Algebra: Chapter 0 Exercises
Chapter 2, Section 1

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Problem 1.3. Prove that $(gh)^{-1} = h^{-1}g^{-1}$ for all elements g, h of a group G.

Proof. We have (by associativity) that $(gh)(g^{-1}h^{-1}) = e$. But $(gh)(gh)^{-1} = e$, so by cancellation $(gh)^{-1} = h^{-1}g^{-1}$. □

Problem 1.4. Suupose that $g^2 = e$ for all elements g of a group G; prove that G is commutative.

Proof. $gh = ghe = gh(hg)^2 = ghhghg = gghg = hg$ □

Problem 1.5. Prove that ever row and every column of the 'multiplication table' of a group contains all elemtns of the group exactly once.

Solution. That every row of a group G's multiplication table is 'sudoku complete' (if you will) is equivalent to the following:

Proposition. For every $g, h \in G$ $g \neq h$, there exists a unique $x \in G$ such that $gx = h$.

Proof. Putting $x = g^{-1}h$, we have $gx = gg^{-1}h = h$. If any y satisfies this property, we have

$$\begin{aligned} gx = h = gy &\implies gx = gy \\ &\implies g^{-1}x = g^{-1}y \\ &\implies x = y \end{aligned}$$

□

The proof for columns is entirely analogous. ■

Problem 1.6. Prove that there is only *one* possible multiplication table for G if G has exactly $1, 2,$ or 3 elements. Analyze the possible multiplication tables for groups with exactly 4 elements, and show that there are *two* distinct tables, up to reordering the elements of G.

Solution. .

1. The proof for $|G| = 1$ is trivial.

2. For $|G| = 2$ and $e, a \in G$, we have $ee = e$, $ea = a$, and $ae = e$. Since each element of a group must have an inverse, we must also have $a = a^{-1}$ (since $e \neq a^{-1}$), so $a^2 = e$.

3. For $|G| = 3$, consider the table:

·	e	a	b
e	e	a	b
a	a	?	?
b	b	?	?

We can complete the table like a sudoku puzzle using problem 1.5. Since $ea = a$, we cannot have $a^2 = a$. Since $eb = b$, we can't have $a^2 = e$ since that would force $ab = b$. Hence, $a^2 = b$.

·	e	a	b
e	e	a	b
a	a	b	?
b	b	?	?

The rest of the table is forced by problem 1.5.

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

■