Algebra: Chapter 0 Exercises Chapter 1, Section 5

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Problem 5.1. A final object in a category C is initial in the opposite category C^{op} .

Proof. Let $T \in \text{Obj}(\mathbf{C})$ be initial and $Z \in \text{Obj}(\mathbf{C})$ be any object. Since $\text{Hom}_{\mathbf{C}}(T, Z)$ is a singleton, $\text{Hom}_{\mathbf{C}^{op}}(Z, T) = \text{Hom}_{\mathbf{C}}(T, Z)$ is also a singleton, making T final in \mathbf{C}^{op}

Problem 5.2. The empty set \emptyset is the *unique* initial object in **Set**.

Proof. Recall that the number of morphisms between any two sets A and B is given by B^A . Thus, the problem of finding the size of an initial object in **Set** boils down to solving $|Z|^{|I|} = 1$ for |I|, where Z is an arbitrary set. The only solution to this is |I| = 0, which is only satisfied by the the null set, given that $\emptyset^{\emptyset} = 0$.

Problem 5.3. Final objects are unique up to isomorphism.

Proof. Let C be a category and $T_1, T_2 \in \text{Obj}(\mathbf{C})$ be final. Because T_1 and T_2 are final, we have, for all $Z \in \text{Hom}_{\mathbf{C}}(Z, T)$:

$$\operatorname{End}(T_1) = \{\operatorname{id}_{T_1}\}$$

$$\operatorname{End}(T_2) = \{\operatorname{id}_{T_2}\}$$

$$\operatorname{Hom}(T_1, T_2) = \{f\} \text{ for some } f$$

$$\operatorname{Hom}(T_2, T_1) = \{g\} \text{ for some } g$$

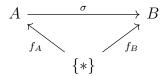
We then have $fg = id_{T_2}$ and $gf = id_{T_2}$, giving us $T_1 \cong T_2$.

Problem 5.4. What are the initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

Solution. For the sake of completeness, a description of this category **Set*** will be included.

Definition. Define **Set*** as follows:

Objects in **Set*** are morphisms $f : \{*\} \to A$ where A is any set, denoted (f, A). Morphisms $\sigma \in \operatorname{Hom}_{\mathbf{Set}^*}((f_A, A), (f_B, B))$ are commutative diagrams:



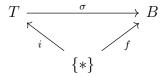
Now, we will answer the question.

Proposition. Initial and final objects in **Set*** are objects (i, T), where T is a singleton and i is the unique function that maps from $\{*\}$ to T.

Proof. We will prove that the objects described are both initial and final in **Set***.

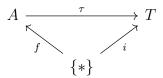
1. Initial:

Consider the object described above. Then, a morphism from this object to another object $(f, B) \in \text{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:



The only choice of σ that makes this diagram commute is defined by $\sigma(i(*)) = f(*)$ (as T is a singleton), so T is initial.

2. Final: Consider the object described above. Then, a morphism to this object from another object $(f, B) \in \text{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:

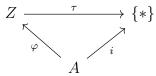


The only choice of τ that makes this diagram commute is defined by $\tau(f(*)) = i(*)$ (as T is a singleton), so T is final.

Problem 5.5. What are the final objects in the category considered in 5.3?

Solution. Final objects in this category (let's denote it \mathbf{C}) are singletons.

Proof. Let $\{*\}$ be a singleton. Then morphisms τ from $(Z, \varphi) \in \text{Obj}(\mathbf{C})$ to $\{*\}$ are commutative diagrams:



Since $\{*\}$ is a singleton, the only τ that satisfies $\tau(\varphi(a)) = \iota(a)$ is defined by $\tau(z) = *$ for all $z \in Z$. Thus $(\{*\}, \tau)$ is final in \mathbb{C} .

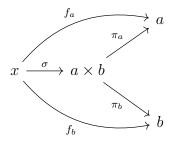
Problem 5.6. Consider the category corresponding to endowing (as in Example 3.3) the set \mathbb{Z}^+ with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category iff d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names?

Solution. Let **Div** denote this category and $a, b \in \text{Obj}(\mathbf{Div})$ be objects.

1. Products

That **Div** has products means the following:

For every $a, b \in \text{Obj}(\mathbf{Div})$, there exists an object $a \times b \in \text{Obj}(\mathbf{Div})$ such that every $(x, f_a, f_b) \in \text{Obj}(\mathbf{Div}_{a,b})$ admits a unique morphism $\sigma \in \text{Hom}_{\mathbf{Div}}(x, a \times b)$ that makes the following diagram commute:



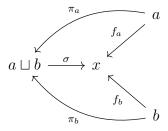
Meaning, for every pair of integers (a, b), there exists a unique factor of a and b, $a \times b$, such that every common factor of a and b divides $a \times b$. This is necessarily the greatest common factor of a and b, since it must be a multiple of every common factor of a and b, and gcf(a, b) is the product of the common factors of a and b.

2. Coproducts

That **Div** has coproducts means the following:

For every $a, b \in \text{Obj}(\mathbf{Div})$, there exists an object $a \sqcup b \in \text{Obj}(\mathbf{Div})$ such that every

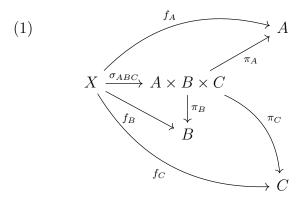
 $(x, f_a, f_b) \in \text{Obj}(\mathbf{Div}^{a,b})$ admits a unique morphism $\sigma \in \text{Hom}_{\mathbf{Div}}(a \sqcup b, x)$ that makes the following diagram commute:



Meaning, for every pair of integers (a, b) there exists a unique multiple of a and b, that divides every common multiple of a and b. Since $a \sqcup b$ is a common multiple of a and b and $a \sqcup b \le x$ for every common multiple x of a and b, $a \sqcup b$ is lcm(a, b).

Problem 5.9. Let \mathbf{C} be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of \mathbf{C} ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property.

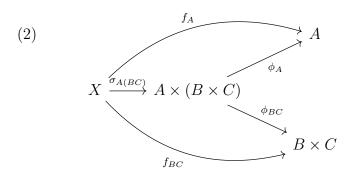
Solution. For objects A, B, C of a category \mathbf{C} , $A \times B \times C$ ought to be universal with respect to the property of mapping to A, B, and C. In other words, every (X, f_a, f_b, f_c) in the category $\mathbf{C}^{A,B,C}$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(X, A \times B \times C)$ such that the following diagram commutes:



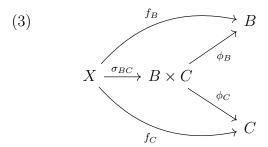
Meaning, $\pi_A \sigma = f_A$, $\pi_B \sigma = f_B$, and $\pi_C \sigma = f_C$.

Proposition. The products $A \times (B \times C)$ and $(A \times B) \times C$ satisfy this property.

Proof. Consider the product $A \times (B \times C) \in \text{Obj}(\mathbf{C})$ and its diagram:



along with the product $B \times C$ and its diagram:



Staring at these diagrams, we can see that $A \times (B \times C)$ satisfies our universal property if we let: $f_{BC} = \sigma_{BC}$,

 $\pi_A = \phi_A$

 $\pi_B = \phi_B \sigma_{ABC}$,

 $\pi_C = \phi_C \sigma_{ABC}$.

This makes the diagram for $A \times B \times C$ commute by the commutativity of (2) and (3), thereby forcing $\sigma_{ABC} = \sigma_{A(BC)}$ since the morphism $A \to A \times (B \times C)$ that makes (2) commute is unique.

The proof for $(A \times B) \times C$ is entirely analogous, giving us that $(A \times B) \times C \cong A \times (B \times C)$ by their satisfaction of the same universal property.

Problem 5.10. Push the envelope farther still, and define products for families (i.e. indexed sets) of objects of a category. Do these exist in **Set**?

Solution. Let \mathbf{C} be a category and S be a family of objects in \mathbf{C} indexed by a set J. The product $P = \prod_J S$ is an object of \mathbf{C} together with morphisms $\pi_j \in \mathrm{Hom}_{\mathbf{C}}(P, S_j)$ for all $j \in J$ such that every $X \in \mathrm{Obj}(\mathbf{C})$ together with morphisms $f_j \in \mathrm{Hom}_{\mathbf{C}}(X, S_j)$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(X, P)$ such that $f_j = \pi_j \sigma$.

That these exist in **Set** is equivalent to the Axiom of Choice.