Algebra: Chapter 0 Exercises

Chapter 3, Section 2

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Problem 1. Prove that if there is a homomorphism from a zero ring to a ring R, then R is a zero ring.

Solution. Let Z denote the zero ring, and let $\varphi : Z \to R$ be a ring homomorphism. Since φ is a homomorphism, it must take the identity in Z to the identity in R, so $\varphi(0) = 1_R$. But 0 is also the *additive* identity in Z, meaning $\varphi(0) = 0$, and so $0 = 1$ in R.

If $r \in R$, we then have $1 \cdot r = 0 \cdot r = 0$, showing that R is the zero ring. ∎

Problem 2. Let R and S be rings, and let $\varphi : R \to S$ be a function preserving both operations $+, \cdot$.

1. Prove that if φ is surjective, then necessarily $\varphi(1_R) = 1_S$.
2. Prove that if $\varphi \neq 0$ and S is an integral domain, then $\varphi(1_R) = 1_S$.

Solution.

1. First suppose φ is surjective. Then, if $s \in S$, then there exists an $r \in R$ such that $\varphi(r) = s$. Note that

$$\begin{aligned} \varphi(1_R) \cdot s &= \varphi(1_R) \cdot \varphi(r) \\ &= \varphi(1_R \cdot r) \\ &= \varphi(r) \\ &= s. \end{aligned}$$

Since this is true for all $s \in S$ (as φ is surjective), this implies that $\varphi(1_R) = 1_S$, as desired.

2. Now, let $\varphi \neq 0$ and suppose $\varphi(1_R) \neq 1_S$. This implies that $\varphi(1_R) - 1_S \neq 0$. Since φ is nonzero, there exists an $r \in R$ with $\varphi(r) \neq 0$. Note, then, that we have:

$$\begin{aligned} \varphi(r) \cdot (\varphi(1_R) - 1_S) &= \varphi(r) \cdot \varphi(1_R) - \varphi(r) \cdot 1_S \\ &= \varphi(r \cdot 1_R) - \varphi(r) \\ &= \varphi(r) - \varphi(r) \\ &= 0, \end{aligned}$$

implying S is not an integral domain since both of the terms in the original product are nonzero. Therefore, if S is an integral domain and $\varphi \neq 0$, then $\varphi(1_R) = 1_S$.

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