Topology and Groupoids Exercises

Chapter 2, Section 8

David Melendez

November 15, 2018

Problem 8.3. Let X, Y be metric spaces (their metrics will both be referred to as d) Show that the following functions define metrics on $X \times Y$ whose metric topology is the product topology.

(a) $D((x,y),(x',y')) = d(x,x') + d(y,y')$

(b) $D((x,y),(x',y')) = \sqrt{d(x,x')^2 + d(y,y')^2}$

Proof. For (a), first let N be a D-neighbourhood of $(a,b) \in X \times Y$. Then there exists an $\varepsilon > 0$ such that $B_D((a,b),\varepsilon) \subseteq N$; that is, if $(x,y) \in B_D((a,b),\varepsilon)$, then

$$\begin{aligned} D((x,y),(a,b)) &= d(x,a) + d(y,b) \\ &< \varepsilon. \end{aligned}$$

Let

$$M = B_d\left(a, \frac{\varepsilon}{2}\right) \times B_d\left(b, \frac{\varepsilon}{2}\right).$$

We then have that if $(x,y) \in M$, then $d(x,a) < \varepsilon/2$ and $d(y,b) < \varepsilon/2$, and so

$$\begin{aligned} D((x,y),(a,b)) &= d(x,a) + d(y,b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

and so we have that $M \subseteq B_D((a,b),\varepsilon) \subseteq N$, and so N is a product neighbourhood of (a,b). Thus every D-neighbourhood of (a,b) is a product neighbourhood of (a,b).

Conversely, suppose N is a product neighbourhood of $(a,b) \in X \times Y$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B_d(a,\varepsilon_1) \times B_d(b,\varepsilon_2) \subseteq N$. If we let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then we have that if $(x,y) \in B_D((a,b),\varepsilon)$, then

$$\begin{aligned} D((a,b),(x,y)) &= d(a,x) + d(b,y) \\ &< \varepsilon; \end{aligned}$$

and so certainly $d(a,x) < \varepsilon$ and $d(b,y) < \varepsilon$. Consequently, we have that $x \in B_d(a,\varepsilon)$ and $y \in B_d(b,\varepsilon)$, and so $(x,y) \in N$. Therefore, we have that $B_D((a,b),\varepsilon) \subseteq N$, and so N is a

D-neighbourhood of (a, b), as desired.

For (b), let $(a, b) \in X \times Y$ and assume N is a D-ball about (a, b); that is, there exists an $\varepsilon > 0$ such that

$$N = B_D\left((a, b), \varepsilon\right).$$

Then, let $\delta = \varepsilon/\sqrt{2}$, and let M be the set

$$M = B_d(a, \delta) \times B_d(b, \delta) \subseteq X \times Y$$

We then have that if $(x, y) \in M$, then $d(x, a) < \delta$ and $d(y, b) < \delta$, and so

$$\begin{aligned} d(x,a)^2 + d(y,b)^2 &< 2\delta^2 \\ &= \varepsilon^2. \end{aligned}$$

Therefore, we have that $D\left((a, b), (x, y)\right) < \varepsilon$, and so $(x, y) \in N$. Consequently, we have that $M \subseteq N$, and so N is a product neighbourhood of (a, b). Thus, every D-neighbourhood is a product neighbourhood.

Suppose conversely that N is a basic product neighbourhood of (a, b), and so $N = B_d(a, \varepsilon_1) \times B_d(b, \varepsilon_2)$ for some $\varepsilon_1, \varepsilon_2 > 0$. Then, let $\delta = \min\{\varepsilon_1, \varepsilon_2, 1\}$. Note that $\delta < 1$, and let $M = B_D\left((a, b), \delta\right)$. We then have that if $(x, y) \in M$, then

$$\begin{aligned} & B_d\left((a,b),(x,y)\right) && < \delta \\ \implies & \sqrt{d(a,x)^2 + d(b,y)^2} && < \delta \\ \implies & d(a,x)^2 + d(b,y)^2 && < \delta^2 \\ \implies & d(a,x)^2 && < \delta^2 - d(b,y)^2 \\ \implies & d(a,x) && < \sqrt{\delta^2 - d(b,y)^2} \\ & && < \sqrt{\varepsilon_1^2 - d(b,y)^2} \\ & && < \sqrt{\varepsilon_1^2} \\ & && = \varepsilon_1. \end{aligned}$$

Similarly, we have that $d(b, y) < \varepsilon_2$. Thus, $(x, y) \in N$, and so we have that $M \subseteq N$, showing that N is a D-neighbourhood of (a, b). Consequently, every product neighbourhood is a D-neighbourhood.

Since product neighbourhoods and D-neighbourhoods coincide, it then follows that the metric topology induced by D is the product topology, as desired. $\square$