Algebra: Chapter 0 Exercises

Chapter 3, Section 6

Products, coproducts, etc. in R-**Mod**

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Problem 6.1. Prove that $R^{\oplus A} \cong F^R(A)$.

Proof. First, define $j : A \to R^{\oplus A}$ by $j(a)(b) = \delta_{ab}$, where δ is the Kronecker delta. We then have, for all $\alpha \in R^{\oplus A}$, that

$$\alpha = \sum_{a \in A} \alpha(a) j(a),$$

since we have for all $x \in A$ that

$$\begin{aligned}
\left(\sum_{a \in A} \alpha(a) j(a) \right)(x) &= \sum_{a \in A} (\alpha(a) j(a))(x) \\
&= \sum_{a \in A} \alpha(a)(j(a)(x)) \\
&= \sum_{a \in A} \alpha(a) \delta_{ax} \\
&= \alpha(x).
\end{aligned}$$

Of course this representation of α as a linear combination of $j(a)$ for all $a \in A$ is unique, as the coefficients are clearly uniquely determined by the image of each $a \in A$ under α.

Thus, if N is an R-module, $f : A \to N$, and $\varphi : R^{\oplus A} \to N$ is an R-module homomorphism such that $\varphi j = f$, we then have, for all $\alpha \in R^{\oplus A}$,

$$\begin{aligned}
\varphi(\alpha) &= \varphi\left(\sum_{a \in A} \alpha(a) j(a) \right) \\
&= \sum_{a \in A} \varphi(\alpha(a) j(a)) \\
&= \sum_{a \in A} \alpha(a) \varphi(j(a)) \\
&= \sum_{a \in A} \alpha(a) f(a);
\end{aligned}$$

thus such a homomorphism is unique, if it exists. Of course, this definition indeed defines a homomorphism that satisfies the desired property, as is easy to verify, and so $R^{\oplus A}$ does satisfy the universal property for the free R-module over A. $\square$

Problem 6.2. Prove or disprove that if R is a ring and M is a nonzero R-module, then M is not isomorphic to $M \oplus M$.

Solution. As a counterexample, let R be a ring and consider the R-module $M = R^{\oplus \mathbb{N}}$ (where $\mathbb{N}$ does not include 0), generated by the set $\{e_1, e_2, \dots\}$. Then, $M \oplus M$ is the cartesian product of M with itself. Consider, then, the function $\varphi : M \to M \oplus M$, defined by

$$\varphi\left(\sum_i r_i e_i\right) = \left(\sum_i r_{2i-1} e_i, \sum_i r_{2i} e_i\right).$$

As can be verified, φ is an R-module homomorphism which is injective and surjective. Hence $M \cong M \oplus M$. $\blacksquare$

Problem 6.3. Let R be a ring, M an R-module, and $p : M \to M$ an R-module homomorphism such that $p^2 = p$ (Such a map is called a *projection*). Prove that $M \cong \ker p \oplus \operatorname{im} p$.

Proof. Define the functions $\varphi : M \to \ker p \oplus \operatorname{im} p$ and $\psi : \ker p \oplus \operatorname{im} p$ by

$$\begin{aligned} \varphi(m) &= (m - p(m), p(m)) \\ \psi(u, v) &= u + v. \end{aligned}$$

Note that $p(m) \in \operatorname{im} p$, and if $m \in M$, then

$$\begin{aligned} p(m - p(m)) &= p(m) - p(p(m)) \\ &= p(m) - p(m) \\ &= 0; \end{aligned}$$

hence $m - p(m) \in \ker p$. Thus the definition of φ makes sense. Past this, it is easy to verify that φ and ψ are R-module homomorphisms and that ψ is a left and right inverse for φ; hence, φ is an isomorphism between M and $\ker p \oplus \operatorname{im} p$. $\square$

Problem 6.5. For any ring R and any two sets A_1, A_2, prove that $\left(R^{\oplus A_1}\right)^{\oplus A_2} \cong R^{\oplus (A_1 \times A_2)}$.

Proof. Let $\varphi : R^{\oplus (A_1 \times A_2)} \to \left(R^{\oplus A_1}\right)^{\oplus A_2}$ be a function defined by

$$\Phi(\varphi)(a)(b) = \varphi(a, b).$$

Then Φ is an R-module isomorphism. $\square$

Problem 6.6. Let R be a ring, and let $F = R^{\oplus n}$ be a finitely generated free R-module. Prove that $\mathrm{Hom}_{R\textbf{-Mod}}(F, R) \cong F$.

Proof. Let $e_1, \ldots, e_n$ be the generators of F, and for $0 \leq i \leq n$, let $\psi_i : F \to R$ be defined by

$$\psi_i\left(\sum_{j=1}^{n} r_j e_j\right) = r_i.$$

Then each ψ_i is well-defined and an R-module homomorphism.

Note, then, that for each $\varphi \in \mathrm{Hom}_{R\textbf{-Mod}}(F, M)$ and $v = \sum_i r_i e_i$, we have that

$$\begin{aligned}\varphi(v) &= \varphi\left(\sum_i r_i e_i\right) \\ &= \sum_i \varphi(r_i e_i) \\ &= \sum_i r_i \varphi(e_i) \\ &= \sum_i \psi_i(v)\varphi(e_i) \\ &= \left(\sum_i \varphi(e_i)\psi_i\right)(v);\end{aligned}$$

thus, if we let $s_i = \varphi(e_i)\psi_i$, then we have that $\varphi = \sum_i s_i \psi_i$, and so $\mathrm{Hom}_{R\textbf{-Mod}}(F, R)$ is generated by $(\psi)_i$ Indeed, each ψ_i is in $\mathrm{Hom}_{R\textbf{-Mod}}(F, R)$, and so the module generated by them is contained within $\mathrm{Hom}_{R\textbf{-Mod}}(F, R)$, as well.

We can then define a function $\Phi : \mathrm{Hom}_{R\textbf{-Mod}}(F, R) \to F$ by

$$\Phi\left(\sum_i r_i \psi_i\right) = \sum_i r_i e_i, \tag{1}$$

It is then easy to show that Φ is an R-module isomorphism. $\square$

Problem 6.7. Let A be any set. For any family $\{M_a\}_{a \in A}$ of modules over a ring R, define the *product* $\prod_{a \in A} M_a$ and coproduct $\bigoplus_{a \in A} M_a$.

Solution. We define the product $P = \prod_{a \in A} M_a$ as follows: We say that P, along with a family of R-module homomorphisms $\{\pi_a : P \to M_a\}_{a \in A}$ is a product of the family $\{M_a\}_{a \in A}$ if for each R-module N and family of morphisms $\{\varphi_a : N \to M_a\}_{a \in A}$, there exists a unique R-module homomorphism $\psi = \prod_{a \in A} \varphi_a : N \to P$ such that for all $a \in A$, we have $\pi_a \psi = \varphi_a$.

In the case where $M_a = R$ for all $a \in A$, we have that the set R^A of functions from A to R, along with the projections $\pi_a(g) = g(a)$ satisfies this universal property. Indeed, if M is an R-module and we have a family of R-module homomorphisms $\{f_a : M \to R\}$, then we have that if $\psi : M \to R^A$ is a function satisfying the condition $\pi_a \psi = f_a$, then

$$\begin{aligned}\psi(m)(a) &= \pi_a(\psi(m)) \\ &= f_a(m);\end{aligned}$$

thus, $\psi(m)$ is the function taking a to $f_a(m)$. It is easy to check that ψ is an R-module homomorphism, and hence that it satisfies the desired universal property.

We define the coproduct $K = \bigoplus_{a \in A} M_a$ as follows: We say that P, along with a family of R-module homomorphisms $\{\iota_a : M_a \to K\}_{a \in A}$ is a coproduct of the family $\{M_a\}_{a \in A}$ if for each R-module N and family of morphisms $\{\varphi_a : M_a \to N\}_{a \in A}$, there exists a unique R-module homomorphism $\psi = \bigoplus_{a \in A} \varphi_a : K \to N$ such that for all $a \in A$, we have $\psi \iota_a = \varphi_a$. ■

Prove that $\mathbb{Z}^{\mathbb{N}} \ncong \mathbb{Z}^{\oplus \mathbb{N}}$. (Hint: Cardinality.)