Algebra: Chapter 0 Exercises Chapter 2, Section 3

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May 22, 2017

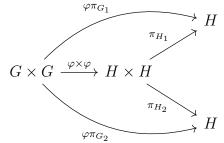
Problem 3.1. Let $\varphi: G \to H$ be a morphism in a category \mathbf{C} with products. Explain why there is a unique morphism

$$(\varphi \times \varphi): G \times G \to H \times H$$

compatible in the evident way with the natural projections.

Solution. I'm going to assume the author means the following:

Since $H \times H$ is a product in \mathbb{C} , we have that $G \times G$ along with two "copies" of $\varphi : G \to H$ admits a unique morphism $(\varphi \times \varphi) : G \times G \to H \times H$ that makes the following diagram commute:



where π_{G_1} and π_{G_2} are the canonical projections of $G \times G$ onto G.

Problem 3.2. Let $\varphi: G \to H$ and $\psi: H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G$, $H \times H$, and $K \times K$ as in Exercise 3.1. Prove that

$$(\psi\varphi)\times(\psi\varphi)=(\psi\times\psi)(\varphi\times\varphi)$$

Solution. To demonstrate this result, first stare at this diagram, mapping $G \times G$ to the product $H \times H$ and $H \times H$ to the product $K \times H$.

Compare it to the following diagram, mapping $G \times G$ straight to $K \times K$:

$$G \xrightarrow{\psi\varphi} K$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$G \times G \xrightarrow{(\psi\varphi)\times(\psi\varphi)} K \times K$$

$$\downarrow \qquad \qquad \downarrow$$

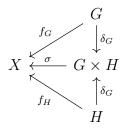
$$G \xrightarrow{\psi\varphi} K$$

Notice that we have morphisms $(\psi\varphi) \times (\psi\varphi)$ and $(\psi \times \psi)(\varphi \times \varphi)$ from $G \times G$ to $K \times K$, both determined by the universal property for products with regards to $K \times K$. These must be equal per Problem 3.1.

Problem 3.3. Show that if G and H are both *abelian* groups, then $G \times H$ satisfies the universal property for coproducts in \mathbf{Ab} .

Solution. That $G \times H$ satisfies the universal property for coproducts in \mathbf{Ab} means the following:

Every triple (X, δ_G, δ_H) (as in the diagram) admits a unique morphism σ such that the following diagram commutes:



That is,

$$\sigma \delta_G = f_G$$
$$\sigma \delta_H = f_H$$

Define $\sigma: G \times H \to X$ by

$$\sigma(g,h) = f_G(g) \cdot f_H(h)$$

Also define $\delta_G: G \to G \times H \ \delta_H: H \to G \times H$ by

$$\delta_G(g) = (g, e_H)$$

$$\delta_G(h) = (e_G, h)$$

The proof that these are homomorphisms is trivial. To show that σ makes the diagram commute, note that

$$\sigma(\delta_G(g)) = \sigma(g, e_H)$$

$$= f_G(g) \cdot f_H(e_H)$$

$$= f_G(h)$$

The same proof applies to H.

This is where our groups being abelian comes in. We will show that σ is a homomorphism:

$$\sigma((g_1, h_1)(g_2, h_2)) = \sigma(g_1 g_2, h_1 h_2)$$

$$= f_G(g_1 g_2) \cdot f_H(h_1 h_2)$$

$$= f_G(g_1) g_G(g_2) f_H(h_1) f_H(h_2)$$

$$= (f_G(g_1) f_H(h_1)) (f_G(g_2) f_h(h_2))$$

$$= \sigma(g_1, h_1) \cdot \sigma(g_2, h_2)$$

 ${\rm done}$