Algebra: Chapter 0 Exercises

Chapter 3, Section 3

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Problem 3.1. Prove that the image of a ring homomorphism $\varphi : R \to S$ is a subring of S. What can you say about φ if its image is an ideal of S? What can you say about φ if its kernel is a subring of R?

Solution. First we'll prove that im φ is a subring of S.

Proof. Suppose $s_1 = \varphi(r_1)$ and $s_2 = \varphi(r_2)$ are elements of im φ. We then have $s_1 + s_2 = \varphi(r_1 + r_2)$ and $s_1 s_2 = \varphi(r_1 r_2)$ since φ is a homomorphism, so both of these are elements of im φ. Additionally, $\varphi(1_R) = 1_S$, making im φ a subring of S. □

If im φ is an ideal of S, then φ is surjective, since the only ideal of S containing the identity 1_S is S itself. If $\ker\varphi$ is a subring of R, then it must contain 1_R, which, combined with the fact that $\ker\varphi$ is an ideal, tells us that $\ker\varphi = R$. Thus φ must be the "zero" morphism $r \mapsto 0$, which isn't actually a ring homomorphism since it does not preserve the identity. ■

Problem 3.2. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of S. Prove that $I = \varphi^{-1}(J)$ is an ideal of R.

Solution. Suppose $x \in I$ and $r \in R$. We then have $\varphi(rx) = \varphi(r)\varphi(x)$, which is in J since J is an ideal and $\varphi(x) \in J$. The same argument applies to xr (as J is a two-sided ideal), so I is an ideal of R. ■

Problem 3.3. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of R.

1. Show that $\varphi(J)$ need not be an ideal of S.

   Proof. Let $R = \mathbb{C}$ and $S = \mathbb{H}$ (the quaternions), and let φ be the inclusion $a + bi \mapsto a + bi$. The whole of $\mathbb{C}$ is of course an ideal of $\mathbb{C}$, but the "copy" of $\mathbb{C}$ in the quaternions $\varphi(\mathbb{C})$ is not an ideal of $\mathbb{H}$, since $(a + bi)j = aj + bk \notin \varphi(\mathbb{C})$. □

2. Assume that φ is surjective; then prove that $\varphi(J)$ *is* an ideal of S.

   Proof. □

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3. Assume that φ is surjective, and let $I = \ker\varphi$; thus we may identify S with R/I. Let $\overline{J} = \varphi(J)$, an ideal of R/I by the previous point. Prove that

$$\frac{R/I}{\overline{J}} \cong \frac{R}{I+J}.$$