Algebra: Chapter 0 Exercises Chapter 3, Section 2

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Problem 1. Prove that if there is a homomorphism from a zero ring to a ring R, then R is a zero ring.

Solution. Let Z denote the zero ring, and let $\varphi: Z \to R$ be a ring homomorphism. Since φ is a homomorphism, it must take the identity in Z to the identity in R, so $\varphi(0) = 1_R$. But 0 is also the *additive* identity in Z, meaning $\varphi(0) = 0$, and so 0 = 1 in R.

If $r \in R$, we then have $1 \cdot r = 0 \cdot r = 0$, showing that R is the zero ring.

Problem 2. Let R and S be rings, and let $\varphi: R \to S$ be a function preserving both operations $+, \cdot$.

- 1. Prove that if φ is surjective, then necessarily $\varphi(1_R) = 1_S$.
- 2. Prove that if $\varphi \neq 0$ and S is an integral domain, then $\varphi(1_R) = 1_S$.

Solution.

1. First suppose φ is surjective. Then, if $s \in S$, then there exists an $r \in R$ such that $\varphi(r) = s$. Note that

$$\varphi(1_R) \cdot s = \varphi(1_R) \cdot \varphi(r)$$

$$= \varphi(1_R \cdot r)$$

$$= \varphi(r)$$

$$= s.$$

Since this is true for all $s \in S$ (as φ is surjective), this implies that $\varphi(1_R) = 1_S$, as desired.

2. Now, let $\varphi \neq 0$ and suppose $\varphi(1_R) \neq 1_S$. This implies that $\varphi(1_R) - 1_S \neq 0$. Since φ is nonzero, there exists an $r \in R$ with $\varphi(r) \neq 0$. Note, then, that we have:

$$\varphi(r) \cdot (\varphi(1_R) - 1_S) = \varphi(r) \cdot \varphi(1_R) - \varphi(r) \cdot 1_S$$
$$= \varphi(r \cdot 1_R) - \varphi(r)$$
$$= \varphi(r) - \varphi(r)$$
$$= 0.$$

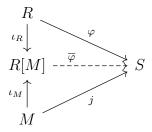
implying S is not an integral domain since both of the terms in the original product are nonzero. Therefore, if S is an integral domain and $\varphi \neq 0$, then $\varphi(1_R) = 1_S$.

Problem 6. Verify the 'extension property' of polynomial rings, stated in Example 2.3.

Solution. I will instead do the more general case, stating and proving a universal property for monoid rings.

Proposition. Let R be a ring, and M a monoid. The monoid ring R[M], as defined in the text, then satisfies the following universal property:

Let $\iota_M:(M,\cdot)\hookrightarrow (R[M],\cdot)$ be the monoid homomorphism $m\mapsto 1_Rm$, and let $\iota_R:R\hookrightarrow R[M]$ be the ring homomorphism $r\mapsto r1_M$. If S is a ring, $\varphi:R\to S$ is a ring homomorphism, and $j:M\to S$ is a monoid homomorphism with respect to multiplication on S such that $a\in \text{im } j$ and $b\in \text{im } \varphi$ implies ab=ba, then there exists a unique ring homomorphism $\overline{\varphi}:R[M]\to S$ such that the following diagram commutes:



Proof. First, we show that $\overline{\varphi}$ is determined by the fact that it must be a ring homomorphism that makes the diagram above commute.

$$\overline{\varphi}\left(\sum_{m\in M} r_m m\right) = \sum_{m\in M} \overline{\varphi}(r_m m)$$

$$= \sum_{m\in M} \overline{\varphi}(\iota_R(r_m) \cdot \iota_M(m))$$

$$= \sum_{m\in M} \overline{\varphi}(\iota_R(r_m)) \cdot \overline{\varphi}(\iota_M(m))$$

$$= \sum_{m\in M} \varphi(r_m) \cdot j(m)$$

Since this function $\overline{\varphi}$ is the only function making the diagram commute, we just have to prove that it is a ring homomorphism. The fact that $\overline{\varphi}$ preserves addition and the identity is clear enough, so we will just show that it preserves multiplication. If we let $p = \sum_{m \in M} a_m m$

and $q = \sum_{m \in M} b_m m$, we then have:

$$\overline{\varphi}(p) \cdot \overline{\varphi}(q) = \left(\sum_{m \in M} \varphi(a_m) \cdot j(m)\right) \left(\sum_{m \in M} \varphi(b_m) \cdot j(m)\right)$$

$$= \sum_{m \in M} \sum_{n \in M} \varphi(a_m) \cdot j(m) \cdot \varphi(b_n) \cdot j(n)$$

$$= \sum_{m \in M} \sum_{n \in M} \varphi(a_m) \cdot \varphi(b_n) \cdot j(m) \cdot j(n)$$

$$= \sum_{m \in M} \sum_{n \in M} \varphi(a_m b_n) \cdot j(mn)$$

$$= \sum_{\ell \in M} \sum_{mn = \ell} \varphi(a_m b_n) \cdot j(\ell)$$

$$= \overline{\varphi}(pq).$$

Hence $\overline{\varphi}$ is a homomorphism, and thus it satisfies the universal property, as desired.

This universal property is a generalization of the universal property for polynomial rings over one indeterminant mentioned in the text, which is really just the monoid ring $R[\mathbb{N}]$. Additionally, much to our pleasure, polynomial rings in n indeterminants (which commute with each other) can be thought of as monoid rings $R[\mathbb{N}^n]$.