Algebra: Chapter 0 Exercises
Chapter 2, Section 3

David Melendez

May 22, 2017

Problem 3.1. Let $\varphi : G \to H$ be a morphism in a category $\mathbf{C}$ with products. Explain why there is a unique morphism
$$(\varphi \times \varphi) : G \times G \to H \times H$$
compatible in the evident way with the natural projections.

Solution. I'm going to assume the author means the following:
Since $H \times H$ is a product in $\mathbf{C}$, we have that $G \times G$ along with two "copies" of $\varphi : G \to H$ admits a unique morphism $(\varphi \times \varphi) : G \times G \to H \times H$ that makes the following diagram commute:

$$\begin{tikzcd}
 & & H \\
G \times G \arrow[r, "\varphi\times\varphi"] \arrow[urr, bend left, "\varphi\pi_{G_1}"] \arrow[drr, bend right, "\varphi\pi_{G_2}"'] & H \times H \arrow[ur, "\pi_{H_1}"] \arrow[dr, "\pi_{H_2}"'] & \\
 & & H
\end{tikzcd}$$

where π_{G_1} and π_{G_2} are the canonical projections of $G \times G$ onto G. ∎

Problem 3.2. Let $\varphi : G \to H$ and $\psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G$, $H \times H$, and $K \times K$ as in Exercise 3.1. Prove that
$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$$

Solution. To demonstrate this result, first stare at this diagram, mapping $G \times G$ to the product $H \times H$ and $H \times H$ to the product $K \times H$.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \arrow[r, "\psi"] & K \\
G \times G \arrow[u] \arrow[d] \arrow[r, "\varphi\times\varphi"] & H \times H \arrow[u] \arrow[d] \arrow[r, "\psi\times\psi"] & K \times K \arrow[u] \arrow[d] \\
G \arrow[r, "\varphi"] & H \arrow[r, "\psi"] & K
\end{tikzcd}$$

Compare it to the following diagram, mapping $G \times G$ straight to $K \times K$:

$$\begin{tikzcd}
G \arrow[r, "\psi\varphi"] & K \\
G \times G \arrow[u] \arrow[d] \arrow[r, "(\psi\varphi)\times(\psi\varphi)"] & K \times K \arrow[u] \arrow[d] \\
G \arrow[r, "\psi\varphi"] & K
\end{tikzcd}$$

Notice that we have morphisms $(\psi\varphi) \times (\psi\sigma)$ and $(\psi \times \psi)(\varphi \times \varphi)$ from $G \times G$ to $K \times K$ determined by the universal property for products with regards to $K \times K$. These must be equal per Problem 3.1. ■

Problem 3.3. Show that if G and H are both *abelian* groups, then $G \times H$ satisfies the universal property for coproducts in **Ab**.

¡++¿