Linear Algebra Done Right Exercises
Chapter 4

David Melendez

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Problem 5. Suppose m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \ldots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m+1$.

Proof. Let $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^X)$ where $X = \{z_1, \ldots, z_{m+1}\}$. Define T by

$$T(p)(x_j) = p(x_j)$$

It is easy to show that this transformation is linear.
Now we compute the null space of T. Note that when $Tp = 0$, we have that $p(x) = 0$ for all $x \in X$, and thus p has at least $|X| = m+1$ distinct zeroes. But since the degree of p is at most m, this must mean that p is the zero polynomial. Hence T is injective and $\dim \operatorname{null} T = 0$. We then have

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_m(\mathbb{F}) - \dim \operatorname{null} T \\ &= m+1 \\ &=^1 \dim \mathbb{F}^X. \end{aligned}$$

This shows that range $T = \mathbb{F}$, and therefore that T is an isomorphism between $\mathcal{P}_m(\mathbb{F})$ and $\mathbb{F}^X$. Since T^{-1} assigns a unique polynomial to each function (i.e. set of ordered pairs distinct in the first slot) as stated in the problem, this completes the proof. □

Problem 6. Suppose $p \in \mathcal{P}(\mathbb{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

Proof. We will prove the contrapositive. Suppose z is a zero of both p and p'. Note that

$$\begin{aligned} p(x) &= (x-z)q(x) \\ p'(x) &= q(x) + (x-z)q'(x) \end{aligned}$$

[1]See the following Stack Exchange post: https://math.stackexchange.com/questions/2288812/finding-dimension-of-a-vector-space-v/2289241#2289241

We then have

$$
\begin{aligned}
0 &= p'(z) \\
&= q(z)
\end{aligned}
$$

and hence z is a zero of q. This means that $(x - z)^2$ is a factor of p, and therefore it can have at most $m - 1$ distinct zeros. $\square$