

# Algebra: Chapter 0 Exercises

## Chapter 1, Section 5

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**Problem 5.1.** A final object in a category  $\mathbf{C}$  is initial in the opposite category  $\mathbf{C}^{op}$ .

*Proof.* Let  $T \in \text{Obj}(\mathbf{C})$  be initial and  $Z \in \text{Obj}(\mathbf{C})$  be any object. Since  $\text{Hom}_{\mathbf{C}}(T, Z)$  is a singleton,  $\text{Hom}_{\mathbf{C}^{op}}(Z, T) = \text{Hom}_{\mathbf{C}}(T, Z)$  is also a singleton, making  $T$  final in  $\mathbf{C}^{op}$   $\square$

**Problem 5.2.** The empty set  $\emptyset$  is the *unique* initial object in  $\mathbf{Set}$ .

*Proof.* Recall that the number of morphisms between any two sets  $A$  and  $B$  is given by  $B^A$ . Thus, the problem of finding the size of an initial object in  $\mathbf{Set}$  boils down to solving  $|Z|^{|I|} = 1$  for  $|I|$ , where  $Z$  is an arbitrary set. The only solution to this is  $|I| = 0$ , which is only satisfied by the the null set, given that  $\emptyset^\emptyset = 0$ .  $\square$

**Problem 5.3.** Final objects are unique up to isomorphism.

*Proof.* Let  $\mathbf{C}$  be a category and  $T_1, T_2 \in \text{Obj}(\mathbf{C})$  be final. Because  $T_1$  and  $T_2$  are final, we have, for all  $Z \in \text{Obj}(\mathbf{C})$ :

$$\begin{aligned}\text{End}(T_1) &= \{\text{id}_{T_1}\} \\ \text{End}(T_2) &= \{\text{id}_{T_2}\} \\ \text{Hom}(T_1, T_2) &= \{f\} \text{ for some } f \\ \text{Hom}(T_2, T_1) &= \{g\} \text{ for some } g\end{aligned}$$

We then have  $fg = \text{id}_{T_2}$  and  $gf = \text{id}_{T_1}$ , giving us  $T_1 \cong T_2$ .  $\square$

**Problem 5.4.** What are the initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

*Solution.* For the sake of completeness, a description of this category  $\mathbf{Set}^*$  will be included.

**Definition.** Define  $\mathbf{Set}^*$  as follows:

Objects in  $\mathbf{Set}^*$  are morphisms  $f : \{*\} \rightarrow A$  where  $A$  is any set, denoted  $(f, A)$ .

Morphisms  $\sigma \in \text{Hom}_{\mathbf{Set}^*}((f_A, A), (f_B, B))$  are commutative diagrams:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ f_A \swarrow & & \searrow f_B \\ & \{*\} & \end{array}$$

Now, we will answer the question.

**Proposition.** Initial and final objects in  $\mathbf{Set}^*$  are objects  $(i, T)$ , where  $T$  is a singleton and  $i$  is the unique function that maps from  $\{*\}$  to  $T$ .

*Proof.* We will prove that the objects described are both initial and final in  $\mathbf{Set}^*$ .

1. Initial:

Consider the object described above. Then, a morphism from this object to another object  $(f, B) \in \text{Obj}(\mathbf{Set}^*)$  is given by the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\sigma} & B \\ i \swarrow & & \searrow f \\ & \{*\} & \end{array}$$

The only choice of  $\sigma$  that makes this diagram commute is defined by  $\sigma(i(*)) = f(*)$  (as  $T$  is a singleton), so  $T$  is initial.

2. Final: Consider the object described above. Then, a morphism to this object from another object  $(f, B) \in \text{Obj}(\mathbf{Set}^*)$  is given by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\tau} & T \\ f \swarrow & & \searrow i \\ & \{*\} & \end{array}$$

The only choice of  $\tau$  that makes this diagram commute is defined by  $\tau(f(*)) = i(*)$  (as  $T$  is a singleton), so  $T$  is final.

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**Problem 5.5.** What are the final objects in the category considered in 5.3?

*Solution.* Final objects in this category (let's denote it  $\mathbf{C}$ ) are singletons.

*Proof.* Let  $\{*\}$  be a singleton. Then morphisms  $\tau$  from  $(Z, \varphi) \in \text{Obj}(\mathbf{C})$  to  $\{*\}$  are commutative diagrams:

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & \{*\} \\ & \swarrow \varphi \quad \searrow i & \\ & A & \end{array}$$

Since  $\{*\}$  is a singleton, the only  $\tau$  that satisfies  $\tau(\varphi(a)) = \iota(a)$  is defined by  $\tau(z) = *$  for all  $z \in Z$ . Thus  $(\{*\}, \tau)$  is final in  $\mathbf{C}$ .  $\square$

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**Problem 5.6.** Consider the category corresponding to endowing (as in Example 3.3) the set  $\mathbb{Z}^+$  with the divisibility relation. Thus there is exactly one morphism  $d \rightarrow m$  in this category iff  $d$  divides  $m$  without remainder; there is no morphism between  $d$  and  $m$  otherwise. Show that this category has products and coproducts. What are their 'conventional' names?

*Solution.* Let  $\mathbf{Div}$  denote this category and  $a, b \in \text{Obj}(\mathbf{Div})$  be objects.

1. Products

That  $\mathbf{Div}$  has products means the following:

For every  $a, b \in \text{Obj}(\mathbf{Div})$ , there exists an object  $a \times b \in \text{Obj}(\mathbf{Div})$  such that every  $(x, f_a, f_b) \in \text{Obj}(\mathbf{Div}_{a,b})$  admits a unique morphism  $\sigma \in \text{Hom}_{\mathbf{Div}}(x, a \times b)$  that makes the following diagram commute:

$$\begin{array}{ccccc} & & & & \\ & & f_a & \searrow & a \\ & & & \pi_a & \\ x & \xrightarrow{\sigma} & a \times b & & \\ & & \pi_b & \searrow & b \\ & & f_b & \swarrow & \end{array}$$

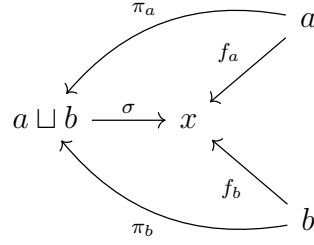
Meaning, for every pair of integers  $(a, b)$ , there exists a unique factor of  $a$  and  $b$ ,  $a \times b$ , such that every common factor of  $a$  and  $b$  divides  $a \times b$ . This is necessarily the greatest common factor of  $a$  and  $b$ , since it must be a multiple of every common factor of  $a$  and  $b$ , and  $\text{gcf}(a, b)$  is the product of the common factors of  $a$  and  $b$ .

2. Coproducts

That  $\mathbf{Div}$  has coproducts means the following:

For every  $a, b \in \text{Obj}(\mathbf{Div})$ , there exists an object  $a \sqcup b \in \text{Obj}(\mathbf{Div})$  such that every

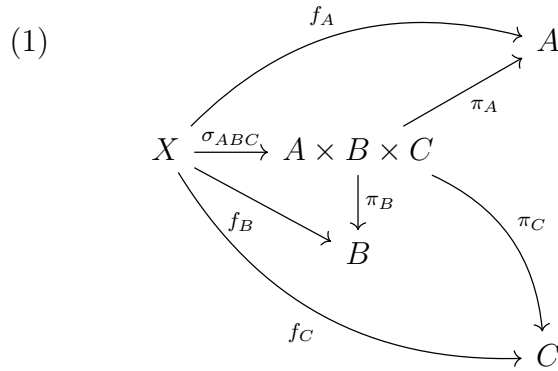
$(x, f_a, f_b) \in \text{Obj}(\mathbf{Div}^{a,b})$  admits a unique morphism  $\sigma \in \text{Hom}_{\mathbf{Div}}(a \sqcup b, x)$  that makes the following diagram commute:



Meaning, for every pair of integers  $(a, b)$  there exists a unique multiple of  $a$  and  $b$ , that divides every common multiple of  $a$  and  $b$ . Since  $a \sqcup b$  is a common multiple of  $a$  and  $b$  and  $a \sqcup b \leq x$  for every common multiple  $x$  of  $a$  and  $b$ ,  $a \sqcup b$  is  $\text{lcm}(a, b)$ . ■

**Problem 5.9.** Let  $\mathbf{C}$  be a category with products. Find a reasonable candidate for the universal property that the product  $A \times B \times C$  of *three* objects of  $\mathbf{C}$  ought to satisfy, and prove that both  $(A \times B) \times C$  and  $A \times (B \times C)$  satisfy this universal property.

*Solution.* For objects  $A, B, C$  of a category  $\mathbf{C}$ ,  $A \times B \times C$  ought to be universal with respect to the property of mapping to  $A, B$ , and  $C$ . In other words, every  $(X, f_A, f_B, f_C)$  in the category  $\mathbf{C}^{A,B,C}$  admits a unique morphism  $\sigma \in \text{Hom}_{\mathbf{C}}(X, A \times B \times C)$  such that the following diagram commutes:



Meaning,  $\pi_A \sigma = f_A$ ,  $\pi_B \sigma = f_B$ , and  $\pi_C \sigma = f_C$ .

**Proposition.** The products  $A \times (B \times C)$  and  $(A \times B) \times C$  satisfy this property.

*Proof.* Consider the product  $A \times (B \times C) \in \text{Obj}(\mathbf{C})$  and its diagram:

$$(2) \quad \begin{array}{ccc} & & A \\ & \nearrow^{f_A} & \\ X & \xrightarrow{\sigma_{A(BC)}} & A \times (B \times C) \\ & \searrow_{f_{BC}} & \\ & & B \times C \end{array} \quad \begin{array}{c} \nearrow^{\phi_A} \\ \searrow_{\phi_{BC}} \end{array}$$

along with the product  $B \times C$  and its diagram:

$$(3) \quad \begin{array}{ccc} & & B \\ & \nearrow^{f_B} & \\ X & \xrightarrow{\sigma_{BC}} & B \times C \\ & \searrow_{f_C} & \\ & & C \end{array} \quad \begin{array}{c} \nearrow^{\phi_B} \\ \searrow_{\phi_C} \end{array}$$

Staring at these diagrams, we can see that  $A \times (B \times C)$  satisfies our universal property if we let:  $f_{BC} = \sigma_{BC}$ ,

$$\pi_A = \phi_A,$$

$$\pi_B = \phi_B \sigma_{ABC},$$

$$\pi_C = \phi_C \sigma_{ABC}.$$

This makes the diagram for  $A \times B \times C$  commute by the commutativity of (2) and (3), thereby forcing  $\sigma_{ABC} = \sigma_{A(BC)}$  since the morphism  $A \rightarrow A \times (B \times C)$  that makes (2) commute is unique.

The proof for  $(A \times B) \times C$  is entirely analogous, giving us that  $(A \times B) \times C \cong A \times (B \times C)$  by their satisfaction of the same universal property. ■

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