Algebra: Chapter 0 Exercises

Chapter 3, Section 2

David Melendez

July 23, 2018

Problem 1. Prove that if there is a homomorphism from a zero ring to a ring R, then R is a zero ring.

Solution. Let Z denote the zero ring, and let $\varphi : Z \to R$ be a ring homomorphism. Since φ is a homomorphism, it must take the identity in Z to the identity in R, so $\varphi(0) = 1_R$. But 0 is also the *additive* identity in Z, meaning $\varphi(0) = 0$, and so $0 = 1$ in R.

If $r \in R$, we then have $1 \cdot r = 0 \cdot r = 0$, showing that R is the zero ring. ■

Problem 2. Let R and S be rings, and let $\varphi : R \to S$ be a function preserving both operations $+, \cdot$.

1. Prove that if φ is surjective, then necessarily $\varphi(1_R) = 1_S$.
2. Prove that if $\varphi \neq 0$ and S is an integral domain, then $\varphi(1_R) = 1_S$.

Solution.

1. First suppose φ is surjective. Then, if $s \in S$, then there exists an $r \in R$ such that $\varphi(r) = s$. Note that

$$\begin{aligned} \varphi(1_R) \cdot s &= \varphi(1_R) \cdot \varphi(r) \\ &= \varphi(1_R \cdot r) \\ &= \varphi(r) \\ &= s. \end{aligned}$$

   Since this is true for all $s \in S$ (as φ is surjective), this implies that $\varphi(1_R) = 1_S$, as desired.

2. Now, let $\varphi \neq 0$ and suppose $\varphi(1_R) \neq 1_S$. This implies that $\varphi(1_R) - 1_S \neq 0$. Since φ is nonzero, there exists an $r \in R$ with $\varphi(r) \neq 0$. Note, then, that we have:

$$\begin{aligned} \varphi(r) \cdot (\varphi(1_R) - 1_S) &= \varphi(r) \cdot \varphi(1_R) - \varphi(r) \cdot 1_S \\ &= \varphi(r \cdot 1_R) - \varphi(r) \\ &= \varphi(r) - \varphi(r) \\ &= 0, \end{aligned}$$

   implying S is not an integral domain since both of the terms in the original product are nonzero. Therefore, if S is an integral domain and $\varphi \neq 0$, then $\varphi(1_R) = 1_S$.

■

Problem 6. Verify the 'extension property' of polynomial rings, stated in Example 2.3.

Solution. I will instead do the more general case, stating and proving a universal property for *monoid* rings.

Proposition. Let R be a ring, and M a monoid. The monoid ring $R[M]$, as defined in the text, then satisfies the following universal property:

Let $\iota_M : (M, \cdot) \hookrightarrow (R[M], \cdot)$ be the monoid homomorphism $m \mapsto 1_R m$, and let $\iota_R : R \hookrightarrow R[M]$ be the ring homomorphism $r \mapsto r 1_M$. If S is a ring, $\varphi : R \to S$ is a ring homomorphism, and $j : M \to S$ is a monoid homomorphism with respect to multiplication on S such that $a \in \operatorname{im} j$ and $b \in \operatorname{im} \varphi$ implies $ab = ba$, then there exists a unique ring homomorphism $\overline{\varphi} : R[M] \to S$ such that the following diagram commutes:

$$\begin{tikzcd}
R \arrow[d, "\iota_R"'] \arrow[dr, "\varphi"] & \\
R[M] \arrow[r, dashed, "\overline{\varphi}"] & S \\
M \arrow[u, "\iota_M"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Proof. First, we show that $\overline{\varphi}$ is determined by the fact that it must be a ring homomorphism that makes the diagram above commute.

$$\begin{aligned}
\overline{\varphi}\left(\sum_{m\in M} r_m m\right) &= \sum_{m\in M} \overline{\varphi}(r_m m) \\
&= \sum_{m\in M} \overline{\varphi}(\iota_R(r_m) \cdot \iota_M(m)) \\
&= \sum_{m\in M} \overline{\varphi}(\iota_R(r_m)) \cdot \overline{\varphi}(\iota_M(m)) \\
&= \sum_{m\in M} \varphi(r_m) \cdot j(m)
\end{aligned}$$

Since this function $\overline{\varphi}$ is the only function making the diagram commute, we just have to prove that it is a ring homomorphism. The fact that $\overline{\varphi}$ preserves addition and the identity is clear enough, so we will just show that it preserves multiplication. If we let $p = \sum_{m\in M} a_m m$

and $q = \sum_{m \in M} b_m m$, we then have:

$$\begin{aligned}
\overline{\varphi}(p) \cdot \overline{\varphi}(q) &= \left(\sum_{m \in M} \varphi(a_m) \cdot j(m) \right) \left(\sum_{m \in M} \varphi(b_m) \cdot j(m) \right) \\
&= \sum_{m \in M} \sum_{n \in M} \varphi(a_m) \cdot j(m) \cdot \varphi(b_n) \cdot j(n) \\
&= \sum_{m \in M} \sum_{n \in M} \varphi(a_m) \cdot \varphi(b_n) \cdot j(m) \cdot j(n) \\
&= \sum_{m \in M} \sum_{n \in M} \varphi(a_m b_n) \cdot j(mn) \\
&= \sum_{\ell \in M} \sum_{mn = \ell} \varphi(a_m b_n) \cdot j(\ell) \\
&= \overline{\varphi}(pq).
\end{aligned}$$

Hence $\overline{\varphi}$ is a homomorphism, and thus it satisfies the universal property, as desired. □

This universal property is a generalization of the universal property for polynomial rings over one indeterminant mentioned in the text, which is really just the monoid ring $R[\mathbb{N}]$. Additionally, much to our pleasure, polynomial rings in n indeterminants (which commute with each other) can be thought of as monoid rings $R\left[\mathbb{N}^n\right]$. ■