Algebra: Chapter 0 Exercises
Chapter 2, Section 7

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Problem 7.4. Prove that the relation defined in Exercise 5.10 on a free abelian group $F = F^{ab}(A)$ by

$$f \sim f' \Leftrightarrow (\exists g \in G): \; f' - f = 2g$$

is compatible with the group structure. Determine the quotient $F/\sim$ as a better known group.

Solution. The relation $\sim$ is compatible with the group structure on F if and only if, considering Proposition 7.3,

$$(\forall f, f', a \in F): \; f \sim f' \implies a + f \sim a + f'$$

(since F is abelian). Suppose then, we have $f \sim f'$, that is $f' - f = 2g$. We then have $(f' + a) - (f + a) = 2g$, and hence $a + f' \sim a + f$.

To determine $F/\sim$ as a better known group, we will establish an isomorphism between the group F and the abelian group $G := (\mathbb{Z}/2\mathbb{Z})^A$.
Define the set function $\kappa : A \to G$ as follows:

$$(\forall a, a' \in A): \; \kappa(a)(a') = [\delta_{a'a}]_2,$$

where δ is the kronecker delta function. We then use this function and the universal property for free (abelian) groups to induce a group homomorphism $\varphi : F \to G$ that makes the following diagram commute:

$$\begin{tikzcd}
F^{ab}(A) \arrow[r, dashed, "\varphi"] & (\mathbb{Z}/2\mathbb{Z})^A \\
A \arrow[u, "j"] \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

where $j : A \to F$ is the usual inclusion. We will now prove two lemmas essential to constructing the desired isomorphism.

Lemma 1. *The homomorphism φ is surjective.*

Proof. Suppose $f \in (\mathbb{Z}/2\mathbb{Z})^A$, let $\bar{a} = j(a) \in F$, and allow $f(x)y$ to be "multiplication" by a coset representitive of $f(x)$, i.e. $[0]_2 y = 0y$ and $[1]_2 y = 1y$. We then have, with the symbol at the top of each sigma representing for clarity the abelian group in which the sum is taking place,

$$\begin{aligned}(\forall a, a' \in A)\varphi\left(\sum_{a\in A}^{F} f(a)\bar{a}\right)(a') &= \sum_{a\in A}^{G} \left(\varphi(f(a)\bar{a})\right)(a') \\ &= \sum_{a\in A}^{G} \left(f(a)(\varphi(\bar{a})(a'))\right) \\ &= \sum_{a\in A}^{\mathbb{Z}/2\mathbb{Z}} \left(f(a)([\delta_{a'a}]_2)\right) \\ &= f(a')\end{aligned}$$

Hence every $f \in G$ is in the image of φ. $\square$

Lemma 2. *The homomorphism φ agrees with the relation $\sim$; that is,*

$$f \sim f' \implies \varphi(f) = \varphi(f')$$

Proof. Suppose $f, f' \in F$ and $f \sim f'$. We then have, for all $a \in A$ and for some $g \in (\mathbb{Z}/2\mathbb{Z})^A$:

$$\begin{aligned}(\varphi(f') - \varphi(f))(a) &= \varphi(f' - f)(a) \\ &= (2g)(a) \\ &= 2(g(a)) \\ &= [0]_2\end{aligned}$$

(as $\mathbb{Z}/2\mathbb{Z}$ has order 2); hence $\varphi(f') = \varphi(f)$, as desired. $\square$

Lemma 3. *If we define $H = [e_F]_\sim$ then we have $F/H = F/{\sim}$, and $\ker(\varphi) = H$.*

Proof. The first statement follows from Proposition 7.4, Proposition 7.7, and the definition of quotient by a normal subgroup.
For the second statement, first suppose that $h \in H$. It then follows, since $h \sim e_F$, that $\varphi(h) = \varphi(e_F) = e_G$, so $h \subseteq F$. For the other direction, suppose $f \in \ker(\varphi)$, and

$$f = \sum_{a\in A} n_a \bar{a}$$

We then have, for all $a' \in A$,

$$\begin{aligned}
\varphi(f)(a') &= \varphi\left(\sum_{a\in A} n_a \bar{a}\right)(a') \\
&= \sum_{a\in A} (n_a \varphi(\bar{a})(a')) \\
&= \sum_{a\in A} (n_a \kappa(a)(a')) \\
&= \sum_{a\in A} (n_a [\delta_{a'a}]_2) \\
&= [n_{a'}]_2
\end{aligned}$$

Since $\varphi(f) = e_G$, we know that each $n_{a'}$ is congruent to 0 modulo two; that is, even. Hence, we have:

$$\begin{aligned}
f - e_F &= f \\
&= 2\sum_{a\in A} \frac{n_a}{2} \bar{a}
\end{aligned}$$

This shows that $f \sim e_f$ and thus $f \in H$, giving us $\ker(\varphi) \subseteq H$. Therefore $\ker(\varphi) = H$, as desired. □

With this all established, we can finally find our isomorphism. We now construct a homomorphism $\widetilde{\varphi} : F/H \to G$ using the universal property for quotient by an equivalence relation:

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G \\
F/H \arrow[ur, dashed, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

Here, π is the quotient map $\pi(f) = [f]_\sim$, and $\widetilde{\varphi}$ is the unique homomorphism making the diagram commute. The first isomorphism theorem shows that $\widetilde{\varphi}$ is an isomorphism, and so $F/{\sim} \cong (\mathbb{Z}/2\mathbb{Z})^A$. ■