# Algebra: Chapter 0 Exercises
Chapter 2, Section 6

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**Problem 6.2.** Prove that the set of upper-triangular matrices form a subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

*Solution.* Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$, and let $\mathcal{M} : \mathcal{L}(V) \to \mathbb{C}^{n,n}$ be the isomorphism that takes a linear operator to its matrix with respect to some basis

$$\mathcal{B} = \{v_1, \ldots, v_n\}.$$

Let $A, B$ be upper-triangular matrices, and let $S, T$ (respectively) be $\mathcal{M}^{-1}(A)$ and $\mathcal{M}^{-1}(B)$. We then know, due to a theorem in Axler, that $\mathrm{span}(v_1, \ldots, v_k)$ is invariant under $S$ and $T$ for $1 \leq k \leq n$ (this property is equivalent to the matrix being upper-triangular). Using this property and the invertibility of $T$, it then follows that

$$\begin{aligned} T^{-1}(a_1 v_1 + \cdots + a_k v_k) &= T^{-1}(T(b_1 v_1 + \cdots + b_k v_k)) \\ &= b_1 v_1 + \cdots + b_k v_k \end{aligned}$$

meaning $\mathrm{span}(v_1, \ldots, v_k)$ is invariant under $T^{-1}$, and hence that $T^{-1}$ is upper-triangular. Similarly,

$$\begin{aligned} (ST^{-1})(a_1 v_1 + \cdots + a_k v_k) &= S(b_1 v_1 + \cdots + b_k v_k) \\ &= c_1 v_1 + \cdots + c_k v_k, \end{aligned}$$

making $ST^{-1}$ upper-triangular and completing the proof. ■

**Problem 6.4.** Let $G$ be a commutative group, and let $n > 0$ be an integer. Prove that $\{g^n | g \in G\}$ is a subgroup of $G$. Prove that this is not necessarily the case if $G$ is not commutative.

*Solution.* Let $G$ be a commutative group, and let $G' = \{g^n | g \in G\}$ where $n$ is any positive integer. To prove that this is a group, suppose $g = g_1^n$ and $h = g_2^n$ are elements of $G'$. We then have:

$$\begin{aligned} gh^{-1} &= (g_1^n)(g_2^n)^{-1} \\ &= (g_1)^n (g_2^{-1})^n \\ &= (g_1 g_2^{-1})^n \\ &\in G' \end{aligned}$$

Hence $G'$ is a subgroup of $G$.
As a counterexample in the case that $G$ is not commutative, let $G = F(\{x, y\})$, the free group generated by $x$ and $y$, and let $n = 2$. In order for $G'$ to be closed under its operation, we would need to have $g \in G$ such that

$$g^2 = x^2 y^2.$$

That such a $g$ does not exist is turning out to be harder to prove than I suspected, so I'll come back to this later. ■

**Problem 6.6.**

1. Let $H, H'$ be subgroups of a group $G$. Prove that $H \cup H'$ is a subgroup of $G$ only if $H \subseteq H'$ or $H' \subseteq H$.

2. On the other hand, let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be subgroups of a group $G$. Prove that $G' = \bigcup_{i \geq 0} H_i$ is a subgroup of $G$.

*Solution.*

1. Suppose $H \cup H'$ is a subgroup of G. We then have, by closure, for all $h \in h$ and $h \in h'$, that $hh' = g$ for some $g \in H \cup H'$. If $g \in H$, we have $h' = h^{-1}g \in H$, meaning $H' \subseteq H$. Alternatively, if $g \in H'$, we have $h = g(h')^{-1} \in H'$ meaning $H \subseteq H'$.

2. Let $h_1 \in H_j$ and $h_2 \in H_k$ (both in $G'$, of course), and assume without loss of generality that $j \leq k$. By the sequence of subset relations, we know that $h_1 \in H_k$, so
$h_1 h_2^{-1} \in H_k \subseteq G'$, completing the proof.

■

**Problem 6.8.** Prove that an abelian group $G$ is finitely generated if and only if there is a surjective homomorphism

$$\bigoplus_{i=1}^{n} \mathbb{Z} \twoheadrightarrow G$$

for some $n$.

*Solution.* First, suppose that an abelian group $G$ is finitely generated. This, by definition, means that there exists a finite subset $A$ of $G$ such that $\langle A \rangle = G$; in other words, $G$ is the image of the homomorphism $\varphi_A$ obtained by applying the universal property for the free abelian group over $A$ as follows:

$$\begin{tikzcd} F^{ab}(A) \arrow[r, dashed, "\varphi_A"] & G \\ A \arrow[u, "j"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

where $\iota$ and $j$ are the inclusion maps into $G$ and $F^{ab}(A)$, respectively. We know by exercise 5.7 that $Z = \bigoplus_{i=1}^{n} \mathbb{Z} \cong F^{ab}(A)$, so we have a surjection

$$Z \xrightarrow{\sim} F^{ab}(A) \xrightarrow{\varphi_A} G$$

For the proof in the other direction, suppose we have a surjective homomorphism $\varphi : Z \to G$. For integers $0 \leq m \leq n$, let $\beta_m$ be the $n$-tuple with 0 in every slot except for the $m$th slot, where there is a 1. Define $A$ to be the set $\{\varphi(\beta_1), \ldots, \varphi(\beta_n)\}$ (since the coproduct and product in **Ab** are the same), and let $a_m$ be the $m$th element of $A$ as listed above. Define the isomorphism $i : F^{ab}(A) \to Z$ by

$$i(m_1 a_1 + \cdots + m_n a_n) = (m_1, \ldots, m_n),$$

and let $f : F^{ab}(A) \to G$ be defined by $f = i\varphi$. Finally, let $j$ and $\iota$ be the standard inclusions into $F^{ab}(A)$ and $G$, respectively.. the following diagram illustrates these morphisms:

$$\begin{tikzcd}
 & Z \arrow[dr, "\varphi"] & \\
F^{ab}(A) \arrow[ur, "i"] \arrow[rr, dashed, "\varphi_A"] & & G \\
A \arrow[u, "j"] \arrow[urr, "\iota"'] & &
\end{tikzcd}$$

Define $\alpha_m$ to be $j(a_m)$, let $a = m_1 a_1 + \cdots + m_n a_n$, and let $\alpha = j(a)$. We then have:

$$\begin{aligned}
(f \circ j)(a) &= f(\alpha) \\
&= (\varphi \circ i)(m_1 \alpha_1 + \cdots + m_n \alpha_n) \\
&= \varphi(m_1, \ldots, m_n) \\
&= m_1 \varphi(\beta_1) + \cdots + m_n \varphi(\beta_n) \\
&= m_1 \iota(a_1) + \cdots + m_n \iota(a_n) \\
&= \iota(a)
\end{aligned}$$

Since the morphism $\varphi_A$ is the only morphism that satisfies this property (by the universality of $F^{ab}(A)$, we know that $f = \varphi_A$. But $f$ (and hence $\varphi_A$) is surjective, giving us $G = \operatorname{im}(\varphi_A) = \langle A \rangle$, as desired. ■