Algebra: Chapter 0 Exercises

Chapter 3, Section 7

Complexes and homology

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Problem 7.1. Assume the complex

$$\cdots \longrightarrow 0 \xrightarrow{f} M \xrightarrow{g} 0 \longrightarrow \cdots$$

is exact. Prove $M \cong 0$.

Proof. By exactness, we have that $M = \ker g = \text{im } f = 0$. $\square$

Problem 7.2. Assume that the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\varphi} M' \longrightarrow 0 \longrightarrow \cdots$$

is exact. Prove that $M \cong M'$.

Proof. By exactness, we have that $\ker\varphi = \text{im } \{0 : 0 \to M\} = 0$ and $\text{im } \varphi = \ker\{0 : M' \to 0\} = M'$ Thus φ is an isomorphism, as desired. $\square$

Problem 7.3. Assume the complex

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\varphi} M' \xrightarrow{\beta} N \longrightarrow 0$$

is exact. Show that up to natural identifications, $L = \ker\varphi$ and $N = \text{coker } \varphi$.

Proof. By exactness, we have that $\ker\varphi = \text{im } \alpha \cong L$ (since α is injective), and $\text{coker } \varphi = M'/\text{im } \varphi = M'/\ker\beta \cong N$, since β is surjective. $\square$

Problem 7.4. Construct short exact sequences of $\mathbb{Z} - modules$

$$(a) \qquad 0 \longrightarrow \mathbb{Z}^{\oplus\mathbb{N}} \xrightarrow{f} \mathbb{Z}^{\oplus\mathbb{N}} \xrightarrow{g} \mathbb{Z} \longrightarrow 0$$

$$(b) \qquad 0 \longrightarrow \mathbb{Z}^{\oplus\mathbb{N}} \xrightarrow{f} \mathbb{Z}^{\oplus\mathbb{N}} \xrightarrow{g} \mathbb{Z}^{\oplus\mathbb{N}} \longrightarrow 0$$

Solution.

(a) Let $f(n_1, n_2, \dots) = (0, n_1, n_2, \dots)$ and $g(n_1, n_2, \dots) = n_1$.

(b) Let $f(n_1, n_2, \dots) = (n_1, 0, n_2, 0, n_3, \dots)$ and $g(n_1, b_2, \dots) = (n_2, n_4, n_6, \dots)$.

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Problem 7.5. Assume that the complex

$$\cdots \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow \cdots$$

is exact and that L and N are Noetherian. Prove that M is Noetherian.

Proof. First, nt hat since L is Noetherian, we have that $\operatorname{im} f$ is Noetherian, since it is the homomorphic image of a Noetherian module. Next, note hat $M/\operatorname{im} f = M/\ker g$ is a submodule of N, and hence is also Noetherian.

Therefore, since $\operatorname{im} f$ and $M/\operatorname{im} f$ are Noetherian, we have that M is Noetherian, as desired. □

Problem 7.7. Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

be a short exact equence of R-modules, and let L be an R-module.

(i) Prove that athere is an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R\textbf{-Mod}}(P, L) \longrightarrow \operatorname{Hom}_{R\textbf{-Mod}}(N, L) \longrightarrow \operatorname{Hom}_{R\textbf{-Mod}}(M, L)$$

(ii) Let M be cyclic R-module, so that $M \cong R/I$ for a (left-)ideal I, and let N be another R-module. Prove that $\operatorname{Hom}_{R\textbf{-Mod}}(M, N) \cong \{n \in N | (\forall a \in I), an = 0\}$. For $a, b \in \mathbb{Z}$, prove that $\operatorname{Hom}_{\textbf{Ab}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}) \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$.