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MIMO Wirers Communication
Assignment 2

1] $L=2$ $T=2$

Spectral Efficiency $(R) = \frac{\log_2 L}{T} = \frac{\log_2(2)}{2}$

$$R = \frac{1}{2} \text{ bit/s/Hz}$$

$$\boxed{R = 0.5 \text{ bit/s/Hz}}$$

2] At time t, the signal received at Rx, j can be given by

$$Y_t^j = \left(\sqrt{\frac{\rho}{M_t}}\right) \sum_{i=1}^{M_t} h_{i,j} c_t^i + n_t^j$$

The average SNR at each receiver antenna is

$$SNR_j = \frac{E\left[\left|\sqrt{\frac{\rho}{M_t}} \sum_{i=1}^{M_t} h_{i,j} c_t^i\right|^2\right]}{E\left[|n_t^j|^2\right]}$$

Since, $n_t^j \in CN(0,1)$

$$E\{|n_t^j|^2\} = \sum_{t=1}^{T} 1/T$$

$$SNR_j = \frac{\sum_{t=1}^{T} E\left[\left|\sqrt{\rho/M_t} \sum_{i=1}^{M_t} h_{i,j} c_t^i\right|^2\right] / T}{1}$$

$M_t = 2$, $M_r = 1$ and $T = 2$

$$E\left[\left|\sqrt{\frac{\rho}{M_t}}\sum_{i=1}^{M_t} h_i c_t^i\right|^2\right] = E\left[\left(\sqrt{\frac{\rho}{M_t}}\sum_{i=1}^{M_t} h_{i,j} c_t^i\right)\left(\sqrt{\frac{\rho}{M_t}}\sum_{i=1}^{M_t} h_{i,j} c_t^i\right)^*\right]$$

$$|a|^2 = a \cdot a^*$$

$$= \frac{\rho}{M_t} \cdot E\left[\sum_{i=1}^{M_t}\sum_{i'=1}^{M_t} h_{i,j}\, h_{i',j}^* \, c_t^i \, c_t^{i'*}\right]$$

$$= \frac{\rho}{M_t} \underbrace{\sum_{i=1}^{M_t} E\left[|h_{i,j}|^2 \cdot |c_t^i|^2\right]}_{\substack{=1 \\ \text{Variance}=1}} + \frac{\rho}{M_t} \underbrace{\sum_{\substack{1 \le i \\ i' \le 2 \\ i \ne i'}} E\left[h_{i,j} h_{i',j}^* c_t^i c_t^{i'*}\right]}_{=0}$$

$$= \frac{\rho}{M_t} \cdot \sum_{i=1}^{M_t} |c_t^i|^2$$

Since $E\left[\left|\sqrt{\frac{\rho}{M_t}}\sum_{i=1}^{M_t} h_i c_t^i\right|^2\right] = \frac{\rho}{M_t}\sum_{i=1}^{M_t} |c_t^i|^2$

$$SNR_j = \frac{\sum_{t=1}^{T} \frac{\rho}{M_t} \sum_{i=1}^{M_t} |c_t^i|^2}{T} = \frac{\rho}{M_t}\sum_{i=1}^{T}\sum_{i=1}^{M_t} |c_t^i|^2 = \frac{\rho}{M_t} \|c\|_F^2$$

$$= \rho$$

which is independent of j and M_t

3] If the codeword C_0 is sent i.e $Y = \sqrt{\frac{\rho}{M_t}}\, C_0 H + N$

for a fixed channel realization H, we have an instantaneous PEP as follows

$$P_r\{(C_0 \to \hat{C}) \mid H\} = Q\left(\sqrt{\frac{\rho}{2M_t}}\, \|(C_0 - \hat{C})H\|_F\right)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt$

$$\|A\|_F^2 = \sum_i \sum_j |a_{i,j}|^2$$

$$\|A\|_F = \left(\sum_i \sum_j |a_{i,j}|^2\right)^{1/2}$$

$$P_r\{C_0 \to C_1 \mid H\} = P_r\left\{\|Y - \sqrt{\tfrac{\rho}{M_t}}\, C_1 H\|_F^2 \leq \|Y - \sqrt{\tfrac{\rho}{M_t}}\, C_0 H\|^2 / H\right\}$$

$$Y = \sqrt{\frac{\rho}{M_t}}\, C_0 H + N$$

$$= P_r\left\{\|\sqrt{\tfrac{\rho}{M_t}}\, (C_0 - C_1)H + N\|_F^2 \leq \|N\|_F^2 \mid H\right\}$$

Denote $D \triangleq (C_0 - C_1)H$

$$= P_r\left\{\|\sqrt{\tfrac{\rho}{M_t}}\, D + N\|_F^2 \leq \|N\|_F^2 \mid H\right\} \to (1)$$

For any matrix A, total power of matrix is

$$\|A\|^2 = tr\{AA^H\} = tr\{A^H A\}$$

Similarly For any matrices A and B

$$\|A+B\|_F^2 = tr\{(A+B)(A+B)^H\} = tr\{AA^H + AB^H + BA^H + BB^H\}$$

$$= tr\{AA^H\} + tr\{AB^H\} + tr\{BA^H\} + tr\{BB^H\}$$

$$= \|A\|_F^2 + 2Re(tr\{AB^H\}) + \|B\|_F^2$$

Therefore

$$Pr\{C_0 \to C_1 | H\} = Pr\left\{\left\|\sqrt{\frac{\rho}{M_t}}D\right\|_F^2 + 2Re\left(tr\left\{\sqrt{\frac{\rho}{M_t}}DN^H\right\}\right) + \|N\|_F^2 \le \|N\|_F^2 \Big| H\right\}$$

$$= Pr\left\{\left\|\sqrt{\frac{\rho}{M_t}}D\right\|_F^2 + 2Re\left(tr\left\{\sqrt{\frac{\rho}{M_t}}DN^H\right\}\right) \le 0 \Big| H\right\} \to (2)$$

Here,

$$2Re(tr\{DN^H\}) \le -\sqrt{\frac{\rho}{M_t}}\|D\|_F^2$$

$$2Re\left(tr\left\{\frac{D}{\|D\|_F}N^H\right\}\right) \le -\sqrt{\frac{\rho}{M_t}}\|D\|_F$$

$T \times M_r$ $\qquad$ $M_r \times T$

$T \times T$ Matrix

Denoting: $\cancel{\left\{\frac{D}{\|D\|_F}N^H\right\}}$

Denoting $\frac{D}{\|D\|_F} \triangleq \{d_{i,j}\}_{T\times M_r}$ $\quad N^H \triangleq \{n_{i,j}\}_{M_r\times T}$

then $tr\left\{\frac{D}{\|D\|_F} N^H\right\} = \sum_{t=1}^{T}\sum_{j=1}^{M_r} d_{t,j}\, n_{j,t} \in CN(0,1)$

$$P_r\{C_0 \to C_1 | H\} = P_r\left\{2\,Re(w) \leq \sqrt{\frac{P}{M_t}}\|D\|_F \,\middle|\, H\right\}$$

$w = w_1 + j w_2 \in CN(0,1)$

$w_1 \in N(0, 1/2)$ $\quad \sqrt{2}\, w_1 \in N(0,1)$

$$P_r\{C_0 \to C_1 | H\} = P_r\left\{\sqrt{2}\, w_1 \leq -\sqrt{\frac{P}{2M_t}}\|D\|_F\right.$$

$$= \int_{-\infty}^{\sqrt{P/M_t}\|D\|_F} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= \int_{\sqrt{P/M_t}\|D\|_F}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= Q\left[\sqrt{\frac{P}{2M_t}}\|D\|_F\right]$$

Here $M_t = 2$

$$P_r\{C_0 \to C_1 | H\} = Q\left(\sqrt{P/4}\,\|D\|_F\right) \quad \text{--- (3)}$$

We know that

$$D \triangleq (C_0 - C_1)H$$

In question

$$C_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad C_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$\|D\|_F = \|(C_0 - C_1)H\|_F$$

$$\|D\|_F = \left\| \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|_F$$

$$= \left\| \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|_F$$

$$= \left\| \begin{matrix} 2h_1 + 2h_2 \\ -2h_1 + 2h_2 \end{matrix} \right\|_F$$

$$\|D\|_F = \sqrt{|2h_1 + 2h_2|^2 + |-2h_1 + 2h_2|^2}$$

$$= \sqrt{8(|h_1|^2 + |h_2|^2)}$$

Substituting $\|D\|_F$ in eq^n 3 we get

$$P_r\{C_0 \to C_1 | H\} = Q\left(\sqrt{P/4} * \sqrt{8(|h_1|^2 + |h_2|^2)}\right)$$

$$= Q\left(\sqrt{2P(|h_1|^2 + |h_2|^2)}\right)$$

— x — Hence Proved — x —

4) $\Pr\{C \rightarrow \hat{C}\} = E_H \left\{ Q\left(\sqrt{P/2M_t}\, \|(C-\hat{C})H\|_F\right) \right\}$

$$\leq E_H \left\{ \tfrac{1}{2} e^{-P/4M_t \|(C-\hat{C})H\|_F^2} \right\}$$

Here, $M_t = 2$, $M_r = 1$

$$\Pr\{C_0 \rightarrow C_1\} \leq E_H \left\{ \tfrac{1}{2} e^{-P/8 \|(C_0 - C_1)\|_F^2} \right\}^{M_r}$$

We know that

$$\|(C-\hat{C})\bar{H}_1\|_F^2 = tr\left\{ [(C-\hat{C})\bar{H}_1][(C-\hat{C})\bar{H}_1] \right\}$$

$$= tr\left\{ \bar{H}_1^H (C-\hat{C})^H (C-\hat{C}) \bar{H}_1 \right\}$$

General Rule

For any matrix A, $A^H A$ has an eigen decomposit^n as follows

$$\cancel{A^H A = U^H \Lambda U}$$

$$A^H A = U^H \Lambda U,$$

Where U is unitary matrix and Λ is diagonal matrix like

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

In which λ_1, λ_2 and λ_3 are non-zero eigen values of $A^H A$ and r is a rank of A

So, applying this general rule

$$(C_0-C_1)^H(C_0-C_1) = U^H \Lambda U$$
$$r = \text{rank}(C_0-C_1)$$

$$\|(C_0-C_1)H\|_F^2 = tr\{H^H(C_0-C_1)^H(C_0-C_1)H\}$$
$$= tr\{H^H U^H \Lambda U H\}$$
$$= tr\{(UH)^H \Lambda (UH)\}$$
$$= tr\{\hat{H}^H \Lambda \hat{H}\}$$
$$\because UH \triangleq \hat{H} = \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \vdots \\ \hat{h}_{H2} \end{bmatrix}$$

here we have $\bar{H} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

$$= \lambda_1|h_1|^2 + \lambda_2|h_2|^2 \quad \text{—} (a)$$

Now, in question 2 we calculated we have

$$\|D\|_F = \|(C_0-C_1)H\|_F = \sqrt{8|h_1|^2 + 8|h_2|^2}$$

From eqn (a) in questn (3)

So, $\|(C_0-C_1)H\|_F^2 = (\sqrt{8|h_1|^2 + 8|h_2|^2}$

~~From eqn (a) in questn (3)~~

$$\|(C_0-C_1)H\|_F^2 = 8|h_1|^2 + 8|h_2|^2$$

$\therefore$ if we see eqn a

we can say that $\lambda_1 = 8$ $\lambda_2 = 8$

So now

$$E_H\left\{e^{-\frac{P}{8}\|(C_0-C_1)H\|_F^2}\right\} = E_H\left\{e^{-\frac{P}{8}(\lambda_1|h_1|^2+\lambda_2|h_2|^2)}\right\}$$

$$= E_h\left\{e^{-P/8\,\lambda_1|h_1|^2}\right\}\cdot E_{h_2}\left\{e^{-P/8\,\lambda_2|h_2|^2}\right\}$$

So here distributⁿ of each random variable is again the same for each column

in which

$$E_{h_1}\left\{e^{-P/8\,\lambda_1|h_1|^2}\right\} = \int_0^\infty e^{-P/8\,\lambda_1 x}\, e^{-x}\, dx$$

$$= \frac{1}{P/8\,\lambda_1+1}$$

Similarly we can calculate for other eigen values

$$E_{h_2}\left\{e^{-P/8\,\lambda_2|h_2|^2}\right\} = \int_0^\infty e^{-P/8\,\lambda_1 x}\, e^{-x}\, dx$$

$$= \frac{1}{P/8\,\lambda_2+1}$$

∴ Putting in

$$P_r\{C_0\to C_1\} \leq \frac{1}{2}\left[\left\{E_h\, e^{-P/8\|C_0-C_1H\|_F^2}\right\}\right]^{M_r}$$

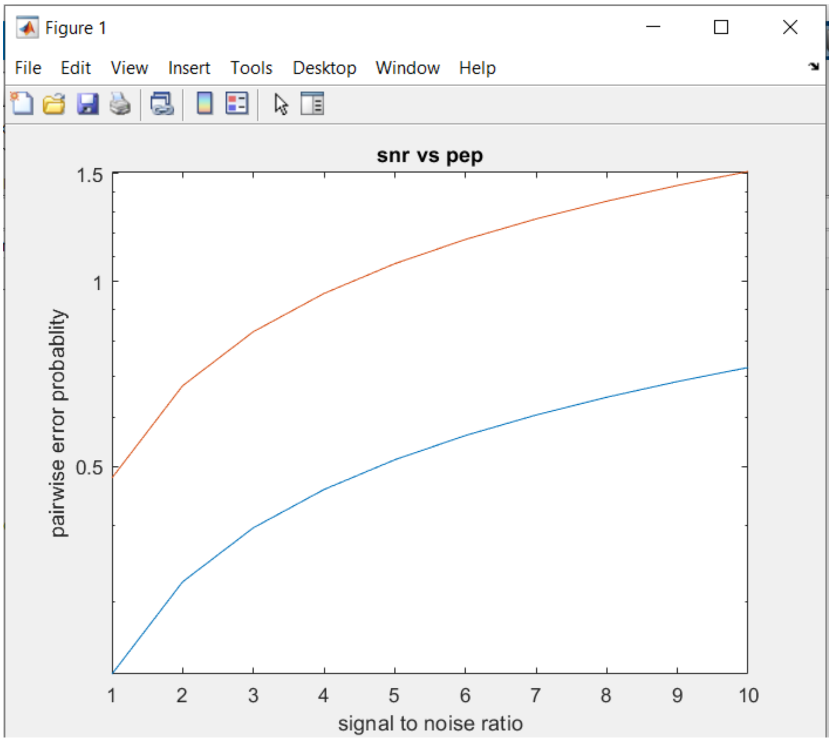
$$\leq \frac{1}{2}\left(\frac{1}{P/8\,\lambda_1+1}\right)\left(\frac{1}{P/8\,\lambda_1+1}\right)$$

and $\lambda_1=\lambda_1= 8$

$$\leq \frac{1}{2}\left(\frac{1}{P/8 \times 8+1}\right)\left(\frac{1}{P/8 \times 8+1}\right)$$

$$\leq \frac{1}{2}\left(\frac{1}{P+1}\right)\left(\frac{1}{P+1}\right)$$

$$P_r(C_0 - C_1) \leq \frac{1}{2}\left(\frac{1}{P+1}\right)^2$$




```
close all;
clc;

c0=[1 1; -1 1];
c1=[-1 -1; 1 -1];
h=[h1 h2];
h1=randn(1);
h2=randn(1);
x=randn(1);
mt=2;
mr=1;
T=2;
rho=1:10;
snr= 10*log10(rho); %is independent of mr and t
syms t
exp=2.718281^(-(t^2)/2);
a= int(exp,t,[x Inf]);
q=(1/(sqrt(2*pi)))*a;
q1=(0.5*(2.718281^(-(x^2/2)))); %for large x
exp1=sqrt(2*rho*((abs(h1)^2)+(abs(h2)^2)));
pep=q*exp1;
pep1=q1*exp1;
figure
semilogy(snr,pep)
xlabel('signal to noise ratio')
```

```
ylabel('pairwise error probablity')
title('snr vs pep')
hold on
semilogy(snr,pep1)
xlabel('signal to noise ratio')
ylabel('pairwise error probablity')
title('snr vs pep')
```