Queues and Wait Times at Canadian Airports

Abstract

By providing efficient and effective **pre-board screening** (PBS), the *Canadian Air Transport Security Authority* (CATSA) ensures that security requirements are met while maintaining an appropriate balance between staffing and the wait time experienced by passengers.

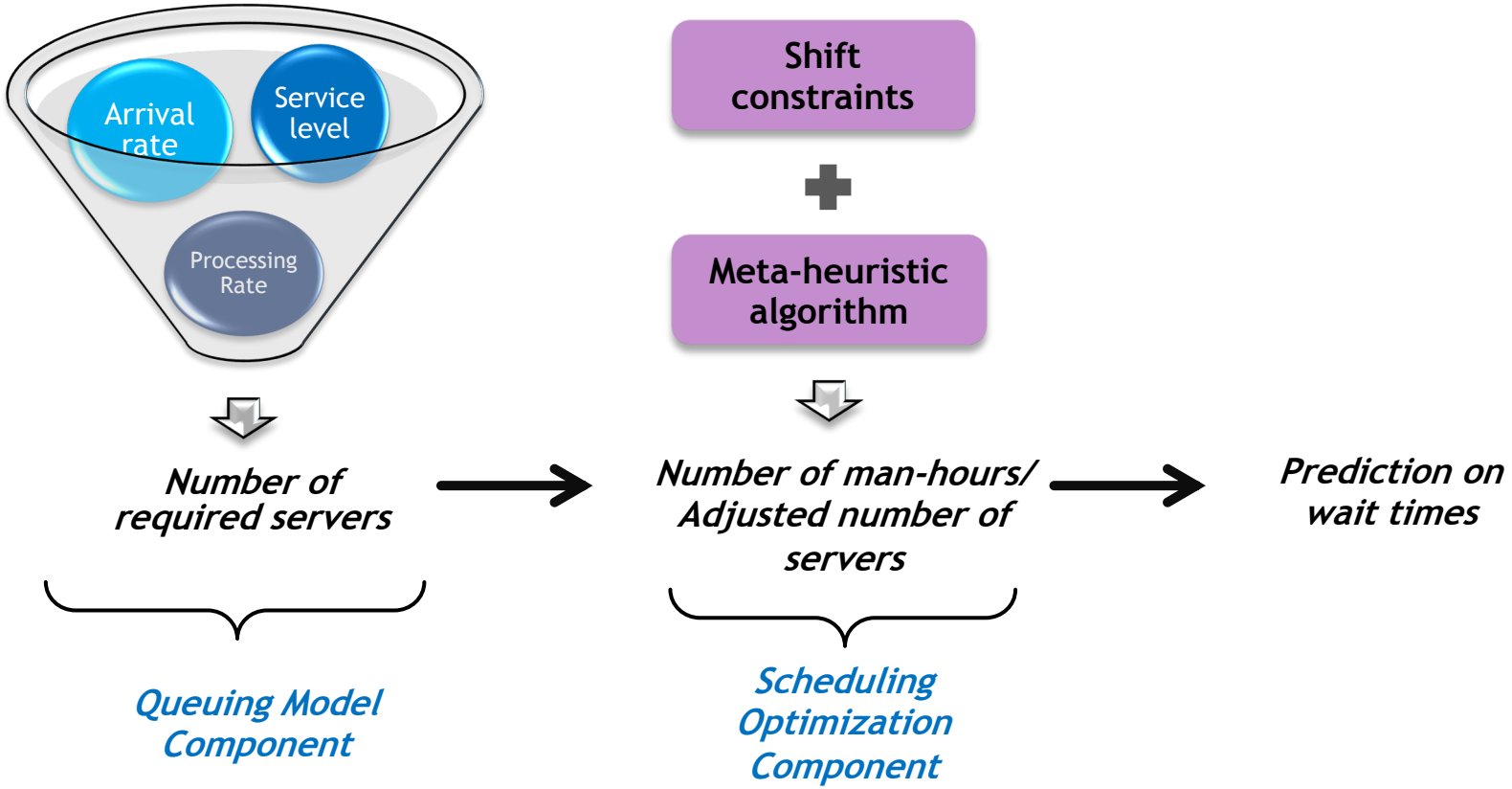
We use **queueing theory** to develop a model which can predict, among other things, the number of servers required to achieve particular service levels based on forecast arrival rates.

In this presentation, we describe the underlying model and discuss some of its possible refinements.

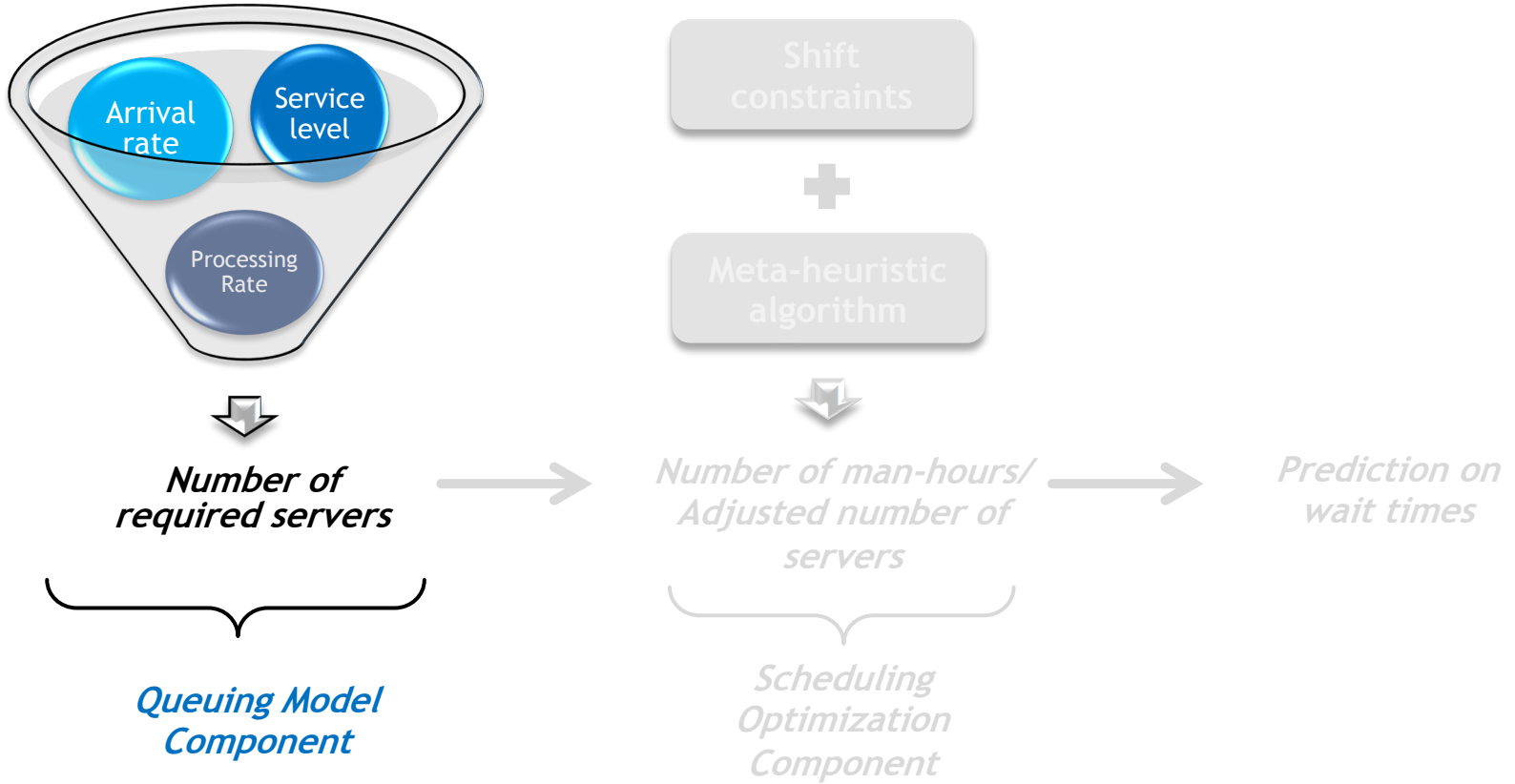
Outline

1. Preliminaries
2. Model Flow
3. $M/M/1$ Queueing Model
4. Regression Model
5. Departure Model
6. Predictions
7. Discussion
8. Post-Mortem

CATSA's Wait Time Impact Model



Presentation Focus



Preliminaries
Objectives

For each combination of checkpoint, time period, day of the week, season (cluster), use field data to provide estimates of:

- passenger arrival rates λ
- processing rates μ
- number of servers c

For each cluster, given λ, μ, and c, calculate the quality of service (QoS) level curve $(p_x(x), x)$ (i.e. percentage p of passengers which wait less than x minutes).

For each cluster, predict the average number of servers c^* required to achieve a prescribed QoS level (p_x, x) given an arrival profile λ^*.

Preliminaries
Definitions

The **Poisson process** is a stochastic process where the time between any two consecutive events is exponentially distributed with parameter λ.

$M/M/c$ queueing model

- arrivals form a single queue governed by a Poisson process
- arriving customers are processed by c servers
- service times are exponentially distributed

Various quantities

- **arrival rate:** rate at which passengers arrive for PBS (i.e. passengers per minute)
- **service rate:** processing rate at a screening line (i.e. maximal potential throughput)
- **number of servers:** number of screening lines
- **service level:** % of people waiting less than a given number of minutes

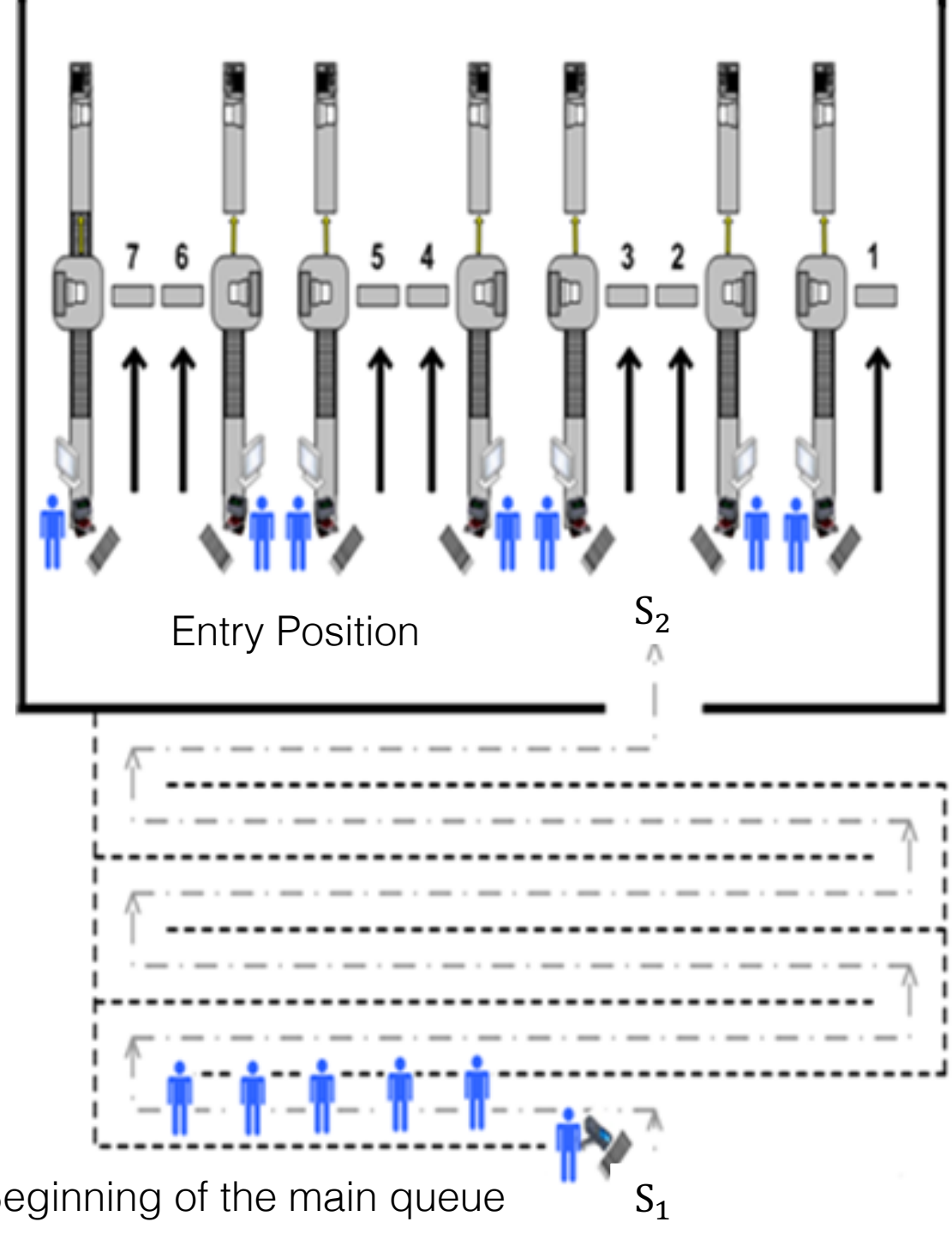
Preliminaries

Description of the PBS Process

Passengers enter the main queue, where their boarding pass may be scanned at S_1.

Once they reach the end of the main queue, their boarding pass is scanned at S_2 and they are sent to one of the active lines for processing.

In practice, it may often happen that only the S_2 reading is available.



Preliminaries

Available Data Sources

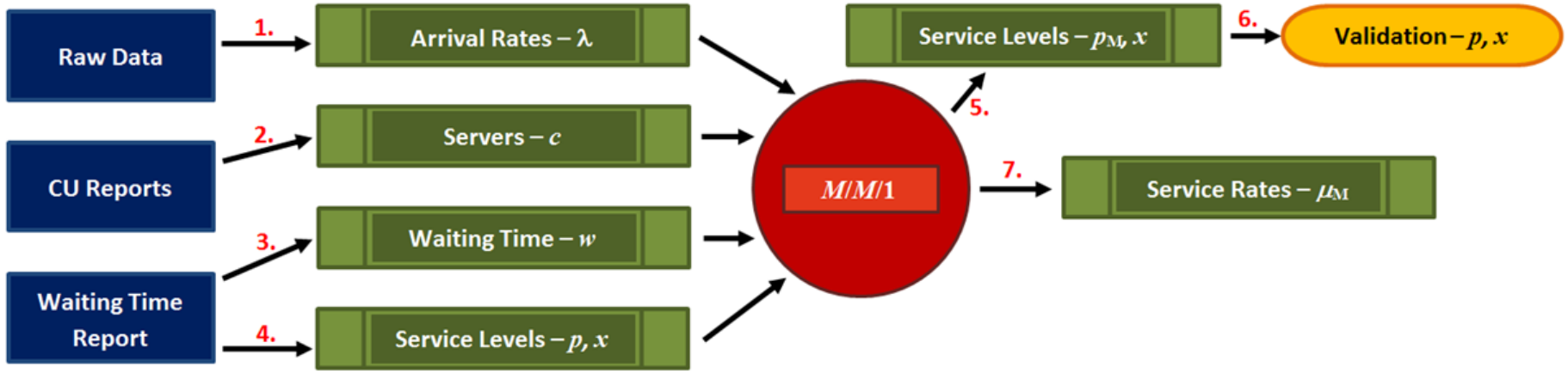
Raw Data: for each passenger reaching the end of the main queue, we have

- the date
- the scan time upon entering the main queue (S_1)
- the scan time upon exiting the main queue (S_2)
- the wait time between S_1 and S_2 (passengers may not have been scanned upon entering the main queue).

Checkpoint Utilization Report: for each day of the year and each 15–minute block, this dataset records the maximum number of open lines (servers).

Waiting Time Report: consists of the subset of Raw Data for both S_1, S_2 are available. Observations for which the wait time exhibits outliers have been removed by CATSA.

Model Flow I
M/M/1



$M/M/1$ Queueing Model
Generalized Servers

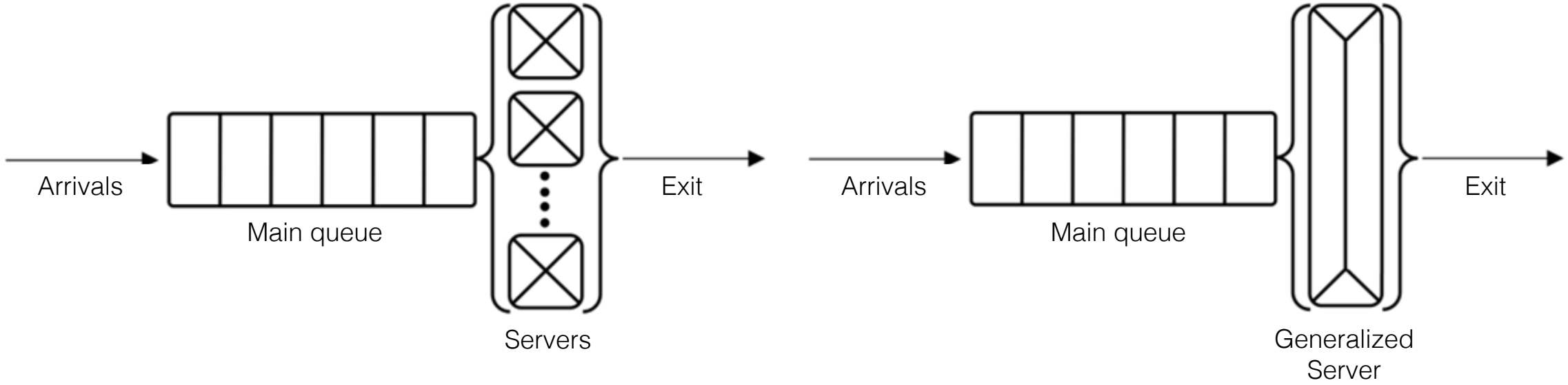
Number of servers varies with time — problematic since service rate estimates with $M/M/c$ depend on the number of open servers:

- there are times when all servers are busy
- others when a number of open servers are idle
- the number of open servers changes according to some vacation policy (difficult to model)

Circumvent this issue (without invoking vacation models):

- $M/M/c$ queue viewed as $M/M/1$ queue where servers are hidden behind a generalized server
- service rates can be estimated independently of the number of servers
- major theorems still hold with $c = 1$, but related quantities are easier to compute

$M/M/1$ Queueing Model
Generalized Servers



However, theory of $M/M/1$ systems is not sufficient to recover the number of servers: need to find a **link between** λ, μ, and c.

$M/M/1$ Queueing Model
"Clusters"

Need to group the data into meaningful "clusters" exhibiting similar properties (i.e. **properties that can be characterized by the same Poisson process**):

- this allows for proper estimation of queuing model parameters (arrival rates, processing rates, etc.)

The selection of the appropriate "cluster" size relies on finding a balancing point between two extremes:

- if clusters cover too long a period of time, the single Poisson process assumption may fail;
- if clusters cover too short a period of time, they are unlikely to exhibit the statistical behaviour of the process.

$M/M/1$ Queueing Model
"Clusters"

Preliminary analysis of the model's accuracy based on:

- **checkpoint**;
- **weekly patterns** (day of week vs wkday/wkend);
- **seasonal patterns** (season vs month), and
- **daily patterns** (2-hr period vs 4-hr period).

The cluster combination that produced the most encouraging queueing results when compared against actual reports was:

checkpoint, weekday/weekend, season, 4 hour-period.

$M/M/1$ Queueing Model

Average Arrival Rate

Some boarding passes are not scanned at S_1 so data cannot be used to derive the cluster arrival rates.

The $S_1 - S_2$ line-up is a birth-death process: the state of the system can only go from n to $n+1$ (when a passenger enters the queue at S_1) or from m to $m-1$ (when a passenger leaves the queue at S_2).

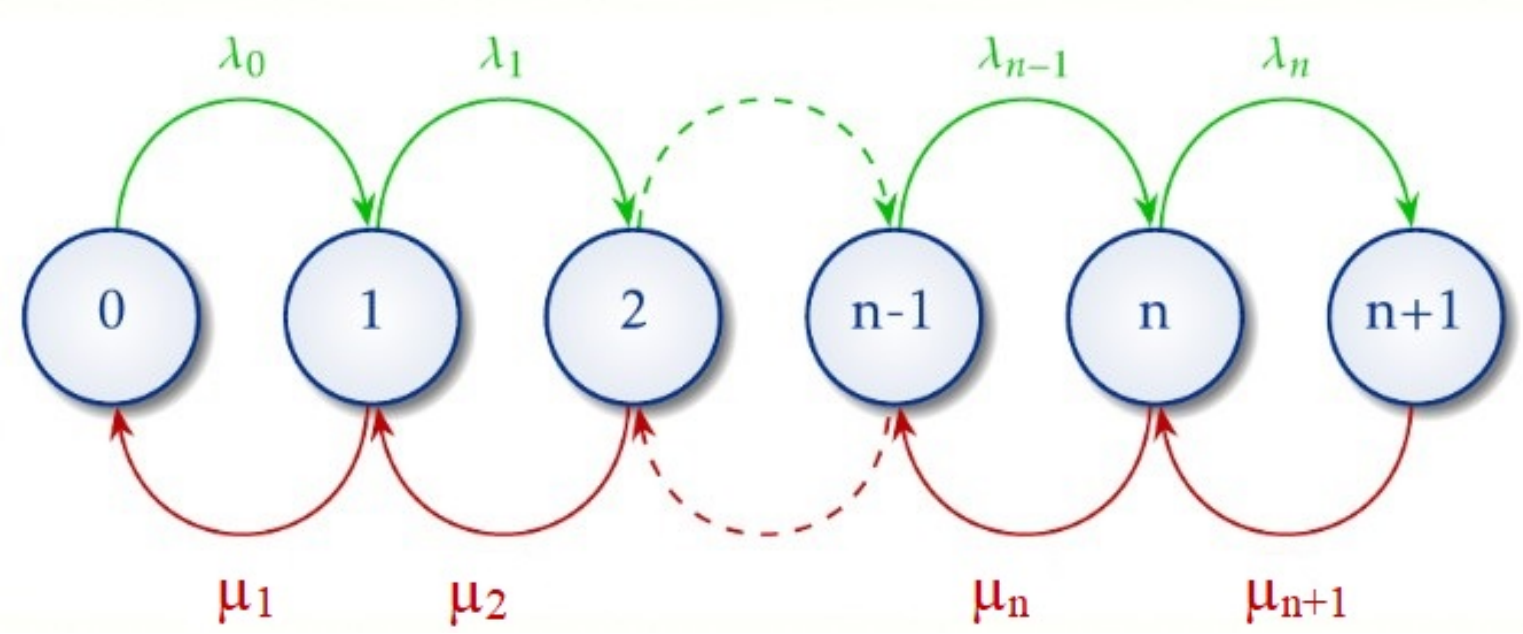
BURKE'S THEOREM for $M/M/c$ queues allows us to compute the arrival rate even in the absence of many records at S_1:

> *Consider an $M/M/c$ queue in the steady state with arrivals modeled by a homogeneous Poisson process with rate parameter λ. Then the departure process is also a homogeneous Poisson process with rate parameter λ.*

$M/M/1$ Queueing Model

Average Arrival Rate

All S_1 arrivals will eventually leave at S_2 and so the fluctuations at S_2 follow the same statistical property governing arrivals to the queue: **arrival rates can be estimated by using data readings at** S_2 within a given cluster.



$M/M/1$ Queueing Model
Average Arrival Rate

Let the number of arrivals in the cluster by time t is denoted by $N(t)$. Then

- $N(t)$ is a **counting process** … (obviously satisfied ✓)
- with **independent** and **stationary** increments, (satisfied with introduction of clusters ✓)
- the number of arrivals in any time interval of length t is **Poisson-distributed** with mean λt, i.e. for all $s, t \geq 0$,

$$P[N(t+s) - N(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0,1, \ldots$$

(holds if the **inter-arrival times** are i.i.d. exponential random variables with the same rate λ … analysis of S_2 in the raw data suggests that this is a decent assumption to make ✓)

Conclusion: in each cluster, arrivals follow a homogeneous (roughly) Poisson process.

Illustration

Arrival Rates

Cluster				# of Hours	Count	Avg Arrival Rate
Week day	0:00	4:00		260	844	0.055
	4:00	8:00		260	129,069	8.274
	8:00	12:00		260	97,949	6.279
	12:00	16:00		260	84,548	5.420
	16:00	20:00		260	78,964	5.062
	20:00	0:00		260	33,061	2.119
Week-end	0:00	4:00		104	1,076	0.172
	4:00	8:00		104	39,674	6.358
	8:00	12:00		104	31,200	5.000
	12:00	16:00		104	26,136	4.188
	16:00	20:00		104	28,129	4.508
	20:00	0:00		104	10,013	1.605

$M/M/1$ Queueing Model

Average Number of Servers

Number of active servers at each checkpoint can be adjusted at any moment during each time period, in order to **accommodate fluctuations in arrivals**.

The CU reports record the **maximum** number of simultaneously active servers during 15-minute block.

Discrepancy between the actual numbers and the reported number is fairly small, due to the short time duration of the blocks.

Data is not available for smaller time scales.

Average is computed over all 15-minute blocks within each cluster.

Illustration

Average Number of Servers

Cluster				Avg # of Servers	Distribution of # of Active Servers								
					0	1	2	3	4	5	6	7	8
	Week day	0:00	4:00	0.14	86.7%	13.0%	0.3%	-	-	-	-	-	-
		4:00	8:00	5.38	-	7.6%	11.3%	4.4%	5.6%	11.3%	21.3%	20.4%	18.1%
		8:00	12:00	4.63	-	-	0.9%	10.3%	35.2%	33.6%	19.0%	1.0%	0.1%
		12:00	16:00	4.19	-	-	3.1%	21.9%	37.6%	27.8%	9.2%	0.4%	-
		16:00	20:00	3.78	-	-	17.7%	30.7%	22.3%	17.6%	9.2%	2.5%	-
		20:00	0:00	0.58	49.4%	42.9%	7.5%	0.2%	-	-	-	-	-
	Week-end	0:00	4:00	0.21	82.5%	13.9%	3.6%	-	-	-	-	-	-
		4:00	8:00	4.56	-	1.9%	9.4%	10.1%	20.7%	32.2%	18.8%	7.0%	-
		8:00	12:00	3.92	-	-	1.0%	31.3%	46.6%	18.0%	2.4%	0.7%	-
		12:00	16:00	3.41	-	-	6.3%	51.9%	36.8%	4.6%	0.5%	0.0%	-
		16:00	20:00	3.60	-	0.7%	17.5%	38.2%	18.8%	15.6%	8.2%	0.5%	0.5%
		20:00	0:00	1.47	0.2%	56.3%	39.7%	3.8%	-	-	-	-	-

$M/M/1$ Queueing Model
Average Wait Time

Not all wait time data is available – if S_1 data is **representative of the overall raw data**, the "real" wait time distribution can be estimated from the subset:

- since the full wait time data is inaccessible, representativeness is hard to prove

Possible reasons why a raw data observation may not be included in the wait time report include:

1. scanned at S_1, but the calculated wait time $w = S_2 - S_1$ is an outlier compared to neighbours
2. not scanned at S_1 because the scanner was overwhelmed by incoming traffic
3. main queue was empty and passenger was processed immediately, leading to $w = 0$

Reason 3 may introduce **bias** if many such observations were removed, which could affect the predicted QoS levels in the small wait time regime.

$M/M/1$ Queueing Model
Average Wait Time

Reason 3 may introduce **bias** if many such observations were removed, which could affect the predicted QoS levels in the small wait time regime.

There is another challenge: it is possible to enter the queue during a period corresponding to a cluster, and to leave it during a period corresponding to another cluster.

For instance, if a cluster ends at noon and the next cluster starts at noon, it can happen that a passenger enters the queue at S_1 at 11:50 and leaves at S_2 at 12:05.

Convention: the cluster in which w is recorded is the cluster in which S_2 falls.

Illustration

Average Wait Time and Performance Levels

Cluster				Count	Avg Wait	Performance					
						5m	10m	15m	20m	25m	30m
	Week day	0:00	4:00	-	-	-	-	-	-	-	-
		4:00	8:00	50,132	6.564	57.1%	79.0%	88.7%	93.5%	96.6%	98.8%
		8:00	12:00	43,033	4.466	68.9%	89.2%	96.5%	99.5%	99.8%	99.9%
		12:00	16:00	32,380	5.374	64.1%	81.8%	92.6%	97.6%	99.3%	99.9%
		16:00	20:00	29,279	5.373	68.0%	81.8%	90.9%	95.8%	97.8%	99.1%
		20:00	0:00	4,511	2.975	86.3%	96.7%	99.9%	100%	100%	100%
	Week-end	0:00	4:00	204	3.992	70.6%	99.5%	100%	100%	100%	100%
		4:00	8:00	14,450	4.520	68.4%	86.4%	96.8%	99.3%	100%	100%
		8:00	12:00	12,638	3.317	82.3%	95.0%	96.8%	98.2%	99.7%	100%
		12:00	16:00	11,938	3.043	83.0%	95.6%	98.5%	99.9%	100%	100%
		16:00	20:00	8,625	5.247	60.5%	80.6%	94.3%	98.9%	100%	100%
		20:00	0:00	1,529	2.382	88.9%	100%	100%	100%	100%	100%

$M/M/1$ Queueing Model

Service Rate

Let W_q be the wait time in the queue. The probability of waiting up to x units of time is

specific to $M/M/1$

$$p(x) = P(W_q \leq x) = 1 - \frac{\lambda}{\mu} e^{-(\mu-\lambda)x} \quad \text{and} \quad \overline{W_q} = \frac{\rho}{\mu-\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

If the arrival rate λ is known and the average wait time $\overline{W_q}$ can be computed by another mean, then it is possible to recover the service rate μ with:

$$\hat{\mu}_M = \frac{\overline{W_q}\lambda + \sqrt{\left(\overline{W_q}\lambda\right)^2 + 4\overline{W_q}\lambda}}{2\overline{W_q}}$$

The QoS levels are thus $\hat{p}_M(x) = 1 - \frac{\lambda}{\hat{\mu}_M} e^{-(\hat{\mu}_M-\lambda)x} \in (0,1)$, if $\lambda < \hat{\mu}_M$.

Illustration

QoS Estimates — $M/M/1$

Cluster				Est Serv Rate	Est ρ	Estimated Performance (M/M/1)					
						5m	10m	15m	20m	25m	30m
	Week day	0:00	4:00	-	-	-	-	-	-	-	-
		4:00	8:00	8.423	0.982	53.5%	78.0%	89.6%	95.1%	97.7%	98.9%
		8:00	12:00	6.495	0.967	67.2%	88.9%	96.2%	98.7%	99.6%	99.9%
		12:00	16:00	5.600	0.968	60.7%	84.0%	93.5%	97.4%	98.9%	99.6%
		16:00	20:00	5.242	0.966	60.7%	84.0%	93.5%	97.3%	98.9%	99.6%
		20:00	0:00	2.414	0.878	79.9%	95.4%	98.9%	99.8%	99.9%	100.0%
	Week-end	0:00	4:00	0.311	0.554	72.3%	86.2%	93.1%	96.5%	98.3%	99.1%
		4:00	8:00	6.572	0.967	66.8%	88.6%	96.1%	98.7%	99.5%	99.8%
		8:00	12:00	5.285	0.946	77.3%	94.5%	98.7%	99.7%	99.9%	100.0%
		12:00	16:00	4.495	0.932	79.8%	95.6%	99.1%	99.8%	100.0%	100.0%
		16:00	20:00	4.691	0.961	61.5%	84.6%	93.8%	97.5%	99.0%	99.6%
		20:00	0:00	1.950	0.823	85.4%	97.4%	99.5%	99.9%	100.0%	100.0%

$M/M/1$ Queueing Model
Validation

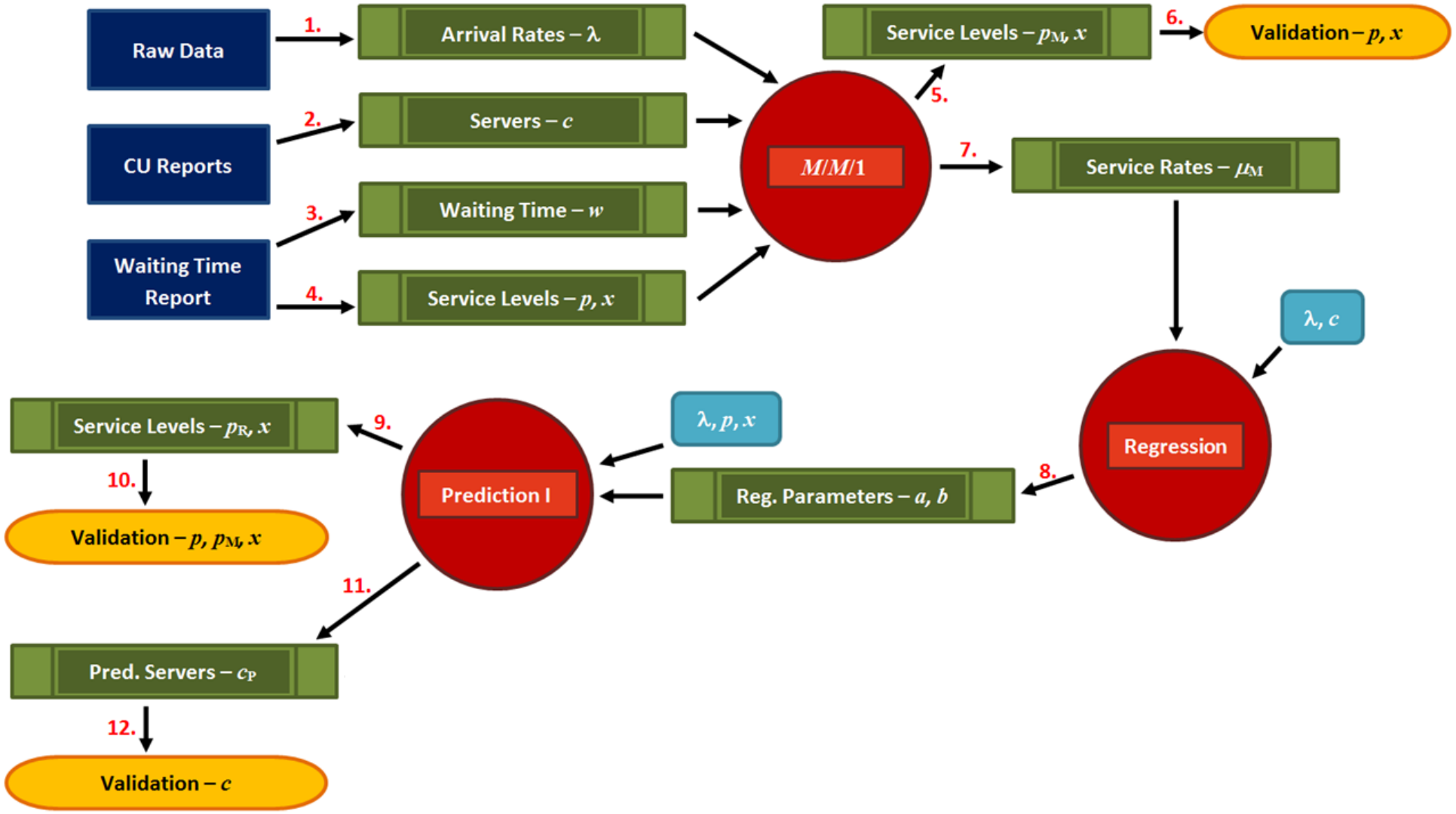
Relations do not hold if the generalized $M/M/1$ hypothesis fails.

At this stage, the simplest way to validate is to compare the wait times generated by the model to those of the empirical data: are the estimated QoS levels $(\hat{p}_{M,x}, x)$ "close to" the empirical QoS levels (p_x, x)?

We found that the generalized $M/M/1$ assumption, while not exact, remains reasonable to make at the checkpoint level.

Still can't extract c without additional assumptions.

Model Flow II
Regression



Regression Model

Linking λ, μ, and c

Regression assumption: on a quarterly level, the cluster mean processing rate μ is a function of

- the number of active servers c (hidden behind the generalized server), and
- the mean arrival rate λ.

Simplest form: $\mu = \mu(\lambda, c) = ac + b\lambda$ (other forms possible as well).

Do quarterly clusters observations $\left(\frac{\lambda}{c}, \frac{\widehat{\mu}_M}{c}\right)$ lie on a line? What would that mean, from a checkpoint perspective?

Quarterly clusters observations are weighted by number of arrivals.

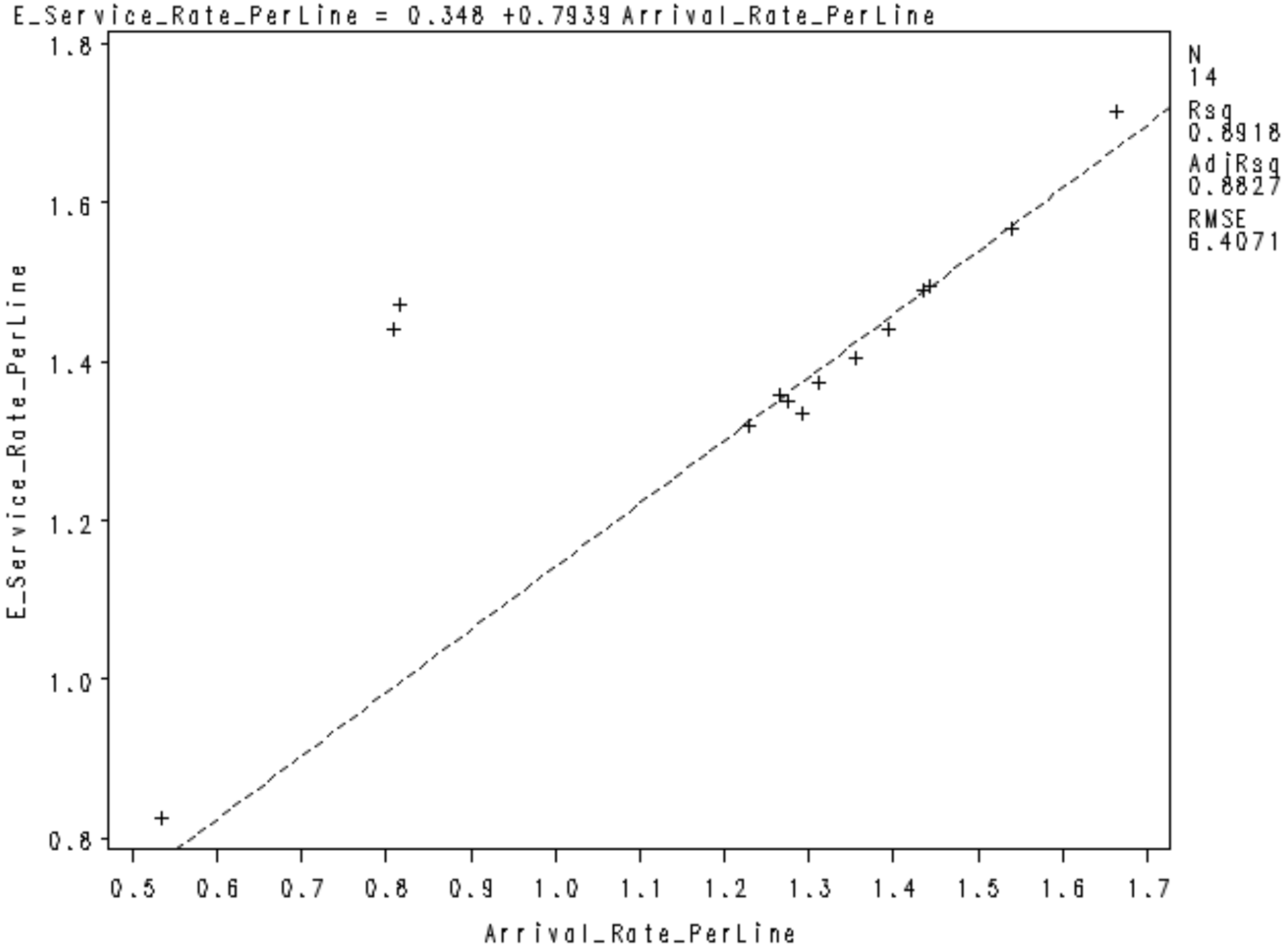
Illustration

Regression

Cluster				Avg # of Servers	Arrival Rate	Est Serv Rate	Arr Rate / Server	Serv Rate / Server
Week day	0:00	4:00		0.14	0.055	0.000	0.405	0.000
	4:00	8:00		5.38	8.274	8.423	1.539	1.567
	8:00	12:00		4.63	6.279	6.495	1.356	1.403
	12:00	16:00		4.19	5.420	5.600	1.292	1.335
	16:00	20:00		3.78	5.062	5.242	1.341	1.388
	20:00	0:00		1.58	2.119	2.414	1.337	1.524
Week-end	0:00	4:00		0.21	0.172	0.311	0.815	1.471
	4:00	8:00		4.56	6.358	6.572	1.394	1.441
	8:00	12:00		3.92	5.000	5.285	1.276	1.349
	12:00	16:00		3.41	4.188	4.495	1.228	1.318
	16:00	20:00		3.60	4.508	4.691	1.253	1.304
	20:00	0:00		1.47	1.605	1.950	1.091	1.326

Illustration

Regression



Regression Model

Service Rate and Performance Levels

Service rate estimates: $\hat{\mu}_R = \hat{a}c + \hat{b}\lambda$, with optimal regression parameters $\hat{a}, \hat{b}$ (quarterly and seasonal estimates).

QoS level curves $(\hat{p}_R(x), x)$ estimates, as long as $\lambda < \hat{\mu}_R$:

$$\hat{p}_R(x) = 1 - \frac{\lambda}{\hat{a}c + \hat{b}\lambda} e^{-(\hat{a}c + \hat{b}\lambda - \lambda)x}$$

Additional assumptions:

- quarterly regressions produce good fits
- there is a quarterly characteristic to the service rate

Illustration

Estimates — $M/M/1$ + Regression

Cluster				Class	Reg Serv Rate	Reg ρ	Estimated Performance (M/M/1 + Reg)					
							5m	10m	15m	20m	25m	30m
	Week day	0:00	4:00	-	-	-	-	-	-	-	-	-
		4:00	8:00	1.5	8.264	1.001	-	-	-	-	-	-
		8:00	12:00	1.4	6.591	0.953	80.0%	95.8%	99.1%	99.8%	100%	100%
		12:00	16:00	1.3	5.804	0.934	86.3%	98.0%	99.7%	100%	100%	100%
		16:00	20:00	1.3	5.338	0.948	76.2%	94.0%	98.5%	99.6%	99.9%	100%
		20:00	0:00	1.3	2.237	0.947	47.5%	70.9%	83.9%	91.1%	95.1%	97.3%
	Week-end	0:00	4:00	-	-	-	-	-	-	-	-	-
		4:00	8:00	1.4	6.600	0.963	71.3%	91.4%	97.4%	99.2%	99.8%	99.9%
		8:00	12:00	1.3	5.383	0.929	86.3%	98.0%	99.7%	100%	100%	100%
		12:00	16:00	1.2	4.584	0.914	87.4%	98.3%	99.8%	100%	100%	100%
		16:00	20:00	1.3	4.892	0.922	86.5%	98.0%	99.7%	100%	100%	100%
		20:00	0:00	1.1	1.852	0.867	74.8%	92.7%	97.9%	99.4%	99.8%	99.9%

Regression Model

Predicted Mean Number of Servers

1. Start with $p = 1 - \frac{\lambda}{ac+b\lambda} e^{-(ac+b\lambda-\lambda)x}$, where $p, \frac{\lambda}{ac+b\lambda} \in (0,1)$;
2. Re-arrange terms $\Longrightarrow (ac + b\lambda)e^{(ac+b\lambda)x} = \frac{\lambda}{1-p} e^{\lambda x}$;
3. Multiply by x on both sides $\Longrightarrow (ac + b\lambda)xe^{(ac+b\lambda)x} = \frac{\lambda x}{1-p} e^{\lambda x}$;
4. Set $y = (ac + b\lambda)x$ and $z = \frac{\lambda x}{1-p} e^{\lambda x} \Longrightarrow ye^{y} = z$;
5. Solve for $y \Longrightarrow (ac + b\lambda)x = y = W_0(z) = W_0\left(\frac{\lambda x}{1-p} e^{\lambda x}\right)$, where W_0 is the Lambert function, and
6. Solve for $c \Longrightarrow c_R = \frac{1}{ax}\left[W_0\left(\frac{\lambda x}{1-p} e^{\lambda x}\right) - b\lambda x\right]$.

The "physics" of the problem also require that $c \in [0, \text{maximum \# of servers}]$

Illustration

Predicted Mean Number of Servers

Cluster				Actual # Servers	Pred # Servers
Week day	0:00	4:00		0.136	-
	4:00	8:00		5.375	5.643
	8:00	12:00		4.629	4.474
	12:00	16:00		4.193	3.829
	16:00	20:00		3.775	3.572
	20:00	0:00		1.585	2.198
Week-end	0:00	4:00		0.212	-
	4:00	8:00		4.560	4.535
	8:00	12:00		3.918	3.650
	12:00	16:00		3.411	3.205
	16:00	20:00		3.599	3.264
	20:00	0:00		1.471	1.701

Regression Model
Validation

Relations do not hold if the combined $\boldsymbol{M/M/1}$ **+ Regression** hypotheses fails.

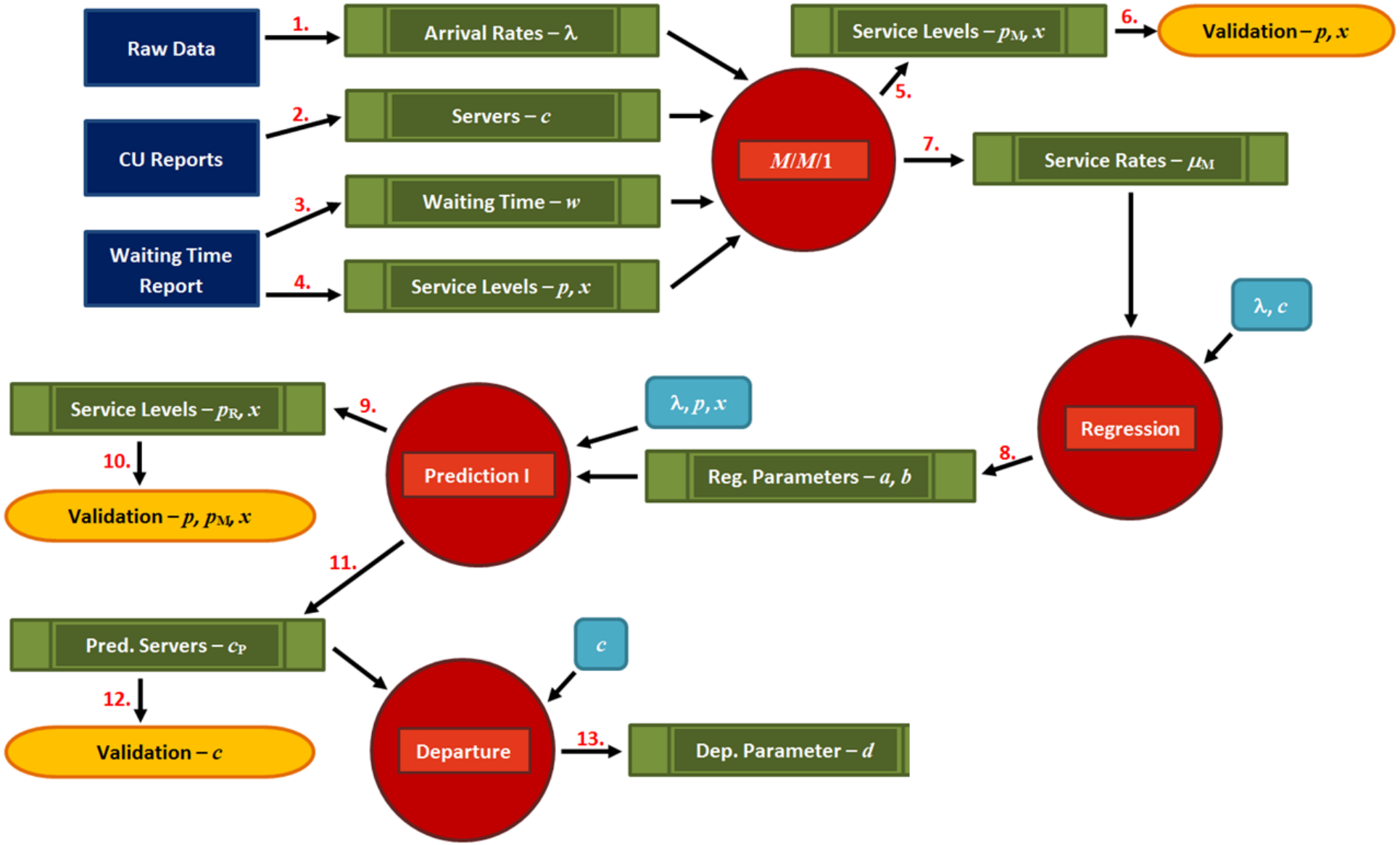
As before, we can compare the estimated QoS levels $(\hat{p}_{R,x}, x)$ to the empirical QoS levels (p_x, x).

We can also compare the predicted # of servers c_R to the empirical # of servers c.

The combined hypotheses, while proving slightly less valid than the $M/M/1$ hypothesis on its own, still provided reasonably close QoS estimates at the quarter and checkpoint levels (regression function $\mu = \mu(\lambda, c)$ adds some uncertainty)

Big advantage: can extract/predict the number of active servers c

Model Flow III
Departure



Departure Model

Predicted Number of Servers vs. Actual Number of Servers

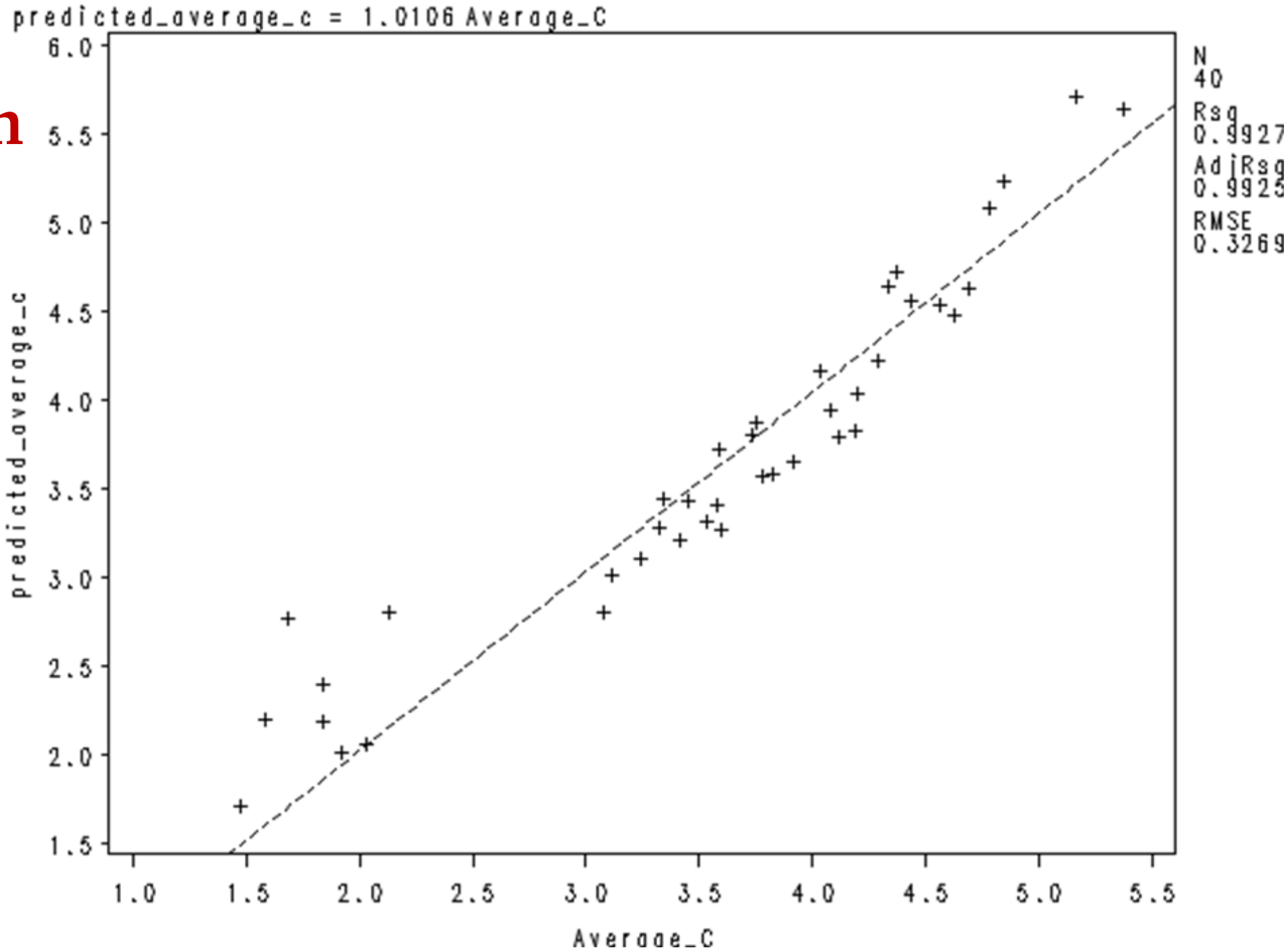
For any given checkpoint, quarter, and cluster, we can compare the actual number of open servers c (given by the CU Report), and the predicted value c_R, given the actual arrival rate λ and the actual QoS level (p, x).

Plotting c_R against c for all clusters strongly suggests that these were linked at the checkpoint level according to $c = \hat{d}c_R$, where $\hat{d}$ is the regression estimate of the **checkpoint departure parameter** d.

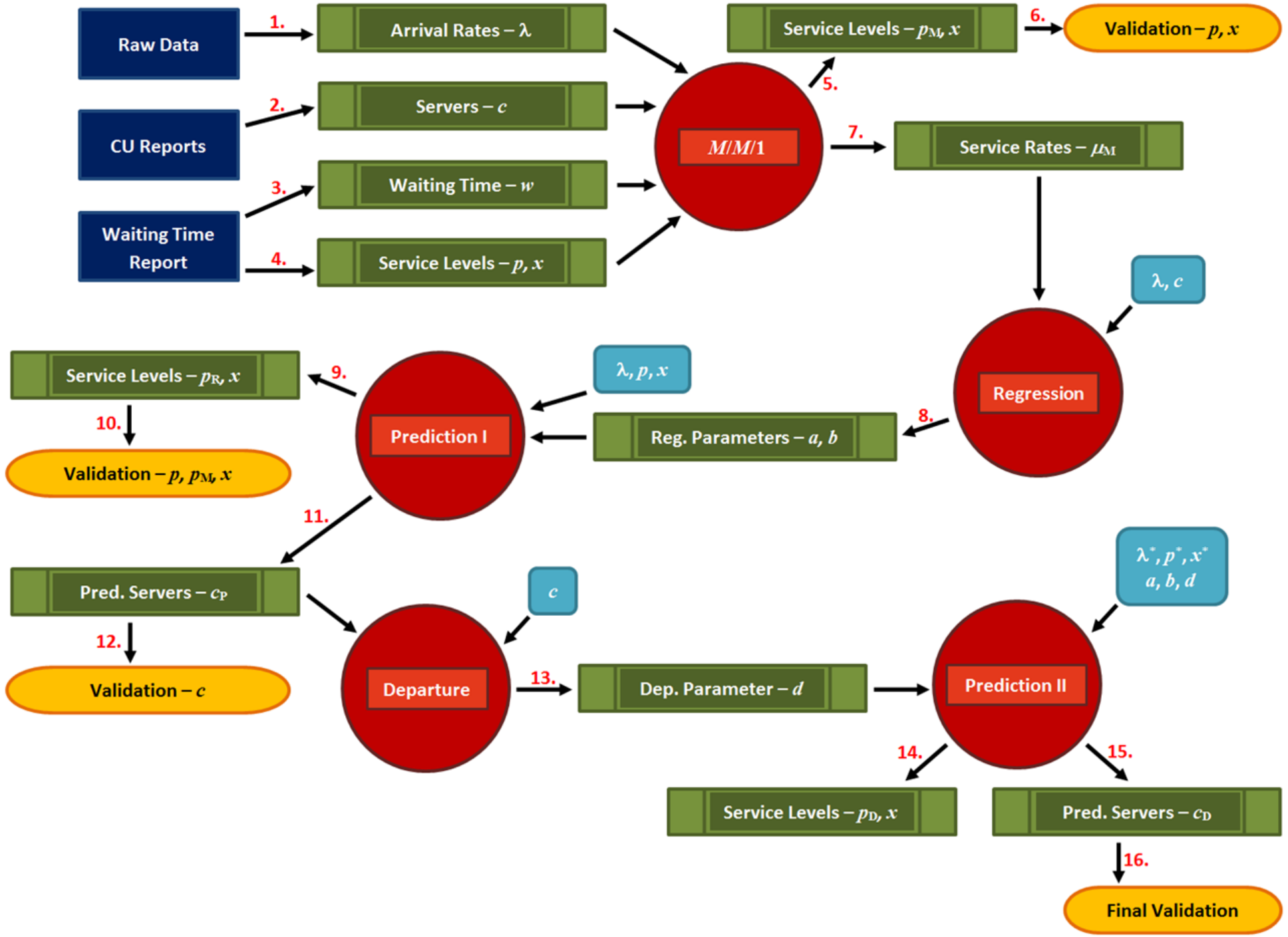
Computed values of d near 1 for nearly all checkpoints further validate the combined model.

Illustration

Departure



Model Flow IV
Final Predictions



Combined Model

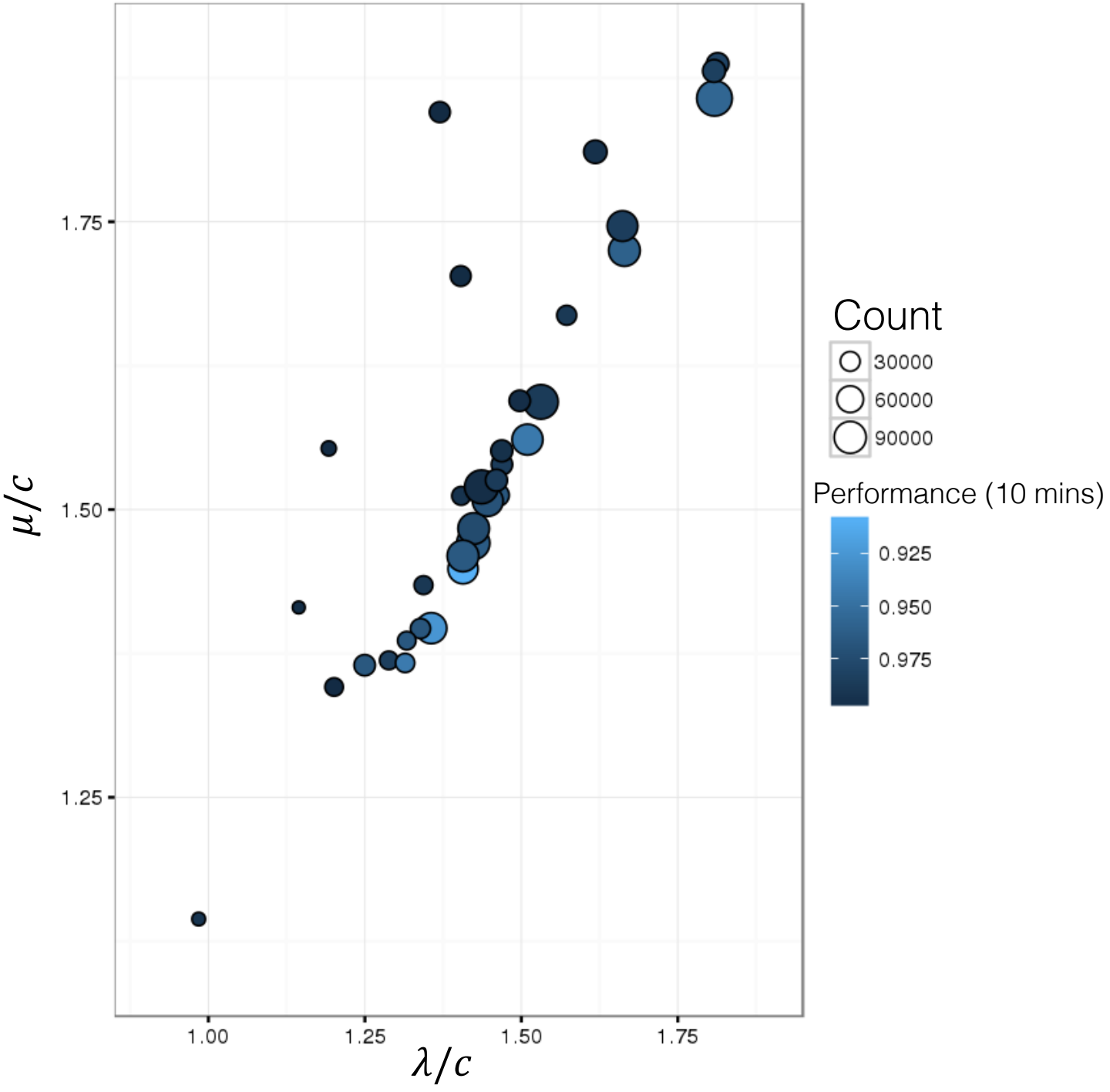
Final Predictions and Validation

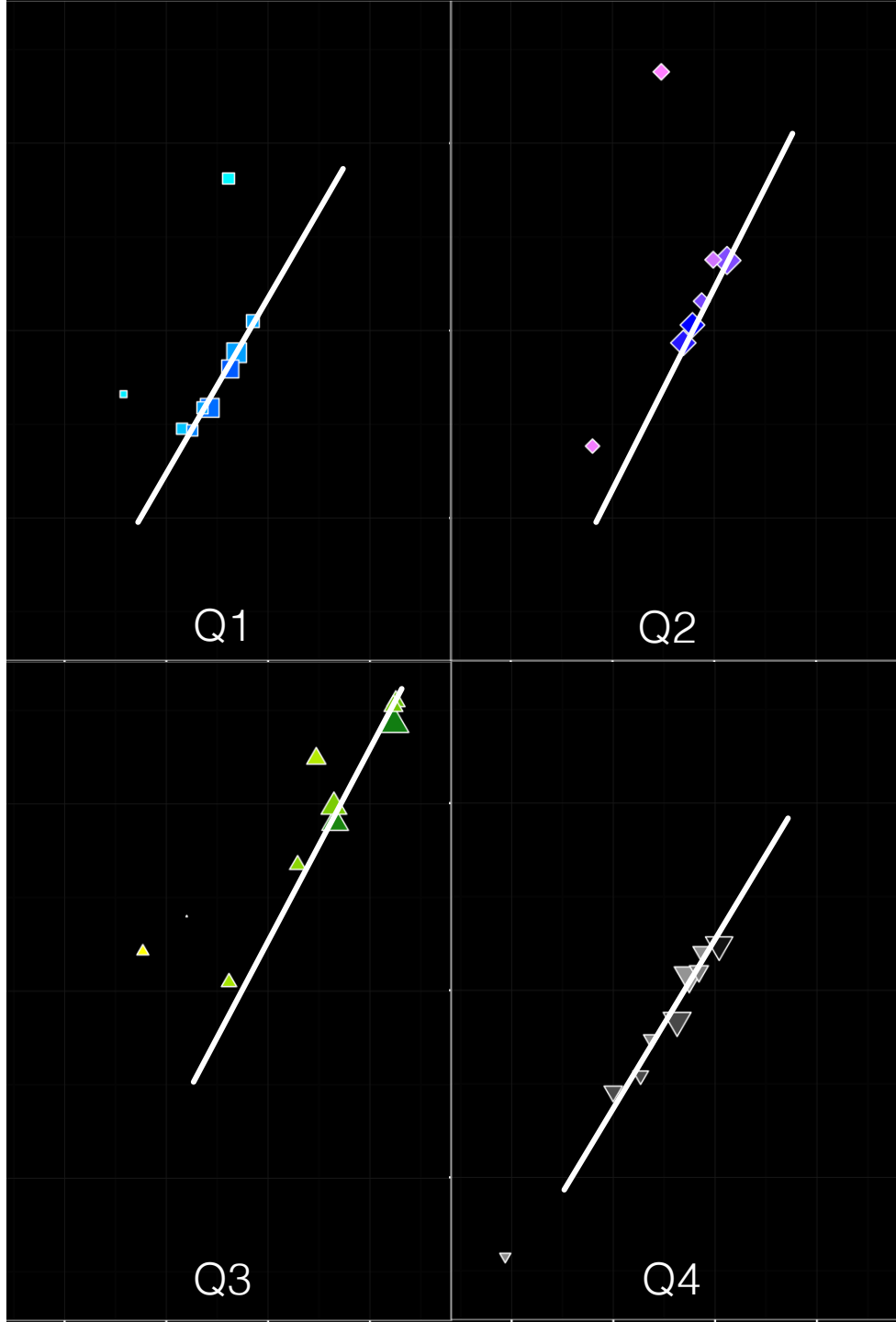
Given an arrival rate λ, a desired QoS level (p, x), and checkpoint parameters (a, b, d), the predicted required number of servers is

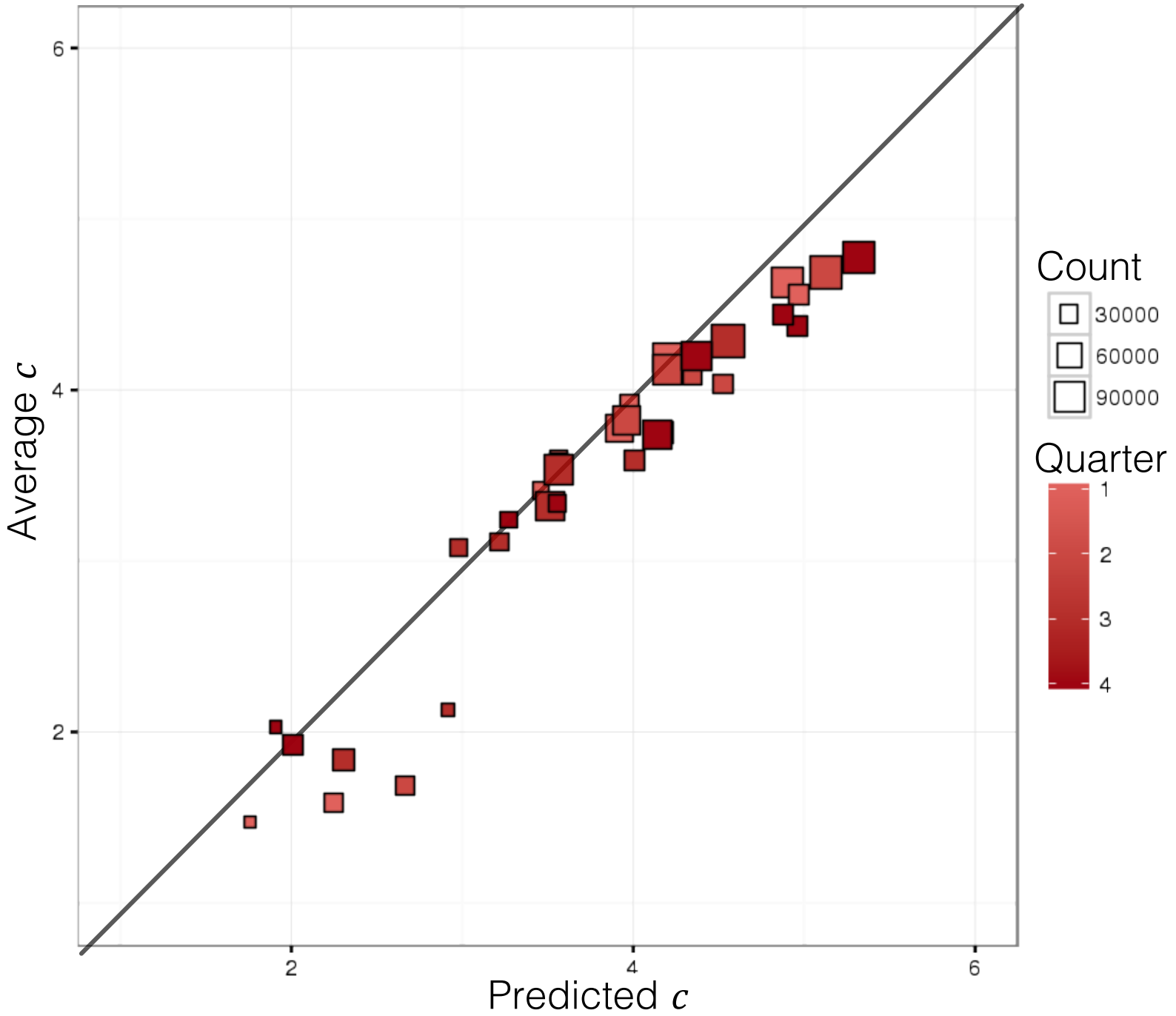
$$c_D = d \cdot c_R = \frac{d}{ax}\left[W_0\left(\frac{\lambda x}{1-p}e^{\lambda x}\right) - b\lambda x\right]$$

It makes little sense to compare the predicted value c_D with the actual number of servers c found in the historical data as the prediction depends not only

- on the forecasted arrival rate (which is likely to be different from the historical rate),
- but also on the attained QoS level (for which an independent forecast is unavailable).


1.75
1.50
1.25
μ/c
1.00
1.25
1.50
1.75
λ/c
Count
30000
60000
90000
Performance (10 mins)
0.925
0.950
0.975


Q1
Q2
Q3
Q4


Average c
Predicted c
6
4
2
2
4
6
Count
30000
60000
90000
Quarter
1
2
3
4

Discussion

Accuracy and Potential Issues

The $\boldsymbol{M/M/1}$ **model** on its own provides the best QoS levels estimates, while the best estimates for the average number of active servers are provided by the **Departure** model.

Some loss of information is inevitable due to the necessity of simplification assumptions.

Possible issues which could affect the model's accuracy:

- Underlying arrival processes are roughly Poisson, wait time distributions are roughly conditionally exponential for each cluster; depending on the distance between the theoretical process and the empirical data, the $M/M/1$ assumption may be inappropriate.

Discussion

Accuracy and Potential Issues

Possible issues which could affect the model's accuracy (continued):

- Wait time distributions may be strongly biased due to missing S_1 scans; no easy way to verify how representative it is
- Server vacation policy is unknown, and may not be uniformly adhered to
- Actual c crudely approximated by maximum number of active lines within a 15 minute block
- Service rates seem to depend on other factors, not just c and the λ

Supplemental Comments

Refinements and Recommendations

Different functional forms $\mu = \mu(\lambda, c)$

Model built using 2 years of data instead of a single year

Number of clusters

Queueing Approach vs. Simulation (of missing S_1 scans)

Consulting Post-Mortem

Clients were mathematicians: knew what they wanted, had a pretty good idea of how to get it.

Clients could have done this on their own, but were not being taken seriously by non-technical stakeholders.

Another approach (simulation) had been used and failed to provide useful results, so there was scepticism on the part of stakeholders.

Academics were not highly regarded, so consultant angle had to be played up.

We were able to help them find a data reporting error through preliminary analysis, which helped solidify our credentials.

Still in use as of 2018; model performs better at the national level than at the airport level.