Algebra: Chapter 0 Exercises

Chapter 2, Section 8

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Problem 8.1. If a group H may be realized as a subgroup of two groups G_1 and G_2, and

$$\frac{G_1}{H} \cong \frac{G_2}{H},$$

does it follow that $G_1 \cong G_2$? Give a proof or counterexample

Solution. No. As a counterexample, take $G_1 = D_6$, $G_2 = C_6$, and $H = C_3$. In this case, we have $D_6/C_3 \cong C_2 \cong C_6/C_3$, but $D_6 \ncong C_3$. ∎

Problem 8.2. Suppose G is a group, and $H \subseteq G$ is a subgroup of index 2. Prove that H is normal in G.

Solution. Consider the function $\varphi : G \to C_2$ defined by

$$\varphi(g) = \begin{cases} 0 & g \in H \\ 1 & g \notin H \end{cases}$$

To check that this is a homomorphism, suppose $g_1, g_2 \notin H$. In particular, $g_2^{-1} \notin H$, so

$$g_1 H = g_2^{-1} H,$$

since there are only two left cosets of H in G, so

$$g_1 g_2 \in H$$

and hence

$$\begin{aligned} \varphi(g_1 g_2) &= 0 \\ &= 1 + 1 \\ &= \varphi(g_1) + \varphi(g_2). \end{aligned}$$

Clearly $\ker\ \varphi = H$, so H is normal in G. ∎