Topology and Groupoids Exercises

Chapter 2, Section 6

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October 29, 2018

Problem 6.1. Let A be a subspace of X and let $\mathcal{B}$ be a base for the neighbourhoods of X. Construct from $\mathcal{B}$ a base of the neighborhoods of A.

Solution. Let $x \in A$ and let M be a neighborhood in A of x. Then there exists a neighborhood N in X of x such that $M = N \cap A$. Then, because N is a neighbourhood in X of x, there exists a neighborhood $P \in \mathcal{B}(x)$ of x such that $P \subseteq N$. We then have that $P \cap A$ is a neighborhood in A of x, and $P \cap A \subseteq N \cap A = M$.

Thus, $\mathcal{B}_A(x) = \{P \cap A : P \in \mathcal{B}(x)\}$ forms a basis for the neighbourhoods of A. ■

Problem 6.2. Let $\mathcal{B}(x), \mathcal{B}'(x')$ be bases for the neighbourhoods of $x \in X, x' \in X'$, respectively. Prove that the sets $M \times N$ for $M \in \mathcal{B}(x), N \in \mathcal{B}'(x)$ form a base for the neighbourhoods of $(x, x') \in X \times X'$, and that the sets $M \times M$ form a base for the neighbourhoods of $(x, x) \in X \times X$.

Proof. Let P be a neighbourhood of $(x, x') \in X \times X'$. Then, there exist neighborhoods $M \subseteq X$ of x and $M' \subseteq X'$ of x' such that $M \times M' \subseteq P$. Further, we then have neighborhoods $N \in \mathcal{B}(x)$ of x and $N' \in \mathcal{B}'(x')$ of x' such that $N \subseteq M$ and $N' \subseteq M'$; consequently, $N \times N' \subseteq M \times M'$ is a neighbourhood of (x, x'). Therefore, the sets $M \times M'$ for $M \in \mathcal{B}(x)$ and $M' \in \mathcal{B}'(x')$ form a basis for the neighbourhoods in $X \times X'$, as desired.

The proof of the second result is very similar. □

Problem 6.3. A topological space X is said to satisfy the *first axiom of countability* if there is a base $\mathcal{B}$ for the neighbourhoods of X such that $\mathcal{B}$ is countable for each $x \in X$. Prove that the following satisfy the first axiom of countability: $\mathbb{R}, \mathbb{Q}$, a discrete space, a space with a countable number of open sets.

Solution. For $\mathbb{R}$ and $\mathbb{Q}$, $\mathcal{B}(x) = \{(x - 1/n, x + 1/n) : n \in \mathbb{N}\}$ works.

For a discrete space, $\mathcal{B}(x) = \{x\}$ works.

For a space with countably many open sets, simply let $\mathcal{B}(x)$ be the set of all open sets containing x. Then, $\mathcal{B}(x)$ is countable, and if N is a neighbourhood of x, then we have $N \supseteq \operatorname{Int} N \in \mathcal{B}(x)$. ■

Problem 6.4. Prove that subspaces and (finite) products of first-countable spaces are also first-countable.

Proof. If A is a subspace of X, then the base for the neighbourhoods of A constructed in Exercise 6.1 is countable. Hence A is first-countable.

If $X_1, \ldots, X_n$ are first countable and $\mathcal{B}_j$ is a countable base for X_j with $1 \leq j \leq n$, then $\mathcal{B} : p \mapsto \prod_j \mathcal{B}_j(p_j)$ is a countable base for the neighbourhoods of $\prod_j X_j$. □

Problem 6.5. A topological space X has a countable base for the neighbourhoods at x. Prove that there is a base for the neighbourhoods of x of sets $B_n, n \in \mathbb{N}$, such that $B_n \supseteq B_{n+1}, n \in \mathbb{N}$.

Proof. Let $\mathcal{B}$ be a countable base for the neighbourhoods of x. Since $\mathcal{B}$ is countable, there exists a bijection $f : \mathbb{N} \to \mathcal{B}$.

Let $B_1 = f(1)$, and for $n > 1$, define B_n by

$$B_n = f(n) \cap \bigcap_{1 \leq i < n} B_i.$$

Then, $B_n \supseteq B_{n+1}$ for $n \in \mathbb{N}$ since each B_{n+1} is an intersection with B_n, and each B_n is a neighbourhood of x by virtue of being an intersection of finitely many neighbourhoods of x.

Note, then, that if M is a neighbourhood of x, then there exists a $k \in \mathbb{N}$ such that $f(k) \subseteq M$. But we also have $B_k \subseteq f(k)$ (since B_k is an intersection involving $f(k)$), and so $B_k \subseteq M$. Thus, for each neighbourhood M of x, there exists a $k \in \mathbb{N}$ such that $B_k \subseteq M$, and so $\{B_i\}_{i \in \mathbb{N}}$ is a base for the neighbourhoods of x, as desired. □

Problem 6.6. Use the conditions for continuity to prove the following:

Let A be a subspace of X, and let $\operatorname{Int}, \operatorname{Int}_A$ denote respectively the interior operators for X, A. If $B \subseteq X$, then

$$(\operatorname{Int} B) \cap A \subseteq \operatorname{Int}_A B \cap A.$$

Proof. Let $\iota : A \to X$ be the inclusion from A into X. Then ι is continuous, and so we have that for all $B \subseteq X$ that

$$\iota^{-1}[\operatorname{Int} B] \subseteq \operatorname{Int}_A \iota^{-1}[B].$$

Note, then, that if $D \subseteq X$, then $\iota(x) \in S$ iff $x \in D$ and $x \in A$, iff $x \in D \cap A$. Thus, $\iota^{-1}[\operatorname{Int} B] = (\operatorname{Int} B) \cap A$, and $\iota^{-1}[B] = B \cap A$, and so we have that

$$(\operatorname{Int} B) \cap A \subseteq \operatorname{Int}_A B \cap A,$$

as desired.

The next part of the problem asks us to prove a similar result for closures. This proof is essentially identical. □

Problem 6.7. Prove that the continuity of $f : X \to Y$ is not equivalent to the condition: if $A \subseteq X$, then $\operatorname{Int} f[A] \subseteq f[\operatorname{Int} A]$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $x \mapsto x$ for $x \leq 1$, and $x \mapsto 2 - x$ for $x > 1$. Then f is clearly continuous.

Then, let $A = (-1, 0) \cup (0, 1) \cup \{2\}$. We then have

$$\begin{aligned} \operatorname{Int} f[A] &= \operatorname{Int} (f[(-1,0)] \cup f[(0,1)] \cup f[\{2\}]) \\ &= \operatorname{Int} (-1,0) \cup (0,1) \cup \{0\} \\ &= (-1, 1). \end{aligned}$$

However, we also have

$$\begin{aligned} f[\operatorname{Int} A] &= f[(-1,0) \cup (0,-1)] \\ &= (-1,0) \cup (0,1), \end{aligned}$$

which does not contain $(-1, 1)$. ■

Problem 6.9. Let $f, g : X \to \mathbb{R}$ be maps. Prove that the sets

$$\begin{aligned} A &= \{x \in X : f(x) \geq g(x)\} \\ B &= \{x \in X : f(x) \leq g(x)\} \end{aligned}$$

are closed.

Proof. Define $h_1 : X \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Then, A is the set of all $x \in X$ such that $h(x) = f(x)$; thus, A is closed by an example from the book. A similar proof where h is a minimum works for B. □

Problem 6.10. Prove the following generalized gluing rule: Let X, Y be topological spaces and let $f : X \to Y$ be a function. If $A_1, \ldots, A_n$ are closed subsets of X such that $X = \bigcup_i A_i$ and $f_i = f|_{A_i}$ is continuous for each i, then f is continuous.

Proof. Let C be a closed set in Y, and let $B_i = f_i^{-1}[C]$. Since each f_i is continuous, we have that each B_i is closed in A_i. Thus, for each i, there exists a $D_i \subseteq X$, closed in X, such that $B_i = D_i \cap A_i$. Consequently, we have that

$$\begin{aligned} f^{-1}[C] &= \bigcup_i f_i^{-1}[C] \\ &= \bigcup_i B_i \\ &= \bigcup_i (D_i \cap A_i), \end{aligned}$$

which is closed in X, since each D_i, A_i is closed, and intersections and finite unions preserve closedness. Therefore, f is continuous, as desired. □