MIMO ASSIGNMENT 3

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1. (1 pt) Determine the normalized coding gain of the following set of space-time signals:

$$C_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

$\longrightarrow$ $L = 4$, $M_t = 2$

Coding gain is given by

$$\zeta = \frac{1}{2\sqrt{M_t}} \min_{\ell \neq \ell'} |\det(C_\ell - C_{\ell'})|^{1/M_t}$$

Since there are four space-time signals we will have to find out the determinant of all possible combinations

$(C_0 - C_1)$, $(C_0 - C_2)$, $(C_0 - C_3)$
$(C_1 - C_2)$, $(C_1 - C_3)$
$(C_2 - C_3)$

$$C_0 - C_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\det(C_0 - C_1) = 0 - (-4) = +4$$

$$C_0 - C_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

$$\det(C_0 - C_2) = 4 - (-4) = +8$$

$$C_0 - C_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(C_0 - C_3) = 4 - 0 = 4$$

$$C_1 - C_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(C_1 - C_2) = 4 - 0 = 4$$

$$C_1 - C_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$\det(C_1 - C_3) = 4 - (-4) = 8$$

$$C_2 - C_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\det(C_2 - C_3) = 0 - (-4) = 4$$

$$|\det(C_0 - C_1)| = 4 \qquad |\det(C_1 - C_2)| = 4$$

$$|\det(C_0 - C_2)| = 8 \qquad |\det(C_1 - C_3)| = 8$$

$$|\det(C_0 - C_3)| = 4 \qquad |\det(C_2 - C_3)| = 4$$

$$\min_{l \neq l'} |\det(C_l - C_{l'})| = 4$$

Normalized Coding Gain (ζ)

$$\zeta = \frac{1}{2\sqrt{M_t}} \min_{l \neq l'} |\det(C_l - C_{l'})|^{1/M_t}$$

$$= \frac{1}{2\sqrt{2}} (4)^{1/2}$$

$$\boxed{\zeta = 1/\sqrt{2} = 0.7071}$$

2. (1 pt) Determine the normalized coding gain of the following set of space-time signals:

$$C_0 = \sqrt{\frac{2}{3}}\begin{bmatrix} j & 1-j \\ -1-j & -j \end{bmatrix}, \quad C_1 = \sqrt{\frac{2}{3}}\begin{bmatrix} -j & -1-j \\ 1-j & j \end{bmatrix},$$

$$C_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} -j & 1+j \\ -1+j & j \end{bmatrix}, \quad C_3 = \sqrt{\frac{2}{3}}\begin{bmatrix} j & -1+j \\ 1+j & -j \end{bmatrix}.$$

→ Similar to the last question

$L = 4 \quad M_t = 2$

$$C_0 - C_1 = \sqrt{\frac{2}{3}}\begin{bmatrix} j & 1-j \\ -1-j & -j \end{bmatrix} - \sqrt{\frac{2}{3}}\begin{bmatrix} -j & -1-j \\ 1-j & j \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}}\begin{bmatrix} 2j & 2 \\ -2 & -2j \end{bmatrix} = \begin{bmatrix} 2j \times \sqrt{\frac{2}{3}} & 2 \times \sqrt{\frac{2}{3}} \\ -2 \times \sqrt{\frac{2}{3}} & -2j \times \sqrt{\frac{2}{3}} \end{bmatrix}$$

$$\det(C_0 - C_1) = 4 \times \frac{2}{3} - \left(-4 \times \frac{2}{3}\right)$$

$$= 8 \times \frac{2}{3} = \frac{16}{3}$$

$$C_0 - C_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} j & 1-j \\ -1-j & -j \end{bmatrix} - \sqrt{\frac{2}{3}}\begin{bmatrix} -j & 1+j \\ -1+j & j \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 2j & -2j \\ -2j & -2j \end{bmatrix}$$

$$\det(C_0 - C_2) = \frac{2}{3}\left(4 - (-4)\right) = \frac{16}{3}$$

$$C_0 - C_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} j & 1-j \\ -1-j & -j \end{bmatrix} - \sqrt{\frac{2}{3}} \begin{bmatrix} j & -1+j \\ 1+j & -j \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & 2-2j \\ -2-2j & 0 \end{bmatrix}$$

$$\det(C_0 - C_3) = 0 - \left[(-2-2j)(2-2j)\right] \times \frac{2}{3}$$

$$= -\left[-4 + \cancel{4j} - \cancel{4j} - 4\right] \times \frac{2}{3}$$

$$= 8 \times \frac{2}{3} = \frac{16}{3}$$

$$C_1 - C_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} -j & -1-j \\ 1-j & j \end{bmatrix} - \sqrt{\frac{2}{3}} \begin{bmatrix} -j & 1+j \\ -1+j & j \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & -2-2j \\ 2-2j & 0 \end{bmatrix}$$

$$\det(C_1 - C_2) = -2/3 \times \left[(2-2j)(-2-2j) \right]$$

$$= -2/3 \times \left[-4 - \cancel{4j} + \cancel{4j} - 4 \right]$$

$$= 16/3$$

$$C_1 - C_3 = \sqrt{2/3}\begin{bmatrix} -j & -1-j \\ 1-j & j \end{bmatrix} - \sqrt{2/3}\begin{bmatrix} j & -1+j \\ 1+j & -j \end{bmatrix}$$

$$= \sqrt{2/3}\begin{bmatrix} -2j & -2j \\ -2j & 2j \end{bmatrix}$$

$$\det(C_1 - C_3) = 2/3\,(4+4) = 16/3$$

$$C_2 - C_3 = \sqrt{2/3}\begin{bmatrix} -j & 1+j \\ -1+j & j \end{bmatrix} - \sqrt{2/3}\begin{bmatrix} j & -1+j \\ 1+j & -j \end{bmatrix}$$

$$= \sqrt{2/3}\begin{bmatrix} -2j & 2 \\ -2 & 2j \end{bmatrix}$$

$$\det(C_2 - C_3) = (4+4) \times 2/3 = 16/3$$

$$\min_{l \neq l'} |\det(C_l - C_{l'})| = 16/3$$

$$\xi = \frac{1}{2\sqrt{M_t}} \min_{l \neq l'} |\det(C_l - C_{l'})|^{1/M_t}$$

$$= \frac{1}{2\sqrt{2}} \times \left(16/3\right)^{1/2}$$

$$= \frac{1}{2\sqrt{2}} \times \sqrt{\frac{16}{3}} = \frac{4}{2\sqrt{2}} \times \frac{1}{\sqrt{3}}$$

$$= \sqrt{\frac{2}{3}}$$

$$\boxed{\xi = 0.8165}$$

3. (4 pts) A MIMO system with $M_t = 4$ transmit antennas uses the following space-time code

$$G_4 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix},$$

where x_1, x_2 and x_3 are independently chosen from a signal constellation.

(a) Show that

$$G_4^H G_4 = (|x_1|^2 + |x_2|^2 + |x_3|^2) I_4.$$

(b) If x_1, x_2 and x_3 are chosen independently from QPSK signals {1, -1, j, -j}, what is the spectral efficiency of the space-time code?

(c) Prove that the space-time code can achieve the full diversity.

(d) Determine the normalized coding gain for the space-time code when x_1, x_2 and x_3 are chosen independently from QPSK signals {1, -1, j, -j}.

(e) Prove that the space-time code has fast ML decoding at the receiver, i.e., x_1, x_2 and x_3 can be decoded separately, not jointly.

(a) Conjugate of $G_4 = \begin{bmatrix} x_1^* & x_2^* & x_3^* & 0 \\ -x_2 & x_1 & 0 & x_3^* \\ -x_3 & 0 & x_1 & -x_2^* \\ 0 & -x_3 & x_2 & x_1^* \end{bmatrix}$

$$G_4^H = \begin{bmatrix} x_1^* & -x_2 & -x_3 & 0 \\ x_2^* & x_1 & 0 & -x_3 \\ x_3^* & 0 & x_1 & x_2 \\ 0 & x_3^* & -x_2^* & x_1^* \end{bmatrix}$$

$$G_4^H G_4 = \begin{bmatrix} x_1^* & -x_2 & -x_3 & 0 \\ x_2^* & x_1 & 0 & -x_3 \\ x_3^* & 0 & x_1 & x_2 \\ 0 & x_3^* & -x_2^* & x_1^* \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1^* x_1 + x_2 x_2^* + x_3 x_3^*) & (x_1^* x_2 - x_2 x_1^*) & (x_1^* x_3 - x_3 x_1^*) & (-x_2 x_3 + x_3 x_2) \\ (x_2^* x_1 - x_1 x_2^*) & (x_2^* x_2 + x_1 x_1^* + x_3 x_3^*) & (x_2^* x_3 - x_3 x_2^*) & (x_1 x_3 - x_3 x_1) \\ (x_3^* x_1 - x_3^* x_1) & (x_3^* x_2 - x_3^* x_2) & (x_3^* x_3 + x_1 x_1^* + x_2 x_2^*) & (-x_1 x_2 + x_2 x_1) \\ (-x_3^* x_2^* + x_2^* x_3^*) & (x_3^* x_1^* - x_1^* x_3^*) & (-x_2^* x_1^* + x_1^* x_2^*) & (x_3^* x_3 + x_2^* x_2 + x_1^* x_1) \end{bmatrix}$$

$$= \begin{bmatrix} |x_1|^2 + |x_2|^2 + |x_3|^2 & 0 & 0 & 0 \\ 0 & |x_1|^2 + |x_2|^2 + |x_3|^2 & 0 & 0 \\ 0 & 0 & |x_1|^2 + |x_2|^2 + |x_3|^2 & 0 \\ 0 & 0 & 0 & |x_1|^2 + |x_2|^2 + |x_3|^2 \end{bmatrix}$$

$$= \left(|x_1|^2 + |x_2|^2 + |x_3|^2\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_4^H G_4 = \left(|x_1|^2 + |x_2|^2 + |x_3|^2\right) I_4$$

Hence Proved

(b)

Spectral Efficiency (R) = $\dfrac{\text{No of bits in a codeword}}{\text{No. of time slots}}$

$$= \frac{6}{4}$$

$$\boxed{R = 3/2}$$

$$\boxed{R = 1.5 \text{ bits/s/Hz}}$$

(c)

$$G_4(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix}$$

$$\Delta G = G_4(x_1, x_2, x_3) - G_4(\check{x}_1, \check{x}_2, \check{x}_3)$$

$$\Delta G = G_4(x_1 - \check{x}_1, x_2 - \check{x}_2, x_3 - \check{x}_3)$$

Using property in Q3(a)

$$\Delta G^H \Delta G =$$

$$G_4^H(x_1 - \check{x}_1, x_2 - \check{x}_2, x_3 - \check{x}_3) \times G_4(x_1 - \check{x}_1, x_2 - \check{x}_2, x_3 - \check{x}_3)$$

$$\Delta G^H \Delta G = \left[(|x_1 - \check{x}_1|)^2 + (|x_2 - \check{x}_2|)^2 + (|x_3 - \check{x}_3|)^2\right] I_4$$

$$\det(\Delta G^H \Delta G) = \left[(|x_1 - \check{x}_1|)^2 + (|x_2 - \check{x}_2|)^2 + (|x_3 - \check{x}_3|)^2\right]^3$$

Therefore, as long as $(x, x_2, x_3) \neq (\check{x}_1, \check{x}_2, \check{x}_3)$ then $\det(\Delta G) \neq 0$ i.e guarantee the Full diversity

(d) Normalized coding gain

$$\eta = \frac{1}{2\sqrt{M_t}} \min_{(x_1,x_2,x_3)\neq(\check{x}_1,\check{x}_2,\check{x}_3)} |\det(\Delta G)|^{1/M_t}$$

Since $M_t = 4$

$$\eta = \frac{1}{2\sqrt{4}} \min_{(x_1,x_2,x_3)\neq(\check{x}_1,\check{x}_2,\check{x}_3)} |\det(\Delta G)|^{1/4}$$

$$\det(\Delta G) = |x_1 - \check{x}_1|^2 + |x_2 - \check{x}_2|^2 + |x_3 - \check{x}_3|^2$$

Possible cases

1. $x_1 \neq \check{x}_1$, $x_2 \neq \check{x}_2$, $x_3 \neq \check{x}_3$
2. $x_1 = \check{x}_1$, $x_2 \neq \check{x}_2$, $x_3 \neq \check{x}_3$
3. $x_1 \neq \check{x}_1$, $x_2 = \check{x}_2$, $x_3 \neq \check{x}_3$
4. $x_1 \neq \check{x}_1$, $x_2 \neq \check{x}_2$, $x_3 = \check{x}_3$
5. $x_1 \neq \check{x}_1$, $x_2 = \check{x}_2$, $x_3 = \check{x}_3$
6. $x_1 = \check{x}_1$, $x_2 \neq \check{x}_2$, $x_3 = \check{x}_3$
7. $x_1 = \check{x}_1$, $x_2 = \check{x}_2$, $x_3 \neq \check{x}_3$
8. $x_1 = \check{x}_1$, $x_2 = \check{x}_2$, $x_3 = \check{x}_3$

Min Value (cases 5–7)

$$\eta = \frac{1}{2\sqrt{4}} \min_{x_1\neq\check{x}_1} \left(|x_1 - \check{x}_1|^2\right)^{1/4}$$

OR

$$\eta = \frac{1}{2\sqrt{4}} \min_{x_2\neq\check{x}_2} \left(|x_2 - \check{x}_2|^2\right)^{2/4}$$

..

$$\epsilon_1 = \frac{1}{2\sqrt{4}} \min_{x_3 \neq \hat{x}_3} \left(|x_3 - \hat{x}_3|^2 \right)^{1/4}$$

$(1, -1)$ $(1, +j)$ $(1, -j)$
$(-1, j)$ $(-1, -j)$
$(j, -j)$

Possible unique combinations

min value is achieved when $(j, -j)$

$$\epsilon = \frac{1}{2\sqrt{4}} (-4)^{1/4}$$

(c) Assume that the channel coefficient matrix H is known at the receiver, then the ML decoding is

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \arg\min_{x_1,x_2,x_3} \|Y - \sqrt{P/2}\, G_4(x_1,x_2,x_3)\|_F^2$$

$$= \arg\min_{x_1,x_2,x_3} tr\{[Y - \sqrt{P/2}\, G_4 H]^H [Y - \sqrt{P/2}\, G_4 H]\}$$

$$= \arg\min_{x_1,x_2,x_3} \left[tr(Y^H Y) - 2Re\{tr[\sqrt{P/2}\, Y^H G_4 H]\} + P/2\, tr\{(G_4 H)^H (G_4 H)\}\right]$$

Since $\|A-B\|_F^2 = \|A\|_F^2 - 2Re\{tr\{A^H B\}\} + \|B\|_F^2$

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \arg\min_{x_1,x_2,x_3} \left[\|Y\|_F^2 - \sqrt{2P}\, Re\{tr(Y^H G_4 H)\} + P/2\, tr\{H^H G_4^H G_4 H\}\right]$$

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \arg\min_{x_1,x_2,x_3} \left[\|Y\|_F^2 - \sqrt{2P}\, Re\{tr(Y^H G_4 H) + P/2\, (|x_1|^2 + |x_2|^2 + |x_3|^2)\, tr\{H^H H\}\right]$$

Ignore $\|Y\|_F^2$ because we just want the combinat^n of x_1, x_2, and x_3 and not the actual minimum value

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \arg\min_{x_1, x_2, x_3} \left[-2\sqrt{P}\, \mathrm{Re}\{ tr(Y^H G_4 H) + \tfrac{P}{2}(|x_1|^2 + |x_2|^2 + |x_3|^2) \| H \|_F^2 \right\}$$

$$tr(Y^H G_4 H) = tr(G_4 H Y^H)$$

$$= tr\left(\begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & o \\ p & q & r & s \end{bmatrix} \right)$$

$$= (ax_1 + ex_2 + lx_3) + (-bx_2^* + fx_1^* + qx_3) + (-cx_3^* + nx_1^* - rx_2)$$
$$+ (-hx_3^* + ox_2^* + sx_1)$$

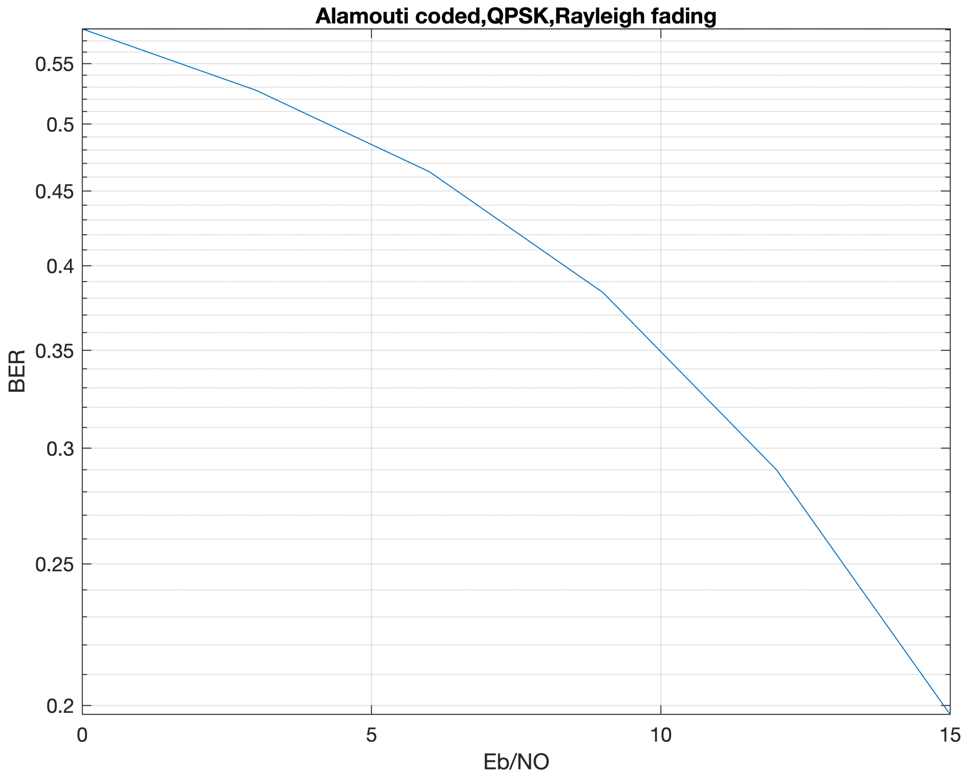
$$(\hat{x}_1, \hat{x}_2, \hat{x}_3) =$$
$$\arg\min_{x_1, x_2, x_3} \left[-2\sqrt{P}\, a x_1 - 2\sqrt{P} f x_1^* - 2\sqrt{P}\, n x_1^* - 2\sqrt{P}\, s x_1 + \tfrac{P}{2} |x_1|^2 \|H\|_F^2 \right.$$
$$-2\sqrt{P}\, e x_2 + 2\sqrt{P}\, b x_2^* + 2\sqrt{P}\, r x_2 - 2\sqrt{P}\, o x_2^* + \tfrac{P}{2} |x_2|^2 \|H\|_F^2$$
$$\left. \cdots \cdots \right]$$

$$= \arg\min_{x_1, x_2, x_3} \left[f_1(x_1) + f_2(x_2) + f_3(x_3) \right]$$

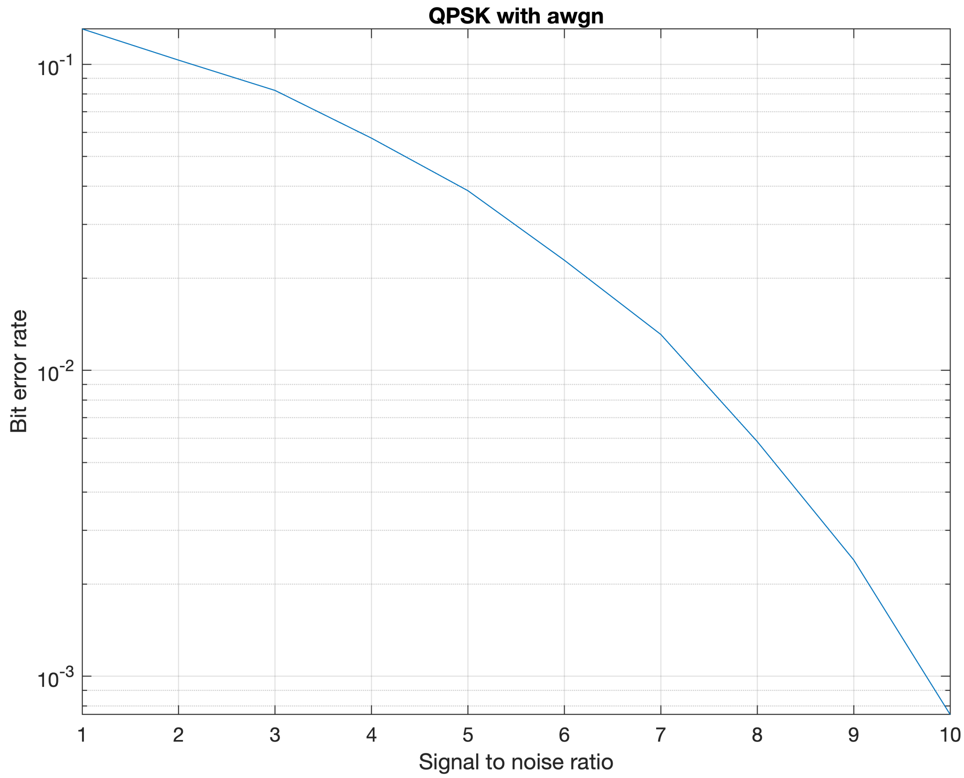
$$= \arg\min_{x_1} f_1(x_1) + \arg\min_{x_2} f_2(x_2) + \arg\min_{x_3} f_3(x_3)$$

i.e x_1, x_2 and x_3 can be decoded seperately, not jointly

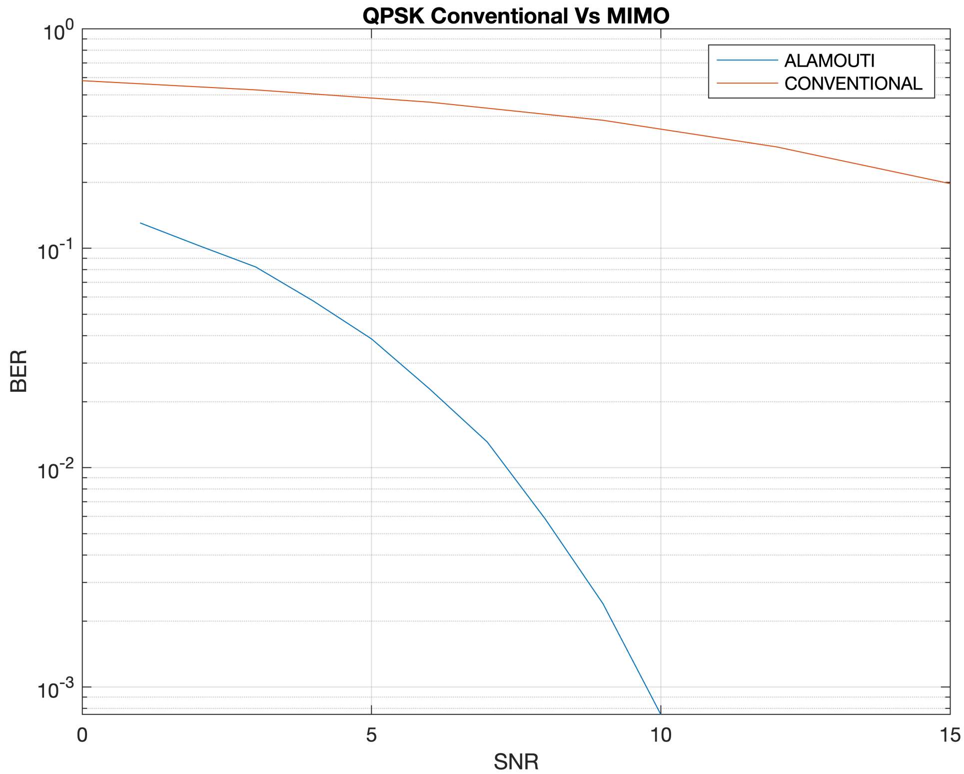
```
clear
EbNOdB_vectro=[0 3 6 9 12 15];
Eb=4;
for snri=1:length(EbNOdB_vectro)
  EbNOdB=EbNOdB_vectro(snri);
  EbNO=10^(EbNOdB/10);
  NO=Eb/EbNO;
  errcnt=0; symcnt=0;
  while errcnt <500000
    s1=sign(rand-0.5);
    s2=sign(rand-0.5);
    hA=sqrt(1/2)*(randn+1i*randn);
    hB=sqrt(1/2)*(randn+1i*randn);
    n1=sqrt(NO/2)*(randn+1i*randn);
    n2=sqrt(NO/2)*(randn+1i*randn);
    r1=hA*s1+hB*s2+n1;
    r2=hA*(-conj(s1)) + hB*(conj(s2)) + n2;
    s1s2pair=[[1,1];[1,-1];[1,1i];[1,-1i];[-1,1];[-1,-1];[-1,1i];[-1,-1i];[1i,1];[1i,-
    N_set=16;
    for k_set=1:N_set
       s1candidate=s1s2pair(k_set,1);
       s2candidate=s1s2pair(k_set,2);
       r1candidate=hA*s1candidate+hB*s2candidate;
       r2candidate= hA*(-conj(s1)) + hB*(conj(s2))  ;
       distance(k_set)=sum(abs([r1, r2]-[r1candidate,r2candidate]).^2);
    end
    [A B]= min(distance);
    s1_hat=s1s2pair(B,1);
    s2_hat=s1s2pair (B,2);
    if(s1_hat~=s1)
      errcnt = errcnt+1;
    end
    if(s2_hat~=s2)
      errcnt=errcnt+1;
    end
    symcnt = symcnt+2;
  end
  BER(snri) = errcnt/symcnt;
end
figure
semilogy(EbNOdB_vectro,BER);
title("Alamouti coded,QPSK,Rayleigh fading");
xlabel("Eb/NO");
ylabel('BER');
grid on
```




```
l=10000;
snrdb=[0 3 6 9 12 15];
snrlin=10.^(snrdb/10);
for snrdb=1:1:10
    si=2*(round(rand(1,l))-0.5);
    sq=2*(round(rand(1,l))-0.5);
    s=si+j*sq;
    w=awgn(s,snrdb,'measured');
    r=w;
    si_=sign(real(r));
    sq_=sign(imag(r));
    ber1=(l-sum(si==si_))/l;
    ber2=(l-sum(sq==sq_))/l;
    ber(snrdb)=mean([ber1 ber2]);
end
snrdb=1:1:10;
semilogy(snrdb,ber)
title('QPSK with awgn');
xlabel('Signal to noise ratio');
ylabel('Bit error rate');
grid on;
```




```
semilogy(snrdb,ber,EbN0dB_vectro,BER)
title('QPSK Conventional Vs MIMO');
xlabel("SNR");
ylabel("BER");
legend(["ALAMOUTI";"CONVENTIONAL"]);
grid on;
```



OBSERVATION : WE GOT MINIMUM BER WITH ALAMOUTI STBC COMPARED TO THE CONVENTIONAL ONE .IF THE CODING GAIN IS HIGH THAN BER WILL BE LESS.