# Algebra: Chapter 0 Exercises

## Chapter 3, Section 4

## Ideals and quotients: Remarks and examples. Prime and maximal ideals

David Melendez

August 18, 2018

**Problem 4.1.** Let $R$ be a ring, and let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals of $R$. We let

$$\sum_{\alpha\in A} I_\alpha = \left\{ \sum_{\alpha\in A} r_\alpha \text{ such that } r_\alpha \in I_\alpha \text{ and } r_\alpha = 0 \text{ for all but finitely many } \alpha \right\}$$

Prove that $J = \sum_\alpha I_\alpha$ is an ideal of $R$ and that it is the smallest ideal containing all of the ideals $I_\alpha$.

*Solution.* First we prove that $J$ is an ideal of R.

*Proof.* Let $a, b \in J$, so that

$$\begin{aligned} a &= \sum_{\alpha\in A} r_\alpha \\ b &= \sum_{\alpha\in A} s_\alpha, \end{aligned}$$

where each $r_\alpha, s_\alpha \in I_\alpha$ and all but finitely many $r_\alpha$ and $s_\alpha$ are nonzero. We then have:

$$\begin{aligned} a + b &= \sum_{\alpha\in A} r_\alpha + \sum_{\alpha\in A} s_\alpha \\ &= \sum_{\alpha\in A} r_\alpha + s_\alpha. \end{aligned}$$

Each term $r_\alpha + s_\alpha$ is in $I_\alpha$ since $r_\alpha, s_\alpha \in I_\alpha$ and $I_\alpha$ is an ideal, and clearly all but finitely many $r_\alpha + s_\alpha$ are nonzero since $(r_\alpha)_{\alpha\in A}$ and $(s_\alpha)_{\alpha\in A}$ both have that property, so $a + b \in J$.

Additionally, if $s \in R$ and $r \in J$ so that $r = \sum_{\alpha\in A} r_\alpha$ (where all but finitely many $r$'s are zero), then we have

$$\begin{aligned} rs &= \left(\sum_{\alpha\in A} r_\alpha\right) s \\ &= \sum_{\alpha\in A} r_\alpha s \\ &\in J, \end{aligned}$$

where the last line is true because each $r_\alpha s \in I_\alpha$ as a result of each $I_\alpha$ being a right-ideal of $R$, and the fact that if $r_\alpha$ is zero then $r_\alpha s$ is also zero, implying that there are cofinitely many zero terms in this resulting sum as well. A similar argument shows that $J$ is a left-ideal of $R$ if each $I_\alpha$ is also a left-ideal. □

Now, we will show that $J = \sum_{\alpha \in A} I_\alpha$ is the smallest ideal of $R$ containing each of the ideals $I_\alpha$ for $\alpha \in A$.

*Proof.* We just proved that $J$ is an ideal of $R$, so now we just need to show that $J$ is a subset of any ideal containing each of the ideals $I_\alpha$. This is immediate: if $r \in J$ is such that $r = \sum_{\alpha \in A} r_\alpha$ for $r_\alpha \in I_\alpha$, then of course any ideal of $R$ containing each $I_\alpha$ contains $r$, since such an ideal is closed under addition. □

■

**Problem 4.2.** Prove that the homomorphic image of a Noetherian ring is Noetherian. That is, prove that if $\varphi : R \to S$ is a surjective ring homomorphism and $R$ is Noetherian, then $S$ is Noetherian.

*Solution.* Suppose $I = (a_1, \ldots, a_n)$ is an ideal of $R$ and $\varphi : R \to S$ is surjective. Then we have

$$\begin{aligned}
\varphi(I) &= \varphi\left(\sum_{i=1}^{n} (a_i)\right) \\
&= \sum_{i=1}^{n} \varphi((a_i)) \\
&= \sum_{i=1}^{n} (\varphi(a_i)),
\end{aligned}$$

and so $\varphi(I)$ is finitely generated.

To see that these operations are justified, note that if $g \in R$ and $J = (g)$ is an ideal, then we have

$$\begin{aligned}
\varphi(J) &= \varphi\left(\{rg : r \in R\}\right) \\
&= \{\varphi(r)\varphi(g) : r \in R\} \\
&= \{r\varphi(g) : r \in R\} \\
&= (\varphi(g)),
\end{aligned}$$

where the third equality follows from the surjectivity of $\varphi$.

Additionally, if $I, J$ are ideals of $R$, then we also have

$$\begin{aligned}
\varphi(I + J) &= \varphi(\{i + j : i \in I, j \in J\}) \\
&= \{\varphi(i) + \varphi(j) : i \in I, j \in J\} \\
&= \varphi(I) + \varphi(J)
\end{aligned}$$

Note, then, that if $J$ is an ideal of $S$, then $\varphi^{-1}(J)$ is an ideal of $R$, allowing us to see that $J = \varphi(\varphi^{-1}(J))$ is finitely generated. Therefore, every ideal of $S$ is finitely generated, and so $S$ is Noetherian. ■

**Problem 4.3.** Prove that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal.

*Solution.* First, we (quite clumsily) compute the ideal $(2, x)$ as follows:

$$\begin{aligned}(2, x) &= \{2p + xq : p, q \in \mathbb{Z}[x]\} \\ &= \{(2a_0 + 2a_1 x + \cdots + 2a_n x^n) + (b_1 x + b_2 x^2 + \cdots + b_m x^m) : a_j, b_j \in \mathbb{Z}\} \\ &= \{2a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_j \in \mathbb{Z}\}.\end{aligned}$$

In other words, the ideal $(2, x)$ consists of all the polynomials in $\mathbb{Z}[x]$ with an even constant term.

Note, then, if $(2, x) = (g)$ for some polynomial $g \in \mathbb{Z}[x]$, then there must be a polynomial $p \in \mathbb{Z}[x]$ such that $2 = gp$, since $2 \in (2, x)$. If this is the case, then we have $\deg g + \deg p = 0$, and so $\deg g = \deg p = 0$. This means that $g$ is constant, and hence is either 1 or 2. In the former case, $(g)$ is the whole ring $\mathbb{Z}[x]$, and in the latter case, $(g)$ is the ideal $2\mathbb{Z}[x]$. Neither of these ideals equal the ideal of $(2, x)$, leading us to conclude that no single polynomial in $\mathbb{Z}[x]$ generates the ideal $(2, x)$. ■

**Problem 4.4.** Prove that if $k$ is a field, then $k[x]$ is a PID. (Hint: Polynomial divison with remainder)

*Solution.* Let $I \subseteq k[x]$ be an ideal. If $I = 0 = (0)$, then clearly it is principal. Otherwise, let $p \in I$ be a monic polynomial of minimal degree $d$ Let $I \subseteq k[x]$ be an ideal. If $I = 0 = (0)$, then clearly it is principal.

Otherwise, let $g \in I$ be a monic polynomial of minimal degree $d$. If $p \in I$, then we can apply division with remainder to find polynomials $q, r \in k[x]$ such that

$$p = gq + r,$$

where $\deg r < d$. Note that since $p \in I$ and $gq \in I$ by (right-) absorption, we then can see that $r = p - gq \in I$, since $I$ is closed under addition. But $d$ is the smallest degree of any nonzero polynomial in $I$ and $\deg r < d$; it then follows that $r = 0$, and so

$$p = gq,$$

showing us that $I \subseteq (g)$.

Of course $(g) \subseteq I$, so we then have $I = (g)$, as desired. ■

**Problem 4.5.** Let $I, J$ be ideals in a commutative ring $R$, such that $I + J = (1)$. Prove that $IJ = I \cap J$.

*Solution.* The simple fact that $IH \subseteq I \cap J$ was already proven in the text, so suppose $r \in I \cap J$. Since $I + J = (1) = R$, we know there exists an $i \in I$ and a $j \in J$ such that $1 = i + j$. Note, then, that:

$$\begin{aligned}r &= r \cdot 1 \\ &= r \cdot (i + j) \\ &= r \cdot i + r \cdot j.\end{aligned}$$

Since $r \in J$ and $R$ is commutative, we know that $ri \in IJ$, and since $r \in J$, we also know that $rj \in IJ$. Hence $r = ri + rj \in IJ$, and so $I \cap J \subseteq IJ$. Therefore, $IJ = I \cap J$, as desired. ■

¡++¿