A clustering framework for conditional extremes models

Environmental and Ecological Statistics Conference 2025 Talk,
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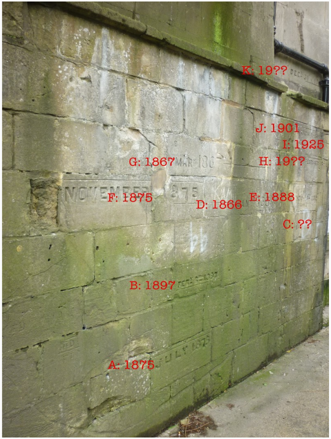
Supervised by Christian Rohrbeck and Jordan Richards (University of Edinburgh)

Introduction


K: 19??
J: 1901
I: 1925
G: 1867
H: 19??
F: 1875
D: 1866
E: 1888
C: ??
B: 1897
A: 1875

Problem

- Often want to estimate

$$\mathbb{P}(X > x, Y > y) = \mathbb{P}(Y > y \mid X > x)\mathbb{P}(X > x)$$

  for large x, y
- "concomitant"/concurrent extreme events for random vector $\mathbf{X}$ often particularly devastating
- Goal: **identify trends by clustering sites with similar tail dependence**

Dependence Modelling

Asymptotic dependence

- Coefficient of extremal dependence $\chi \in [0, 1]$,

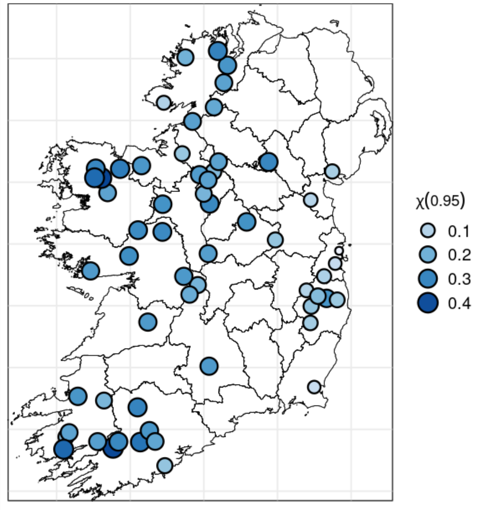
$$\chi = \lim_{u \to 1} \mathbb{P}\big[F_1(X_1) > u \mid F_2(X_2) > u\big]$$

- (Increasingly strong) asymptotic dependence for $\chi > 0$.
- However, χ only gives summary; inference requires **dependence model**.

Coles, J. Heffernan, and J. Tawn, *"Dependence measures for extreme value analyses"* (1999)

Ireland

- Precipitation[1] & wind speed[2] data for 59 sites across Ireland, Winter months (Oct-Mar) 1990-2020

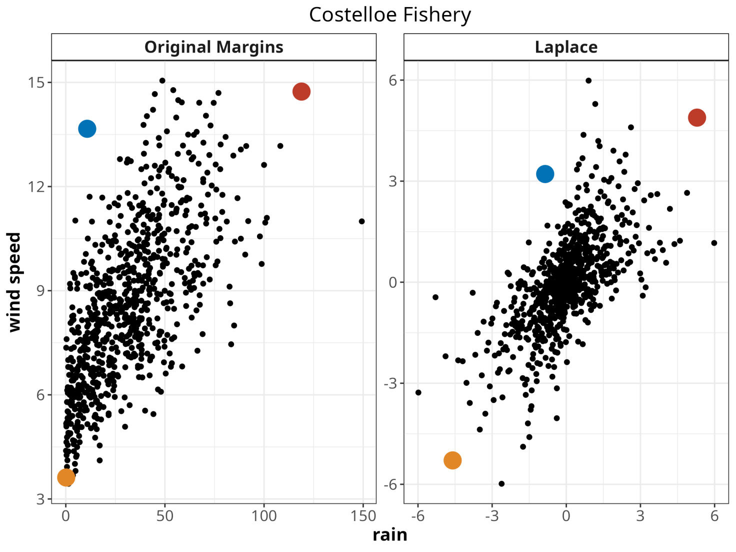


[1] Met Éireann weekly aggregate

[2] ERA5 reanalysis weekly mean of daily maxima

Conditional extremes

Marginal transformation



Conditional extremes

- Heteroskedastic regression dependence model:

$$\left(Y \mid X = x\right) = \alpha x + x^{\beta} Z, \text{ for } x > u$$

  - slope parameter α for Y given large X,
  - "spread" parameter $\beta \in (-\infty, 1]$ controls stochasticity of relationship between Y and large X.

Conditional extremes

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$$\left(\boldsymbol{Y}_{-i} \mid Y_i = y_i\right) = \boldsymbol{\alpha}_{|i} y_i + y_i^{\boldsymbol{\beta}_{|i}} \boldsymbol{Z}_{|i}, \text{ for } y_i > u_{Y_i}$$

  - slope parameter $\alpha_{j|i} \in [-1, 1]$ for Y_j given large Y_i,
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J. E. Heffernan and J. A. Tawn, *"A Conditional Approach for Multivariate Extreme Values (with Discussion)"* (2004)

Conditional extremes

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- Key assumptions:
  - Residuals $\mathbf{Z}_{|i} \sim N(\boldsymbol{\mu}_{|i}, \boldsymbol{\Sigma}_{|i})$
  - $\mathbf{Z}_{|i}, Y_i$ conditionally independent for large Y_i

- Special cases:

$$\begin{aligned} \boldsymbol{\alpha}_{|i} = 0, \boldsymbol{\beta}_{|i} = 0 \quad &\Longrightarrow \mathbf{Y}_{-i}, \mathbf{Y}_i \text{ independent,} \\ \boldsymbol{\alpha}_{|i} = -1/1, \boldsymbol{\beta}_{|i} = 0 \quad &\Longrightarrow \text{ perfect positive/negative dependence,} \\ -1 < \boldsymbol{\alpha}_{|i} < 1 \quad &\Longrightarrow \text{ asymptotic independence} \end{aligned}$$

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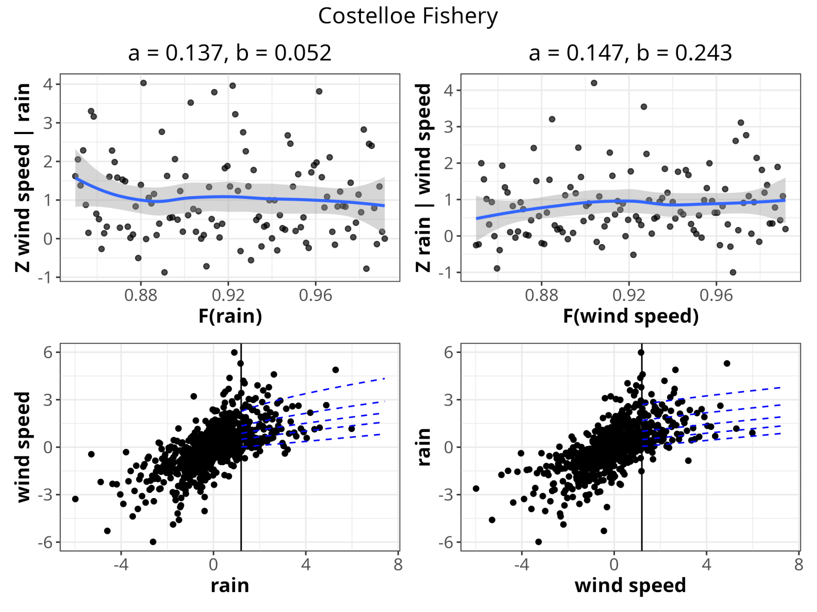
Conditional extremes



Inference

Inference assumes conditional distribution follows a multivariate Normal (MVN) distribution:

$$\left(\boldsymbol{Y}_{-i} \mid Y_i = y_i\right) \sim N\left(\boldsymbol{\alpha}_{|i} y_i + y_i^{\boldsymbol{\beta}_i} \boldsymbol{\mu}_{|i}, y_i^{\boldsymbol{\beta}_i} \boldsymbol{\Sigma}_{|i}\right), \text{ for } Y_i > u_{Y_i}$$

$\Longrightarrow$ **dependence structures at different sites can be compared using their MVN distributions**

Clustering

skew-geometric Jensen-Shannon divergence

- Kullback-Leibler divergence $KL(X \mid\mid Y)$ measures (**asymmetric**) distance from X to Y
- $KL(X \mid\mid Y)$ has closed form for two MVNs
- $KL(X \mid\mid Y) + KL(Y \mid\mid X)$ symmetric but does not satisfy triangle inequality $\implies$ **not a true metric**.

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- weighted product G_α of two exponential family members is in **same family**.
- **Skew-geometric Jensen-Shannon divergence**:

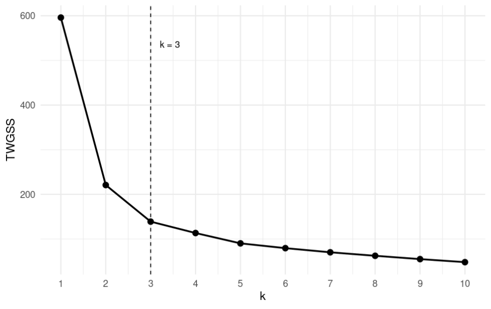
$$JS^{G_\alpha}(X \,||\, Y) = \frac{1}{2}\left\{KL(X \,||\, G_\alpha(X, Y)) + KL(Y \,||\, G_\alpha(X, Y))\right\}$$

Nielsen, *"On the Jensen–Shannon Symmetrization of Distances Relying on Abstract Means"* (2019)

Clustering

- Uses **k-medoids** to cluster over JS_{G_α} dissimilarity matrix between sites
- **Elbow plot**: k chosen using **Total Within Group Sum of Squares** between sites x within clusters C_i with medoids m_i:

$$\mathrm{TWGSS}(k) = \sum_{i=1}^{k} \sum_{x \in C_i} JS^{G_\alpha}\left(x \,||\, m_i\right)$$

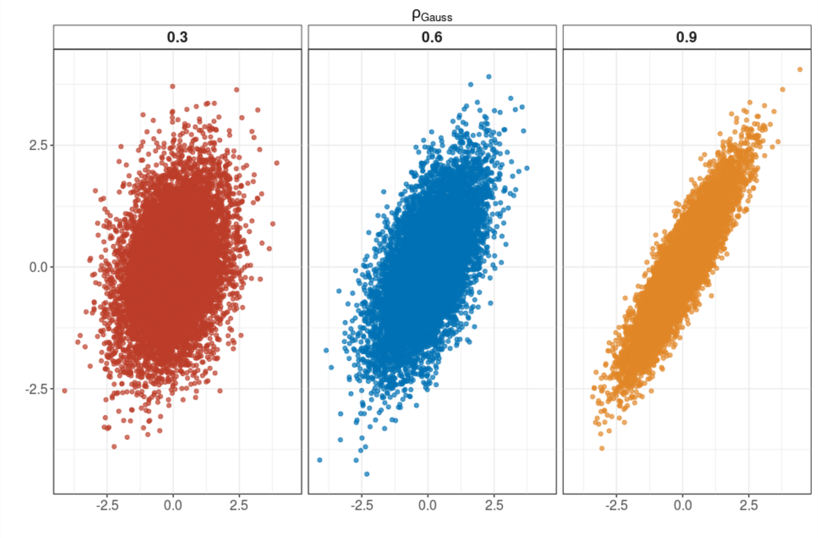


Simulations

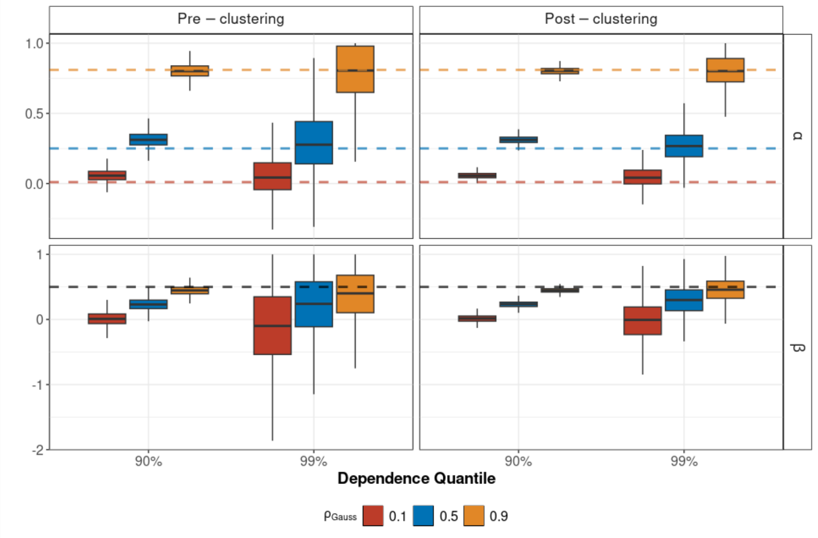
Gaussian copula

- For single "site", generate data from bivariate Gaussian copula with correlation ρ_{Gauss}.
- (asymptotic) CE parameters are $\alpha = \rho^2_{\text{Gauss}}, \beta = 1/2$
- Example: 12 "sites" with $10{,}000$ observations of 2 variables
- 3 clusters of 4 locations defined by respective ρ_{Gauss} values of $0.3, 0.6, 0.9$.

Gaussian copula



Gaussian copula - results



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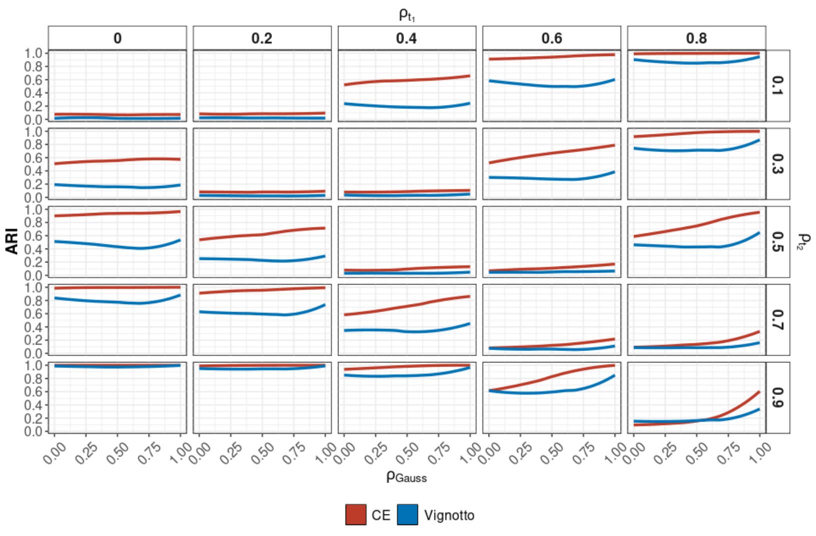
- 2 clusters each of 6 sites
- Grid search performed over $\rho_{\text{Gauss}}, \rho_{t_1}, \rho_{t_2} \in [0, 1]$.
- Clustering compared to competing Vignotto algorithm using Adjusted Rand Index ARI $\in [0, 1]$

Vignotto, Engelke, and Zscheischler,
"Clustering bivariate dependencies of compound precipitation and wind extremes over Great Britain and Ireland" (2021)

Mixture simulation



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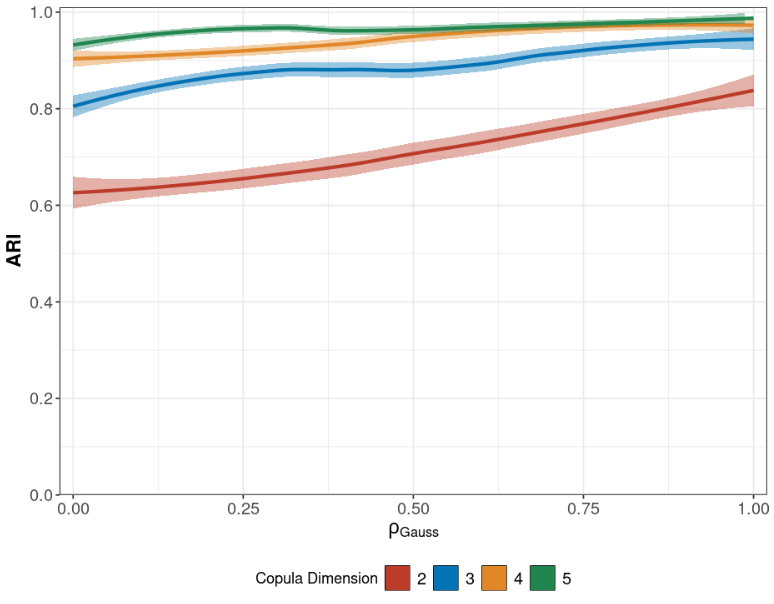
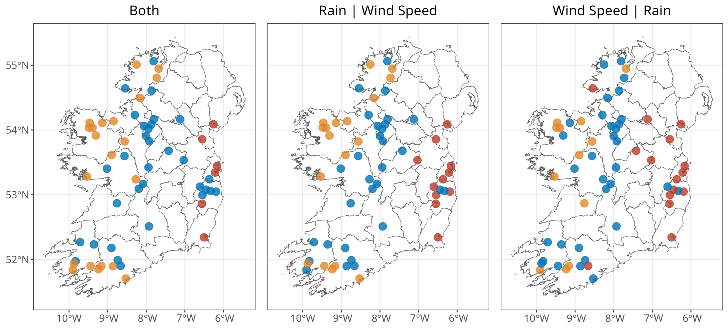


Ireland meteorological data

Ireland clustering



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- Conclusion:
  - Principled & effective clustering framework for CE models
  - Simulations & application show clustering can group sites with similar tail dependence, which may aid interpretation

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- **Conclusion**:
  - Principled & effective clustering framework for CE models
  - Simulations & application show clustering can group sites with similar tail dependence, which may aid interpretation
- **Limitations**:
  - **CE**: Gaussian assumption for $\boldsymbol{Z}_{|i}$
  - **Clustering**: Uncertainty in CE fits is ignored in clustering

Thank you! For slides and supplementary materials, please scan the QR code below:

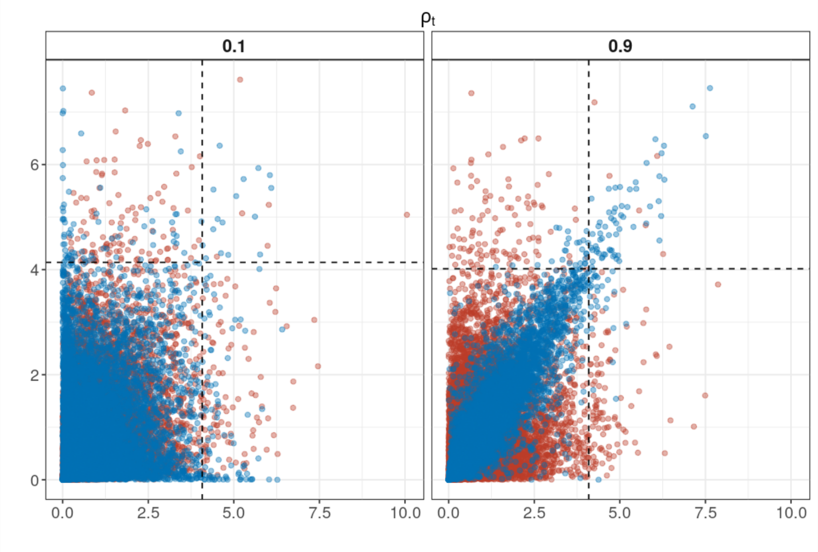
Email: *pot23@bath.ac.uk*

References

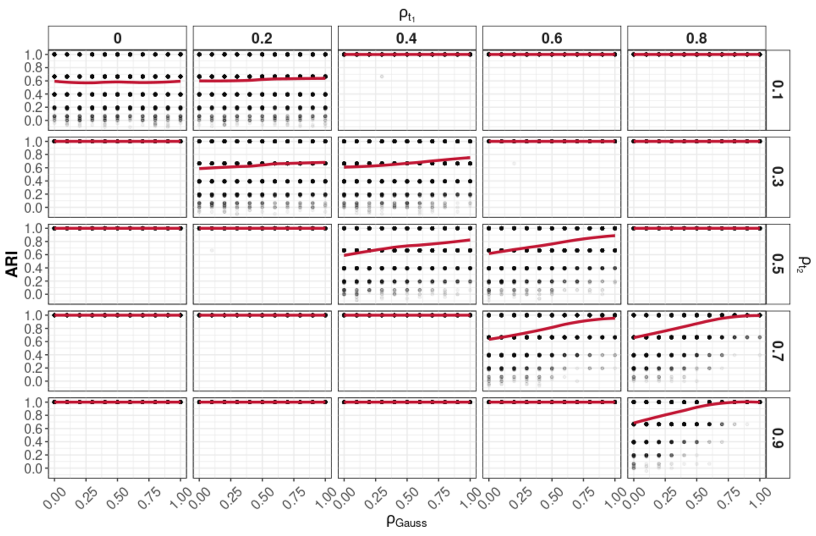
- Coles, Stuart, Janet Heffernan, and Jonathan Tawn (1999). "Dependence measures for extreme value analyses". In: *Extremes (Boston)* 2.4, pp. 339–365.
- Heffernan, Janet E. and Jonathan A. Tawn (July 2004). "A Conditional Approach for Multivariate Extreme Values (with Discussion)". In: *Journal of the Royal Statistical Society Series B: Statistical Methodology* 66.3, pp. 497–546. ISSN: 1467-9868. DOI: `10.1111/j.1467-9868.2004.02050.x`. URL: `http://dx.doi.org/10.1111/j.1467-9868.2004.02050.x`.
- Nielsen, Frank (May 2019). "On the Jensen–Shannon Symmetrization of Distances Relying on Abstract Means". In: *Entropy* 21.5, p. 485. ISSN: 1099-4300. DOI: `10.3390/e21050485`. URL: `http://dx.doi.org/10.3390/e21050485`.
- Vignotto, Edoardo, Sebastian Engelke, and Jakob Zscheischler (2021). "Clustering bivariate dependencies of compound precipitation and wind extremes over Great Britain and Ireland". In: *Weather and Climate Extremes* 32, p. 100318.

Appendix

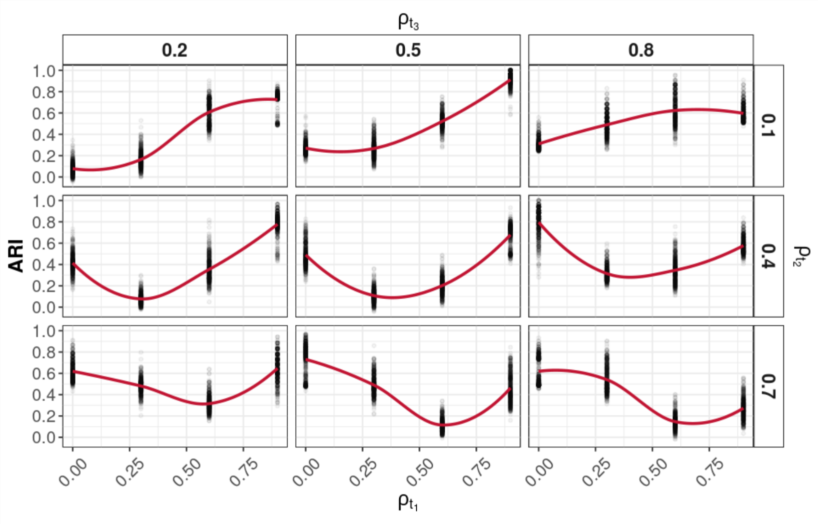
Mixture simulation



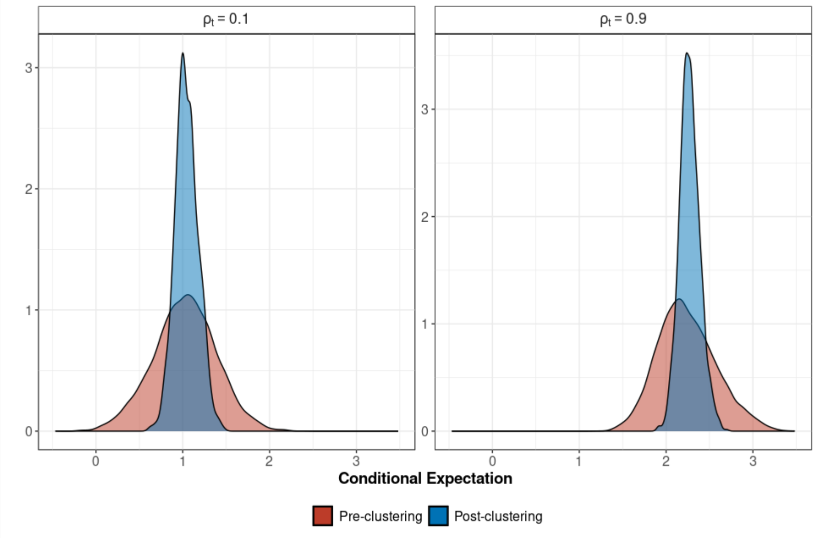
Mixture simulation (three variables)



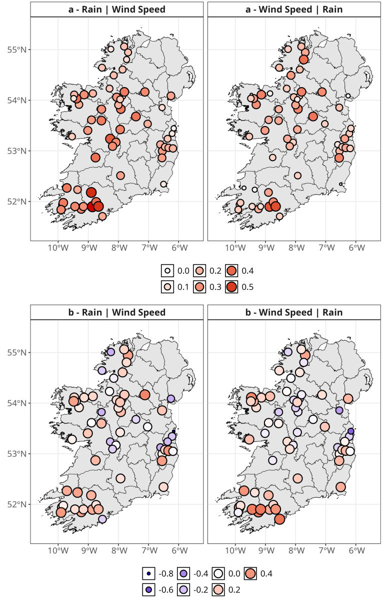
More realistic example



Uncertainty Reduction



α, β pre-clustering



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