Algebra: Chapter 0 Exercises

Chapter 2, Section 1

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Problem 2.2. If $d \leq n$, then S_n contains elements of order d.

Proposition. Let c_d, called a *d-cycle* in S_n, be defined as follows:

$$c_d(m) = \begin{cases} d & m = 1 \\ d-1 & 1 < m \leq d \\ m & m > d \end{cases}$$

For example, if we're working in S_6, then $c_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 3 & 5 & 6 \end{pmatrix}$.
Then $|c_d| = d$ for $1 \leq d \leq n$.

Proof. Note that if $0 < k < d$, then $c_d^k(d) = d - k \geq 1$ (c_d^k never "reaches" the point at which it cycles from 1 to d since $k < d$), so $|c_d| \geq d$. Then, we have, for $m \leq d$,

$$\begin{aligned} c_d^d(m) &= (c_d^m \cdot c_d^{d-m})(m) \\ &= c_d^{d-m}(d) \\ &= d - (d - m) \\ &= m \end{aligned}$$

Clearly $c_d^d(m) = m$ if $m > d$, so c_d^d is the identity, as desired. □

Problem 2.5. Describe generators and relations for all dihedral groups D_{2n}.

Solution. We will define the dihedral group D_{2n} as follows:

$$D_{2n} = \langle x, y \mid x^2 = y^n = (xy)^2 = e \rangle$$

Proposition. With this definition of D_{2n}, every combination $x^{i_1}y^{i_2}x^{i_4}y^{i_5}\cdots$ equals $x^i y^j$ for some $0 \leq i \leq 1, 0 \leq j < n$.

Proof. We will use induction on m, the number of elements we're composing.
The cases for $0 \leq m \leq 2$ are obvious.
Suppose this reduction holds for m. Then, if m is odd, we have

$$\begin{aligned}\left(x^{k_1}y^{k_2}\cdots x^{k_m}\right)y^{k_{m+1}} &= x^i y^j y^{k_{m+1}}\\ &= x^i y^{j+k_{m+1}}\end{aligned}$$

The case where m is even is more interesting. First, we will establish the following based off of the third relation:

$$\begin{aligned}(xy)^2 = e \implies xyxy &= e\\ \implies x(yxy) &= e\\ \implies x^{-1} &= yxy\\ \implies yx &= x^{-1}y^{-1}\\ &= xy^{n-1}\end{aligned}$$

Now, suppose m is even. We then have, with $\leq i \leq 1$, $0 \leq j < n$, and $0 \leq k \leq 1$:

$$\left(x^{k_1}y^{k_2}\cdots x^{k_{m-1}}y^{k_m}\right)x^{k_{m+1}} = x^i y^j x^k$$

Since every other case is trivial, we will assume $0 \neq j \neq n$ and $k = 1$. Additionally, we will assume wlog that $j < n$. Then, we have

$$\begin{aligned}x^i y^j x^k &= x^i y^j x\\ &= x^i y^{j-1}(yx)\\ &= x^i y^{j-1} x y^{n-1}\\ &= x^i y^{j-2} x y^{2n-2}\\ &= x^i y^{j-2} x y^{n-2}\\ &= x^i y^{j-3} x y^{n-3}\\ &= \cdots\\ &= x^i y^0 x y^{n-j}\\ &= x^{i+1} y^{n-j}\end{aligned}$$

as desired. ■

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