

Topology and Groupoids Exercises

Chapter 2, Section 4

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Problem 1. Prove that the relation ' X is a subspace of Y ' is a partial order relation for topological spaces.

Solution. Denote by \leq the relation 'is a subspace of'. Of course ' X is a subset of Y ' is a partial order relation for sets, so we simply need to verify that the topologies line up as they should. We will be working with open set topologies.

First, consider (X, τ_1) as a subspace of (X, τ) , where τ_1 is the relative topology on X . If $U \in \tau_1$, then there exists a set $V \in \tau$ such that $U = V \cap X$. Since X and V are τ -open, $U = V \cap X$ is also τ -open, so $U \in \tau$. Hence $\tau_1 \subseteq \tau$. Conversely, if $U \in \tau$, then of course $U = U \cap X \in \tau_1$, and so $\tau_1 = \tau$. Therefore $X \leq X$, and so \leq is reflexive.

Next, suppose $X \leq Y$ and $Y \leq X$. Then, in particular, $X \subseteq Y$ and $Y \subseteq X$, so $X = Y$, and so \leq is antisymmetric.

Finally, suppose $(X, \tau_X) \leq (Y, \tau_Y)$ and $(Y, \tau_Y) \leq (Z, \tau_Z)$. Then $U \in \tau_X$ implies there exists some $V \in \tau_Y$ such that $U = V \cap X$. Further, since $V \in \tau_Y$, and Y is a subspace of Z , there exists a τ_Z -open W such that $V = W \cap Y$. We then have $U = W \cap Y \cap X = W \cap X$ (since $X \subseteq Y$ implies $Y \cap X = X$, and so U is open in the relative topology on X with respect to Z).

Conversely, suppose U is open in Z . Then note that $U \cap X = U \cap (X \cup Y) = (U \cap Y) \cap X$ is open in X , since $Y \leq Z$ implies $U \cap Y$ is τ_Y -open, and $X \leq Y$ implies $(U \cap Y) \cap X$ is τ_X -open. Thus \leq is transitive, completing the proof that \leq is a partial order of topological spaces. ■

Problem 2. Prove that the set $I = \{x \in \mathbb{Q} : -\sqrt{2} \leq x \leq \sqrt{2}\}$ is both open and closed in \mathbb{Q} .

Solution. Note that $I = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$, where the first interval is open in \mathbb{R} and the second is closed in \mathbb{R} . The result then follows from the conditions on open and closed sets in a relative topology. ■

Problem 3. Let A be the subspace of \mathbb{R} of point $1/n$ for $n \in \mathbb{Z} \setminus \{0\}$. Prove that A is discrete, but that the subspace $A \cup \{0\}$ of \mathbb{R} is not discrete.

Solution. Note that if $1/n \in A$, then we have $\left\{\frac{1}{n}\right\} = \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cap A$, where the interval is an open interval in \mathbb{R} . Hence all singletons in A are open, implying that all subsets

of A are open since the union of any family of open sets of A is also open. This means A is discrete.

Now, denote by Y the subspace $A \cup \{0\}$ of X . Take A as a subset of Y , and note that $\text{Cl}_Y A = \overline{A} \cap Y = Y \neq A$ (note that every \mathbb{R} -neighborhood of 0 meets A , and so $0 \in \overline{A}$); hence A is not closed in Y , and its complement $\{0\}$ is not open in Y . Therefore Y does not have the discrete topology. ■

Problem 4. Prove that a subspace of a discrete space is discrete, and a subspace of an indiscrete space is indiscrete.

Solution. Let Z be a discrete topological space, and let $A \subseteq Z$ be a subspace. Then $x \in A$ implies $\{x\} = \{x\} \cap A$, where the singleton on the right side is open in Z ; hence $\{x\}$ is open in A , implying every set in A is open, and so A is discrete.

On the other hand, let X be an indiscrete topological space, and let $A \subseteq X$ be a subspace. If $U \subseteq A$ is open, then there exists an open set V in X such that $U = V \cap A$. Since X is indiscrete, V is either \emptyset or X , and so U is either $\emptyset \cap A = \emptyset$ or $X \cap A = A$; hence A is indiscrete. ■

Problem 5. Let A be the subspace $[0, 2] \setminus \{1\}$ of \mathbb{R} . Prove that $I = [0, 1)$ is both open and closed in A .

Solution. This follows from the fact that $I = (-1, 1) \cap A = [-1, 1] \cap A$. ■

Problem 6. Let $x \in X$ and let A be a neighborhood (in X) of x . Prove that the neighborhoods in A of x are exactly the neighborhoods in X of x which are contained in A .

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