Developing clustering algorithms for conditional extremes models

Thesis formulation report (TFR) presentation
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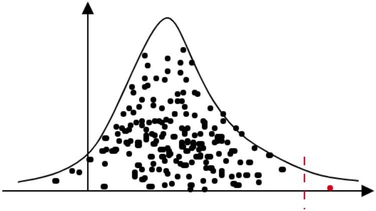
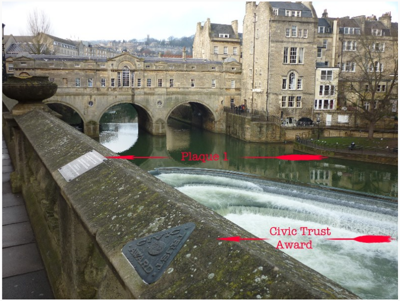
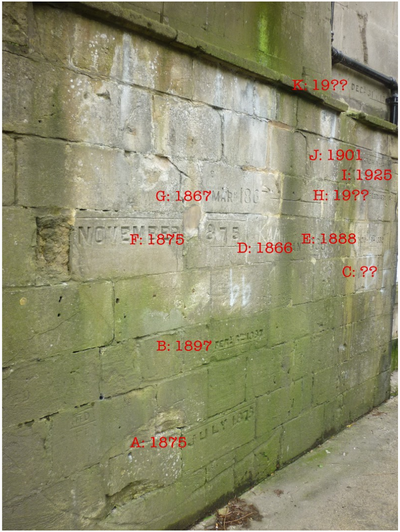
Introduction

Introduction

- Extreme Value Theory models the extreme tails of distributions X
- Only known method that can reliably predict beyond observations in extremal context
- Application fields include finance and insurance, and environmental data, such as extreme precipitation and/or wind speeds.



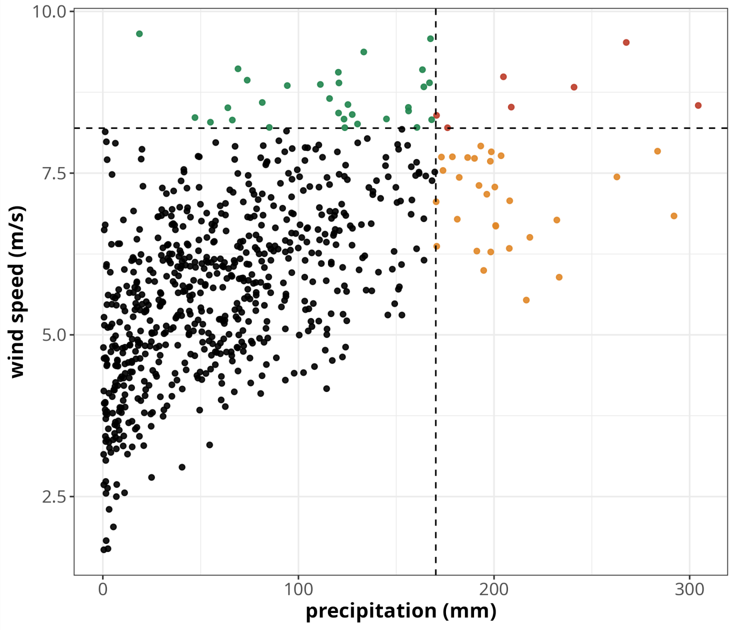

Plaque 1
Civic Trust
Award


K: 19??
J: 1901
I: 1925
G: 1867
H: 19??
F: 1875
D: 1866
E: 1888
C: ??
B: 1897
A: 1875

Multivariate extremes

- Concurrently occurring extremal events can be particularly destructive
- $\mathbb{P}(X \in C) = \sum_{i=1}^{d} \mathbb{P}(X_i \in C_i)$, for some extreme set C_i corresponding to each vector X_i.
- This may refer to different spatial locations, or different variables
- For example, offshore platforms must be built to withstand extreme wind speed and wave height conditions at sea
- Storm defences must withstand extreme rainfall and wind speed conditions, and insurance premiums must take into account these particularly destructive events

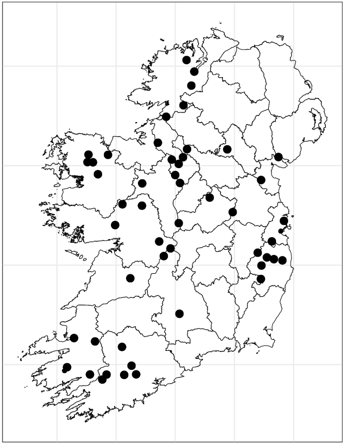
Multivariate extremes



Motivating example

Motivating example - Ireland

- Extreme precipitation and wind speed in Ireland, 1990-2020
- Precipitation data from 52 Met Éireann weather sites across country, wind data from ERA5 reanalysis dataset.
- Take weekly sum of precipitation and mean of daily wind speed maxima for Winter only (Oct-Mar), in line with Vignotto et. al. (2021) [1].



Univariate extremes

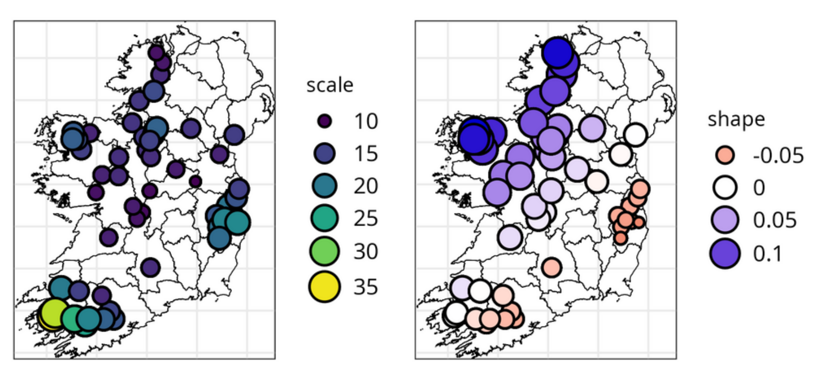
Generalised Pareto distribution

The Generalised Pareto distribution (GPD) has survival function:

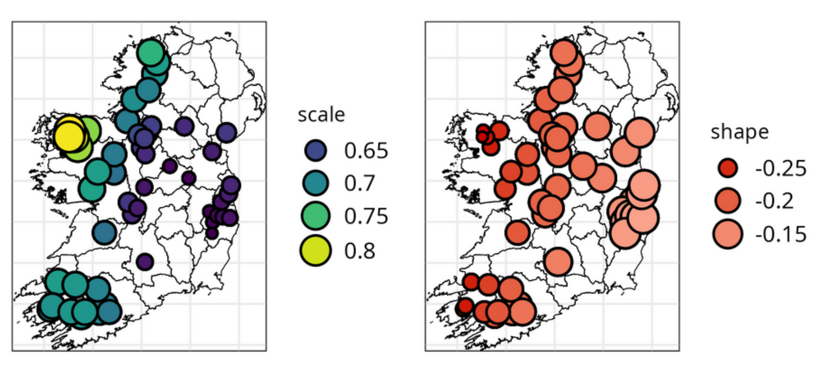
$$\mathbb{P}(X > x + u \mid X > u) = \left(1 + \xi \frac{x}{\sigma}\right)_+^{-1/\xi},$$

- $(x)_+ = \max(0, x)$
- Scale σ, shape ξ, threshold u
- For $\xi < 0$, (increasingly small) finite upper end point
- Can model parameters as function of spatial location, i.e. $\sigma(s), \xi(s)$, with $s =$ (longitude, latitude)
- Threshold often taken as high quantile of data, can also model as $u(s)$

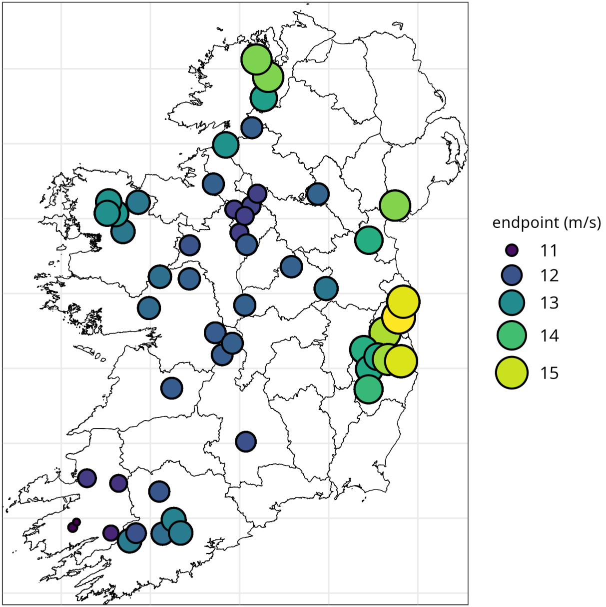
Motivating example - Precipitation



Motivating example - Wind speed



Motivating example - Wind speed



Conditional extremes

Introduction

- Gets name from modelling *conditional* on single variable, e.g. precipitation conditional on observing extreme wind speed
- Has simple marginal and dependence components
- Can model all varieties of extremal dependence, from asymptotic independence (Probability of X_i being extreme is not effected by X_j being extreme) to perfect dependence (X_i is extreme only where X_j is extreme)
- Some definitions:
    - X_{-i}: Vector X without i^{th} component
    - $\alpha_{|i}$: Vector of parameters $\alpha_{j|i}$ conditional on X_i, $j \neq i$

Univariate

Univariate component of model: semi-parametric piecewise function of empirical distribution below threshold and GPD (from previous section) above threshold:

$$\hat{F}_{X_i}(x) = \begin{cases} 1 - \{1 - \tilde{F}_{X_i}(u_{X_i})\}\{1 + \xi_i(x - u_{X_i})/\sigma_i\}_+^{-1/\xi_i} & \text{if } x > u_{X_i} \\ \tilde{F}_{X_i}(x) & \text{if } x \leq u_{X_i}, \end{cases}$$

where $\tilde{F}_{X_i}$ is the empirical distribution function of X_i.

Marginal transformation

Marginals transformed to Laplace margins using probability integral transform:

$$Y_i = \begin{cases} \log\{2F_{X_i}(X_i)\} & \text{for } X_i < F_{X_i}^{-1}(0.5), \\ -\log\{2(1 - F_{X_i}(X_i))\} & \text{for } X_i \geq F_{X_i}^{-1}(0.5), \end{cases}$$

- transformation denoted Y_i to differentiate from original marginals X
- Both tails exponential with mean 1

Multivariate

Dependence component of model:

$$Y_{-i} = \alpha_{|i} y_i + y_i^{\beta_{|i}} Z_{|i}, \text{ for } Y_i = y_i > u_{Y_i}.$$

- $\alpha_{j|i} \in [-1, 1]$ is slope parameter for Y_j conditional on Y_i, $\beta_{j|i} \in (-\infty, 1]$ is spread parameter
- Residuals $Z_{|i}$ said to have distribution $G_{|i}$
- $\beta_{|i}$ controls level of stochasticity of relationship between Y_{-i} and large Y_i.
- Special cases:
  - $\alpha_{|i} = 0, \beta_{|i} = 0 \implies Y_{-i}, Y_i$ independent
  - $\alpha_{|i} = -1/1, \beta_{|i} = 0 \implies$ perfect positive/negative dependence
  - $-1 < \alpha_{|i} < 1 \implies$ asymptotic independence

Estimation

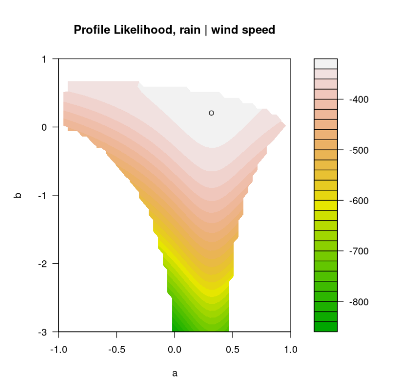
Common estimation procedure:

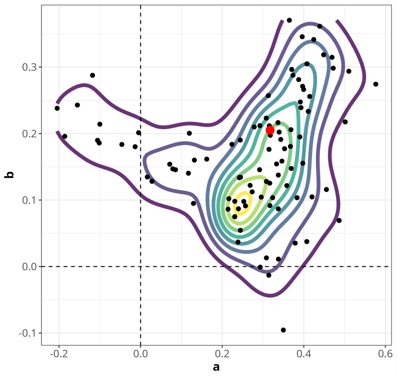
1. First assume $Z_{|i} \sim N(\mu_{|i}, \sigma_{|i})$, generate residuals from this distribution
2. Use likelihood methods to estimate $\hat{\alpha}_{|i}, \hat{\beta}_{|i}$, and nuisance parameters $\hat{\mu}_{|i}, \hat{\sigma}_{|i}$
3. Estimate $\hat{G}_{|i}$ as the empirical distribution of

$$Z_{|i} = \frac{Y_{-i} - \hat{\alpha}_{|i} Y_i}{Y_i^{\hat{\beta}_{|i}}}.$$

This procedure is very simple, and can easily be used to generate MC samples to calculate desired probabilities

Motivating example – Uncertainty

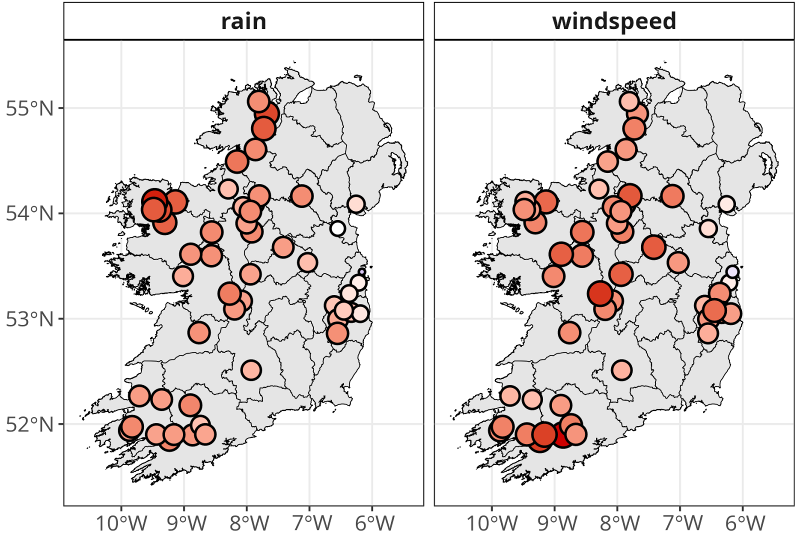




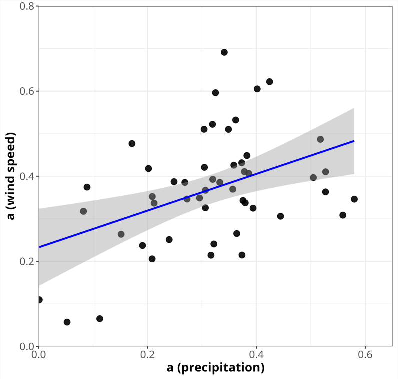
Fixing β

- Problem: Difficult to interpret results while varying both α, β due to uncertainty (and negative, unlikely estimates for β)
- "Hack": Fix $\beta = 0.1$ (allowing some stochasticity), estimate only α
- Idea: By estimating only one parameter, should contain all information about extremal dependence, and be more easily interpretable

Results



Interpretation



- High values of α for wind speed seem to be positively correlated with high values for rainfall, indicating locations where concurrent extremal events are more common
- Extreme rainfall is more likely to occur along with high wind speeds the further West you travel

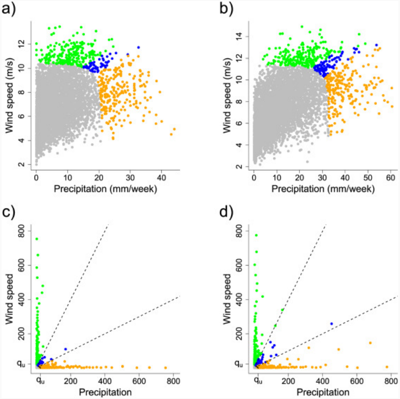
Clustering

Clustering

- There are extensions to CE model which seek to overcome shortcomings of “vanilla” model, such as spatial and spatio-temporal versions.
- Alternative/additional approach: clustering
- Clustering done for two main reasons: to enhance the explainability and interpretation of data, and to improve parameter estimation

Explanatory clustering

- Mainly derives distance metric to perform clustering like k-means, k-mediods, and have user-defined cluster number
- Bernard et. al. (2013) [2] uses the F-madogram (extremal variogram) as distance measure
- Vignotto et. al. (2021) [1] computes KL divergence for risk functions of transformed bivariate rainfall and wind speed observations for different raster locations in UK and Ireland



Hierarchical clustering

- Seeks to find (latent, data-driven) groupings over which parameter inference is optimised
- In Frequestist setting, EM algorithm often used to sequentially maximise likelihood over group membership and within-group parameters (Carreau et. al. (2017) [3], Dupuis et. al. (2023) [4])
- In Bayesian setting, one method includes using Reversible Jump MCMC algorithm which estimates number of clusters and cluster membership as latent variables (Rohrbeck, Tawn (2021) [5])
- Need for priors can be seen as both a strength and a weakness, but the use of distributions improves uncertainty estimation

Conclusions and future work

Conclusions and future work

Conclusions

- Conditional extremes model explained and shown with motivating example
- "Hacks" required to get reasonably interpretable results with "vanilla" model
- Lit review of clustering within extremes , some promising methods found

Future Work

- Perform k-mediods similar to [1] on conditional extremes regression line, where KL divergence appears to be appropriate distance metric, compare results
- Derive Bayesian clustering algorithm similar to [5]
  - Sensible initial prior distribution for Z is the multivariate Normal distribution, will assess suitability in practice
- If successful, expand schemes to extended conditional extremes models

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