Algebra: Chapter 0 Exercises

Chapter 3, Section 3

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Problem 3.1. Prove that the image of a ring homomorphism $\varphi : R \to S$ is a subring of S. What can you say about φ if its image is an ideal of S? What can you say about φ if its kernel is a subring of R?

Solution. First we'll prove that im φ is a subring of S.

Proof. Suppose $s_1 = \varphi(r_1)$ and $s_2 = \varphi(r_2)$ are elements of im φ. We then have $s_1 + s_2 = \varphi(r_1 + r_2)$ and $s_1 s_2 = \varphi(r_1 r_2)$ since φ is a homomorphism, so both of these are elements of im φ. Additionally, $\varphi(1_R) = 1_S$, making im φ a subring of S. □

If im φ is an ideal of S, then φ is surjective, since the only ideal of S containing the identity 1_S is S itself. If $\ker \varphi$ is a subring of R, then it must contain 1_R, which, combined with the fact that $\ker \varphi$ is an ideal, tells us that $\ker \varphi = R$. Thus φ must be the "zero" morphism $r \mapsto 0$, which isn't actually a ring homomorphism since it does not preserve the identity. ■

Problem 3.2. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of S. Prove that $I = \varphi^{-1}(J)$ is an ideal of R.

Solution. Suppose $x \in I$ and $r \in R$. We then have $\varphi(rx) = \varphi(r)\varphi(x)$, which is in J since J is an ideal and $\varphi(x) \in J$. The same argument applies to xr (as J is a two-sided ideal), so I is an ideal of R. ■

Problem 3.3. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of R.

1. Show that $\varphi(J)$ need not be an ideal of S.

   Proof. Let $R = \mathbb{C}$ and $S = \mathbb{H}$ (the quaternions), and let $\iota : \mathbb{C} \to \mathbb{H}$ be the inclusion $a + bi \mapsto a + bi$. The whole of $\mathbb{C}$ is of course an ideal of $\mathbb{C}$, but the "copy" of $\mathbb{C}$ in the quaternions $\iota(\mathbb{C})$ is not an ideal of $\mathbb{H}$, since $(a + bi)j = aj + bk \notin \iota(\mathbb{C})$. □

2. Assume that φ is surjective; then prove that $\varphi(J)$ *is* an ideal of S.

Proof. We already know that $\varphi(J)$ is a subgroup of S since J is a subgroup of R, so let $s \in S$ and $i \in \varphi(J)$. There then exists a $j \in J$ such that $i = \varphi(j)$, and since φ is surjective, there exists an $r \in R$ such that $s = \varphi(r)$. Note, then, that

$$\begin{aligned} si &= \varphi(r)\varphi(j) \\ &= \varphi(rj) \\ &\in \varphi(J), \end{aligned}$$

since rj is in J due to the fact that J is an ideal; hence $\varphi(J)$ is a left-ideal in S. A similar argument shows that $\varphi(J)$ is also a right-ideal in S. $\square$

3. Assume that φ is surjective, and let $I = \ker\varphi$; thus we may identify S with R/I. Let $\overline{J} = \varphi(J)$, an ideal of R/I by the previous point. Prove that

$$\frac{R/I}{\overline{J}} \cong \frac{R}{I+J}.$$

Proof. Denote by ψ the surjective ring homomorphism $R \to \dfrac{S}{\overline{J}}$ defined by the following chain of homomorphisms:

$$R \twoheadrightarrow \frac{R}{I} \twoheadrightarrow \frac{R/I}{\widetilde{\varphi}^{-1}(\overline{J})} \xrightarrow{\bar{\iota}} \frac{S}{\overline{J}}$$

where $\widetilde{\varphi}$ is the isomorphism $r + I \mapsto \varphi(r)$, and $\bar{\iota}$ is the isomorphism $(r + I) + \widetilde{\varphi}^{-1}(\overline{J}) \mapsto \widetilde{\varphi}(r + I) + \widetilde{\varphi}(\widetilde{\varphi}^{-1}(\overline{J})) = \varphi(r) + \overline{J}$. Hence ψ is defined by $\psi(r) = \varphi(r) + J$. Note, then, that $r \in \ker\psi$ if and only if $\varphi(r) \in \varphi(J)$, if and only if there exists a $j \in R$ such that $\varphi(r) = \varphi(j)$, or equivalently $\varphi(r - j) = 0$, which is true if and only if there exists some $\nu \in \ker\varphi = I$ such that $r - j = \nu$ (equivalently $r = \nu + j$), if and only if $r \in I + J$.
Thus, by the first isomorphism theorem for rings, we have:

$$\frac{R}{I+J} \cong \frac{S}{\overline{J}} \cong \frac{R/I}{\widetilde{\varphi}^{-1}(\overline{J})}.$$

If we identify $\widetilde{\varphi}^{-1}(\overline{J})$ with $\overline{J}$ in the last quotient ring (such an identification can be done in good conscience since doing so using any isomorphism between R/I and S yields isomorphic quotient rings), we can then say that

$$\frac{R}{I+J} \cong \frac{R/I}{\overline{J}}.$$

$\square$

Problem 3.4. Let R be a ring such that every subgroup of $(R, +)$ is in fact an ideal of R. Prove that $R \cong \mathbb{Z}/n\mathbb{Z}$, where n is the characteristic of R.

Solution. Since every subgroup of R is an ideal of R, note that in particular, the subgroup $I = \langle 1_R \rangle$ generated by the identity element is an ideal of R. Note, then, that for all $r \in R$, we have $r1_R = r \in I$, since $1_R \in I$, and so R is actually cyclic, with order equal to the order of 1_R; in other words, the characteristic n of R. The unique map $\varepsilon : \mathbb{Z} \to R$ is then surjective (since R is generated by 1_R as a group) and has kernel $n\mathbb{Z}$; hence, by the first isomorphism theorem for rings, we have $R \cong \mathbb{Z}/n\mathbb{Z}$. ■

Problem 3.5. Let J be a two-sided ideal of the ring $\mathcal{M}_n(R)$ of $n \times n$ matrices over a ring R. Prove that a matrix $A \in \mathcal{M}_n(R)$ belongs to J if and only if the matrices obtained by placing any entry of A in any position, and 0 elsewhere, belong to J.

Solution. First suppose that $A \in J$. For natural numbers i, j, a, b less than n, We will "find" the matrix B in J with A_{ij} at position a, b.

Let $\eta(p, q)$ the matrix with 1 in the entry at position (q, p) and 0 elsewhere, and let $B = \eta(a, i)A\eta(j, b)$. Let δ be the kronecker delte, and note, then, that

$$\begin{aligned}
B_{xy} &= \sum_{k=1}^{n} \eta(a, i)_{xk}(A\eta(j, b))_{ky} \\
&= \delta_{xa}(A\eta(j, b))_{iy} \\
&= \delta_{xa} \sum_{k=1}^{n} A_{ik}\eta(j, b)_{ky} \\
&= \delta_{xa}\delta_{yb}A_{ij};
\end{aligned}$$

hence B is the matrix with A_{ij} at position (a, b) and 0 elsewhere. Since B was obtained by multiplying A on the left and the right by other matrices, it is an element of J, as J is a two-sided ideal. This completes the proof in one direction.

For the proof in the other direction, suppose the matrices obtained by placing any entry of A in ny position, and 0 elsewhere, belong to J. Then, of course, A is the sum of the matrices that have A_{ij} at position (i, j) where i, j range from 1 to $n - 1$; since J is a subgroup of R, this matrix is in J. ■

Problem 3.6. Let J be a two-sided ideal of the ring $\mathcal{M}_n(R)$ of $n \times n$ matrices over a ring R, and let $I \subseteq R$ be the set of $(1, 1)$ entries in J. Prove that I is a two-sided ideal of R and J consists precisely of those matrices whose entries all belong to I.

Solution. First we will prove that I is a two-sided ideal of R. Suppose $r \in I$, and $a \in R$. By exercise 3.5, then, the matrix $r \cdot \eta(1, 1)$ is in J, and so $(r \cdot \eta(1, 1))(a \cdot \eta(1, 1)) = (ra \cdot \eta(1, 1)) \in J$ since J is a right-ideal of $\mathcal{M}_n(R)$, and so $ra \in I$ by the definition of I. Therefore I is a right ideal of R. The same argument can be used to conclude that I is also a left-ideal of R, since J is a left-ideal of $\mathcal{M}_n(R)$.

For the second part of the exercise, suppose first that $A \in J$. Then, by exercise 3.5, we know that for any integers i, j between 1 and $n-1$, there is a matrix in J with A_{ij} at entry $(1,1)$. Thus, $A_{ij} \in I$.

Conversely, suppose A is a matrix whose entries all belong to I. Then, for each entry A_{ij} of A, the matrix $A_{ij} \cdot \eta(i,j)$ is in J by the definition of I and exercise 3.5, so their sum A must also be in J as J is closed under addition (due to it being an ideal). Therefore J consists precisely of those matrices whose entries all belong to I. ■

Problem 3.7. Let R be a ring, and let $a \in R$. Prove that Ra is a left-ideal of R and aR is a right-ideal of R. Prove that a is a left-, resp. right-, unit if and only if $R = aR$, resp $R = Ra$.

Solution. First we will prove that Ra is a left-ideal of R. Suppose $x \in Ra$ so that $x = ra$ for some $r \in R$. Then if $s \in R$, we have $sx = sra = (sr)a \in Ra$. Hence Ra is a left ideal of R. A similar argument shows that aR is a right-ideal of R.

For the second question, note that $R = aR$ (resp. $R = Ra$) if and only if left- resp. right-multiplication by a is surjective, if and only if a is a left- resp. right- ideal of R. ■

Problem 3.8. Prove that a ring R is a division ring if and only if the only left-ideals and right-ideals are $\{0\}$ and R.

In particular, a commutative ring R is a field if and only if the only ideals of R are $\{0\}$ and R.

Solution. Suppose R is a division ring, and I is a right-ideal of R. Of course $\{0\}$ is a right-ideal of R, so suppose $r \neq 0$ is an element of I. Then since R is a division ring, r has a two-sided inverse r^{-1}. Note, then, that since I is an ideal of R, we have $rr^{-1} = 1_R \in I$, and so $I = R$. The same argument applies if I is a left-ideal of R, completing the proof in one direction.

Conversely, suppose the only left- and right-ideals of R are $\{0\}$ and R itself. Then it follows from exercise 3.7 that for all nonzero $a \in R$, we have $aR = R$ and $Ra = R$ (since aR and Ra are nonzero ideals of R), and so a is a left- and right-unit in r; hence every element of R is a two-sided unit, and R is a divison ring. ■

Problem 3.9. Counterpoint to Exercise 3.8: It is *not* true that a ring R is a division ring if and only if its only two-sided ideals are $\{0\}$ and R. A nonzero ring with this property is said to be *simple*; by Exercise 3.8, fields are the only simple *commutative* rings.

Prove that $\mathcal{M}_n(\mathbb{R})$ is simple. (Use Exercise 3.6).

Solution. Suppose J is a nonzero two-sided ideal of $\mathcal{M}_n(\mathbb{R})$. Let α be a nonzero entry of a matrix in J. Then, by exercise 3.5, the matrix with α at the position $(1,1)$ and 0 elsewhere is in J. If we then multiply this matrix with the matrix that has α^{-1} at position $(1,1)$, we then find (using the fact that J is an ideal) that the matrix with 1 at $(1,1)$ and zero elsewhere is in J. Applying 3.5 again and the fact that J is closed under addition, we find that the identity matrix is in J, and so J is the whole of $\mathcal{M}_n(\mathbb{R})$. Therefore $\mathcal{M}_n(\mathbb{R})$ is simple. ■

Problem 3.10. Let $\varphi : k \to R$ be a ring homomorphism, where k is a field and R is a nonzero ring. Prove that φ is injective.

Solution. Suppose $\nu \in k$ is nonzero and $\varphi(\nu) = 0$. Then we have

$$\begin{aligned} 0 &= \varphi(\nu) \\ &= \varphi(\nu)\varphi(\nu^{-1}) \\ &= \varphi(\nu\nu^{-1}) \\ &= \varphi(1) \\ &= 1, \end{aligned}$$

which is a contradiction since R is nonzero. Hence $\nu = 0$ and φ is injective. ∎

Problem 3.11. Let R be a ring containing $\mathbb{C}$ as a subring. Prove that there are no ring homomorphisms $R \to \mathbb{R}$.

Solution. Since $\mathbb{C}$ is a subring of R, the element i is then in R. Note, then, that $i^4 = 1$, and so if φ is to be a homomorphism $R \to \mathbb{R}$, we must then have $\varphi(i^4) = 1$. Since this implies $\varphi(i)^4 = 1$, we then know that $\varphi(i)$ is either 1 or -1, since the only fourth roots of 1 in $\mathbb{R}$ are 1 and -1. Either way, we then have, since φ is a homomorphism, that $\varphi(i^2) = \varphi(i)^2 = 1$. But we also have $\varphi(i^2) = \varphi(-1) = -\varphi(1) = -1$, which implies that $1 = -1$. Since this is not true in R, we then know that φ is not a homomorphism, and so there are no ring homomorphisms from R to $\mathbb{R}$. ∎