is mono as a set-function, and in particular as an R-module homomorphism. Thus (c) implies (a). Additionally, we know that an R-module homomorphism has trivial kernel if and only if it is injective, and so in particular (b) implies (c).

Assume, then, that φ is a monomorphism; that is, for all R-modules P and R-module homomorphisms $\alpha_1, \alpha_2 : P \to M$, we have that $\varphi\alpha_1 = \varphi\alpha_2$ implies $\alpha_1 = \alpha_2$. Let $P = \ker \varphi$, and consider $\alpha_1 = \iota$, the inclusion into M, and $\alpha_2 = 0$, the trivial homomorphism. We then have that $\varphi \circ 0 = \varphi \circ \iota$, and so $\iota = 0$; thus, $\ker \varphi = \text{im } \iota = 0$. Therefore, (a) implies (b), and we have the desired equivalence.

For the second part, first note that if φ is surjective as a set function, then φ is epi as a set-function, and in particular as an R-module homomorphism. Thus (c) implies (a). Additionally, if φ has trivial cokernel, then we have that $N/\text{im } \varphi = 0$, and so $\text{im } \varphi = N$; thus, φ is surjective, giving us that (b) implies (c).

Now, assume that φ is an epimorphism, so that $\alpha_1\varphi = \alpha_2\varphi$ implies $\alpha_1 = \alpha_2$ for all R-modules P and homomorphisms $\alpha_1, \alpha_2 : N \to P$. In particular, let $P = \text{coker } \varphi$, $\alpha_1 = \pi$ be the projection, and $\alpha_2 = 0$ be the zero homomorphism. Then $\pi \circ \varphi = 0 = 0 \circ \varphi$, and so $\pi = 0$; hence $\text{coker } \varphi = \pi(N) = 0(N) = 0$. Therefore, (a) implies (b), and we have the desired equivalence. ■

Problem 6.13. Prove that every homomorphic image of a finitely generated module is finitely generated.

Solution. By definition, an R-module M is finitely generated if and only if there exists a finite set A and a function $\iota : A \to M$ such that the R-module homomorphism $\gamma : F^R(A) \to M$ induced by ι is surjective.

Suppose, then, that M is finitely generated so that we have such a set A and a function ι which induce a surjection γ, and let $\varphi : M \to N$ be a surjective R-module homomorphism. Stare at the following diagram

$$\begin{tikzcd} F^R(A) \arrow[r, "\gamma"] & M \arrow[r, "\varphi"] & N \\ A \arrow[u, "j"] \arrow[ur, "\iota"'] & & \end{tikzcd}$$

and note that since $\gamma j = \iota$, we then have that $(\varphi\gamma)j = \varphi\iota$; thus, by the uniqueness clause of the universal property for free modules, $\varphi\gamma$ is the unique R-module homomorphism $F^R(A) \to N$ induced by $\varphi\iota$. Since γ and φ are surjective, we then have that $\varphi\gamma$ is surjective, and so N is finitely generated, as desired. ■

Problem 6.14. Prove that the ideal $I = (x_1, x_2, \dots)$ of the ring $R = \mathbb{Z}[x_1, x_2, \dots]$ is not finitely generated (as an ideal, i.e. as an R-module).

Solution. Assume I is finitely generated by a set $G = \{g_1, \dots, g_n\} \subseteq I$, and assume without loss of generality that each g_i is a monomial. Let N be the largest integer such that some g_i is divisible by x_N, and consider the ring homomorphism $\varphi : R \to R$ (induced by the universal property for polynomial rings) that maps each integer to itself, $x_i \mapsto 0$ for $i < N$, and $x_i \mapsto x_i$ for $i \geq N$. If $x_N = \sum a_i g_i$ for some polynomials a_i, then we have

$$\begin{aligned} x_N &= \varphi(x_N) \\ &= \varphi\left(\sum_i a_i g_i\right) \\ &= \sum_i \varphi(a_i)\varphi(g_i) \\ &= 0, \end{aligned}$$

where the last equality follows from g_i not being divisibly by x_N. This is a contradiction; hence I is not finitely generated. ■

Problem 6.15. Let R be a commutative ring. Prove that a commutative R-algebra S is finitely generated as an algebra over R if and only if it is finitely generated as a commutative algebra over R.

Proof. First suppose S is finitely generated as a commutative R-algebra. Then, employing the universal property for free R-algebras, there exists a finite set A and a set-function

$\gamma : A \to S$ such that the unique R-algebra homomorphism $R[A] \to S$ induced by γ is a surjection.

Additionally, the natural inclusion $A \hookrightarrow R[A]$ induces an R-algebra homomorphism $R\langle A\rangle \to R[A]$ which is clearly surjective. Consequently, we consider the following diagram,

$$\begin{tikzcd}
R\langle A\rangle \arrow[r, two heads, "\varphi"] & R[A] \arrow[r, two heads, "\psi"] & S \\
A \arrow[u, hook, "j"] \arrow[ur, hook, "j'"'] \arrow[urr, bend right, "\gamma"'] & &
\end{tikzcd}$$

where $\varphi j = j'$ and $\psi j' = \gamma$. Note, then that we have $\psi\varphi j = \psi j' = \gamma$, and so $\psi\varphi$ is the unique R-algebra homomorphism $R\langle A\rangle \to R[A]$ induced by γ. Since this homomorphism is the composition of two surjections, it itself is a surjection, and so S is finitely generated as an R-algebra, as desired.

Suppose conversely that S is finitely generated as an R-algebra. Let $\mathfrak{c}$ be the centralizer ideal of $R\langle A\rangle$–that is, the ideal generated by all $ab - ba$ for $a, b \in R\langle A\rangle$. We then have that $R\langle A\rangle/\mathfrak{c} \cong R[A]$ (proof?). Observe, then, the following diagram,

$$\begin{tikzcd}
\dfrac{R\langle A\rangle}{\mathfrak{c}} \arrow[dr, "\widetilde{\varphi}"] & \\
R\langle A\rangle \arrow[u, "\pi"] \arrow[r, "\varphi"] & S \\
A \arrow[u, "j"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where π is the projection, φ is induced by γ, and $\widetilde{\varphi}$ is induced by the universal property for quotient algebras.

If we think of $R\langle A\rangle/\mathfrak{c}$ as $R[A]$, then πj is the inclusion $A \hookrightarrow R[A]$, and we have that $\widetilde{\varphi}\pi j = \varphi j = \gamma$, and so $\widetilde{\varphi}$ is the morphism induced by γ and the universal property for free commutative R-algebras. Since $\widetilde{\varphi}$ is surjective (because φ is surjective), it then follows that S is finitely generated as a commutative R-algebra, as desired. $\square$

Problem 6.16. Let R be a ring. A (left-)R module is cyclic if $M = \langle m\rangle$ for some $m \in M$. Prove that simple modules are cyclic. Prove that an R-module M is cyclic if and only if $M \cong R/I$ for some (left-)ideal I. Prove that every quotient of a cyclic module is cyclic.

Solution. Recall that a simple module is a module with only trivial (0 and itself) submodules. Suppose, then, that M is a simple R-module. Let m be any nonzero element of M, and let $N = \langle m\rangle$. Certainly $m \in N$ since $1m = m$, so N is nonempty. Since M is simple, it then follows that $N = M$ and so $M = \langle m\rangle$ as desired.

For the next part, suppose that M is cyclic, so that $M = \langle m\rangle$. Define an R-module homomorphism $\varphi : R \to M$ by $\varphi(r) = rm$. Since M is generated by m, we have that φ is surjective, and so $R/\ker\varphi \cong M$. Thus simple R modules are quotients of R.

Conversely, suppose $M \cong R/I$ as R-modules for some ideal I of R. We then have an isomorphism $\widetilde{\varphi} : R/I \to M$. Define, then, a homomorphism $\varphi : R \to I$ by $\varphi(r) = \widetilde{\varphi}(r + I)$. Clearly φ is surjective, and so for every $m \in M$, we have that there exists an $r \in R$ such that $\varphi(r) = m$. Note, then, that we have $m = \varphi(r) = r \cdot \varphi(1)$, and so M is generated by $\varphi(1)$, showing that M is cyclic as desired.

For the last part, it's enough to note that if M is generated by m, then $\pi(m)$ generates quotients of M. ■

Problem 6.17. Let M be a cyclic R-module, so that $M \cong R/I$ for a (left-)ideal I, and let N be another R-module.

(a) Prove that $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N) \cong \{n \in N : (\forall a \in I), an = 0\}$.

(b) For $a, b \in \mathbb{Z}$, prove that $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}) \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$.

Solution. For (a), let $P = \{n \in N : (\forall a \in I), an = 0\}$, and define a function $\psi : P \to \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ by $\psi(n)([r]) = rn$. The function ψ is well-defined as a result of the condition on P, and is an R-module homomorphism. Note that $n \in \ker\psi \implies (\forall r \in R)rn = 0$, and so in particular, $1 \cdot n = n = 0$. Thus, ψ is injective.

Note additionally that if $\varphi \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$, then for all $r \in R$, we have $\varphi([r]) = \varphi(r \cdot [1]) = r\varphi([1])$; thus, φ is entirely determined by where it takes $[1]$. Therefore, if $\varphi([r]) = rn$ for some $n \in N$, then we have $\varphi = \psi(n)$, and so ψ is surjective, as desired.

The second result follows immediately if we let $M = \mathbb{Z}/a\mathbb{Z}$ and $N = \mathbb{Z}/b\mathbb{Z}$. ■

Problem 6.18. Let M be an R-module, and let N be a submodule of N. Prove that if N and M/N are both finitely generated, then M is finitely generated.

Proof. Suppose N is finitely generated by $n_1, \ldots, n_k$, and M/N is finitely generated by $[m_1], \ldots, [m_\ell]$ for some $m_1, \ldots, m_\ell \in M$. We then have that $m = \sum_i m_i + N$. Note that $m - \sum_i m_i \in N$ since it is in the kernel of the projection $M \to M/N$, and so we have

$$\begin{aligned} m &= \sum_i m_i - \left(m + \sum_i m_i\right) \\ &= \sum +im_i - \sum_i n_i, \end{aligned}$$

since N is finitely generated. Therefore, M is generated by $m_1, \ldots, m_\ell, n_1, \ldots, n_k$, as desired. □

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