Algebra: Chapter 0 Exercises

Chapter 1, Section 5

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Problem 5.1. A final object in a category $\mathbf{C}$ is initial in the opposite category $\mathbf{C}^{op}$.

Proof. Let $T \in \text{Obj}(\mathbf{C})$ be initial and $Z \in \text{Obj}(\mathbf{C})$ be any object. Since $\text{Hom}_{\mathbf{C}}(T, Z)$ is a singleton, $\text{Hom}_{\mathbf{C}^{op}}(Z, T) = \text{Hom}_{\mathbf{C}}(T, Z)$ is also a singleton, making T final in $\mathbf{C}^{op}$ $\square$

Problem 5.2. The empty set $\emptyset$ is the *unique* initial object in **Set**.

Proof. Recall that the number of morphisms between any two sets A and B is given by B^A. Thus, the problem of finding the size of an initial object in **Set** boils down to solving $|Z|^{|I|} = 1$ for $|I|$, where Z is an arbitrary set. The only solution to this is $|I| = 0$, which is only satisfied by the the null set, given that $\emptyset^\emptyset = 0$. $\square$

Problem 5.3. Final objects are unique up to isomorphism.

Proof. Let $\mathbf{C}$ be a category and $T_1, T_2 \in \text{Obj}(\mathbf{C})$ be final. Because T_1 and T_2 are final, we have, for all $Z \in \text{Hom}_{\mathbf{C}}(Z, T)$:

$$\begin{aligned}
\text{End}(T_1) &= \{\text{id}_{T_1}\} \\
\text{End}(T_2) &= \{\text{id}_{T_2}\} \\
\text{Hom}(T_1, T_2) &= \{f\} \text{ for some } f \\
\text{Hom}(T_2, T_1) &= \{g\} \text{ for some } g
\end{aligned}$$

We then have $fg = \text{id}_{T_2}$ and $gf = \text{id}_{T_2}$, giving us $T_1 \cong T_2$. $\square$

Problem 5.4. What are the initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

Solution. For the sake of completeness, a description of this category $\mathbf{Set}^*$ will be included.

Definition. Define $\mathbf{Set}^*$ as follows:
Objects in $\mathbf{Set}^*$ are morphisms $f : \{*\} \to A$ where A is any set, denoted (f, A).
Morphisms $\sigma \in \mathrm{Hom}_{\mathbf{Set}^*}((f_A, A), (f_B, B))$ are commutative diagrams:

$$\begin{tikzcd} A \arrow[rr, "\sigma"] & & B \\ & \{*\} \arrow[ul, "f_A"] \arrow[ur, "f_B"'] & \end{tikzcd}$$

Now, we will answer the question.

Proposition. Initial and final objects in $\mathbf{Set}^*$ are objects (i, T), where T is a singleton and i is the unique function that maps from $\{*\}$ to T.

Proof. We will prove that the objects described are both initial and final in $\mathbf{Set}^*$.

1. Initial:
   Consider the object described above. Then, a morphism from this object to another object $(f, B) \in \mathrm{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:

   $$\begin{tikzcd} T \arrow[rr, "\sigma"] & & B \\ & \{*\} \arrow[ul, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

   The only choice of σ that makes this diagram commute is defined by $\sigma(i(*)) = f(*)$ (as T is a singleton), so T is initial.

2. Final: Consider the object described above. Then, a morphism to this object from another object $(f, B) \in \mathrm{Obj}(\mathbf{Set}^*)$ is given by the following commutative diagram:

   $$\begin{tikzcd} A \arrow[rr, "\tau"] & & T \\ & \{*\} \arrow[ul, "f"] \arrow[ur, "i"'] & \end{tikzcd}$$

   The only choice of τ that makes this diagram commute is defined by $\tau(f(*)) = i(*)$ (as T is a singleton), so T is final.

□

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Problem 5.5. What are the final objects in the category considered in 5.3?

Solution. Final objects in this category (let's denote it $\mathbf{C}$) are singletons.

Proof. Let $\{*\}$ be a singleton. Then morphisms τ from $(Z, \varphi) \in \mathrm{Obj}(\mathbf{C})$ to $\{*\}$ are commutative diagrams:

$$\begin{tikzcd} Z \arrow[rr, "\tau"] & & \{*\} \\ & A \arrow[ul, "\varphi"] \arrow[ur, "i"'] & \end{tikzcd}$$

Since $\{*\}$ is a singleton, the only τ that satisfies $\tau(\varphi(a)) = \iota(a)$ is defined by $\tau(z) = *$ for all $z \in Z$. Thus $(\{*\}, \tau)$ is final in $\mathbf{C}$. $\square$

$\blacksquare$

Problem 5.6. Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category iff d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names?

Solution. Let $\mathbf{Div}$ denote this category and $a, b \in \mathrm{Obj}(\mathbf{Div})$ be objects.

1. Products
   That $\mathbf{Div}$ has products means the following:
   For every $a, b \in \mathrm{Obj}(\mathbf{Div})$, there exists an object $a \times b \in \mathrm{Obj}(\mathbf{Div})$ such that every $(x, f_a, f_b) \in \mathrm{Obj}(\mathbf{Div}_{a,b})$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{Div}}(x, a \times b)$ that makes the following diagram commute:

$$\begin{tikzcd} & & a \\ x \arrow[r, "\sigma"] \arrow[urr, bend left, "f_a"] \arrow[drr, bend right, "f_b"'] & a \times b \arrow[ur, "\pi_a"] \arrow[dr, "\pi_b"'] & \\ & & b \end{tikzcd}$$

   Meaning, for every pair of integers (a, b), there exists a unique factor of a and b, $a \times b$, such that every common factor of a and b divides $a \times b$. This is necessarily the greatest common factor of a and b, since it must be a multiple of every common factor of a and b, and $\mathrm{gcf}(a, b)$ is the product of the common factors of a and b.

2. Coproducts
   That $\mathbf{Div}$ has coproducts means the following:
   For every $a, b \in \mathrm{Obj}(\mathbf{Div})$, there exists an object $a \sqcup b \in \mathrm{Obj}(\mathbf{Div})$ such that every

$(x, f_a, f_b) \in \mathrm{Obj}(\mathbf{Div}^{a,b})$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{Div}}(a \sqcup b, x)$ that makes the following diagram commute:

$$\begin{tikzcd}
 & & a \arrow[dl, "f_a"'] \arrow[dll, bend right, "\pi_a"'] \\
a \sqcup b \arrow[r, "\sigma"] & x & \\
 & & b \arrow[ul, "f_b"] \arrow[ull, bend left, "\pi_b"]
\end{tikzcd}$$

Meaning, for every pair of integers (a, b) there exists a unique multiple of a and b, that divides every common multiple of a and b. Since $a \sqcup b$ is a common multiple of a and b and $a \sqcup b \leq x$ for every common multiple x of a and b, $a \sqcup b$ is $\mathrm{lcm}(a, b)$.

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Problem 5.9. Let $\mathbf{C}$ be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of $\mathbf{C}$ ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property.

Solution. For objects A, B, C of a category $\mathbf{C}$, $A \times B \times C$ ought to be universal with respect to the property of mapping to A, B, and C. In other words, every (X, f_a, f_b, f_c) in the category $\mathbf{C}^{A,B,C}$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(X, A \times B \times C)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & A \\
X \arrow[r, "\sigma_{ABC}"] \arrow[urr, bend left, "f_A"] \arrow[dr, "f_B"'] \arrow[ddrr, bend right, "f_C"'] & A \times B \times C \arrow[ur, "\pi_A"'] \arrow[d, "\pi_B"] \arrow[ddr, bend left, "\pi_C"] & \\
 & B & \\
 & & C
\end{tikzcd} \tag{1}$$

Meaning, $\pi_A \sigma = f_A$, $\pi_B \sigma = f_B$, and $\pi_C \sigma = f_C$.

Proposition. The products $A \times (B \times C)$ and $(A \times B) \times C$ satisfy this property.

Proof. Consider the product $A \times (B \times C) \in \mathrm{Obj}(\mathbf{C})$ and its diagram:

(2)
$$
\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f_A"] \arrow[r, "\sigma_{A(BC)}"] \arrow[drr, bend right, "f_{BC}"'] & A \times (B \times C) \arrow[ur, "\phi_A"'] \arrow[dr, "\phi_{BC}"] & \\
 & & B \times C
\end{tikzcd}
$$

along with the product $B \times C$ and its diagram:

(3)
$$
\begin{tikzcd}
 & & B \\
X \arrow[urr, bend left, "f_B"] \arrow[r, "\sigma_{BC}"] \arrow[drr, bend right, "f_C"'] & B \times C \arrow[ur, "\phi_B"'] \arrow[dr, "\phi_C"] & \\
 & & C
\end{tikzcd}
$$

Staring at these diagrams, we can see that $A \times (B \times C)$ satisfies our universal property if we let: $f_{BC} = \sigma_{BC}$,
$\pi_A = \phi_A$,
$\pi_B = \phi_B \sigma_{ABC}$,
$\pi_C = \phi_C \sigma_{ABC}$.
This makes the diagram for $A \times B \times C$ commute by the commutativity of (2) and (3), thereby forcing $\sigma_{ABC} = \sigma_{A(BC)}$ since the morphism $A \to A \times (B \times C)$ that makes (2) commute is unique.
The proof for $(A \times B) \times C$ is entirely analogous, giving us that $(A \times B) \times C \cong A \times (B \times C)$ by their satisfaction of the same universal property. ■

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Problem 5.10. Push the envelope farther still, and define products for families (i.e. indexed sets) of objects of a category. Do these exist in **Set**?

Solution. Let $\mathbf{C}$ be a category and S be a family of objects in $\mathbf{C}$ indexed by a set J. The product $P = \prod_J S$ is an object of $\mathbf{C}$ together with morphisms $\pi_j \in \mathrm{Hom}_{\mathbf{C}}(P, S_j)$ for all $j \in J$ such that every $X \in \mathrm{Obj}(\mathbf{C})$ together with morphisms $f_j \in \mathrm{Hom}_{\mathbf{C}}(X, S_j)$ admits a unique morphism $\sigma \in \mathrm{Hom}_{\mathbf{C}}(X, P)$ such that $f_j = \pi_j \sigma$.
That these exist in **Set** is equivalent to the Axiom of Choice.

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