Topology and Groupoids Exercises

Chapter 2, Section 2

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Problem 1. What are the open sets of X hen X is discrete, that is, has the discrete topology?, is indiscrete, that is, has the indescrete topology? What is the closure of $\{x\}, x \in X,$ in these cases?

Solution. Recall that under the discrete topology, a set $N \subseteq X$ is a neighborhood of a point $x \in X$ if and only if $x \in N$; that is, $\mathrm{Int}N = N$. Hence, under the discrete topology, every set is open.

On the other hand, under the indescrete toplogy, a set $N \subseteq X$ is a neighborhood of a point $x \in X$ if and only if $N = X$ and $x \in N$. Here, the only open sets are and X itself. ■

Problem 2. Let X be a topological space and let $A \subseteq X$. Prove that $\mathrm{Int}A$ is the union of all open sets U such that $U \subseteq A$ and $\overline{A}$ is the intersection of all closed sets C such that $A \subseteq C$.

Proof. First, note that $x \in \mathrm{Int}A$ if and only if there exists some $U \subseteq A$ such that $x \in U$, if and only if $x \in \mathcal{U}$, where $\mathcal{U}$ is the family of all open sets containing A.

For closed sets, first suppose $x \in \overline{A}$, and so every neighborhood of x meets A. If C is a closed subset of X containing A, then every neighborhood of x must also meet C, implying $x \in \overline{C} = C$ since C is closed. Hence x is in the intersection of all closed sets containing A, completing the inclusion in one direction.

Conversely, if $x \notin \overline{A}$, then $\overline{A}$ itself is a closed set containing A that does not contain x, so x is certainly not in the intersection of all closed sets containing A. Thus, $\overline{A}$ is the intersection of all closed sets in X that contain A as a subset. □

The previous result essentially means that the interior of A is the largest open set within A, and the closure of A is the smallest closed set containing A.

Problem 3. Let X be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A if each neighborhood of x contains points of A other than x. The set of limit points of A is written $\widehat{A}$. Prove that $\overline{A} = A \cup \widehat{A}$, and that A is closed iff $\widehat{A} \subseteq A$. Give examples of non-empty subsets A of $\mathbb{R}$ such that:

(i) $\widehat{A} = \varnothing$

(ii) $\widehat{A} \neq \varnothing$ and $\widehat{A} \subseteq A$

(iii) A is a proper subset of $\widehat{A}$

(iv) $\widehat{A} \neq \varnothing$ but $A \cap \widehat{A} = \varnothing$

Solution. First we will prove that $\overline{A} = A \cup \widehat{A}$.

Proof. First suppose that $x \in \overline{A}$. Then, by definition, every neighborhood of x meets A. If x is not in A, then that every neighborhood of x meets A means that every neighborhood of x contains points of A that aren't x, meaning $x \in \widehat{A}$. Hence $\overline{A} \subseteq A \cup \widehat{A}$.

Conversely, suppose $x \in A \cup \widehat{A}$. If $x \in A$, then every neighborhood of x contains $x \in A$. If $x \in \widehat{A}$, then every neighborhood of x contains a point in A. Hence $A \cup \widehat{A} \subseteq \overline{A}$, and so $\overline{A} = A \cup \widehat{A}$. □

Next, we will prove that A is closed iff $\widehat{A} \subseteq A$.

Proof. A is closed iff $A = \overline{A}$, meaning $A = A \cup \widehat{A}$, whence $\widehat{A} \subseteq A$. □

Now, we produce each of the examples requested:

(i) Let A be the singleton set $\{0\}$. Obviously every neighborhood of 0 contains 0, so 0 is not a limit point of A. If $x \neq 0$, then the open interval $(x - |x|, x + |x|)$ does not contain 0, so x is not a limit point of A.

(ii) Let $A = [0, 1] \cup \{2\}$. Then $\widehat{A} = [0, 1] \subseteq A$.

(iii) Let $A = (0, 1)$. Then $\widehat{A} = [0, 1] \supset A$.

(iv) Let $A = \{1/n \mid n \in \mathbb{N}\}$. Then for any $1/n \in A$, the interval $\left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right)$ with $\delta = \frac{1}{n} - \frac{1}{n+1}$ contains only $1/n \in A$, and so $A \cap \widehat{A} = \varnothing$. However, by the Archimedean property of the real numbers, there exists for every $\varepsilon > 0$ an $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Hence every open interval containing 0 also contains a point in A, and so $0 \in \widehat{A}$.

■

Problem 4. Let X be a topological space and let $A \subseteq B \subseteq X$. We say that A is *dense* in B if $B \subseteq \overline{A}$, and A is *dense* if $\overline{A} = X$. Prove that if A is dense in X and U is open then

$$U \subseteq \overline{A \cap U}.$$

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