Algebra: Chapter 0 Exercises

Chapter 1, Section 4

David Melendez

February 9, 2017

Problem 4.1. Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad etc.$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses.

Solution. Let $Z_m \in \mathrm{Obj}(C)$ and $f_m \in \mathrm{Hom}(Z_{m+1}, Z_m)$ for every $m \in \mathbb{N}$. Let n be the number of morphisms we're composing. We will use induction on n.

Base case: Suppose $n = 3$. Then, since C is a category, we have $f_1(f_2 f_3) = (f_1 f_2) f_3$.

Induction: Suppose that all parenthesizations of $f_1, \dots f_{j-1}$ under composition are equivalent for all $1 \leq j < n$. Then, for some $1 < k \leq n$, let α be some parenthesization of $f_1, \dots, f_{k-1}$, and let β be some parenthesization of $f_k, \dots, f_n$. Any parenthesization of $f_1, \dots, f_n$ will then be of the form $\alpha\beta$. By associativity and our inductive hypothesis, we have $\alpha = ((f_k \dots f_{n-1}) f_n)$, and so

$$\begin{aligned}
\alpha\beta &= (f_1 \dots f_{k-1}) \left((f_k \dots f_{n-1}) f_n\right) \\
&= \left((f_1 \dots f_{k-1})(f_k \dots f_{n-1})\right) f_n \\
&= ((\dots((f_1 f_2) f_3) \dots) f_n
\end{aligned}$$

as desired. ∎

Problem 4.2. In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid?

Solution. Recall that a *groupoid* is a category in which every morphism is an isomorphism. ■