Algebra: Chapter 0 Exercises

Chapter 1, Section 4

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Problem 4.1. Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad etc.$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses.

Solution. Let $Z_m \in \mathrm{Obj}(C)$ and $f_m \in \mathrm{Hom}(Z_{m+1}, Z_m)$ for every $m \in \mathbb{N}$. Let n be the number of morphisms we're composing. We will use induction on n.

Base case: Suppose $n = 3$. Then, since C is a category, we have $f_1(f_2 f_3) = (f_1 f_2) f_3$.

Induction: Suppose that all parenthesizations of $f_1, \ldots f_{j-1}$ under composition are equivalent for all $1 \leq j < n$. Then, for some $1 < k \leq n$, let α be some parenthesization of $f_1, \ldots, f_{k-1}$, and let β be some parenthesization of $f_k, \ldots, f_n$. Any parenthesization of $f_1, \ldots, f_n$ will then be of the form $\alpha\beta$. By associativity and our inductive hypothesis, we have $\alpha = ((f_k \ldots f_{n-1}) f_n)$, and so

$$\begin{aligned}\alpha\beta &= (f_1 \ldots f_{k-1})\,((f_k \ldots f_{n-1}) f_n) \\ &= ((f_1 \ldots f_{k-1})(f_k \ldots f_{n-1}))\, f_n \\ &= ((\ldots((f_1 f_2) f_3) \ldots) f_n\end{aligned}$$

as desired. ■

Problem 4.2. In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid?

Solution. Recall that a *groupoid* is a category in which every morphism is an isomorphism. Let C be a category as defined in Example 3.3, and let $(S, \sim)$ be the category's designated set and relation. C is a groupoid if $\sim$ is symmetric.

Proof. Let (a, b) be a morphism from a to b in C. By our definition of C, we have $a \sim b$. Since $\sim$ is symmetric, we then have $b \sim a$, and so (b, a) is also a morphism in C (from b to a). Composing these, we have $(a, b)(b, a) = (b, b) = \mathrm{id}_b$. Similarly, we also have $(b, a)(a, b) = (a, a) = \mathrm{id}_a$, making (a, b) an isomorphism as desired. $\blacksquare$

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