# Topology and Groupoids Exercises

Chapter 2, Section 5

Continuity

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October 5, 2018

**Problem 5.3.** Let $X = A \cup B, Y$ be topological spaces and let $f : X \to Y$ be a function such that $f|A, f|B$ are continuous Prove that $f$ is continuous if

$$\overline{(A \setminus B)} \cap (B \setminus A) = \varnothing$$
$$(A \setminus B) \cap \overline{(B \setminus A)} = \varnothing.$$

*Proof.* First, note that

$$\begin{aligned} X \setminus B &= (A \cap B) \setminus B \\ &= A \setminus B, \end{aligned}$$

and so we have that $x \in \overline{(A \setminus B)} = \overline{(X \setminus B)}$ if and only if every neighborhood of $x$ meets $X \setminus B$; that is, if and only if no neighborhood of $x$ is entirely contained within $B$, if and only if $x \notin \text{Int}B$, if and only if $x \in (X \setminus \text{Int}B)$. Hence, $\overline{(A \setminus B)} = X \setminus \text{Int}B$. Consequently, we have

$$\begin{aligned} \varnothing &= \overline{(A \setminus B)} \cap (B \setminus A) \\ &= (X \setminus \text{Int}B) \cap (B \setminus A). \end{aligned}$$

It then follows that if $x \in (B \setminus A)$, then $x \notin (X \setminus \text{Int}B)$; that is, $x \in \text{Int}B$. Therefore, $(B \setminus A) \subseteq \text{Int}B$. A similar argument shows that $(A \setminus B) \subseteq \text{Int}A$.

It then follows from the "gluing rule" (2.5.11) that $f$ is continuous. $\square$

**Problem 5.4.** Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. Prove that the following functions $x \to \mathbb{R}$ are maps.

(a) $x \mapsto |f(x)|$

(b) $x \mapsto f(x)/g(x)$ (if $g(x)$ is never 0)

(c) $x \mapsto \max\{f(x), g(x)\}, x \mapsto \min\{f(x), g(x)\}$

*Solution.*

(a) First, consider the function $h : \mathbb{R} \to \mathbb{R}$ that maps $x \mapsto |x|$. Let $A = (-\infty, 0]$ and $B = [0, \infty)$. Then note that $A$ and $B$ are both closed, $\mathbb{R} = A \cap B$, and the restrictions of $h$ to $A$ and $B$ ($h|_A(x) = -x, h|_B(x) = x$) are continuous. It then follows from the "gluing rule" that $h$ is continuous. Note, then, that the function $x \mapsto |f(x)|$ is just the composition $hf$, and so the function is continuous since $h$ and $f$ are continuous.

(b) Note that the function $x \mapsto 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$, and so the function $x \mapsto f(x)/g(x)$ is continuous since it can be written as a composition of continuous maps

$$\begin{aligned} x &\mapsto (f(x), g(x)) \\ &\mapsto \left(f(x), \frac{1}{g(x)}\right) \\ &\mapsto (f(x))\left(\frac{1}{g(x)}\right) \\ &= \frac{f(x)}{g(x)}. \end{aligned}$$

$\blacksquare$