Algebra: Chapter 0 Exercises

Chapter 1, Section 4

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Problem 4.1. Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad etc.$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses.

Solution. Let $Z_m \in \mathrm{Obj}(C)$ and $f_m \in \mathrm{Hom}(Z_{m+1}, Z_m)$ for every $m \in \mathbb{N}$. Let n be the number of morphisms we're composing. We will use induction on n.

Base case: Suppose $n = 3$. Then, since C is a category, we have $f_1(f_2f_3) = (f_1f_2)f_3$.

Induction: Suppose that all parenthesizations of $f_1, \ldots f_{j-1}$ under composition are equivalent for all $1 \leq j < n$. Then, for some $1 < k \leq n$, let α be some parenthesization of $f_1, \ldots, f_{k-1}$, and let β be some parenthesization of $f_k, \ldots, f_n$. Any parenthesization of $f_1, \ldots, f_n$ will then be of the form $\alpha\beta$. By associativity and our inductive hypothesis, we have $\alpha = ((f_k \ldots f_{n-1})f_n)$, and so

$$\begin{aligned}
\alpha\beta &= (f_1 \ldots f_{k-1})\left((f_k \ldots f_{n-1})f_n\right) \\
&= \left((f_1 \ldots f_{k-1})(f_k \ldots f_{n-1})\right) f_n \\
&= ((\ldots((f_1f_2)f_3)\ldots)f_n
\end{aligned}$$

as desired. ∎

Problem 4.2. In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid?

Solution. Recall that a *groupoid* is a category in which every morphism is an isomorphism. Let C be a category as defined in Example 3.3, and let $(S, \sim)$ be the category's designated set and relation. C is a groupoid if $\sim$ is symmetric.

Proof. Let (a, b) be a morphism from a to b in C. By our definition of C, we have $a \sim b$. Since $\sim$ is symmetric, we then have $b \sim a$, and so (b, a) is also a morphism in C (from b to a). Composing these, we have $(a, b)(b, a) = (b, b) = \mathrm{id}_b$. Similarly, we also have $(b, a)(a, b) = (a, a) = \mathrm{id}_a$, making (a, b) an isomorphism as desired. □

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Problem 4.3. Let A, B be objects of a category C, and $f \in \mathrm{Hom}_C(A, B)$ a morphism.

Solution. .

1. If f has a right-inverse, then f is an epimorphism.

   Proof. Let $f \in \mathrm{Hom}_C(A, B)$ be a morphism, $g \in \mathrm{Hom}_C(B, A)$ its right-inverse, and $\alpha_1, \alpha_2 \in \mathrm{Hom}_C(A, Z)$ morphisms for some $Z \in \mathrm{Obj}(C)$ with $\alpha_1 f = \alpha_2 f$. We then have

$$\begin{aligned} \alpha_1 &= \alpha_1(fg) \\ &= (\alpha_1 f)g \\ &= (\alpha_2 f)g \\ &= \alpha_2(fg) \\ &= \alpha_2 \end{aligned}$$

   making f an epimorphism. □

2. The converse of 1 does not hold; that is, there exists in some category C an epimorphism that does not have a right-inverse.

   Proof. The category obtained by endowing $\mathbb{Z}$ with the relation $\leq$ contains morphisms that satisfy this property. Let **C** be this category; $a, b \in \mathrm{Obj}(\mathbf{C})$ such that $a \neq b$ (so $a < b$); $f \in \mathrm{Hom}(a, b)$; $z \in \mathrm{Obj}(\mathbf{C})$ such that $b \leq z$; and $\alpha_1, \alpha_2 \in \mathrm{Hom}(b, z)$. That $\alpha_1 f = \alpha_2 f$ implies $\alpha_1 = \alpha_2$ is trivially true since Hom(b, z) has exactly one morphism, so f is an epimorphism.
   However, since $a < b$, $b > a$, meaning Hom(b,a) has no morphisms. Thus, f has no right-inverse.

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Problem 4.4. Prove that the composition of two morphisms is a monomorphism. Deduce that one can define a subcategory $\mathbf{C}_{\text{mono}}$ of a category $\mathbf{C}$ by taking the same objects as in $\mathbf{C}$, and defining $\text{Hom}_{\mathbf{C}_{\text{mono}}}(A, B)$ to be the subset of $\text{Hom}_{\mathbf{C}}(A, B)$ consisting of monomorphisms, for all objects A, B. Do the same for epimorphisms. Can you define a subcategory $\mathbf{C}_{\text{nonmono}}$ of $\mathbf{C}$ by restricting to morphisms that are *not* monomorphisms?

Solution. Let $\mathbf{C}$ be a category; $A, B, C \in \text{Obj}(\mathbf{C})$ be objects in C; and $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$ be morphisms in C. If f and g are monic, then gf is also monic.

Proof. Since f and g are monic, we have, for all $Z_1, Z_2 \in \text{Obj}(\mathbf{C})$, $\alpha_1, \beta_1 \in \text{Hom}(B, Z_1)$, and $\alpha_2, \beta_2 \in \text{Hom}(C, Z_2)$:

$$\begin{aligned} \alpha_1 f = \beta_1 f &\implies \alpha_1 = \beta_1 \\ \alpha_2 g = \beta_2 g &\implies \alpha_2 = \beta_2 \end{aligned}$$

We then have:

$$\begin{aligned} \alpha_2(gf) = \beta_2(gf) &\implies (\alpha_2 g)f = (\beta_2 g)f \\ &\implies \alpha_2 g \quad = \beta_2 g \\ &\implies \alpha_2 \quad\;\; = \beta_2 \end{aligned}$$

making gf monic as desired. □

With this, we can define a category $\mathbf{C}_{\text{mono}}$ by

$$\text{Obj}(\mathbf{C}_{\text{mono}}) = \text{Obj}(\mathbf{C})$$

and

$$\text{Hom}_{\mathbf{C}_{\text{mono}}}(A, B) = \{f \in \text{Hom}_{\mathbf{C}}(A, B) \ \mid f \text{ is monic}\}$$

for all $A, B \in \text{Obj}(\mathbf{C}_{\text{mono}})$. Composition of morphisms is defined as normal (since we've proved monomorphisms are closed under composition, and the indentities are those in $\mathbf{C}$ since identities are trivially monic.
The proof for epimorphisms is analogous.
Non-monomorphisms, however, do not form a category:

Proposition. Let $\mathbf{C}$ be a category; A, B, C, Z be objects in $\mathbf{C}$, and $f \in \text{Hom}_{\mathbf{C}}(A, B)$ and $g \in \text{Hom}_{\mathbf{C}}(B, C)$ be nonmonomorphisms. Then there exist morphisms $\alpha_1, \alpha_2 \in \text{Hom}_{\mathbf{C}}(C, Z)$ such that $\alpha_1(gf) = \alpha_2(gf)$ but $\alpha_1 \neq \alpha_2$.

Proof. Since g is not monic, there exist $\alpha_1, \alpha_2 \in \text{Hom}_{\mathbf{C}}(C, Z)$ such that $a = \alpha_1 g = \alpha_2 g$ but $\alpha_1 \neq \alpha_2$. Let α_1 and α_2 be these morphisms. Then, we have

$$\begin{aligned} \alpha_1(gf) &= (\alpha_1 g)f \\ &= (\alpha_2 g)f \\ &= \alpha_2(gf) \end{aligned}$$

but $\alpha_1 \neq \alpha_2$. Thus, gf is not monic. □

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