Topology and Groupoids Exercises
Chapter 2, Section 4

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Problem 1. Prove that the relation 'X is a subspace of Y' is a partial order relation for topological spaces.

Solution. Denote by $\leq$ the relation 'is a subspace of'. Of course 'X is a subset of Y' is a partial order relation for sets, so we simply need to verify that the topologies line up as they should. We will be working with open set topologies.

First, consider (X, τ_1) as a subspace of (X, τ), where τ_1 is the relative topology on X. If $U \in \tau_1$, then there exists a set $V \in \tau$ such that $U = V \cap X$. Since X and V are τ-open, $U = V \cap X$ is also τ-open, so $U \in \tau$. Hence $\tau_1 \subseteq \tau$. Conversely, if $U \in \tau$, then of course $U = U \cap X \in \tau_1$, and so $\tau_1 = \tau$. Therefore $X \leq X$, and so $\leq$ is reflexive.

Next, suppose $X \leq Y$ and $Y \leq X$. Then, in particular, $X \subseteq Y$ and $Y \subseteq X$, so $X = Y$, and so $\leq$ is antisymmetric.

Finally, suppose $(X, \tau_X) \leq (Y, \tau_Y)$ and $(Y, \tau_Y) \leq (Z, \tau_Z)$. Then $U \in \tau_X$ implies there exists some $V \in \tau_Y$ such that $U = V \cap X$. Further, since $V \in \tau_Y$, and Y is a subspace of Z, there exists a τ_Z-open W such that $V = W \cup Y$. We then have $U = W \cup Y \cup X = W \cup X$ (since $X \subseteq Y$ implies $Y \cup X = X$, and so U is open in the relative topology on X with respect to Z.

Conversely, suppose U is open in Z. Then note that $U \cup X = U \cup (X \cup Y) = (U \cup Y) \cup X$ is open in X, since $Y \leq Z$ implies $U \cup Y$ is τ_Y-open, and $X \leq Y$ implies $(U \cup Y) \cup X$ is $\tau_X - open$. Thus $\leq$ is transitive, completing the proof that $\leq$ is a partial order of topological spaces. ∎

Problem 2. Prove that the set $I = \{x \in \mathbb{Q} : -\sqrt{2} \leq x \leq \sqrt{2}\}$ is both open and closed in $\mathbb{Q}$.

Solution. Note that $I = \left(-\sqrt{2}, \sqrt{2}\right) \cap \mathbb{Q} = \left[-\sqrt{2}, \sqrt{2}\right] \cap \mathbb{Q}$, where the first interval is open in $\mathbb{R}$ and the second is closed in $\mathbb{R}$. The result then follows from the conditions on open and closed sets in a relative topology. ∎

Problem 3. Let A be the subspace of $\mathbb{R}$ of point $1/n$ for $n \in \mathbb{Z} \setminus \{0\}$. Prove that A is discrete, but that the subspace $A \cup \{0\}$ of $\mathbb{R}$ is not discrete.

Solution. Note that if $1/n \in A$, then we have $\left\{\dfrac{1}{n}\right\} = \left(\dfrac{1}{n+1}, \dfrac{1}{n-1}\right) \cap A$, where the interval is an open interval in $\mathbb{R}$. Hence all singletons in A are open, implying that all subsets of A are open since the union of any family of open sets of A is also open. This means A is discrete.

Now, denote by Y the subspace $A \cup \{0\}$ of X. Take A as a subset of Y, and note that $\mathrm{Cl}_Y A = \overline{A} \cap Y = Y \neq A$ (note that every $\mathbb{R}$-neighborhood of 0 meets A, and so $0 \in \overline{A}$); hence A is not closed in Y, and its complement $\{0\}$ is not open in Y. Therefore Y does not have the discrete topology. ■

Problem 4. Prove that a subspace of a discrete space is discrete, and a subspace of an indiscrete space is indiscrete.

Solution. Let Z be a discrete topological space, and let $A \subseteq Z$ be a subspace. Then $x \in A$ implies $\{x\} = \{x\} \cap A$, where the singleton on the right side is open in Z; hence $\{x\}$ is open in A, implying every set in A is open, and so A is discrete.

On the other hand, let X be an indiscrete topological space, and let $A \subseteq X$ be a subspace. If $U \subseteq A$ is open, then there exists an open set V in X such that $U = V \cap A$. Since X is indiscrete, V is either $\varnothing$ or X, and so U is either $\varnothing \cap A = \varnothing$ or $X \cap A = A$; hence A is indiscrete. ■

Problem 5. Let A be the subspace $[0, 2] \setminus \{1\}$ of $\mathbb{R}$. Prove that $I = [0, 1)$ is both open and closed in A.

Solution. This follows from the fact that $I = (-1, 1) \cap A = [-1, 1] \cap A$. ■

Problem 6. Let $x \in X$ and let A be a neighborhood (in X) of x. Prove that the neighborhoods in A of x are exactly the neighborhoods in X of x which are contained in A.

Solution. First, suppose N is a neighborhood in X of x contained in A. Then there exists an open set U in X such that $x \in U \subseteq N$. Note, then, that $U \cap A \subseteq N \cap A = N$ is an open set in A that contains x, and so N is a neighborhood of x in A.

Conversely, suppose N is a neighborhood of x in A, and so there exists an open set U in A such that $x \in U \cap A$. Since U is open in A, there exists a set V open in X such that $U = V \cap A$. Hence, we have $V \cap A \subseteq N$ is a neighborhood of x in X, and so N is a neighborhood of x in X contained in A. ■

Problem 7. Let $\leq$ be an order relation on the set X. If $A \subseteq X$ then the restriction of $\leq$ is an order relation on A. Show that it is not necessarily true that if A, X have the order topologies, then A is a subspace of X. What is the order topology on $\mathbb{Q}$?