Algebra: Chapter 0 Exercises

Chapter 3, Section 4

Ideals and quotients: Remarks and examples. Prime and maximal ideals

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Problem 4.1. Let R be a ring, and let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals of R. We let

$$\sum_{\alpha \in A} I_\alpha = \left\{ \sum_{\alpha \in A} r_\alpha \text{ such that } r_\alpha \in I_\alpha \text{ and } r_\alpha = 0 \text{ for all but finitely many } \alpha \right\}$$

Prove that $J = \sum_\alpha I_\alpha$ is an ideal of R and that it is the smallest ideal containing all of the ideals I_α.

Solution. First we prove that J is an ideal of R.

Proof. Let $a, b \in J$, so that

$$a = \sum_{\alpha \in A} r_\alpha$$
$$b = \sum_{\alpha \in A} s_\alpha,$$

where each $r_\alpha, s_\alpha \in I_\alpha$ and all but finitely many r_α and s_α are nonzero. We then have:

$$\begin{aligned} a + b &= \sum_{\alpha \in A} r_\alpha + \sum_{\alpha \in A} s_\alpha \\ &= \sum_{\alpha \in A} r_\alpha + s_\alpha. \end{aligned}$$

Each term $r_\alpha + s_\alpha$ is in I_α since $r_\alpha, s_\alpha \in I_\alpha$ and I_α is an ideal, and clearly all but finitely many $r_\alpha + s_\alpha$ are nonzero since $(r_\alpha)_{\alpha \in A}$ and $(s_\alpha)_{\alpha \in A}$ both have that property, so $a + b \in J$.

Additionally, if $s \in R$ and $r \in J$ so that $r = \sum_{\alpha \in A} r_\alpha$ (where all but finitely many r's are zero), then we have

$$\begin{aligned} rs &= \left(\sum_{\alpha \in A} r_\alpha \right) s \\ &= \sum_{\alpha \in A} r_\alpha s \\ &\in J, \end{aligned}$$

where the last line is true because each $r_\alpha s \in I_\alpha$ as a result of each I_α being a right-ideal of R, and the fact that if r_α is zero then $r_\alpha s$ is also zero, implying that there are cofinitely many zero terms in this resulting sum as well. A similar argument shows that J is a left-ideal of R if each I_α is also a left-ideal. □

Now, we will show that $J = \sum_{\alpha \in A} I_\alpha$ is the smallest ideal of R containing each of the ideals I_α for $\alpha \in A$.

Proof. We just proved that J is an ideal of R, so now we just need to show that J is a subset of any ideal containing each of the ideals I_α. This is immediate: if $r \in J$ is such that $r = \sum_{\alpha \in A} r_\alpha$ for $r_\alpha \in I_\alpha$, then of course any ideal of R containing each I_α contains r, since such an ideal is closed under addition. □

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Problem 4.2. Prove that the homomorphic image of a Noetherian ring is Noetherian. That is, prove that if $\varphi : R \to S$ is a surjective ring homomorphism and R is Noetherian, then S is Noetherian.

Solution. Suppose $I = (a_1, \ldots, a_n)$ is an ideal of R and $\varphi : R \to S$ is surjective. Then we have

$$\begin{aligned} \varphi(I) &= \varphi\left(\sum_{i=1}^{n}(a_i)\right) \\ &= \sum_{i=1}^{n}\varphi((a_i)) \\ &= \sum_{i=1}^{n}(\varphi(a_i)), \end{aligned}$$

and so $\varphi(I)$ is finitely generated.

To see that these operations are justified, note that if $g \in R$ and $J = (g)$ is an ideal, then we have

$$\begin{aligned} \varphi(J) &= \varphi\left(\{rg : r \in R\}\right) \\ &= \{\varphi(r)\varphi(g) : r \in R\} \\ &= \{r\varphi(g) : r \in R\} \\ &= (\varphi(g)), \end{aligned}$$

where the third equality follows from the surjectivity of φ.

Additionally, if I, J are ideals of R, then we also have

$$\begin{aligned} \varphi(I + J) &= \varphi(\{i + j : i \in I, j \in J\}) \\ &= \{\varphi(i) + \varphi(j) : i \in I, j \in J\} \\ &= \varphi(I) + \varphi(J) \end{aligned}$$

Note, then, that if J is an ideal of S, then $\varphi^{-1}(J)$ is an ideal of R, allowing us to see that $J = \varphi(\varphi^{-1}(J))$ is finitely generated. Therefore, every ideal of S is finitely generated, and so S is Noetherian. ■

Problem 4.3. Prove that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal.

Solution. First, we (quite clumsily) compute the ideal $(2, x)$ as follows:

$$\begin{aligned}(2, x) &= \{2p + xq : p, q \in \mathbb{Z}[x]\} \\ &= \{(2a_0 + 2a_1x + \cdots + 2a_nx^n) + (b_1x + b_2x^2 + \cdots + b_mx^m) : a_j, b_j \in \mathbb{Z}\} \\ &= \{2a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_j \in \mathbb{Z}\}.\end{aligned}$$

In other words, the ideal $(2, x)$ consists of all the polynomials in $\mathbb{Z}[x]$ with an even constant term.

Note, then, if $(2, x) = (g)$ for some polynomial $g \in \mathbb{Z}[x]$, then there must be a polynomial $p \in \mathbb{Z}[x]$ such that $2 = gp$, since $2 \in (2, x)$. If this is the case, then we have $\deg g + \deg p = 0$, and so $\deg g = \deg p = 0$. This means that g is constant, and hence is either 1 or 2. In the former case, (g) is the whole ring $\mathbb{Z}[x]$, and in the latter case, (g) is the ideal $2\mathbb{Z}[x]$. Neither of these ideals equal the ideal of $(2, x)$, leading us to conclude that no single polynomial in $\mathbb{Z}[x]$ generates the ideal $(2, x)$. ■

Problem 4.4. Prove that if k is a field, then $k[x]$ is a PID. (Hint: Polynomial divison with remainder)

Solution. Let $I \subseteq k[x]$ be an ideal. If $I = 0 = (0)$, then clearly it is principal. Otherwise, let $p \in I$ be a monic polynomial of minimal degree d Let $I \subseteq k[x]$ be an ideal. If $I = 0 = (0)$, then clearly it is principal.

Otherwise, let $g \in I$ be a monic polynomial of minimal degree d. If $p \in I$, then we can apply division with remainder to find polynomials $q, r \in k[x]$ such that

$$p = gq + r,$$

where $\deg r < d$. Note that since $p \in I$ and $gq \in I$ by (right-) absorption, we then can see that $r = p - gq \in I$, since I is closed under addition. But d is the smallest degree of any nonzero polynomial in I and $\deg r < d$; it then follows that $r = 0$, and so

$$p = gq,$$

showing us that $I \subseteq (g)$.

Of course $(g) \subseteq I$, so we then have $I = (g)$, as desired. ■

Problem 4.5. Let I, J be ideals in a commutative ring R, such that $I + J = (1)$. Prove that $IJ = I \cap J$.

Solution. The simple fact that $IH \subseteq I \cap J$ was already proven in the text, so suppose $r \in I \cap J$. Since $I + J = (1) = R$, we know there exists an $i \in I$ and a $j \in J$ such that $1 = i + j$. Note, then, that:

$$\begin{aligned}r &= r \cdot 1 \\ &= r \cdot (i + j) \\ &= r \cdot i + r \cdot j.\end{aligned}$$

Since $r \in J$ and R is commutative, we know that $ri \in IJ$, and since $r \in J$, we also know that $rj \in IJ$. Hence $r = ri + rj \in IJ$, and so $I \cap J \subseteq IJ$. Therefore, $IJ = I \cap J$, as desired. ■

Problem 4.6. Let I, J be ideals in a commutative ring R. Assume that $R/(IJ)$ is reduced (that is, it has no nonzero nilpotent elements). Prove that $IJ = I \cap J$.

Solution. We will proceed by proving the contrapositive. Since we already know that $IJ \subseteq I \cap J$ for any ideals I, J, assume that $I \cap J \not\subseteq IJ$. There then exists an $r \in I \cap J$ with $r \notin IJ$; that is, such that the coset $r + (IJ)$ is nonzero in $R/(IJ)$. Note, then, that $r^2 = rr \in IJ$, since $r \in I$ and $r \in J$, and so $(r + IJ)^2 = 0$ in the ring $R/(IJ)$. Hence $R/(IJ)$ is not reduced, as desired.

Therefore, if $R/(IJ)$ is reduced, then $I \cap J \subseteq IJ$, and therefore $I \cap J = IJ$. ■

Problem 4.7. Let $R = k$ be a fiels. Prove that every nonzero (principal) ideal in $k[x]$ is generated by a unique monic polynomial.

Solution. Let I be an ideal of $k[x]$. Then, by exercise 4.4, there is a monic polynomial $p_1 \in k[x]$ such that $I = (p_1)$. Let $p_2 \in k[x]$ be such that $(p_2) = (p_1) = I$. Then, since $p_1 \in (p_2)$ and $p_2 \in (p_2)$, there exist q_1 and q_2 such that

$$p_1 = q_1 p_2$$

and

$$p_2 = q_2 p_1.$$

We then have

$$p_1 = q_1 q_2 p_1,$$

and hence $q_1 q_2 = 1$, as $k[x]$ is an integral domain.

Note, then, that

$$\begin{aligned} 0 &= \deg(q_1 q_2) \\ &=^1 \deg(q_1) + \deg(q_2), \end{aligned}$$

where equality (1) follows from $k[x]$ being an integral domain and thus having no nonzero zero divisors. Therefore, q_1 and q_2 are both degree 0, that is, constants.

It then follows that if p_2 is monic, then $q_1 = 1$ and so $p_1 = p_2$, showing that I is generated by a unique *monic* polynomial as desired. ■