

Algebra: Chapter 0 Exercises

Chapter 3, Section 3

David Melendez

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Problem 3.1. Prove that the image of a ring homomorphism $\varphi : R \rightarrow S$ is a subring of S . What can you say about φ if its image is an ideal of S ? What can you say about φ if its kernel is a subring of R ?

Solution. First we'll prove that $\text{im } \varphi$ is a subring of S .

Proof. Suppose $s_1 = \varphi(r_1)$ and $s_2 = \varphi(r_2)$ are elements of $\text{im } \varphi$. We then have $s_1 + s_2 = \varphi(r_1 + r_2)$ and $s_1 s_2 = \varphi(r_1 r_2)$ since φ is a homomorphism, so both of these are elements of $\text{im } \varphi$. Additionally, $\varphi(1_R) = 1_S$, making $\text{im } \varphi$ a subring of S . \square

If $\text{im } \varphi$ is an ideal of S , then φ is surjective, since the only ideal of S containing the identity 1_S is S itself. If $\ker \varphi$ is a subring of R , then it must contain 1_R , which, combined with the fact that $\ker \varphi$ is an ideal, tells us that $\ker \varphi = R$. Thus φ must be the "zero" morphism $r \mapsto 0$, which isn't actually a ring homomorphism since it does not preserve the identity. \blacksquare

Problem 3.2. Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of S . Prove that $I = \varphi^{-1}(J)$ is an ideal of R .

Solution. Suppose $x \in I$ and $r \in R$. We then have $\varphi(rx) = \varphi(r)\varphi(x)$, which is in J since J is an ideal and $\varphi(x) \in J$. The same argument applies to xr (as J is a two-sided ideal), so I is an ideal of R . \blacksquare

Problem 3.3. Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of R .

1. Show that $\varphi(J)$ need not be an ideal of S .

Proof. Let $R = \mathbb{C}$ and $S = \mathbb{H}$ (the quaternions), and let $\iota : \mathbb{C} \rightarrow \mathbb{H}$ be the inclusion $a + bi \mapsto a + bi$. The whole of \mathbb{C} is of course an ideal of \mathbb{C} , but the "copy" of \mathbb{C} in the quaternions $\iota(\mathbb{C})$ is not an ideal of \mathbb{H} , since $(a + bi)j = aj + bk \notin \iota(\mathbb{C})$. \square

2. Assume that φ is surjective; then prove that $\varphi(J)$ is an ideal of S .

Proof. We already know that $\varphi(J)$ is a subgroup of S since J is a subgroup of R , so let $s \in S$ and $i \in \varphi(J)$. There then exists a $j \in J$ such that $i = \varphi(j)$, and since φ is surjective, there exists an $r \in R$ such that $s = \varphi(r)$. Note, then, that

$$\begin{aligned} si &= \varphi(r)\varphi(j) \\ &= \varphi(rj) \\ &\in \varphi(J), \end{aligned}$$

since rj is in J due to the fact that J is an ideal; hence $\varphi(J)$ is a left-ideal in S . A similar argument shows that $\varphi(J)$ is also a right-ideal in S . \square

3. Assume that φ is surjective, and let $I = \ker \varphi$; thus we may identify S with R/I . Let $\bar{J} = \varphi(J)$, an ideal of R/I by the previous point. Prove that

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I+J}.$$

Proof. Denote by ψ the surjective ring homomorphism $R \rightarrow \frac{S}{\bar{J}}$ defined by the following chain of homomorphisms:

$$R \longrightarrow \frac{R}{I} \longrightarrow \frac{R/I}{\tilde{\varphi}^{-1}(\bar{J})} \xrightarrow{\tilde{\iota}} \frac{S}{\bar{J}}$$

where $\tilde{\varphi}$ is the isomorphism $r + I \mapsto \varphi(r)$, and $\tilde{\iota}$ is the isomorphism $(r + I) + \tilde{\varphi}^{-1}(\bar{J}) \mapsto \tilde{\varphi}(r + I) + \tilde{\varphi}(\tilde{\varphi}^{-1}(\bar{J})) = \varphi(r) + \bar{J}$. Hence ψ is defined by $\psi(r) = \varphi(r) + \bar{J}$. Note, then, that $r \in \ker \psi$ if and only if $\varphi(r) \in \varphi(J)$, if and only if there exists a $j \in R$ such that $\varphi(r) = \varphi(j)$, or equivalently $\varphi(r - j) = 0$, which is true if and only if there exists some $\nu \in \ker \varphi = I$ such that $r - j = \nu$ (equivalently $r = \nu + j$), if and only if $r \in I + J$.

Thus, by the first isomorphism theorem for rings, we have:

$$\frac{R}{I+J} \cong \frac{S}{\bar{J}} \cong \frac{R/I}{\tilde{\varphi}^{-1}(\bar{J})}.$$

If we identify $\tilde{\varphi}^{-1}(\bar{J})$ with \bar{J} in the last quotient ring (such an identification can be done in good conscience since doing so using any isomorphism between R/I and S yields isomorphic quotient rings), we can then say that

$$\frac{R}{I+J} \cong \frac{R/I}{\bar{J}}.$$

\square

Problem 3.4. Let R be a ring such that every subgroup of $(R, +)$ is in fact an ideal of R . Prove that $R \cong \mathbb{Z}/n\mathbb{Z}$, where n is the characteristic of R .

Solution. Since every subgroup of R is an ideal of R , note that in particular, the subgroup $I = \langle 1_R \rangle$ generated by the identity element is an ideal of R . Note, then, that for all $r \in R$, we have $r1_R = r \in I$, since $1_R \in I$, and so R is actually cyclic, with order equal to the order of 1_R ; in other words, the characteristic n of R . The unique map $\varepsilon : \mathbb{Z} \rightarrow R$ is then surjective (since R is generated by 1_R as a group) and has kernel $n\mathbb{Z}$; hence, by the first isomorphism theorem for rings, we have $R \cong \mathbb{Z}/n\mathbb{Z}$. ■

Problem 3.5. Let J be a two-sided ideal of the ring $\mathcal{M}_n(R)$ of $n \times n$ matrices over a ring R . Prove that a matrix $A \in \mathcal{M}_n(R)$ belongs to J if and only if the matrices obtained by placing any entry of A in any position, and 0 elsewhere, belong to J .

Solution. First suppose that $A \in J$. For natural numbers i, j, a, b less than n , We will "find" the matrix B in J with A_{ij} at position a, b .

Let $\eta(p, q)$ the matrix with 1 in the entry at position (q, p) and 0 elsewhere, and let $B = \eta(a, i)A\eta(j, b)$. Let δ be the kronecker delte, and note, then, that

$$\begin{aligned} B_{xy} &= \sum_{k=1}^n \eta(a, i)_{xk} (A\eta(j, b))_{ky} \\ &= \delta_{xa} (A\eta(j, b))_{iy} \\ &= \delta_{xa} \sum_{k=1}^n A_{ik} \eta(j, b)_{ky} \\ &= \delta_{xa} \delta_{yb} A_{ij}; \end{aligned}$$

hence B is the matrix with A_{ij} at position (a, b) and 0 elsewhere. Since B was obtained by multiplying A on the left and the right by other matrices, it is an element of J , as J is a two-sided ideal. This completes the proof in one direction.

For the proof in the other direction, suppose the matrices obtained by placing any entry of A in ny position, and 0 elsewhere, belong to J . Then, of course, A is the sum of the matrices that have A_{ij} at position (i, j) where i, j range from 1 to $n - 1$; since J is a subgroup of R , this matrix is in J . ■

Problem 3.6. Let J be a two-sided ideal of the ring $\mathcal{M}_n(R)$ of $n \times n$ matrices over a ring R , and let $I \subseteq R$ be the set of $(1, 1)$ entries in J . Prove that I is a two-sided ideal of R and J consists precisely of those matrices whose entries all belong to I .

Solution. First we will prove that I is a two-sided ideal of R . Suppose $r \in I$, and $a \in R$. By exercise 3.5, then, the matrix $r \cdot \eta(1, 1)$ is in J , and so $(r \cdot \eta(1, 1))(a \cdot \eta(1, 1)) = (ra \cdot \eta(1, 1)) \in J$ since J is a right-ideal of $\mathcal{M}_n(R)$, and so $ra \in I$ by the definition of I . Therefore I is a right ideal of R . The same argument can be used to conclude that I is also a left-ideal of R , since J is a left-ideal of $\mathcal{M}_n(R)$.

For the second part of the exercise, suppose first that $A \in J$. Then, by exercise 3.5, we

know that for any integers i, j between 1 and $n - 1$, there is a matrix in J with A_{ij} at entry $(1, 1)$. Thus, $A_{ij} \in I$.

Conversely, suppose A is a matrix whose entries all belong to I . Then, for each entry A_{ij} of A , the matrix $A_{ij} \cdot \eta(i, j)$ is in J by the definition of I and exercise 3.5, so their sum A must also be in J as J is closed under addition (due to it being an ideal). Therefore J consists precisely of those matrices whose entries all belong to I . ■

Problem 3.7. Let R be a ring, and let $a \in R$. Prove that Ra is a left-ideal of R and aR is a right-ideal of R . Prove that a is a left-, resp. right-, unit if and only if $R = aR$, resp. $R = Ra$.

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