Mathematical Analysis

Chapter 4
Limits and Continuity

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Overview

The main objects of study in analysis are functions. In this chapter, we

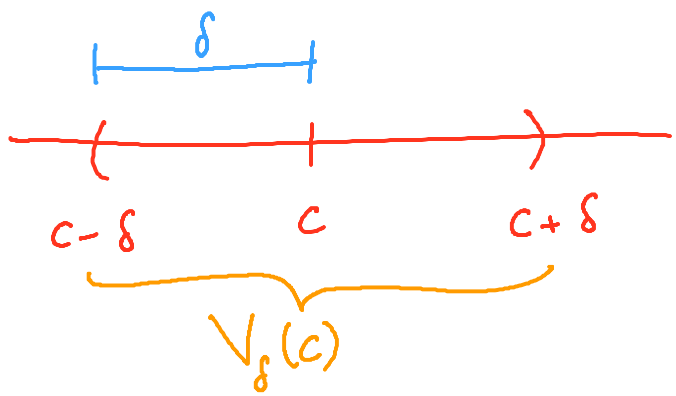
- introduce the $\varepsilon - \delta$ definition of the limit of a function,
- provide results that help to compute such limits,
- identify two types of continuity, and
- present some of the heavy-hitting theorems that form the basis of analytical endeavours.

Outline

4.1 – Limit of a Function

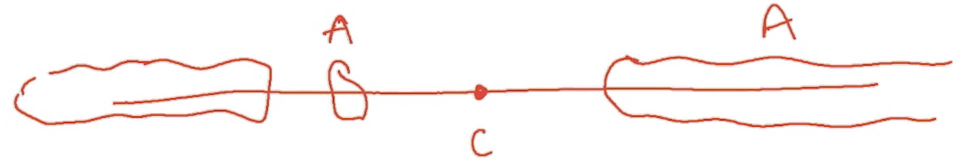
The objects we have studied thus far are functions of $\mathbb{N}$ into $\mathbb{R}$. However, most of calculus deals with functions of $\mathbb{R}$ into $\mathbb{R}$. How do we generalize the concepts and results we have derived for sequences to functions?

Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. The **neighbourhood** $V_\delta(c)$, where $\delta > 0$, is the interval $\{x \in \mathbb{R} \mid |x - c| < \delta\} = (c - \delta, c + \delta)$.

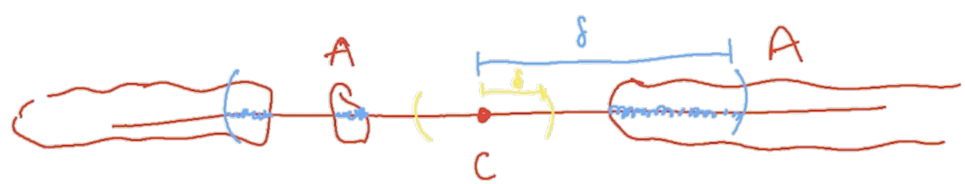


The point $c \in \mathbb{R}$ is a **limit point** (or **cluster point**) of A if every neighbourhood $V_\delta(c)$ contains at least one point $x \in A$ other than c.

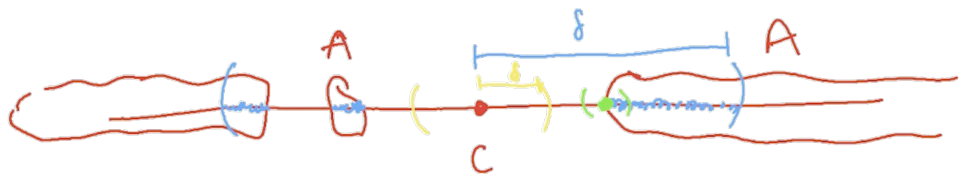
Consider the set $A \subseteq \mathbb{R}$ drawn below.



The $V_\delta(c)-$neighbourhood in blue contains points in A other than c, but c is not a limit point of A since the $V_\delta(c)-$neighbourhood in yellow does not contain points of A.



The point at the center of the green interval is a limit point of A, however.



The set of all limit points of A is denoted by $\overline{A}$; a limit point of A does not have to be in A.

Example: What are the limit points of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$?

Solution. Let $n \in \mathbb{N}$. The distance between a point $\frac{1}{n}$ and its immediate successor/predecessor $\frac{1}{n\pm1}$ is

$$\frac{1}{n} - \frac{1}{n \pm 1} = \frac{1}{n(n \pm 1)} > \frac{1}{3n^2}.$$

Let $\delta = \frac{1}{3n^2}$. Then

$$V_\delta(\tfrac{1}{n}) = (\tfrac{1}{n} - \tfrac{1}{3n^2}, \tfrac{1}{n} + \tfrac{1}{3n^2}) \subseteq (\tfrac{1}{n-1}, \tfrac{1}{n+1}),$$

so the only point of A in $V_\delta(\frac{1}{n})$ is $\frac{1}{n}$. Thus $\frac{1}{n} \notin \overline{A}$.

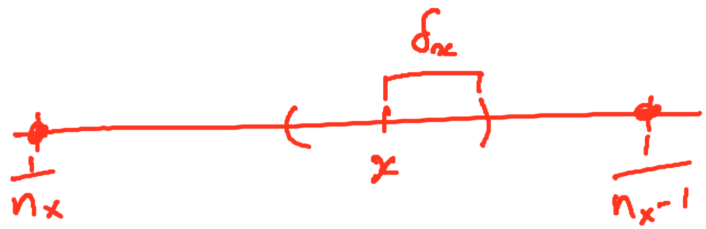
No negative real number is a limit point of A; indeed, if $x < 0$, set $\delta = \frac{|x|}{2}$. Then $V_\delta(x) \subseteq (-\infty, 0)$ and so contains no point of A.

Similarly, no real number > 1 is a limit point of A.

Hence $\overline{A} \subseteq [0,1] \setminus A$.

Let $x \in (0,1] \setminus A$. According to the Archimedean Property, $\exists n_x \in \mathbb{N}$ s.t. $n_x > \frac{1}{x} > n_x - 1$, so $\frac{1}{n_x} < x < \frac{1}{n_x - 1}$. Set $\delta_x = \frac{1}{2}\min\{|x - \frac{1}{n_x}|, |x - \frac{1}{n_x-1}|\}$.

Then $V_{\delta_x}(x)$ contains none of the points of A.



The only remaining possibility is $x = 0$. Let $\delta > 0$. By the Archimedean Property, $\exists N_\delta$ such that $\frac{1}{N_\delta} < \delta$. But $0 \neq \frac{1}{N_\delta} \in A$, Thus

$$\varnothing \neq \left\{\frac{1}{N_\delta}\right\} \subseteq V_\delta \cap A = (-\delta, \delta) \cap A,$$

so $x = 0$ is the only limit point of A: $\overline{A} = \{0\}$. ■

Directly determining the limit points of a set is a time-intensive endeavour. Thankfully, there is a link between limit points and convergent sequences.

Theorem 24. *A point $c \in \mathbb{R}$ is a limit point of A if and only if there is a sequence $(a_n) \subseteq A$, with $a_n \neq c$ for $n \in \mathbb{N}$, such that $a_n \to c$.*

Proof. Suppose c is a limit point of A. By definition, the neighbourhood $V_{\frac{1}{n}}(c)$ must contain a point $a_n \neq c \in A$, for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ s.t. $\frac{1}{N_\varepsilon} < \varepsilon$. Thus

$$n > N_\varepsilon \implies 0 < |a_n - c| < \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon, \quad \text{i.e. } a_n \to c.$$

Conversely, suppose that there is a sequence $(a_n) \subseteq A$, with $a_n \neq c$ for all $n \in \mathbb{N}$, such that $a_n \to c$. Let $\delta > 0$. By definition, $\exists N_\delta \in \mathbb{N}$, such that $0 < |a_n - c| < \delta$ for all $n > N_\delta$.

Then $a_n \in V_\delta(c)$ and $a_n \neq c$ for all $n > N_\delta$. Thus any neighbourhood of c contains at least one $a_n \neq c$, so $c \in \overline{A}$. ■

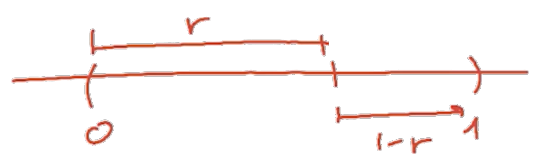
Any limit point of A is in fact the limit of a sequence in A, and *vice-versa*.

Example: Let $A = [0,1] \cap \mathbb{Q}$. What are the limit points of A?

Solution. Any convergent sequence $(a_n) \subseteq A$ is such that $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$, so its limit must also lie in $[0,1]$, according to Theorem 15.

Theorem 24 tells us that any limit point of A is the limit of a sequence of rationals in $[0,1]$. The sequences $(\frac{1}{n})$ and $(1-\frac{1}{n})$ lie in A. Since $\frac{1}{n} \to 0$ and $1-\frac{1}{n} \to 1$, then $0, 1 \in \overline{A}$.

Now, let $r \in (0,1)$. Set $\eta = \min\{r, 1-r\}$.



Then $\eta > 0$ and $\frac{1}{\eta} > 0$. By the Archimedean Property, $\exists M \in \mathbb{N}$ s.t. $M > \frac{1}{\eta}$. Then

$$0 \leq r - \eta < r - \frac{1}{M} > r + \frac{1}{M} < r + \eta \leq 1,$$

since $\eta = r$ if $r \leq 1/2$ and $\eta = 1 - r$ if $r \geq 1/2$. So

$$n > M \implies 0 < r - \frac{1}{n} < r + \frac{1}{n} < 1.$$

But the Density Theorem states that for all $n > M$, $\exists a_n \neq r \in \mathbb{Q}$ such that $r - \frac{1}{n} < a_n < r + \frac{1}{n}$.

The sequence (a_n) thus constructed converges to r. Indeed, let $\varepsilon > 0$. According to the Archimedean Property, $\exists N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$.

Set $N_\varepsilon = \max\{M, N\}$. Then

$$n > N_\varepsilon \implies 0 < |a_n - r| < \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

and so $a_n \to r$ and $r \in \overline{A}$. Consequently, $\overline{A} = [0, 1]$. ■

Intuitively, a limit of a function f at c is a value L towards which $f(x)$ "approaches" as x gets closer to c, if it exists. But what does that actually mean? What would need to happen for the value not to exist?

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in \overline{A}$: $L \in \mathbb{R}$ is the **limit of f at c** if

$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0$ such that $0 < |x-c| < \delta_\varepsilon$ and $x \in A \implies |f(x)-L| < \varepsilon$.

This situation is denoted by $\lim_{x \to c} f(x) = L$, or by $f(x) \to L$ when $x \to c$.

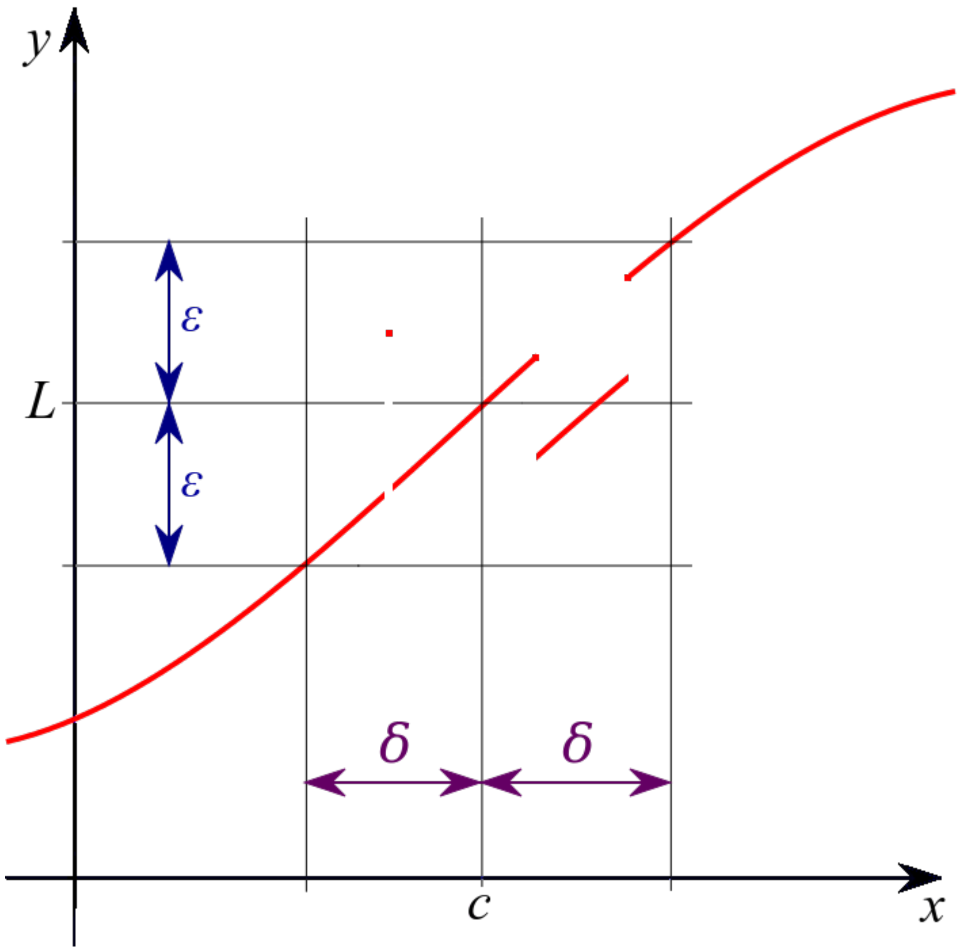
The limit of f at c is **not** $L \in \mathbb{R}$ if

$$\exists \varepsilon_0 > 0,\ \forall \delta > 0,\ \exists x_\delta \in A \text{ such that } 0 < |x_\delta - c| < \delta_\varepsilon \text{ and } |f(x_\delta) - L| \geq \varepsilon_0,$$

which we denote by $\lim_{x \to c} f(x) \neq L$, or by $f(x) \not\to L$ when $x \to c$.

It is the same principle as that of the limit of a sequence: given $\varepsilon > 0$, we need to find a $\delta_\varepsilon > 0$ which satisfies the definition.

Graphically, this is equivalent to putting a horizontal strip of width 2ε around the line $y = L$, and showing that there is a neighbourhood $V_{\delta_\varepsilon}(c)$ such that $f(x)$ is in the strip for any $x \in V_{\delta_\varepsilon}$.



y
L
ε
ε
δ
δ
c
x

Examples:

1. Let $f:[0,1)\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}2, & x\in(0,1)\\ 3, & x=0\end{cases}$$

Show $\lim\limits_{x\to 0} f(x)=2$.

Proof. Let $\varepsilon>0$. Set $\delta_\varepsilon=1$. Then

$$x\in[0,1) \text{ and } 0<|x-c|<\delta_\varepsilon \implies |f(x)=2|=0<0\cdot\delta<\varepsilon,$$

which completes the proof. ■

2. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{x^2+2x+2}{x+1}$. Show $\lim_{x\to 2} f(x) = \frac{10}{3}$.

Proof. Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then

$$\left|\frac{x^2+2x+2}{x+1} - \frac{10}{3}\right| = \left|\frac{3(x^2+2x+2)-10(x+1)}{x+1}\right| = \left|\frac{3x^2-4x-4}{3x+3}\right|$$
$$= \underbrace{\left|\frac{3x+2}{3x+3}\right|}_{<1} |x-2| < |x-2| < \delta_\varepsilon = \varepsilon$$

when $x \geq 0$ and $0 < |x-2| < \delta_\varepsilon$. ■

3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

Proof. Note that $c \in A = \mathbb{R} \setminus \{0\}$. We can only use the definition of the limit if $c \in \overline{A}$. That it does so is a given, as $(\frac{1}{n}) \subseteq A$ and $\frac{1}{n} \to 0$, with $\frac{1}{n} \neq 0$ for all $n \in \mathbb{N}$, according to Theorem 24.

Let $\varepsilon > 0$ and set $\delta_\varepsilon = \sqrt{\varepsilon}$. Then

$$\left|x^2 \cos(1/x) - 0\right| = |x|^2 \underbrace{|\cos(1/x)|}_{\leq 1} \leq |x|^2 = |x-0|^2 < \delta_\varepsilon^2 < \varepsilon,$$

whenever $x \in \mathbb{R} \setminus \{0\}$ and $0 < |x - 0 < \delta_\varepsilon$. ■

As is the case with sequences, a function has at most one limit at any of its limit points c.

Theorem 25. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then f has at most one limit at c.*

Proof. Suppose that

$$\lim_{x\to c} f(x) = L' \quad \text{and} \quad \lim_{x\to c} f(x) = L'', \quad \text{where } L' < L''.$$

Let $\varepsilon = \frac{L''-L'}{3} > 0$. By definition, $\exists \delta'_\varepsilon, \delta''_\varepsilon$ such that $|f(x) - L'| < \varepsilon$ and $|f(x) - L''| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta'_\varepsilon$, $0 < |x - c| < \delta''_\varepsilon$.

Set $\delta_\varepsilon = \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$. Then, whenever $x \in A$ and $0 < |x - c| < \delta_\varepsilon$, we have

$$\begin{aligned} f(x) < L'' + \varepsilon = L' + \frac{L''-L'}{3} &= \frac{2L'+L''}{3} = \frac{L'+L''}{3} + \frac{L'}{3} \\ &< \frac{L'+L''}{3} + \frac{L''}{3} < \frac{2L''+L'}{3} = L'' - \frac{L''-L'}{3} = L'' - \varepsilon < f(x), \end{aligned}$$

a contradiction, hence $L' \not< L''$. The proof that $L'' \not< L'$ is identical. ■

As is the case with sequences, the definition is useless if we do not have a candidate for L beforehand. The next result allows us to get such a candidate before using the definition (if required).

Theorem 26. (Sequential Criterion) *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then*

$$\lim_{x\to c} f(x) = L \quad \textit{if and only if} \quad \lim_{n\to\infty} f(x_n) = L$$

for any sequence $(x_n) \subseteq A$, with $x_n \neq c$ for all $n \in \mathbb{N}$, such that $x_n \to c$.

Proof. Assume $\lim_{x\to c} f(x) = L$. Let $\varepsilon > 0$. Then $\exists \delta_\varepsilon > 0$ such that

$$x \in A \text{ and } 0 < |x - c| < \delta_\varepsilon \implies |f(x) - L| < \varepsilon.$$

Suppose $(x_n) \subseteq A$ is such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$.

Then $\exists M_{\delta_\varepsilon} > 0$ such that $0 < |x_n - c| < \delta_\varepsilon$ whenever $n > M_{\delta_\varepsilon}$.

Let $N_\varepsilon = M_{\delta_\varepsilon}$. Then

$$x_n \neq c \in A \text{ and } n > N_\varepsilon \implies 0 < |x_n - c| < \delta_\varepsilon \implies |f(x_n) - L| < \varepsilon,$$

which is to say $f(x_n) \to L$.

Conversely, if $\lim_{x\to c} f(x) \neq L$, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ but $|f(x) - L| \geq \varepsilon_0$. Thus, for $n \in \mathbb{N}$ and $\delta = \frac{1}{n}$, $\exists x_n = x_\delta$ as above.

The sequence $(x_n) \subseteq A$ is such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$. According to the Squeeze Theorem, $x_n \to c$, with $|f(x_n) - L| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Thus $f(x_n) \not\to L$. ■

Examples:

1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=3x^3+x+1$. Compute $\lim_{x\to 7} f(x)$.

   Solution. Let (x_n) be any sequence converging to 7 with $x_n\neq 7$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned}\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(3x_n^2+x_n+1) = 3\left(\lim_{n\to\infty} x_n\right)^2+\lim_{n\to\infty} x_n+1 \\ =3\cdot 7^3+7+1=1037.\end{aligned}$$

   Since (x_n) was arbitrary, $f(x)\to 1037$ when $x\to 7$, according to the Sequential Criterion. ■

2. Let $f:(2,\infty)\to\mathbb{R}$, $f(x)=\frac{(x-1)(x-2)}{(x-2)}$. Compute $\lim\limits_{x\to 2} f(x)$.

Solution. Let (x_n) be any sequence converging to 2 with $x_n\neq 2$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned}\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\frac{(x_n-1)(x_n-2)}{(x_n-2)} = \lim_{n\to\infty}(x_n-1) = \lim_{n\to\infty} x_n - 1 \\ =2-1=1.\end{aligned}$$

Since (x_n) was arbitrary, $f(x)\to 1$ when $x\to 2$, according to the Sequential Criterion. ■

3. Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x)=x^2\cos(1/x)$. Show that $\lim\limits_{x\to 0} f(x)=0$.

Proof. Let $(x_n)\subseteq\mathbb{R}\setminus\{0\}$ be any sequence converging to 0 (with $x_n\neq 0$ for all $n\in\mathbb{N}$). Then

$$0\leq |x_n^2\cos(1/x_n)|\leq |x_n^2| = |x_n|^2.$$

However, since $x_n\to 0$, then both $|x_n|\to 0$ and $|x_n|^2\to 0$, which is to say that

$$\lim_{n\to 0} |x_n^2\cos(1/x_n)| = 0$$

according to the Squeeze Theorem. Thus $x_n^2\cos(1/x_n)\to 0$. Since (x_n) was arbitrary, $f(x)\to 0$ when $x\to 0$, according to the Sequential Criterion. ■

4. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}0, & x\in\mathbb{Q}\\ 1, & x\notin\mathbb{Q}\end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Define $(x_n),(y_n)$ by $x_n=\frac{1}{n}$, $y_n=\sqrt{2}n$ for all $n\in\mathbb{N}$. Then $(x_n)\subseteq\mathbb{Q}$ and $(y_n)\subseteq\mathbb{R}\setminus\mathbb{Q}$. Furthermore, $x_n,y_n\to 0$, with $x_n,y_n\neq 0$ for all $n\in\mathbb{N}$. But $f(x_n)=0$ and $f(y_n)=1$ for all $n\in\mathbb{N}$, so

$$\lim_{n\to\infty} f(x_n)=0\neq 1=\lim_{n\to\infty} f(y_n),$$

thus $\lim_{x\to 0} f(x)$ does not exist. ■

5. Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Show that $\lim_{x \to 0} (x + \operatorname{sgn}(x))$ does not exist.

Proof. Define $(x_n), (y_n)$ by $x_n = \frac{1}{n}$, $y_n = -\sqrt{1}n$ for all $n \in \mathbb{N}$. Then $x_n, y_n \to 0$, with $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$.

But $f(x_n) = \frac{1}{n} + \operatorname{sgn}\left(\frac{1}{n}\right) = \frac{1}{n} + 1$, and $f(y_n) = -\frac{1}{n} + \operatorname{sgn}\left(-\frac{1}{n}\right) = -\frac{1}{n} - 1$

for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left(\frac{1}{n} + 1\right) \neq -1 = \lim_{n\to\infty} \left(\frac{1}{n} + 1\right) = \lim_{n\to\infty} f(x_n),$$

thus $\lim_{x\to 0} f(x)$ does not exist. ■

To show that the limit does not exist, it is enough to show that two specific sequences $(x_n), (y_n) \subseteq A$, with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$ and $x_n, y_n \to c$, exist such that $f(x_n) \to L_1$, $f(y_n) \to L_2$, $L_1 \neq L_2$.

But it is not sufficient to find two sequences $(x_n), (y_n) \subseteq A$ with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$, $x_n, y_n \to c$, and $f(x_n), f(y_n) \to L$ to show that the limit exist.

Note that at no point have needed to use the graph of the functions.

4.2 – Properties of Limits

Theorem 27. (Operations on Limits)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and c a limit point of A. Suppose $f(x) \to L$ and $g(x) \to M$ when $x \to c$. Then

1. $\lim_{x \to c} |f(x)| = |L|$;
2. $\lim_{x \to c} (f(x) + g(x)) = L + M$;
3. $\lim_{x \to c} f(x)g(x) = LM$;
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, *if $g(x) \neq 0$ for all $x \in A$ and if $M \neq 0$.*

Proof. This result is an easy consequence of Theorems 14 and 26. Let $(x_n) \subseteq A$ with $x_n \neq c$ and $x_n \to c$ for all $n \in \mathbb{N}$. Then $f(x_n) \to L$ and $g(x_n) \to M$.

1. $\lim_{x\to c} |f(x)| = \lim_{n\to\infty} |f(x_n)| = \left| \lim_{n\to\infty} f(x_n) \right| = L.$

2. $\lim_{x\to c}\left[f(x)+g(x)\right] = \lim_{n\to\infty} \left[f(x_n)+g(x_n)\right] = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L+M.$

3. $\lim_{x\to c}\left[f(x)g(x)\right] = \lim_{n\to\infty} \left[f(x_n)g(x_n)\right] = \lim_{n\to\infty} f(x_n) \cdot \lim_{n\to\infty} g(x_n) = LM.$

4. $\lim_{x\to c}\left[\dfrac{f(x)}{g(x)}\right] = \lim_{n\to\infty}\left[\dfrac{f(x_n)}{g(x_n)}\right] = \dfrac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \dfrac{L}{M}$, if $g(x) \neq 0$ for $x \in A$ and if $M = 0$. ■

There is also a Squeeze Theorem for functions, but it is not nearly as useful as the corresponding result for sequences.

Theorem 28. (Squeeze Theorem for Functions)
Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, *and* c *a limit point of* A. *If* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in A$ *and if* $f(x), h(x) \to L$ *when* $x \to c$, *then* $g(x) \to L$ *when* $x \to c$.

Proof. Let $(x_n) \subseteq A$, with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. According to the Sequential Criterion,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = L.$$

Since $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} g(x_n) = L$, by the Squeeze Theorem (for sequences). Since (x_n) was arbitrary, we conclude that $g(x) \to L$, again by the Sequential Criterion. ■

Examples:

1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=k$, $k\in\mathbb{R}$. Show that $\lim_{x\to c} f(x)=k$ for all $c\in\mathbb{R}$.

   Proof. Let $\varepsilon>0$. Set $\delta_\varepsilon=\varepsilon$. Then $|f(x)-k|=|k-k|=0<\varepsilon$, when $0<|x-c|<\delta_\varepsilon=\varepsilon$. ■

2. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x$. Show that $\lim_{x\to c} f(x)=f(c)$ for all $c\in\mathbb{R}$.

   Proof. Let $\varepsilon>0$. Set $\delta_\varepsilon=\varepsilon$. Then $|f(x)-c|=|x-c|<\delta_\varepsilon=\varepsilon$, when $0<|x-c|<\delta_\varepsilon=\varepsilon$. ■

3. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{x^3+2x-4}{x^2+1}$. Compute $\lim_{x\to 3} f(x)$.

Solution. According to Theorem 27, and the preceding examples,

$$\lim_{x\to 3}(x^3+2x+4) = \left(\lim_{x\to 3} x\right)^3 + 2\left(\lim_{x\to 3} x\right) + \lim_{x\to 3} 4 = 3^2+2(3)+3 = 37$$

$$\lim_{x\to 3}(x^2+1) = \left(\lim_{x\to 3} x\right)^2 + 1 = 3^2+1 = 10,$$

and so $\lim\limits_{x\to 3}\dfrac{x^3+2x-4}{x^2+1} = \frac{10}{3}$, **because** $x^2+1\neq 0$ for all $x\in\mathbb{R}$. ■

4. Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x)=x^2\cos(1/x)$. Show that $\lim\limits_{x\to 0} f(x)=0$.

Proof. We cannot use the multiplication component of Theorem 27 to compute the limit since $\lim\limits_{x\to 0}\cos(1/x)$ does not exist.

Indeed, let $(x_n), (y_n) \subseteq \mathbb{R}\backslash\{0\}$ be such that $x_n = \frac{1}{(2n-1)\pi}$, and $y_n = \frac{1}{2n\pi}$ for all $n \in \mathbb{N}$. Then $x_n, y_n \to 0$. But

$$\cos\left(\frac{1}{x_n}\right) = \cos((2n-1)\pi) = -1 \quad \text{and} \quad \cos\left(\frac{1}{y_n}\right) = \cos(2n\pi) = 1$$

for all $n \in \mathbb{N}$. Then

$$\cos(1/x_n) \to -1 \neq 1 \leftarrow \cos(1/y_n).$$

This does not mean that

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$$

does not exist, only that we cannot use Theorem 27 to compute it. ■

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \overline{A}$. The function f is **bounded on some neighbourhood of** c if $\exists \delta > 0$ and $M > 0$ are such that $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

Theorem 29. *If $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in \overline{A}$, and $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$, then f is bounded on some neighbourhood of c.*

Proof. Let $\varepsilon = 1$. By definition, $\exists \delta_1 > 0$ such that $|f(x) - L| < 1$ whenever $x \in A$ and $0 < |x - c| < \delta_1$.

Since

$$|f(x)| - |L| < |f(x) - L|,$$

then $|f(x)| - |L| \leq 1$ whenever $x \in A$ and $0 < |x - c| < \delta_1$.

If $c \notin A$, set $M = |L| + 1$. If $c \in A$, set $M = \max\{|f(c)|, |L| + 1\}$. In either case, $|f(x)| \leq M$ whenever $x \in A$ and $0 < |x - c| < \delta_1$. ■

4.3 – Continuous Functions

Functions like polynomials, or trigonometric functions, are continuous.

Intuitively, a function is continuous at a point if the graph of the function at that point can be traced without lifting the pen. The notion of "continuity" is fundamental is calculus.

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A$; f **is continuous at** c if

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that $|x - c| < \delta_\varepsilon$ and $x \in A \implies |f(x) - f(c)| < \varepsilon$.

When computing the limit of f at c, we are interested in the behaviour of the function near c, but not at c. When we are dealing with continuity, we also include the behaviour at c.

When c is a limit point of A, this definition actually means that

$$\lim_{x\to c} f(x) = f(c).$$

If $c \notin \overline{A}$, the expression $\lim_{x\to c} f(c)$ is meaningless since no sequence $(x_n) \subseteq A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ converges to c.

In that case, f is automatically continuous at c. Indeed, there will then be a $\delta > 0$ such that $V_\delta(c)$ contains no point of A but c.

Then for $\varepsilon > 0$, whenever $x \in A$ and $|x - c| < \delta$ (i.e. whenever $x = c$), we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$$

The definition contains 3 statements: a function f is continuous at c if

1. $f(c)$ is defined;
2. $\lim_{x\to c} f(x)$ exists, and
3. $\lim_{x\to c} f(x) = f(c)$.

Let $B \subseteq A$. If f is continuous for all $c \in B$, then f **is continuous on** B.

Examples:

- Let $f : [0,\infty) \to \mathbb{R}$, $f(x) = \frac{x^2+2x+2}{x+1}$. Is f continuous at $c = 2$?

Solution. Since 2 is a limit point of $[0,\infty)$, we need only verify if $\lim_{x\to 2} f(x) = f(2)$. But we have already seen that

$$\lim_{x\to 2} f(x) = \frac{10}{3} = f(2),$$

so f is continuous at $c = 2$. ■

- Let $f : [0,1) \to \mathbb{R}$,

$$f(x) = \begin{cases} 2, & x \in (0,1) \\ 3, & x = 0 \end{cases}$$

Is f continuous at $c = 0$?

Solution. Since 0 is a limit point of $[0,1)$, we need only verify if $\lim_{x\to 0} f(x) = f(0)$. But we have already seen that

$$\lim_{x\to 0} f(x) = 2 \neq 3 = f(0),$$

so f is not continuous at $c = 0$. ■

- Let $f:\mathbb{R}\to\mathbb{R}$, $f(x) = 3x^3 + x + 1$. Is f continuous at $c = 7$?

Solution. Since 7 is a limit point of $\mathbb{R}$, we need only verify if $\lim_{x\to 7} f(x) = f(7)$. But we have already seen that

$$\lim_{x\to 7} f(x) = 1037 = f(7),$$

so f is continuous at $c = 7$. ■

- Let $f : (2, \infty) \to \mathbb{R}$, $f(x) = \frac{(x-1)(x-2)}{(x-2)}$. Is f continuous at $c = 2$?

  Solution. Since f is not defined at $c = 2$ and since $2 \notin A$, f is not continuous at $c = 2$. ■

- Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

  Is f continuous at $c = 0$?

  Solution. As $f(0) = 0$, we only need to verify if $\lim_{x \to 0} f(x) = f(0)$. But we have already seen that $\lim_{x \to 0} f(x)$ does not exist, so f is not continuous at $c = 0$. ■

- Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=k$, $k\in\mathbb{R}$. Is f continuous on $\mathbb{R}$?

  Solution. Since all $c\in\mathbb{R}$ are limit points of $\mathbb{R}$, we need only verify if $\lim_{x\to c} f(x)=f(c)$. But we have already seen that

$$\lim_{x\to c} f(x)=k=f(c),\quad \text{for all } c\in\mathbb{R},$$

  so f is continuous on $\mathbb{R}$. ■

- Let $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} x, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$$

  Is f continuous at $c=0$?

Solution. Since $f(0) = 0$, we need to verify if $\lim_{x\to 0} f(x) = f(0)$.

Let $\varepsilon > 0$ and set $\delta_\varepsilon > 0$. Then

$$|x - 0| < \delta_\varepsilon \implies |f(x) - f(0)| = |f(x)| \leq |x| = |x - 0| < \delta_\varepsilon = \varepsilon,$$

so f is continuous at $c = 0$. ■

- Same function, but at $c \neq 0$?

Solution. Let $n \in \mathbb{N}$. According to the Density Theorem, $\exists x_n \in \mathbb{Q}$, $y_n \notin \mathbb{Q}$ such that

$$c < x_n + c + \frac{1}{n} \quad \text{and} \quad c < y_n < c + \frac{1}{n}.$$

According to the Squeeze Theorem (for sequences), $x_n, y_n \to c$. But $f(x_n) = x_n$ and $f(y_n) = 0$ for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0.$$

Since $c \neq 0$, these limits are different, and so $\lim\limits_{x\to c} f(x)$ does not exist, according to the Sequential Criterion. ■

- Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Is f continuous on $[0, \infty)$?

Solution. Let $\varepsilon > 0$. If $c = 0$, set $\delta_\varepsilon = \varepsilon$. Then

$$x \geq 0 \text{ and } |x-0| < \delta_\varepsilon \implies f(x)-f(0)| = \sqrt{x} = \sqrt{|x-0|} < \sqrt{\delta_\varepsilon = \varepsilon},$$

so f is continuous at $c = 0$.

If $c > 0$, set $\delta_\varepsilon = \sqrt{c}\varepsilon$. Then

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{|x-c|}{\sqrt{c}} < \frac{\delta_\varepsilon}{\sqrt{c}} = \varepsilon$$

whenever $x \geq 0$ and $|x - c| < \delta_\varepsilon$. Hence f is continuous at $c > 0$, and so also on $[0, \infty)$. ■

- Let $A = \{x \in \mathbb{R} \mid x > 0\}$. Consider the function $f : A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q}, \text{ with } \gcd(m,n) = 1 \end{cases}$$

Where is f is continuous?

Solution. We consider two types of limit points of A: $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$.

If $0 < a \in \mathbb{Q}$, let $(x_n) \subseteq A \cap \mathbb{Q}^{\complement}$ be such that $x_n \to a$. Then $f(x_n) \to 0$. But $f(a) > 0$ (as $a \in \mathbb{Q}$), so $\lim_{x \to a} f(x) \neq f(a)$ according to the Sequential Criterion.

If $0 < b \notin \mathbb{Q}$, let $\varepsilon > 0$. By the Archimedean Property, there exists an integer $N_0 > \frac{1}{\varepsilon}$. There can only be a finite set of rationals with denominator $< N_0$ in the interval $(b-1, b+1)$.

Indeed, suppose $n < N_0$ and $\frac{m}{n} \in (b-1, b+1)$. Then whenever $|k| > 2n$, we have

$$\left|\frac{m+k}{n} - \frac{m}{n}\right| = \frac{|k|}{n} > 2 \implies \frac{m+k}{n} \notin (b-1, b+1).$$

Consequently, $\exists \delta > 0$ such that there are no rational number $\frac{m}{n}$ with denominator $< N_0$ in $(b-\delta, b+\delta)$, which is to say that for all $x \in (b-\delta, b+\delta)$, either $f(x) = 0$ (when x is irrational) or $f(x) = \frac{1}{n} \leq \frac{1}{N_0}$ (when x is rational).

Thus, if $|x-b| < \delta$ and $x \in A$, we have

$$|f(x) - f(b)| = |f(x) - 0| = |f(x)| \leq \frac{1}{N_0} < \varepsilon,$$

so $\lim_{x \to b} f(x) = f(b)$.

Thus f is continuous at every irrational in A and discontinuous at every rational in A. ■

Continuity behaves very nicely with respect to elementary operations on functions.

Theorem 30. (Operations on Continuous Functions)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and $c \in A$. If f, g are continuous at c, then

1. *$|f|$ is continuous at c;*
2. *$f + g$ is continuous at c;*
3. *fg is continuous at c;*
4. *$\frac{f}{g}$ is continuous at c if $g \neq 0$ on A.*

Proof. If c $\not{\overline{A}}$, there is nothing to prove.

If $c \in \overline{A}$, then

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c).$$

We can then apply Theorem 27 directly with $L = f(c)$ and $M = g(c)$. ■

Corollary. The same results holds if we replace "continuous at c" with "continuous at A".

Since constants and the identity function are continuous on $\mathbb{R}$ (see preceding examples), so are polynomial functions. Furthermore, rational functions are continuous on their domain.

The **composition** of the functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$, with $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Theorem 31. (Composition of Continuous Functions)
Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $c \in A$. *If* f *is continuous at* c, g *is continuous at* $f(c)$, *and* $f(A) \subseteq B$, *then* $g \circ f : A \to B$ *is continuous at* c.

Proof. Let $\varepsilon > 0$. As g is continuous at $f(c)$, $\exists \delta_\varepsilon > 0$ such that

$$y \in B \text{ and } |y - f(c)| < \delta_\varepsilon \implies |g(y) - g(f(c))| < \varepsilon.$$

Since f is continuous at c, $\exists \eta_{\delta_\varepsilon} = \eta_\varepsilon > 0$ such that

$$x \in A \text{ and } |x - c| < \eta_{\delta_\varepsilon} \implies |f(x) - f(c)| < \delta_\varepsilon \implies$$

$x \in A$ and $|x-c| < \eta_\varepsilon \implies |(g\circ f)(x)-(g\circ f)(c)| = |g(f(x))-g(f(c))| < \varepsilon$,
which completes the proof. ■

Corollary. The same results holds if we replace "continuous at c" with "continuous at A".

Example: Let $f : [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{3x^3 + x + 1}$. Show that f is continuous on $[0,\infty)$.

Proof. We can write $f = g \circ h$, where $g : [0,\infty) \to \mathbb{R}$, $g(y) = \sqrt{y}$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = 3x^2 + x + 1$. Since g and h are both continuous on their domains and $h(\mathbb{R}) \subseteq [0,\infty)$, g is continuous on $[0,\infty)$, according to Theorem 31. ■

An **algebraic** function is a function obtained via the (possibly repeated) composition of rational functions and root functions. The class of algebraic functions is continuous on it domain. The same goes for trigonometric, exponential, and logarithmic functions, *via* their power series definition.

4.4 – Max/Min Theorem

We now begin our study of the classical theorems of calculus.

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. The function $f : A \to \mathbb{R}$ is **bounded** on A if $\exists M > 0$ such that $|f(x)| < M$ for all $x \in A$.

Examples:

1. $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$, is bounded on $[0,1]$ as $|f(x)| < 2, \forall x \in [0,1]$.

2. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, is not bounded on $\mathbb{R}$ Indeed, suppose $\exists M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$. Then $|x^2| = |x|^2 < M$ for all $x \in \mathbb{R}$, i.e. $|x| < \sqrt{M}$ for all $x \in \mathbb{R} \implies M$ is an upper bound of $\mathbb{R}$. But there is no such bound, $\therefore g$ is not bounded on $\mathbb{R}$.

3. $f : (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not bounded on $(0,1]$, but it is bounded on $[a,1]$ for all $a \in (0,1]$.

Theorem 32. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is bounded on $[a,b]$.*

Proof. Suppose f is not bounded on $[a,b]$. Hence, for all $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ such that $|f(x_n)| > n$. However, $(x_n) \subseteq [a,b]$ so that (x_n) is bounded.

By the BW Theorem, $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to \hat{x} \in [a,b]$, since

$$a \leq x_{n_k} \leq b \quad \text{for all } k.$$

Since f is continuous, we have

$$f(\hat{x}) = \lim_{x\to\hat{x}} f(x) = \lim_{k\to\infty} f(x_{n_k}),$$

so $(f(x_{n_k}))$ is bounded, being a convergent sequence. But this contradicts the assumption that $|f(x_{n_k})| > n_k \geq k$ for all k.

Hence f is bounded on $[a,b]$. ■

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. We say that f **reaches a global maximum on** A if $\exists x^* \in A$ such that $f(x^*) \geq f(x)$ for all $x \in A$.

Similarly, f **reaches a global minimum on** A if $\exists x_* \in A$ such that $f(x_*) \leq f(x)$ for all $x \in A$.

Continuous functions on closed, bounded sets have a useful property.

Theorem 33. (MAX/MIN THEOREM)
If $f : [a,b] \to \mathbb{R}$ is continuous, then f reaches a global maximum and a global minimum of $[a,b]$.

Proof. Let $f([a,b]) = \{f(x) \mid x \in [a,b]\}$. According to Theorem 32, $f([a,b])$ is bounded as f is continous, and so, by completeness of $\mathbb{R}$,

$$s^* = \sup\{f(x) \mid x \in [a,b]\} \quad \text{and} \quad s_* = \inf\{f(x) \mid x \in [a,b]\}$$

both exist.

We need only show $\exists x^*, x_* \in [a,b]$ such that $f(x^*) = s^*$ and $f(x_*) = s_*$.

Since $s^* - \frac{1}{n}$ is not an upper bound of $f([a,b])$ for every $n \in \mathbb{N}$, $\exists x_n \in [a,b]$

with
$$s^* - \frac{1}{n} < f(x_n) \leq s^*, \quad \text{for all } n \in \mathbb{N}.$$
According to the Squeeze Theorem, we must have $f(x_n) \to s^*$ (this says nothing about whether x_n converges or not, however).

But $(x_n) \subseteq [a,b]$ is bounded, so applying the Bolzano-Weierstrass Theorem, we find that $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to x^* \in [a,b]$.

As f is continuous,

$$s^* = \lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x^*).$$

The existence of $x_* \in [a,b]$ such that $f(x_*) = s_*$ is shown similarly.

Examples:

1. The function $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$, reaches its maximum and minimum on $[0,1]$ since f is continuous, being a polynomial.

2. Let $f : [0,1) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 2, & x \in (0,1) \\ 3, & x = 0 \end{cases}$$

   Then f is not continuous on $[0,1)$, and $[0,1)$ is not closed and bounded, so we cannot use the Max/Min Theorem to conclude that f reaches its global max/global min on $[0,1)$... although it does: 3 at $x^* = 0$ and 2 at any $x_* \in (0,1)$.

⚠ The hypotheses of a theorem have to be satisfied in order to use it to draw its conclusion, but the conclusion is not necessarily false if the hypotheses are not met.

3. The function $f:[a,1]\to\mathbb{R}$, $a\in(0,1]$, defined by $f(x)=\frac{1}{x}$ reaches its global max/global min on $[a,1]$ as f is continuous on $[a,1]$, being a rational function there.

4. The function $f:(0,1]\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}$ is continuous on $(0,1]$, but we cannot use the Max/Min Theorem as $(0,1]$ is not closed.

   In this case, f has no global maximum, but it does hav a global minimum at $x=1$.

4.5 – Intermediate Value Theorem

The following result has many implications; it can notably be used to locate the roots of a function.

Theorem 34. *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\exists \alpha, \beta \in [a, b]$ such that $f(\alpha)f(\beta) < 0$, then $\exists \gamma \in (a, b)$ such that $f(\gamma) = 0$.*

Proof. We prove that the results holds for $f(\alpha) < 0 < f(\beta)$; the other case having a similar proof.

Write $\alpha_1 = \alpha$, $\beta_1 = \beta$, $I_1 = [\alpha_1, \beta_1]$, and $\gamma_1 = \frac{\alpha_1+\beta_1}{2}$. There are 3 possibilities:

i. if $f(\gamma_1) = 0$, set $\gamma = \gamma_1$; then $\gamma \in (\alpha_1, \beta_1)$ and the theorem is proven;

ii. if $f(\gamma_1) > 0$, set $\alpha_2 = \alpha_1$, $\beta_2 = \gamma_1$;

iii. if $f(\gamma_1) < 0$, set $\alpha_2 = \gamma_1$, $\beta_2 = \beta_1$.

In the last two cases, set $I_2 = [\alpha_2, \beta_2]$. Then $I_1 \supseteq I_2$, length$(I_1) = \frac{\beta_1 - \alpha_1}{2^0}$ and

$$f(\alpha_2) < 0 < f(\beta_2).$$

This is the base case $n = 1$ of an induction process, which can be extended for all $n \in \mathbb{N}$. One of two things can occur: either

1. $\exists n \in \mathbb{N}$ such that $f(\gamma_n) = 0$, with $\gamma_n \in (\alpha_n, \beta_n) \subseteq (\alpha, \beta)$, in which case the theorem is proven, or

2. there is a chain of nested intervals

$$I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq I_{k+1} \supseteq \cdots$$

where $I_n = [\alpha_n, \beta_n]$, length$(I_n) = \frac{\beta_n - \alpha_n}{2^{n-1}}$, $f(\alpha_n) < 0 < f(\beta_n)\ \forall n \in \mathbb{N}$.

According to the Nested Intervals Theorem, since

$$\inf_{n \in \mathbb{N}} \{\text{length}(I_n)\} = \lim_{n \to \infty} \frac{\beta_n - \alpha_n}{2^{n-1}} = 0,$$

$\exists c \in [\alpha, \beta] \subseteq [a, b]$ such that $\bigcap_{n \in \mathbb{N}} I_n = \{c\}$.

It remains to show that $f(c) = 0$.

Note that the sequences $(\alpha_n), (\beta_n)$ both converge to c. Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon > \log_2(\frac{\beta-\alpha}{\varepsilon}) + 1$.

Since $c \in I_n$ for all $n \in \mathbb{N}$, then

$$|\alpha_n - c| < \text{length}(I_n) = \frac{\beta - \alpha}{2^{n-1}} < \varepsilon$$

whenever $n > N_\varepsilon$. The proof that $\beta_n \to c$ is identical.

Since f is continuous on $[a, b]$, it is also continuous at c. Thus,

$$\lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} f(\beta_n) = f(c).$$

But $f(\alpha_n) < 0$ for all n, so, Theorem 15:

$$f(c) = \lim_{n\to\infty} f(\alpha_n) \leq 0.$$

Using the same Theorem, we have $f(c) \geq 0$. Then $f(c) = 0$.

Lastly, note that $c \neq \alpha, \beta$; otherwise, $f(\alpha)f(\beta) = 0$.

This concludes the proof, with $\gamma = c$. ■

Example: Show that $\exists x \in \mathbb{R}^+$ such that $x^2 = 2$.

Proof. The function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^2 - 2$ is continuous on $[0, 2]$. As $f(0) = 0^2 - 2 = -2 < 0$ and $f(2) = 2^2 - 2 = 2 > 0$, $\exists \gamma \in (0, 2)$ such that $\gamma^2 - 2 = 0$, so $\gamma^2 = 2$, according to Theorem 34. ■

This result easily generalizes to the following.

Theorem 35. (Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ *be continuous. If* $\exists \alpha < \beta \in [a, b]$ *s.t.* $f(\alpha) < k < f(\beta)$ *or* $f(\alpha) > k > f(\beta)$*, then* $\exists \gamma \in (a, b)$ *such that* $f(\gamma) = k$*.*

Proof. Assume that $f(\alpha) < k < f(\beta)$; the proof for the other case is similar.

Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - k$. Theorem 30 shows that g is continuous on $[a, b]$. Furthermore,

$$g(\alpha) = f(\alpha) - k < k - k = 0 < f(\beta) - k = g(\beta).$$

According to Theorem 34, $\exists \gamma \in (\alpha, \beta)$ such that $g(\gamma) = f(\gamma) - k = 0$. Thus $f(\gamma) = k$. ■

The following result combines the Max/Min Theorem and the Intermediate Value Theorem.

Theorem 36. *If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is a closed and bounded interval.*

Proof. Let $m = \inf\{f[a,b]\}$ and $M = \sup\{f[a,b]\}$.

According to the Max/Min Theorem, $\exists \alpha, \beta \in [a,b]$ such that $f(\alpha) = m$ and $f(\beta) = M$.

If $m = M$, then f is constant and $f([a,b]) = [m,m] = [M,M]$.

If $m < M$, then $\alpha \neq \beta$. Furthermore, $m \leq f(x) \leq M$ for all $x \in [a,b]$, so that $f([a,b]) \subseteq [m,M]$.

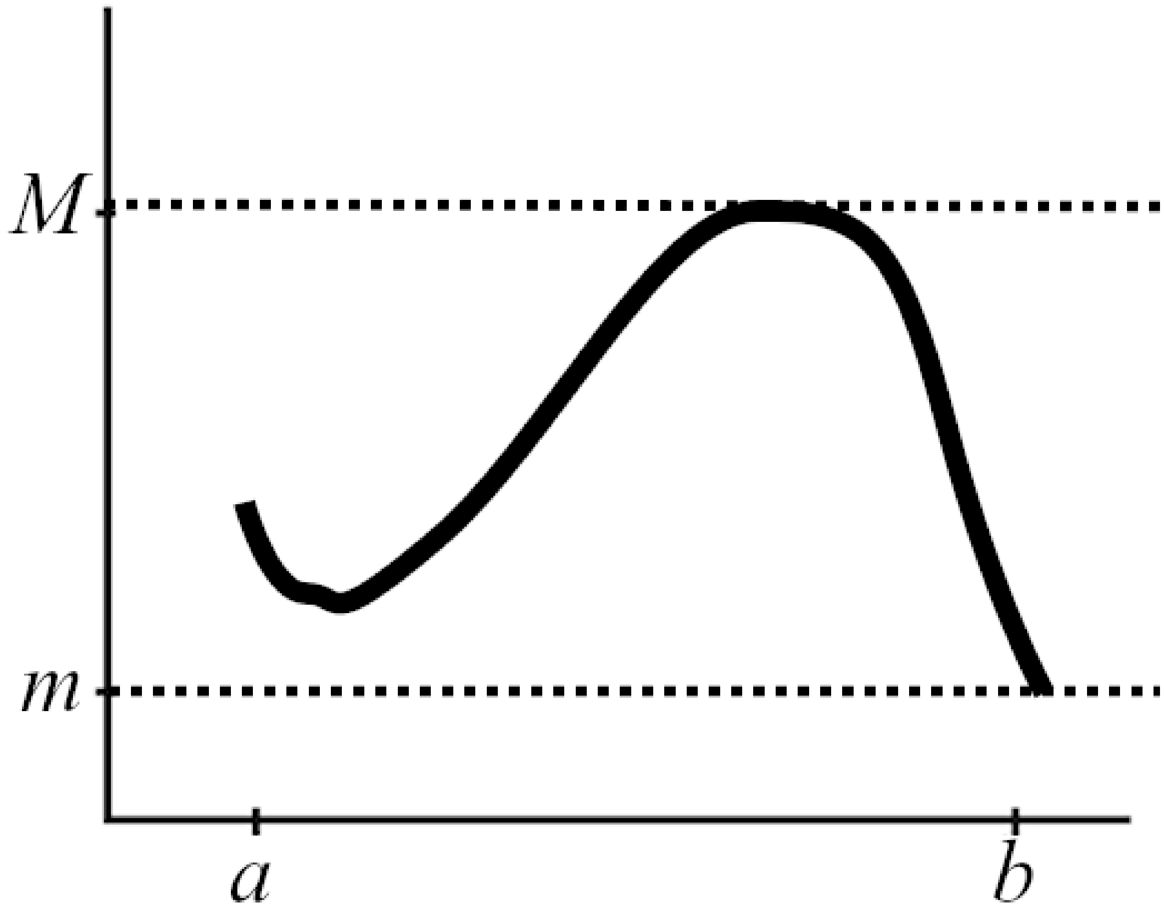
Now, let $k \in [m, M]$. According to the Intermediate Value Theorem, $\exists \gamma$ between α and β such that $f(\gamma) = k$. Hence $k \in f([a, b])$ and so $[m, M] \subseteq f([a, b])$. Consequently, $f([a, b]) = [m, M]$. ■

The image of any interval by a continuous function is always an interval, but the only time that we know for a fact that image is of the same type as the original is when the original is closed and bounded.

Examples:

1. Let $f : [0, 1] \to \mathbb{R}$, $f(x) = 2x - 1$. Then $f([0, 1])$ is closed and bounded (in fact, $f([0, 1]) = [-1, 1]$, but that is not given by the Theorem).

2. The function $f : (0, 2\pi) \to \mathbb{R}$ defined by $f(x) = \sin x$ is continuous and $f((0, 2\pi)) = [-1, 1]$, but Theorem 36 does not apply.


M
m
a
b

4.6 – Uniform Continuity

Let $f : A \to \mathbb{R}$ be continuous (on A). For $\varepsilon > 0$ and $c \in A$, the $\delta_\varepsilon > 0$ that is used to show continuity of f at c depends generally on ε **and** c. But there are instances when δ_ε depends only on ε.

The function f is **uniformly continuous** on A if

$$x, y \in A \text{ and } |x - y| < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon.$$

The notion of uniform continuity is more restrictive than that of (simple) continuity.

Theorem 37. *If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.*

Proof. Let $c \in A$ and $\varepsilon > 0$. As f is uniformly continuous on A, $\exists \delta_\varepsilon > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta_\varepsilon \text{ and } x, y \in A.$$

In particular, if $y = c$ then

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever } |x - c| < \delta_\varepsilon \text{ and } x \in A.$$

As c is arbitrary, f is continuous on A. ■

Example: Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.

Proof. That f is continuous on $(0, \infty)$ is immediate, as it is a rational function.

Let $(x_n) = (\frac{1}{n}) \subseteq (0, \infty)$. Clearly, (x_n) is a Cauchy sequence as it is a convergent sequence. But $f(x_n) = \frac{1}{1/n} = n$ for all $n \in \mathbb{N}$, so $(f(x_n))$ is not a Cauchy sequence in $\mathbb{R}$ (as it is not bounded, and thus divergent).

According to a Lemma that we prove next, f cannot be uniformly continuous on $(0, \infty)$. ■

Lemma. If f is uniformly continuous on A and $(x_n) \subseteq A$ is a Cauchy sequence, then $f(x_n)$ is a Cauchy sequence.

Proof. If $(x_n) \subseteq A$ is a Cauchy sequence and $\delta > 0$, $\exists N_\delta \in \mathbb{N}$ such that $|x_m - x_n| < \delta$ whenever $m, n > N_\delta$.

But f is uniformly continuous on A, so that $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon.$$

Combining these two statements, with $N_\varepsilon = M_{\delta_\varepsilon}$, yields

$$m, n > N_\varepsilon \implies |x_m - x_n| < \delta_\varepsilon \implies |f(x_m) - f(x_n)| < \varepsilon,$$

and so $(f(x_n))$ is a Cauchy sequence. ■

While continuous functions are not generally uniformly continuous, there is a specific class of functions for which continuity is equivalent to uniform continuity.

Theorem 38. *Let $f : [a, b] \to \mathbb{R}$. Then f is uniformly continuous on $[a, b]$ if and only if f is continuous on $[a, b]$.*

Proof. Theorem 38 shows that if f is uniformly continuous on $[a, b]$, it is continuous on $[a, b]$.

Now, assume f is continuous on $[a, b]$. If f is not uniformly continuous, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta, y_\delta \in [a, b]$ with

$$|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0 \text{ and } |x_\delta - y_\delta| < \delta.$$

For $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$. The corresponding sequences $(x_{\delta_n}), (y_{\delta_n})$ lie in $[a, b]$, with

$$|x_{\delta_n} - y_{\delta_n}| < \delta_n = \frac{1}{n} \quad \text{and} \quad |f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

As (x_{δ_n}) is bounded, $\exists (x_{\delta_{n_k}}) \subseteq (x_{\delta_n})$ such that $x_{\delta_{n_k}} \to z$ with $k \to \infty$, according to the Bolazano-Weierstrass Theorem.

Furthermore, $z \in [a, b]$ according to Theorem 15.

The corresponding sequence $(y_{\delta_{n_k}})$ also converges to z subce

$$0 \leq |y_{\delta_{n_k}} - z| \leq |y_{\delta_{n_k}} - x_{\delta_{n_k}}| + |x_{\delta_{n_k}} - z| < \frac{1}{n_k} + |x_{\delta_{n_k}} - z|$$

according to the Squeeze Theorem, as both $\frac{1}{n_k}, |x_{\delta_{n_k}} - z| \to 0$ with $k \to \infty$.

But f is continuous, both $(f(x_{\delta_{n_k}})), (f(y_{\delta_{n_k}})) \to f(z)$. But that is impossible as $|f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}$.

Thus f must be uniformly continuous. ■

Example: Show $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{1}{1+x^2}$ is uniformly continuous on $[0,1]$.

Proof. Let $\varepsilon>0$. Set $\delta_\varepsilon=\varepsilon$. Then

$$\begin{aligned}|f(x)-f(y)| &= \left|\frac{1}{1+x^2}-\frac{1}{1+y^2}\right| = \left|\frac{y^2-x^2}{(1+x^2)(1+y^2)}\right| = \left|\frac{x+y}{(1+x^2)(1+y^2)}\right||x-y| \\ &\leq \left(\left|\frac{y}{1+y^2}\right|\cdot\underbrace{\frac{1}{1+x^2}}_{\leq 1}+\left|\frac{x}{1+x^2}\right|\cdot\underbrace{\frac{1}{1+y^2}}_{\leq 1}\right)|x-y| \\ &\leq \left(\underbrace{\left|\frac{y}{1+y^2}\right|}_{\leq 1/2}+\underbrace{\left|\frac{x}{1+x^2}\right|}_{\leq 1/2}\right)|x-y| \leq |x-y| < \delta_\varepsilon = \varepsilon\end{aligned}$$

whenever $|x-y|<\delta_\varepsilon$. ■

Note that $\forall z\in\mathbb{R}$, $0\leq(|z|-1)^2=z^2-2|z|+1 \implies 2|z|\leq 1+z^2$.

4.7 – Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to 0} f(x+c) = L$.
2. Show $\lim_{x\to c} x^3 = c^3$ for any $c \in \mathbb{R}$.
3. Use either the $\varepsilon - \delta$ definition of the limit or the Sequential Criterion for limits to establish the following limits:
   (a) $\lim_{x\to 2} \dfrac{1}{1-x} = -1$;
   (b) $\lim_{x\to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$;
   (c) $\lim_{x\to 0} \dfrac{x^2}{|x|} = 0$, and
   (d) $\lim_{x\to 1} \dfrac{x^2-x+1}{x+1} = \dfrac{1}{2}$

4. Show that the following limits do not exist:
   (a) $\lim_{x\to 0} \dfrac{1}{x^2}$, with $x > 0$;
   (b) $\lim_{x\to 0} \dfrac{1}{\sqrt{x}}$, with $x > 0$;
   (c) $\lim_{x\to 0}(x + \operatorname{sgn}(x))$, and
   (d) $\lim_{x\to 0} \sin(1/x^2)$, with $x > 0$.
5. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} (f(x))^2 = L$. Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$. Show that if $L \neq 0$, then f may not have a limit at c.
6. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a closed interval in $\mathbb{R}$ and let $c \in J$. If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show the converse is not necessarily true.

7. Determine the following limits and state which theorems are used in each case.

   (a) $\lim_{x\to 2}\sqrt{\dfrac{2x+1}{x+3}}$, $(x>0)$;

   (b) $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$, $(x>0)$;

   (c) $\lim_{x\to 0}\sqrt{\dfrac{(x+1)^2-1}{x}}$, $(x>0)$, and

   (d) $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$, $(x>0)$.

8. Give examples of functions f and g such that f and g do not have limits at point c, but both $f+g$ and fg have limits at c.

9. Determine whether the following limits exist in $\mathbb{R}$:
   (a) $\lim\limits_{x\to 0} \sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;
   (b) $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;
   (c) $\lim\limits_{x\to 0} \operatorname{sgn}\sin\left(\frac{1}{x}\right)$, with $x \neq 0$, and
   (d) $\lim\limits_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right)$, with $x > 0$.
10. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume $\lim\limits_{x\to 0} f(x) = L$ exists. Prove that $L = 0$ and that f has a limit at every point $c \in \mathbb{R}$.
11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

12. Let $f : (0,1) \to \mathbb{R}$ be bounded and s.t. $\lim_{x\to 0} f(x)$ does not exist. Show that there are two convergent sequences $(x_n), (y_n) \subseteq (0,1)$ with $x_n, y_n \to 0$ and $f(x_n) \to \xi$, $f(y_n) \to \zeta$, but $\xi \neq \zeta$.
13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $P = \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighbourhood $V_\delta(c) \subseteq P$.
14. Prove that if an additive function is continuous at some point $c \in \mathbb{R}$, it is continuous on $\mathbb{R}$.
15. If f is a continuous additive function on $\mathbb{R}$, show that $f(x) = cx$ for all $x \in \mathbb{R}$, where $c = f(1)$.
16. Let $I = [a,b]$ and $f : I \to \mathbb{R}$ be a continuous function on I s.t. $\forall x \in I$, $\exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Show $\exists c \in I$ s.t. $f(c) = 0$.
17. Show that every polynomial with odd degree has at least one real root.
18. Let $f : [0,1] \to \mathbb{R}$ be continuous and s.t. $f(0) = f(1)$. Show $\exists c \in [0, \frac{1}{2}]$ s.t. $f(c) = f(c + \frac{1}{2})$.

19. Determine whether the following limits exist in $\mathbb{R}$:
   (a) $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x^2}\right)$, with $x \neq 0$;
   (b) $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x^2}\right)$, with $x \neq 0$;
   (c) $\lim\limits_{x\to 0} \operatorname{sgn}\sin\left(\dfrac{1}{x}\right)$, with $x \neq 0$, and
   (d) $\lim\limits_{x\to 0} \sqrt{x}\sin\left(\dfrac{1}{x^2}\right)$, with $x > 0$.
20. If $f(x) = x$ and $g(x) = \sin x$, show that f and g are both uniformly continuous on $\mathbb{R}$ but that their product is not uniformly continuous on $\mathbb{R}$.
21. Let $A \subseteq \mathbb{R}$ and suppose that f has the following property: $\forall \varepsilon > 0$, $\exists g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A with $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Show f is uniformly continuous on A.
22. Prove that a continuous $p-$periodic fonction on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

23. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

Prove that g is continuous at 0.

24. Let $f : \mathbb{R} \to \mathbb{R}$. The **pre-image** of a subset $B \subseteq \mathbb{R}$ under f is

$$f^{-1}(B) = \{a \in A \mid f(a) \in B\}.$$

Prove that f is continuous if and only if the pre-image of every open subset of $\mathbb{R}$ is an open subset of $\mathbb{R}$.

25. A function $f : A \to \mathbb{R}$ is said to be **Lipschitz** if there is a positive number M such that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in A.$$

Show that a Lipschitz function must be uniformly continuous, but that uniformly continuous functions do not have to be Lipschitz.

Solutions

1. **Proof.**

$$\lim_{x\to c} f(x) = L$$

$$\Updownarrow$$

$$\forall\varepsilon > 0, \exists\delta_\varepsilon > 0 \text{ s.t. } |f(x) - L| < \varepsilon \text{ when } 0 < |x - c| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\text{Set } x = y + c: \ \forall\varepsilon > 0, \exists\delta_\varepsilon > 0 \text{ s.t. } |f(y+c) - L| < \varepsilon \text{ when } 0 < |y| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\forall\varepsilon > 0, \exists\delta_\varepsilon > 0 \text{ s.t. } |f(y+c) - L| < \varepsilon \text{ when } 0 < |y - 0| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\lim_{y\to 0} f(y+c) = L. \qquad \blacksquare$$

2. **Proof.** If $|x-c|<1$, then $|x|<|c|+1$.

Let $\varepsilon>0$. Set $\delta_\varepsilon=\min\{1,\frac{\varepsilon}{3|c|^2+3|c|+1}\}$.

Then

$$\begin{aligned}
|x^3-c^3| &= |x-c||x^2+cx+c^2| \\
&\leq |x-c|\left(|x|^2+|c||x|+|c|^2\right) \\
&< |x-c|\left((|c|+1)^2+|c|(|c|+1)+|c|^2\right) \\
&= |x-c|\left(3|c|^2+3|c|+1\right) \\
&< \delta_\varepsilon\cdot\left(3|c|^2+3|c|+1\right)\leq\frac{\varepsilon}{3|c|^2+3|c|+1}\cdot\left(3|c|^2+3|c|+1\right)=\varepsilon,
\end{aligned}$$

whenever $0<|x-c|<\delta_\varepsilon$ and $x\in\mathbb{R}$. ■

3. **Proof.**

(a) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. Then

$$0 < |x-2| < \delta_\varepsilon \implies |x-2| < \frac{1}{2} \iff \frac{3}{2} < x < \frac{5}{2}$$
$$\iff \frac{1}{2} < x-1 < \frac{3}{2} \iff \frac{1}{x-1} < 2.$$

Thus

$$\left|\frac{1}{1-x} - (-1)\right| = \frac{1}{|x-1|}|x-2| = \frac{1}{x-1}|x-2| < 2\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-2| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-2| < \delta_\varepsilon$, we've seen that $x > \frac{3}{2}$ and so that $|x-1| = x-1$. This explains why we have gotten rid of the absolute values above.)

(b) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, 3\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \implies |x-1| < \frac{1}{2} \iff \frac{1}{2} < x < \frac{3}{2}$$
$$\iff 3 < 2(x+1) < 5 \iff \frac{1}{2(x+1)} < \frac{1}{3}.$$

Thus

$$\left|\frac{x}{1+x} - \frac{1}{2}\right| = \frac{1}{2|x+1|}|x-1| = \frac{1}{2(x+1)}|x-1| < \frac{1}{3}\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-1| < \delta_\varepsilon$, we've seen that $2(x+1) > 3$ and so that $2|x+1| = 2(x+1)$. This explains why we have gotten rid of the absolute values above.)

(c) Let $(x_n) \subseteq \mathbb{R}$ be a sequence s.t. $x_n \to 0$ and $x_n \neq 0$ for all n. Then

$$\frac{x_n^2}{|x_n|} = \frac{|x_n|^2}{|x_n|} = |x_n| \to 0,$$

by theorem 14. By another theorem, the limit must be thus 0.

(d) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{3}{2}\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \implies |2x-1| < 2 \text{ and } \left|\frac{1}{2(x+1)}\right| < \frac{1}{3}.$$

Thus ,whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$, we have

$$\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-1}{2(x+1)}\right| |x-1| < \frac{2}{3}|x-1| < \frac{2}{3}\delta_\varepsilon < \varepsilon. \quad \blacksquare$$

4. **Proof.** In each instance, we only give some sequence(s) for which theorem 26 shows the limit does not exist.

(a) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/n^2} = n^2 \to \infty$.

(b) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/\sqrt{n}} = \sqrt{n} \to \infty$.

(c) $x_n = \frac{1}{n}, y_n = -\frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{n} + 1 \to 1, f(y_n) = -\frac{1}{n} - 1 \to -1$.

(d) $x_n = \sqrt{\frac{2}{(4n+1)\pi}}, y_n = \sqrt{\frac{2}{(4n+3)\pi}} \to 0$ but

$$f(x_n) = \sin\left(\frac{4n+1}{2}\pi\right) \to 1,\; f(y_n) = \sin\left(\frac{4n+3}{2}\pi\right) \to -1. \quad \blacksquare$$

5. **Proof.** If $\lim_{x\to c}(f(x))^2 = 0$ then $\forall \eta > 0$, $\exists \delta_\eta > 0$ such that

$$|f(x)|^2 = \left|(f(x))^2 - 0\right| < \eta$$

whenever $0 < |x - c| < \delta_\eta$. Let $\varepsilon > 0$.

By definition of the real numbers, $\exists \eta_\varepsilon > 0$ such that $\varepsilon = \sqrt{\eta_\varepsilon}$. Set $\delta_\varepsilon = \delta_{\eta_\varepsilon}$. Then

$$|f(x) - 0| = |f(x)| = \sqrt{|f(x)|^2} < \sqrt{\eta_\varepsilon} = \varepsilon$$

whenever $0 < |x - c| < \delta_\varepsilon$.

Now, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0\end{cases}.$$

Then $(f(x))^2\equiv 1$ and

$$\lim_{x\to 0}(f(x))^2=\lim_{x\to 0}1=1.$$

But $\lim_{x\to 0}f(x)$ does not exist since $(x_n)=(\frac{1}{n})$, $(y_n)=(-\frac{1}{n})$ are sequences such that $x_n,y_n\to 0$, $x_n,y_n\neq 0$ for all n and

$$f(x_n)=-1\to -1\neq 1\leftarrow 1=f(y_n). \quad \blacksquare$$

6. **Proof.** Suppose $\lim_{x\to c} f(x) = L$ exists.

Then, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$.

But $f_2(x) = f(x)$ for all $x \in J \subseteq \mathbb{R}$.

Then, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ (exactly as above) s.t. $|f_2(x)-L| = |f(x)-L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$ and $x \in J$, and so $\lim_{x\to c} f_2(x) = L$.

Now consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \cup (1, \infty) \\ 1 & \text{if } x \in [0, 1] \end{cases},$$

with $J = [0, 1]$ and $f_2 = f|_J$. Then $\lim_{x\to 1} f_2(x) = 1$ but $\lim_{x\to 1} f(x)$ does not exist. ■

7. **Proof.** We will do (c) and just give the answers to the others.

Consider the sequence $(x_n) = (\frac{1}{n})$. Then $x_n \to 0$, $x_n \neq 0$ $\forall n \in \mathbb{N}$, and

$$\frac{(x_n+1)^2-1}{x_n} = \frac{\left(\frac{1}{n}+1\right)^2-1}{\frac{1}{n}} = \frac{1}{n}+2 \to 2.$$

Hence, if $\lim\limits_{x\to 0}\dfrac{(x+1)^2-1}{x}$ exists, its value must be 2, by theorem 26.

Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then when $0 < |x-0| < \delta_\varepsilon$ and $x > 0$, we have

$$\left|\frac{(x+1)^2-1}{x}-2\right| = \left|\frac{x^2+2x+1-1-2x}{x}\right| = \left|\frac{x^2}{x}\right| = |x| = |x-0| < \delta_\varepsilon = \varepsilon.$$

(a) 1 (b) 4 (d) $\frac{1}{2}$ ■

8. **Proof.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

and $g(x) = -f(x)$ for all $x \in \mathbb{R}$.

Then $f(x) + g(x) \equiv 0$ and $f(x)g(x) \equiv -1$. As a result,

$$\lim_{x \to 0} (f+g)(x) = 0 \quad \text{and} \lim_{x \to 0} (fg)(x) = -1,$$

but the limits of f and g don't exist at 0 (see problem 5). ■

9. **Proof.**

(a) Let $(x_n) = (\frac{1}{\sqrt{n\pi}})$ and $(y_n) = (\sqrt{\frac{2}{(4n+1)\pi}})$ for all $n \in \mathbb{N}$.

Then $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. But

$$\sin\left(\frac{1}{x_n^2}\right) = \sin(n\pi) = 0 \quad \text{and} \quad \sin\left(\frac{1}{y_n^2}\right) = \sin\left(\frac{(4n+1)\pi}{2}\right) = 1$$

for all $n \in \mathbb{N}$.

Then $\sin(1/x_n^2) \to 0$ and $\sin(1/y_n^2) \to 1$. As $0 \neq 1$, $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ doesn't exist.

(b) Consider the sequence $(x_n) = (\frac{1}{\sqrt{n\pi}})$. Then $x_n \to 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore,

$$x_n \sin\left(\frac{1}{x_n^2}\right) = \frac{1}{\sqrt{n\pi}} \sin(n\pi) = \frac{1}{\sqrt{n\pi}} \cdot 0 \to 0.$$

As a result, if $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right)$ exists, it must take the value 0. Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then

$$\left| x \sin\left(\frac{1}{x^2}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x^2}\right) \right| \leq |x| = |x - 0| < \delta_\varepsilon = \varepsilon$$

whenever $0 < |x - 0| < \delta_\varepsilon$ and $x > 0$. Hence $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0$.

(c) Let $(x_n) = \left(\frac{2}{(2n+1)\pi}\right)$. Then $x_n \to 0$, $x_n \neq 0$ for all $n \in \mathbb{N}$ and

$$\operatorname{sgn}\left(\sin\left(\frac{1}{x_n}\right)\right) = \operatorname{sgn}\left((-1)^n\right) = (-1)^n,$$

which does not converge. Hence $\lim_{x\to 0} \operatorname{sgn}\left(\sin\left(\frac{1}{x}\right)\right)$ does not exist.

(d) $\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right) = 0$, with the same proof as (b), save for $\delta_\varepsilon = \varepsilon^2$. ■

10. **Proof.** As f is additive, $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$, so

$$L = \lim_{y\to 0} f(y) = \lim_{2x\to 0} f(2x) = \lim_{x\to 0} f(2x) = \lim_{x\to 0} 2f(x) = 2\lim_{x\to 0} f(x) = 2L;$$

hence $L = 2L$ and $L = 0$, i.e. $\lim_{x\to 0} f(x) = 0$.

Now, let $c \in \mathbb{R}$. Then

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c} (f(x-c) + f(c)) = \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c) \\ &= \lim_{y\to 0} f(y) + f(c) = 0 + f(c) = f(c).\end{aligned}$$

As f is defined on all of $\mathbb{R}$, $f(c)$ exists for all $c \in \mathbb{R}$, and so $\lim_{x\to c} f(x) = f(c)$ exists for all $c \in \mathbb{R}$. ■

11. **Proof.** Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Set $\delta_\varepsilon = \frac{\varepsilon}{K}$. Then

$$|f(x) - f(c)| \leq K|x - c| < K\delta_\varepsilon < K\frac{\varepsilon}{K} = \varepsilon$$

whenever $|x - c| < \delta_\varepsilon$. ■

12. **Proof.** For $n \in \mathbb{N}$, let $I_n = (0, 1/n)$ and set

$$s_n = \sup f(I_n) \quad \text{and} \quad t_n = \inf f(I_n).$$

These are well defined as $f(I_n)$ is bounded. By construction, (s_n) is decreasing and (t_n) is increasing. Since

$$s_1 \geq s_n = \sup f(I_n) \geq \inf f(I_n) = t_n \geq t_1,$$

(s_n) is bounded below by t_1 and (t_n) is bounded above by s_1. Hence $s_n \to s$ and $t_n \to t$ exist, by the Monotone Convergence theorem.

For $n \in \mathbb{N}$, let $x_n, y_n \in I_n$ be s.t.

$$|f(x_n) - s_n| < \frac{1}{n} \quad \text{and} \quad |f(y_n) - t_n| < \frac{1}{n}.$$

This can always be done as $s_n - \frac{1}{n}$ and $t_n + \frac{1}{n}$ are not the supremum and the infimum, respectively, of $f(I_n)$.

Then, $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore, $f(x_n) \to s$ and $f(y_n) \to t$, by the Squeeze Theorem; indeed, $s_n - \frac{1}{n} < f(x_n) \leq s_n$, $t_n \leq f(y_n) < t_n + \frac{1}{n}$, $s_n \to s$, and $t_n \to t$, and the statement follows.

Now, suppose that $s = t = \ell$. Then $s_n, t_n \to \ell$. Let $\varepsilon > 0$. $\exists N_1, N_2 \in \mathbb{N}$ s.t. $|s_n - \ell| < \varepsilon$ whenever $n > N_1$ and $|t_n - \ell| < \varepsilon$ whenever $n > N_2$.

Set $N_\varepsilon = \max\{N_1, N_2\}$. Then

$$\ell - \varepsilon < t_n \leq s_n < \ell - \varepsilon$$

whenever $n > N_\varepsilon$. Set $\delta_\varepsilon = \frac{1}{N_\varepsilon}$. Then

$$\ell - \varepsilon < t_{N_\varepsilon} = \inf f(I_{N_\varepsilon}) \leq f(x) \leq \sup f(I_{N_\varepsilon}) \leq s_{N_\varepsilon} < \ell + \varepsilon,$$

i.e. $|f(x)-\ell|<\varepsilon$ whenever $0<|x-0|<\frac{1}{N_\varepsilon}=\delta_\varepsilon$. Hence $\lim_{x\to 0} f(x)=\ell$, which contradicts the hypothesis that the limit does not exist.

As a result, $s\neq t$, which completes the proof. ■

13. **Proof.** Let $c \in P$. Then $f(c) > 0$ and $\exists \varepsilon_0 > 0$ s.t. $f(c) - \varepsilon_0 > 0$.

By continuity of f, $\exists \delta_{\varepsilon_0}$ s.t. $|f(x) - f(c)| < \varepsilon_0$ whenever $|x - c| < \delta_{\varepsilon_0}$.

Thus, $0 < f(c) - \varepsilon_0 < f(x)$ for all $x \in V_{\delta_{\varepsilon_0}}$, i.e. $V_{\delta_{\varepsilon_0}} \subseteq P$. ■

14. **Proof.** In the light of a previous question on the topic, it is sufficient to show that if $\lim_{x\to c} f(x) = f(c)$ for some $c \in \mathbb{R}$, then $\lim_{x\to 0} f(x) = 0$.

Let f be continuous at c. Then

$$\begin{aligned} f(c) &= \lim_{x\to c} f(x) = \lim_{x\to c} \left(f(x-c) + f(c)\right) \\ &= \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c) = \lim_{y\to 0} f(y) + f(c), \end{aligned}$$

hence $\lim_{y\to 0} f(y) = 0$, which completes the proof. ■

15. **Proof.** Let $n \in \mathbb{N}$. Then

$$f(1) = f\left(\frac{n}{n}\right) = f\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{1}{n}\right) = nf\left(\frac{1}{n}\right),$$

hence $\frac{1}{n} f(1) = f\left(\frac{1}{n}\right)$.

Set $c = f(1)$. Let $y \in \mathbb{Q}$. Then $y = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}^{\times}$, and

$$f(y) = f\left(\frac{m}{n}\right) = mf\left(\frac{1}{n}\right) = m\frac{1}{n} f(1) = yc.$$

Let $x \in \mathbb{R}$. Since x is a limit point of $\mathbb{Q}$, $\exists (x_n) \subseteq \mathbb{Q}$ s.t. $x_n \to x$, with $x_n \neq x$ for all $n \in \mathbb{N}$. But $f(x_n) \to f(x)$, by continuity, so $f(x_n) = cx_n \to cx$, by the above discussion. Hence, $f(x) = cx$. ■

16. **Proof.** Let $x_1 \in I$. By hypothesis, $\exists x_2 \in I$ s.t.

$$|f(x_2)| \leq \frac{1}{2}|f(x_1)|.$$

Since $x_2 \in I$, $\exists x_3 \in I$ s.t.

$$|f(x_3)| \leq \frac{1}{2}|f(x_2)| \leq \frac{1}{2}\left(\frac{1}{2}|f(x_1)|\right) = \frac{1}{2^2}|f(x_1)|,$$

and so on. The sequence $(x_n) \subseteq I$ thusly built satistfies

$$0 \leq |f(x_n)| \leq \frac{1}{2^{n-1}}|f(x_1)|,$$

by induction (can you show this?).

Then $\lim_{n\to\infty} |f(x_n)| = 0$, by the Squeeze Theorem, and so $f(x_n) \to 0$.

As (x_n) is bounded, it has a convergent subsequence (x_{n_k}) (by the Bolzano-Weierstrass Theorem) whose limit c is in I (because $a \leq x_n \leq b$ for all n).

Since $\big(f(x_{n_k})\big)$ is a subsequence of $(f(x_n))$, then

$$\lim_{k\to\infty} f(x_{n_k}) = 0.$$

However,

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c),$$

as f is continuous. Hence $f(c) = 0$. ■

17. **Proof.** Let

$$f(x)=a_{2n+1}x^{2n+1}+a_{2n}x^{2n}+\cdots+a_1x+a_0,$$

where $a_i\in\mathbb{R}$ for $i=0,\ldots,2n+1$. Assume that $a_{2n}\neq 0$ (if that is not the case, the proof will proceed in a similar fashion, but a_{2n} will be replaced by the first a_i that is non-zero, starting with a_{2n-1}; if all coefficients are 0, then the real root is 0).

Let

$$M=\max\left\{(2n+1)\frac{|a_{2n}|}{|a_{2n+1}|},\left(\frac{|a_{2n-k}|}{|a_{2n}|}\right)^{1/k};k=1,\ldots,2n\right\}.$$

Then, whenever $|x| \geq M$,

- $|a_{2n}||x^{2n}| \geq |a_{2n}||x^{2n}|$;
- $|a_{2n}||x^{2n}| \geq |a_{2n-1}||x^{2n-1}|$;
- $\cdots$;
- $|a_{2n}||x^{2n}| \geq |a_1||x|$, and
- $|a_{2n}||x^{2n}| \geq |a_0|$,

and so

$$\begin{aligned}|a_{2n}x^{2n} + \cdots a_0| \leq |a_{2n}||x^{2n}| + \cdots + |a_0| \leq |a_{2n}||x^{2n}| + \cdots + |a_{2n}||x^{2n}| \\ = (2n+1)|a_{2n}||x^{2n}| \leq |a_{2n+1}||x^{2n+1}| = |a_{2n+1}x^{2n+1}|.\end{aligned}$$

Then $f(M+1)f(-M-1) < 0$. As f is continuous on $[-M-1, M+1]$, $\exists c \in [-M-1, M+1]$ s.t. $f(c) = 0$, by the IVT. ■

18. **Proof.** Let $g:[0,\frac{1}{2}]\to\mathbb{R}$ be defined by $g(x)=f(x)-f(x+\frac{1}{2})$. By construction, g is continuous on $[0,\frac{1}{2}]$. If $g(0)=g(1/2)=0$, there is nothing else to show.

Otherwise,

$$g(0)=f(0)-f(1/2) \quad \text{and} \quad g(1/2)=f(1/2)-f(1)=f(1/2)-f(0);$$

hence $g(0)g(\frac{1}{2})<0$.

By the IVT, $\exists c\in[0,\frac{1}{2}]$ s.t. $g(c)=0$, that is $f(c)-f(c+\frac{1}{2})=0$. This completes the proof. ■

19. **Proof.** If $x, y \in A$, then $x, y \geq 1$. In particular, $|x| = x$ and $|y| = y$, and $\frac{1}{x^2y}, \frac{1}{xy^2} \leq 1$.

Let $\varepsilon > 0$ and set $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{|y+x||y-x|}{x^2y^2} \\
&= |y - x|\left(\frac{y}{x^2y^2} + \frac{x}{x^2y^2}\right) \\
&= |x - y|\left(\frac{1}{x^2y} + \frac{1}{xy^2}\right) \leq 2|x - y| < 2\delta_\varepsilon = \varepsilon
\end{aligned}$$

whenever $|x - y| < \delta_\varepsilon$ and $x, y \in A$.

We show that the negation of the definition of uniform continuity holds on B.

Let $\varepsilon = 1$ and $\delta > 0$. Then, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N^2} < \delta$. Set $x_N = \frac{1}{N}$ and $y_N = \frac{1}{N+1}$. Clearly, $x_N, y_N \in B$ and

$$|x_N - y_N| = \left|\frac{1}{N} - \frac{1}{N+1}\right| = \frac{1}{N(N+1)} \leq \frac{1}{N^2} < \delta.$$

However,

$$|f(x_N) - f(y_N)| = |N^2 - (N+1)^2| = 2N + 1 > \varepsilon,$$

that is, f is not uniformly continuous on B. ■

20. **Proof.** Let $\varepsilon > 0$ and set $\delta_\varepsilon = \varepsilon$. Then

$$|f(x) - f(y)| = |x - y| < \delta_\varepsilon = \varepsilon$$

and

$$\begin{aligned}|g(x) - g(y)| = |\sin x - \sin y| &= 2\left|\sin\left(\frac{1}{2}(x-y)\right)\cos\left(\frac{1}{2}(x+y)\right)\right| \\ &\leq 2\frac{1}{2}|x-y| \cdot 1 = |x-y| < \delta_\varepsilon = \varepsilon\end{aligned}$$

(the second-last inequality can be obtained using Taylor's theorem on $\sin$, see chapter 5), whenever $|x - y| < \delta_\varepsilon$ and $x, y \in \mathbb{R}$. Hence f and g are both uniformly continuous.

Set $h(x) = x \sin x$. Let $\varepsilon = 1$ and $\delta > 0$. Them $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$ and $K \in \mathbb{N}$ s.t.

$$K > \frac{1}{4}\left(1 - \cos\frac{1}{N}\right)^{-1} + 3.$$

Define

$$x_K = \frac{4K-3}{2}\pi \quad \text{and} \quad y_K = \frac{4K-3}{2}\pi - \frac{1}{N}.$$

Then $|x_K - y_K| = \frac{1}{N} < \delta$ and

$$|h(x_K) - h(y_K)| \geq \frac{4K-3}{2}\pi\left(1 - \cos\frac{1}{N}\right) > \frac{\pi}{2} > 1 = \varepsilon,$$

and so h is not uniformly continuous. ■

21. **Proof.** Let $\varepsilon > 0$. Then $\frac{\varepsilon}{3} > 0$ and there exists $g_{\varepsilon/3}$ as in the hypothesis: hence $\exists \eta_{\varepsilon/3} > 0$ s.t. $|g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \eta_{\varepsilon/3}$ and $x, y \in A$.

Set $\delta_\varepsilon = \eta_{\varepsilon/3}$. Then

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - g_{\varepsilon/3}(x) + g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y) + g_{\varepsilon/3}(y) - f(y)| \\ &\leq |f(x) - g_{\varepsilon/3}(x)| + |g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| + |g_{\varepsilon/3}(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

whenever $|x - y| < \delta_\varepsilon$ and $x, y \in A$.

Hence, f is uniformly continuous on A. ■

22. **Proof.** Since f is continuous, then $|f|$ is also continuous, being the composition of two continuous functions.

As f is $p-$periodic, $\exists c \in [0,p]$ s.t.

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in[0,p]} |f(x)| = |f(c)|,$$

by the Max/Min Theorem. Hence f is bounded by $|f(c)|$ on $\mathbb{R}$.

Now, let $\varepsilon > 0$.

By hypothesis, f is continuous on the closed interval $[-1,p+1]$, which implies that that f is uniformly continuous on $[-1,p+1]$ (Theorem 38).

Then, $\exists\delta_\varepsilon > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_\varepsilon$ and $x, y \in [-1, p+1]$.

Without loss of generality, we can assume that $\delta_\varepsilon < 1$ (why?). Let $x, y \in \mathbb{R}$ s.t. $|x - y| < \delta_\varepsilon$.

Then $\exists k \in \mathbb{Z}$ and $\alpha, \beta \in [-1, p+1]$ s.t. $x = \alpha + kp$ and $y = \beta + kp$.

Thus $|\alpha - \beta| = |x - y| < \delta_\varepsilon$ and $|f(x) - f(y)| = |f(\alpha) - f(\beta)| < \varepsilon$, since f is uniformly continuous on $[-1, p+1]$; consequently, f is uniformly continuous. ■

23. **Proof.** Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then,

$$\left|\frac{1}{n} - 0\right| < \delta \implies \left|g\left(\frac{1}{n}\right) - g(0)\right| = \left|\frac{1}{n}\right| = \left|\frac{1}{n} - 0\right| < \delta_\varepsilon = \varepsilon$$

whenever $|1/n - 0| < \delta_\varepsilon$, so g is continuous at 0. ■

24. **Proof.** Suppose f is continuous and let $B \subseteq \mathbb{R}$ be open. Choose $a \in f^{-1}(B)$. Thus $f(a) \in B$. Since B is open, there exists $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq B$.

Since f is continuous, there exists $\delta > 0$ such that

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq B.$$

Thus $B(a, \delta) \subseteq f^{-1}(B)$. So $f^{-1}(B)$ is open.

Now suppose that the pre-image of every open subset of $\mathbb{R}$ is open. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then $B(f(a), \varepsilon)$ is an open subset of $\mathbb{R}$. Therefore, by assumption, $f^{-1}(B(f(a), \varepsilon))$ is open. Since $a \in f^{-1}(B(f(a), \varepsilon))$, this means that there exists $\delta > 0$ such that

$$B(a, \delta) \subseteq f^{-1}(B(f(a), \varepsilon)) \implies f(B(a, \delta)) \subseteq B(f(a), \varepsilon).$$

Thus f is continuous at a. Since a was arbitrary, f is continuous. ■

The pre-image of closed sets by a continuous function is also closed. Note that

$$f^{-1}(B)^{\complement} = \{a \in A \mid f(a) \notin B\} = \{a \in A \mid f(a) \in B^{\complement}\} = f^{-1}\left(B^{\complement}\right).$$

Hence

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow f^{-1}(B) \text{ is open for all open } B \Leftrightarrow f^{-1}(B)^{\complement} \text{ is closed for all open } B \\ &\Leftrightarrow f^{-1}\left(B^{\complement}\right) \text{ is closed for all open } B \Leftrightarrow f^{-1}(C) \text{ is open for all closed } C, \end{aligned}$$

where in the last if and only if statement we let $C = B^{\complement}$ (so C is closed if and only if B is open).

25. **Proof.** We will prove the statement in the general multi-dimensional case. The one-dimensional case will then simply be a special case of the more general result.

Suppose f is Lipschitz and $\mathbf{a} \in A$. Let $\varepsilon > 0$. Set $\delta = \varepsilon/M$. Then

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\| < M\varepsilon/M = \varepsilon.$$

Thus f is uniformly continuous.

Consider the function $g : [0,1] \to \mathbb{R}$, $g(x) = \sqrt{x}$. The function g is continuous on the compact interval $[0,1]$, hence it is uniformly continuous by Theorem 38. Assume that g is Lipschitz.

Then $\exists M > 0$ such that

$$|h(x) - h(0)| \leq M|x - 0| \quad \forall\, x \in [0,1] \implies \sqrt{x} \leq Mx, \quad \forall x \in [0,1]$$
$$\implies M \geq \frac{1}{\sqrt{x}} \quad \forall\, x \in [0,1].$$

This contradicts the fact that $1/\sqrt{x} \to \infty$ as $x \to 0^+$. Hence, g cannot be Lipschitz. ■