Mathematical Analysis

Chapter 3
Sequences

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Overview

A large chunk of analysis concerns itself with problems of convergence. In this chapter, we

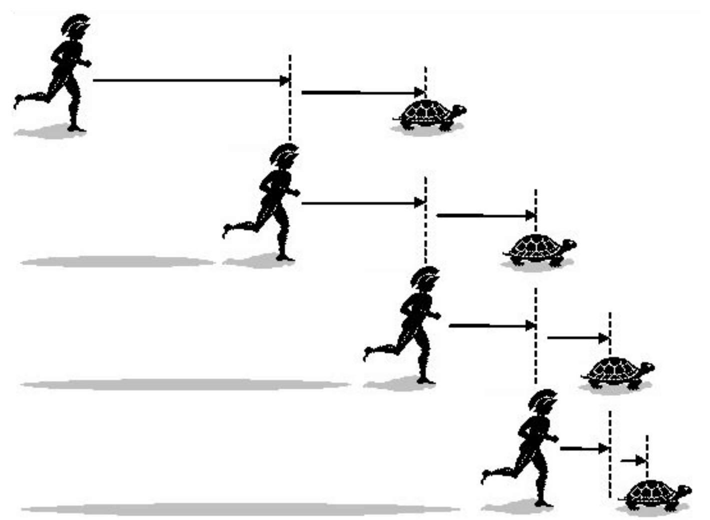
- introduce sequences and limits,
- provide results that help to compute such limits, and
- identify situations when the limit can be shown to exist without having to compute it.

Outline

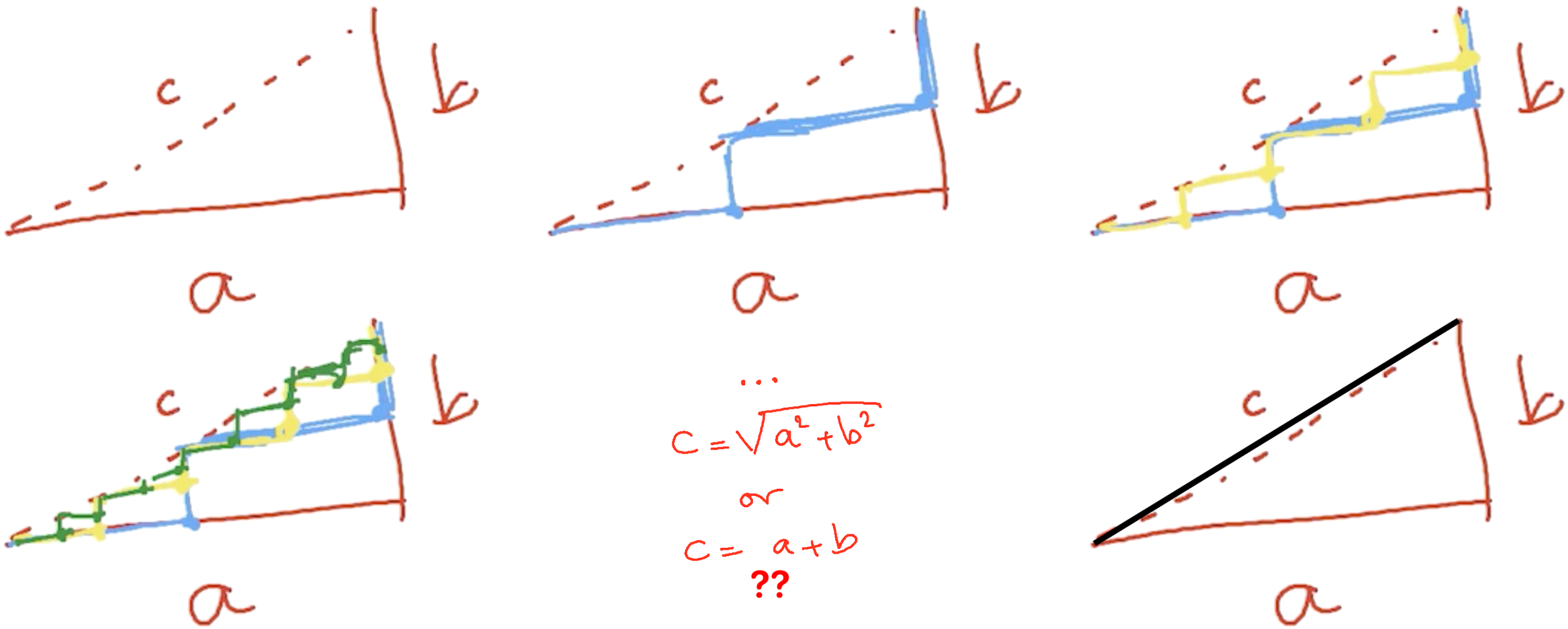
3.1 – Infinity vs. Intuition

When dealing with infinity, our intuition sometimes falters.

Example: (Zeno's Paradox)
Achilles pursues a turtle. When he reaches her starting point, she has moved a certain distance. When he crosses that distance, she has moved yet another distance, and so forth. Achilles is always trailing the turtle, so he cannot catch her. Is this what happens in reality?



Example: (Anti-Pythagorean Theorem)
Consider a right-angle triangle with base a, height b, and hypotenuse c. We can build staircase structures that each have the same constant length as $a + b$, while increasing the number of stairs (see image below).



Example: (Infinite Sum I)
Let $S = 1 + (-1) + 1 + (-1) + \cdots$. Then

$$S = (1 + (-1)) + (1 + (-1)) + \cdots = 0 + 0 + \cdots = 0$$
$$S = 1 - (1 + (-1) + 1 + (-1) + \cdots) = 1 + S \implies S = 1/2$$
$$S = 1 + ((-1) + 1) + ((-1) + 1) + \cdots = 1 + 0 + 0 + \cdots = 1$$

Therefore $0 = \frac{1}{2} = 1$. Does this make sense?

Example: (Infinite Sum II)
Let $S = 1 + 2 + 4 + 8 + \cdots$. Then

$$S = 1 + 2(1 + 2 + 4 = 8 + \cdots) = 1 + 2S \implies S = -1.$$

Can a sum of positive terms yield a negative result?

3.2 – Limit of a Sequence

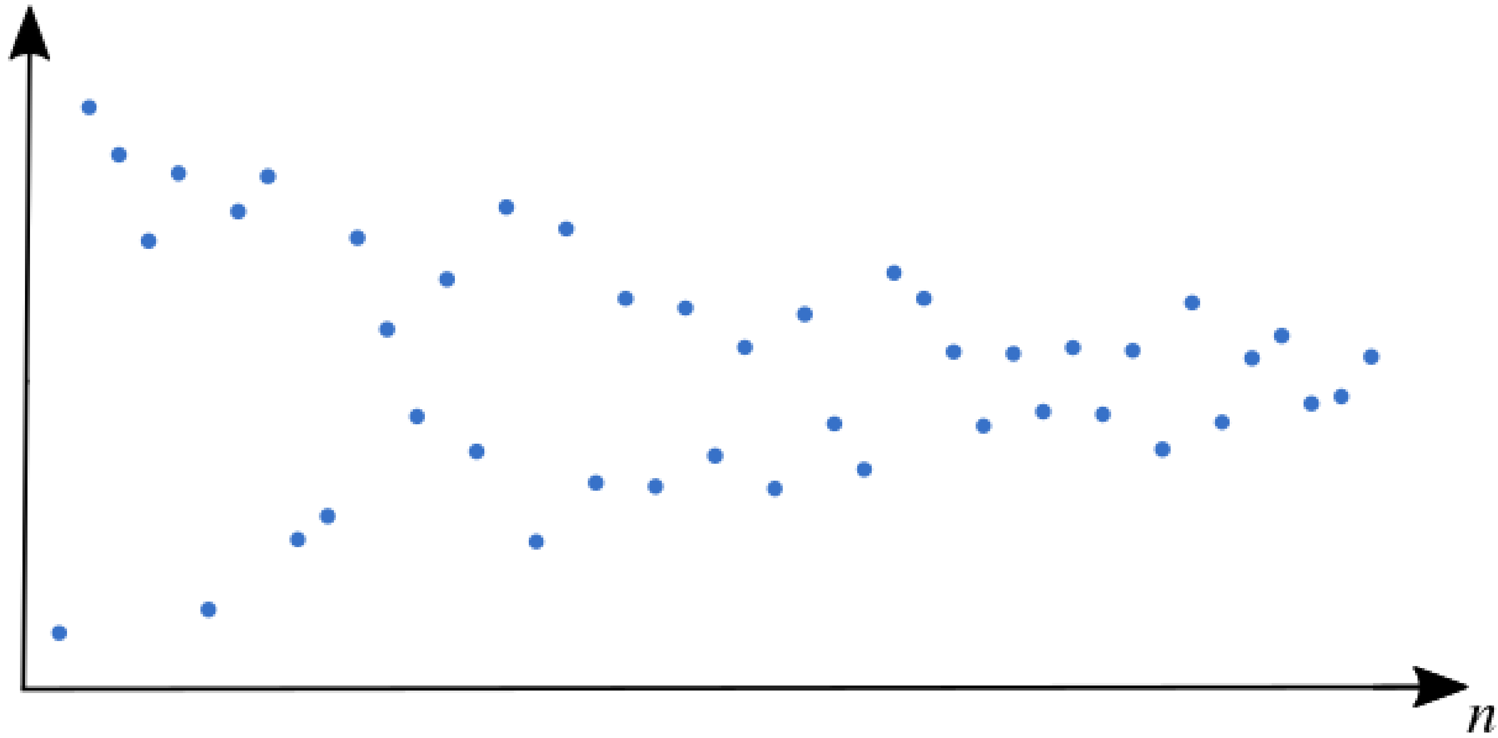
A **sequence** of real numbers is a function $X : \mathbb{N} \to \mathbb{R}$ defined by $X(n) = a_n$, where $a_n \in \mathbb{R}$. We denote the sequence X by $(a_n)_{n\in\mathbb{N}}$ or simply by (a_n).

Examples:

1. $X : \mathbb{N} \to \mathbb{R}$, $n \mapsto 2n$ is the sequence with $X(1) = 2$, $X(2) = 4$, etc. We may write $(a_n) = (2, 4, 6, \ldots)$.

2. $X : \mathbb{N} \to \mathbb{R}$, $n \mapsto \frac{1}{n}$ is the sequence with $X(1) = \frac{1}{2}$, $X(2) = \frac{1}{2}$, etc. We may write $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$.

In general, we let $\mathbb{N}$ stand for whatever countable subset of $\mathbb{N}$ is required for the definition of the sequence to make sense.



A sequence as a function on $\mathbb{N}$.

The notation used for sequences varies from one resource to the next.

We will mostly use round brackets:

$$(a_n) \text{ where } a_n = \frac{1}{n^2}, \quad \left(\frac{1}{n^2}\right), \quad \left(1, \frac{1}{4}, \frac{1}{9}, \ldots\right)$$

all denote the same sequence.

A sequence is an ordered set of **terms** a_n, that is, a set of **indexed values**. The set of all values taken by the sequence (a_n) is called the **range** of (a_n) and we denote it by $\{a_n\}$.

A sequence and its range are two different notions.

Examples:

1. The terms of the sequence $(\frac{1}{n^2})$ are $(1, \frac{1}{4}, \frac{1}{9}, \ldots)$, while its range is $\{1, \frac{1}{4}, \frac{1}{9}, \ldots\}$.

2. The terms of the sequence $(\frac{1+(-1)^n}{n})$ are $(0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots)$, while its range is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

Certain sequences are defined with the help of a **recurrence relation**: the first few terms are given, and the subsequent terms are computed using the preceding terms.

Example: The Fibonacci sequence given by $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$ is the classic example: $(1, 1, 2, 3, 5, 8, 13, \ldots)$.

We now examine in detail the sequence $(x_n) = (\frac{1}{2n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{78}, \ldots)$.

As the index n increases, the values of x_n **approach** 0. But what does this mean, mathematically?

Let $\varepsilon > 0$. In theory, ε could take on any positive value, but in practice we are interested in small values $\varepsilon \ll 1$. Then the real number $\frac{1}{2\varepsilon}$ is positive, i.e. $\frac{1}{2\varepsilon} > 0$.

According to the Archimedean Property, there exists a **threshold** $N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon > \frac{1}{2\varepsilon}$.

Different values of ε lead to different thresholds: for instance, if $\varepsilon = \frac{1}{100}$, then $N_\varepsilon > \frac{1}{2(1/100)} = 50$. If $\varepsilon = \frac{1}{1000}$, then $N_\varepsilon > 500$, and so forth.

No matter what value $\varepsilon > 0$ takes, if we look at indices past the threshold (i.e. when $n > N_\varepsilon$), we have

$$n > N_\varepsilon > \frac{1}{2\varepsilon} \implies n > \frac{1}{2\varepsilon} \iff \varepsilon > \frac{1}{2n}.$$

Thus, for all indices n **after** the threshold N_ε (i.e. $\forall n > N_\varepsilon$),

$$|x_n - 0| = |x_n| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \varepsilon \implies 0 - \varepsilon < x_n < 0 + \varepsilon.$$

The interval $(-\varepsilon, \varepsilon)$ thus contains **all** the terms of the sequence **after** the N_εth term, which is to say $x_n \in (-\varepsilon, \varepsilon)$ for all $n > N_\varepsilon$.

Another way to say this is that the interval $(-\varepsilon, \varepsilon)$ contains all the terms of the sequence (x_n), except maybe for a finite number of terms $x_1, \ldots, x_{N_\varepsilon}$.

If $\varepsilon = \frac{1}{100}$, according to the Archimedean Property, $\exists N_{1/100} > \frac{1}{2(1/100)} = 50$ ($N_{1/100} = 51$ does the trick) such that

$$n > 51 \implies |x_n - 0| = |x_n| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{2(51)} = \frac{1}{102} < \frac{1}{100}.$$

In other words, the interval $(-1/100, 1/100)$ contains all the terms of the sequence from $n = 52$ onward.

The threshold $N_{1/100} = 51$ does not work for ε values smaller than $1/100$, however.

If $\varepsilon = 1/1000$, for instance, we need $N_{1/1000} > \frac{1}{2(1/1000)} = 500$ to guarantee that all the terms after the threshold fall in the interval $(-1/1000, 1/1000)$, and so on.

A sequence (x_n) of real numbers is said to **converge** to a **limit** $L \in \mathbb{R}$, denoted by

$$x_n \to L \quad \text{or} \quad \lim_{n\to\infty} x_n = L$$

if

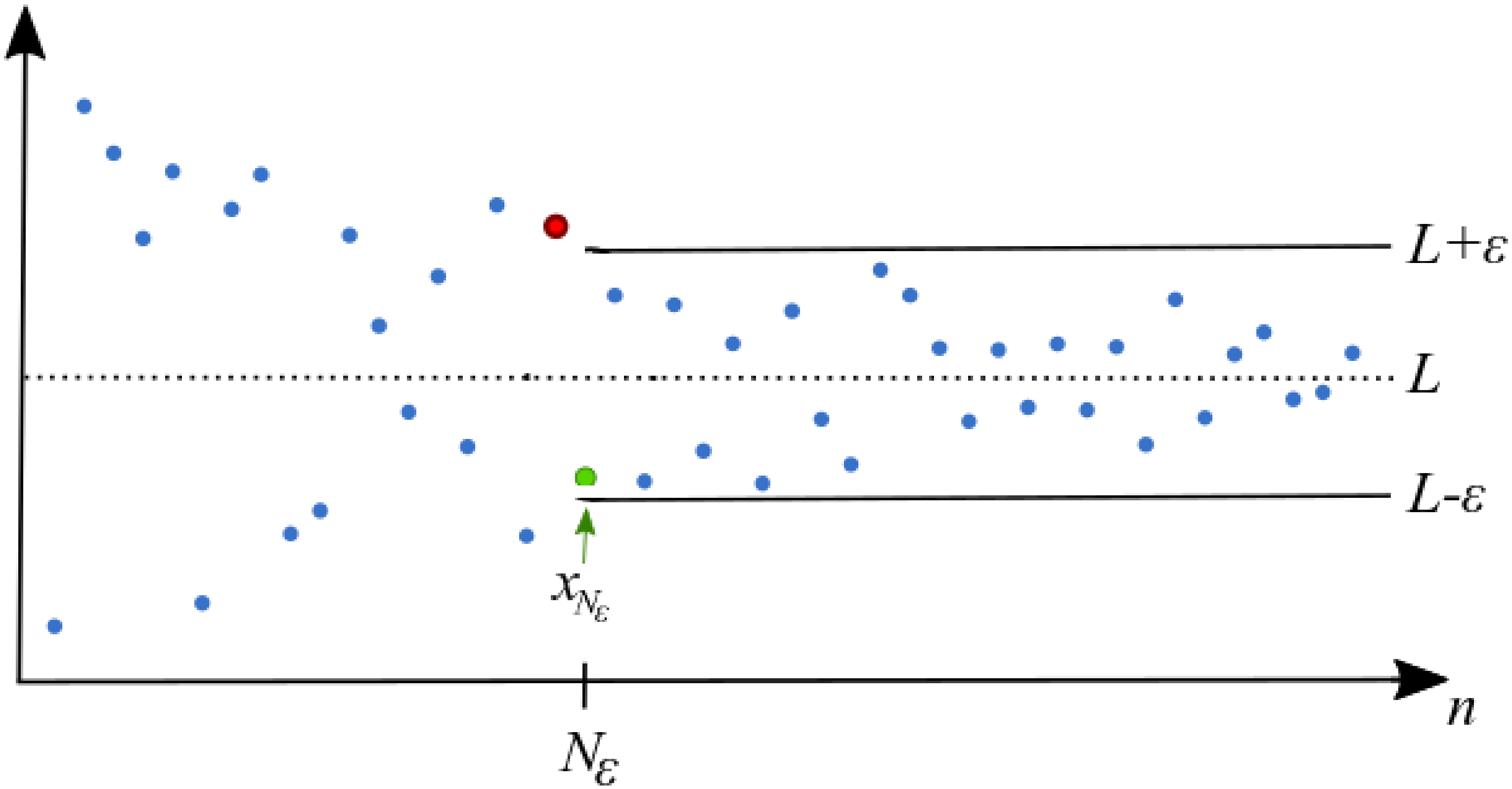
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \implies |x_n - L| < \varepsilon.$$

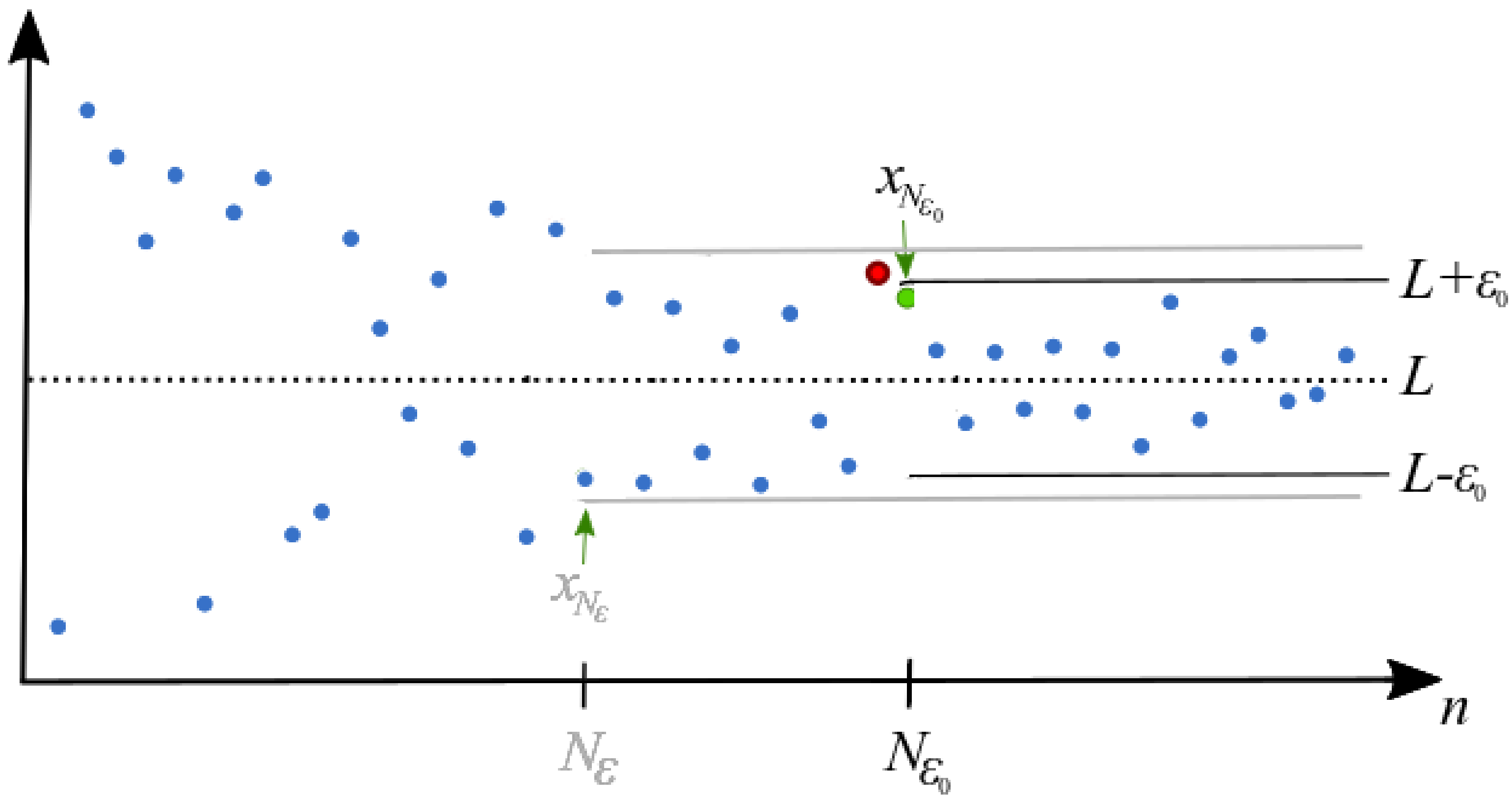
A sequence (x_n) which does not converge to a limit is said to be **divergent**:

$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L.$$

There is only one way for a sequence to converge: its values are getting closer and closer to the limit. But there is more than one way for a sequence to diverge.

Can you think of some?


$L+\varepsilon$
L
L-ε
x_{N_ε}
N_ε
n


$x_{N_{\varepsilon_0}}$
$L+\varepsilon_0$
L
$L-\varepsilon_0$
x_{N_ε}
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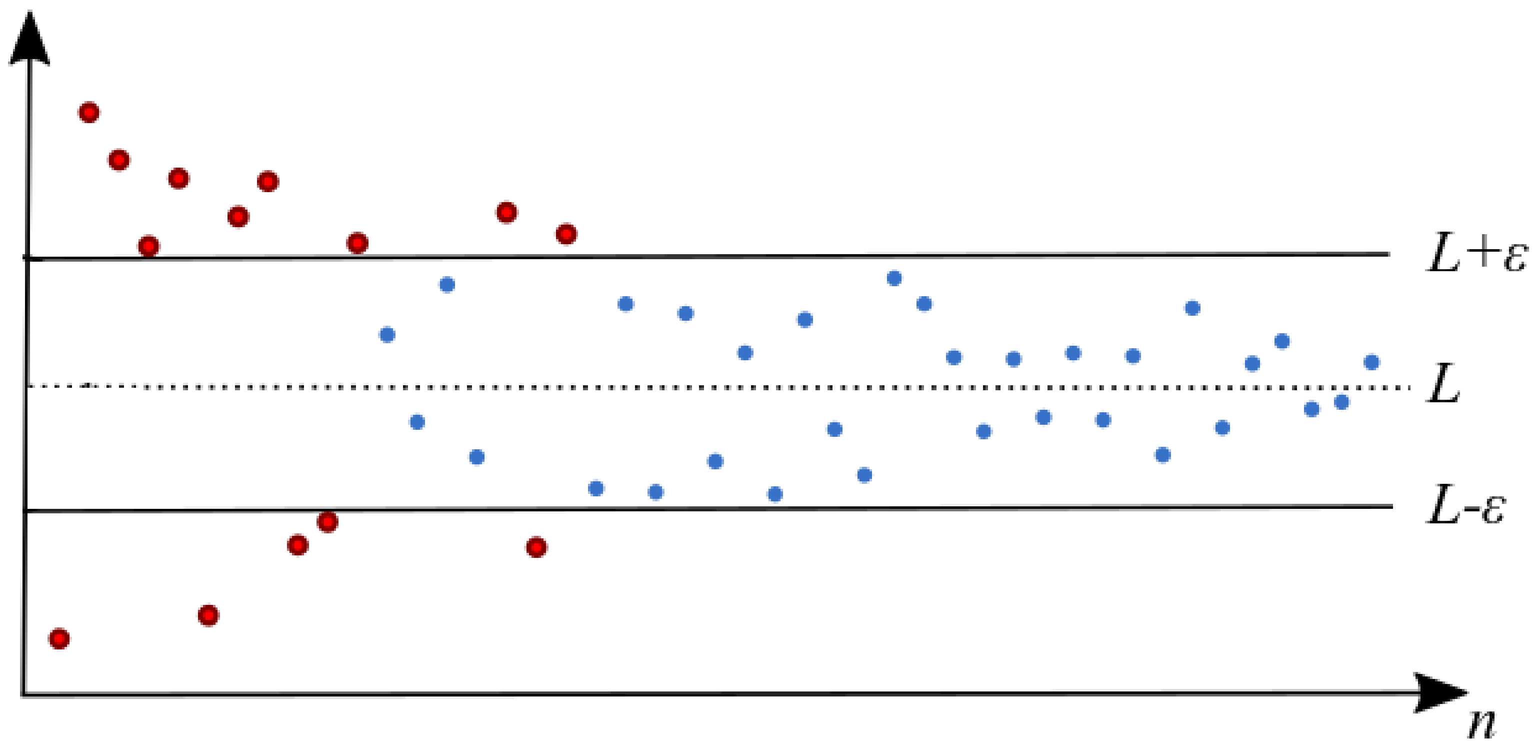
N_ε
N_{ε_0}


$L+\varepsilon$
L
L-ε
n

Examples:

1. Show that $\frac{1}{n} \to 0$.

   Proof. Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$, so $\varepsilon > \frac{1}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

   $$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon.$$

   This completes the proof. ■

2. Show that $\frac{n+1}{n^2+1} \to 0$.

Proof. Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$, so $\varepsilon > \frac{2}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

$$\left|\frac{n+1}{n^2+1} - 0\right| = \frac{n+1}{n^2+1} \leq \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon.$$

This completes the proof. ■

3. Show that $\frac{4-2n-3n^2}{2n^2+n} \to -\frac{3}{2}$.

Proof. Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$, so $\varepsilon > \frac{2}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

$$\left|\frac{4-2n-3n^2}{2n^2+n} - \left(-\frac{3}{2}\right)\right| = \left|\frac{2(4-2n-3n^2)+3(2n^2+n)}{2(2n^2+n)}\right| = \frac{|8-n|}{4n^2+2n}.$$

Note that $8-n \leq 8n$ if $1 \leq n \leq 8$, and that $n-8 \leq 8n$ if $n \geq 8$, so that $|8-n| \leq 8n$ for all $n \geq 1$. Thus

$$\frac{|8-n|}{4n^2+2n} \leq \frac{8n}{4n^2+2n} < \frac{8n}{4n^2} = \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon$$

when $n > N_\varepsilon$, which completes the proof. ■

4. Show that (n) is divergent.

Proof. Suppose instead that (x_n) converges to $a \in \mathbb{R}$. Let $\varepsilon > 0$. By definition, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - a| = |n-a| < \varepsilon$ whenever $n > N_\varepsilon$, which implies that $n < a+\varepsilon$ for all $n > N_\varepsilon$, or that $a+\varepsilon$ is a an upper bound for $\mathbb{N}$. This contradicts the Archimedean Property, so the sequence (n) must diverge. ■

The main benefit of the formal definition of the limit of a sequence is that it does not call on infinity: we write $n \to \infty$, but that is a merely a notation convenience.

On the flip side, the formal definition has 2 major inconveniences:

1. it cannot be used to **determine the limit** of a convergent sequence – it can only be used to verify that a given candidate is (or is not) a limit of a sequence;

2. it can seem artificial to some extent, especially upon a first encounter.

The goal is simple: we must **determine a threshold** N_ε that does the trick. This often requires **backtracking** from the end of the string of inequalities rather than to proceed directly from "Let $\varepsilon > 0$".

We have been careful to refer to "a" limit when the sequence converges, but we should really be talking about "the" limit in such cases.

Theorem 12. (UNIQUE LIMIT) *A convergent sequence* (x_n) *of real numbers has exactly one limit.*

Proof. Suppose that $x_n \to x'$ and $x_n \to x''$. Let $\varepsilon > 0$. Then there exist 2 integers $N'_\varepsilon, N''_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x'| < \varepsilon \text{ whenever } n > N'_\varepsilon \quad \text{and} \quad |x_n - x''| < \varepsilon \text{ whenever } n > N''_\varepsilon.$$

Set $N_\varepsilon = \max\{N'_\varepsilon, N''_\varepsilon\}$. Then whenever $n > N_\varepsilon$, we have

$$0 \leq |x' - x''| = |x' - x_n + x_n - x''| \leq |x_n - x'| + |x_n - x''| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $0 \leq \frac{|x'-x''|}{2} < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\frac{|x'-x''|}{2} = 0$ and $x' = x''$. ■

A sequence $(x_n) \subseteq \mathbb{R}$ is **bounded** by $M > 0$ if $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 13. *Any convergent sequence (x_n) of real numbers is bounded.*

Proof. Let $(x_n) \subseteq \mathbb{R}$ converge to $x \in \mathbb{R}$. Then for $\varepsilon = 1$, say, $\exists N \in \mathbb{N}$ such that

$$|x_n - x| < 1 \quad \text{when } n > N.$$

Thanks to the reverse triangle inequality, we also have

$$|x_n| - |x| \leq |x_n - x| < 1 \quad \text{when } n > N,$$

so that $|x_n| < |x| + 1$ when $n > N$.

Now, set $M = \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$. Then $|x_n| \leq M$ for all n and so (x_n) is bounded. ■

We can prove theorems **directly**, as in Theorem 13, by induction, as in Bernouilli's Inequality, or by **contradiction**, as in the Archimedean Property.

The **contrapositive** of a statement $P \implies Q$ is $\neg Q \implies \neg P$. These two statements are **logically equivalent** to one another; it may be that it is easier to demonstrate the contrapositive than the original statement.

The **converse** of a statement $P \implies Q$ is $Q \implies P$. There is no general link between a statement and its converse: sometimes they are both true, sometimes they are both false, sometimes only of them is true.

Example: The contrapositive of Theorem 13 is "Any unbounded sequence is divergent", which is valid since Theorem 13 is true. Its converse is "Any bounded sequence is convergent" – we have to try to prove it (if we think it is true), or to find a **counter-example** (if we think it is false).

3.3 – Operations on Sequences

The following result removes the need to use the formal definition.

Theorem 14. (Operations on Convergent Sequences)
Let $(x_n), (y_n)$ *be convergent, with* $x_n \to x$ *and* $y_n \to y$. *Let* $c \in \mathbb{R}$. *Then*

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ *and* $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, *if* $y_n, y \neq 0$ *for all* n.

Proof. We show each part using the definition of the limit of a sequence.

1. Let $\varepsilon > 0$. As $x_n \to x$, $\exists N'_\varepsilon$ such that $|x_n - x| < \varepsilon$ whenever $n > N'_\varepsilon$. But $||x_n| - |x|| \leq |x_n - x|$, according to Theorem 6. Hence, for $\varepsilon > 0$, $\exists N_\varepsilon = N'_\varepsilon$ such that

$$||x_n| - |x|| \leq |x_n - x| < \varepsilon$$

whenever $n > N_\varepsilon$, i.e. $|x_n| \to |x|$.

2. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. As $x_n \to x$ and $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \tag{1}$$

whenever $n > N^x_{\frac{\varepsilon}{2}}$ and $n > N^y_{\frac{\varepsilon}{2}}$ respectively. Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}\right\}$.

Then, whenever $n > N_\varepsilon$ (so whenever n is strictly larger than $N^x_{\varepsilon/2}$ and $N^y_{\varepsilon/2}$ *at the same time*),

$$\begin{aligned} |(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ \boxed{\text{by (1)}} &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e. $(x_n + y_n) \to (x + y)$.

3. According to Theorem 13, (x_n) and (y_n) are bounded since they are convergent sequences. Then $\exists M_x, M_y \in \mathbb{N}$ such that

$$|x_n| < M_x \quad \text{and} \quad |y_n| < M_y$$

for all n.

Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M_x}, \frac{\varepsilon}{2M_y} > 0$. As $x_n \to x$, $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2M_y}}, N^y_{\frac{\varepsilon}{2M_x}} \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2M_y} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2M_x} \tag{2}$$

whenever $n > N^x_{\frac{\varepsilon}{2M_y}}$ and $n > N^y_{\frac{\varepsilon}{2M_x}}$ respectively. Moreover, $|y| \leq M_y$ (otherwise $\frac{|y|-M_y}{2} > 0$. Then, for $\varepsilon = \frac{|y|-M_y}{2}$, we get

$$|y_n - y| \geq ||y| - |y_n|| \geq |y| - M_y = 2\varepsilon > \varepsilon$$

for all $n \in \mathbb{N}$, which contradicts the definition of $y_n \to y$).

Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2M_x}}, N^y_{\frac{\varepsilon}{2M_y}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned}
|x_ny_n - xy| = |x_ny_n - x_ny + x_ny - xy| &= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n||y_n - y| + |y||x_n - x| \\
&< M_x|y_n - y| + M_y|x_n - x| \\
\boxed{\text{by (2)}} \quad &< M_x\frac{\varepsilon}{2M_x} + M_y\frac{\varepsilon}{2M_y} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

i.e. $x_ny_n \to xy$.

Furthermore, if the sequence (y_n) is given by $y_n = c$ for all n, then the preceding result yields $cx_n \to cx$, since $y_n = c \to c$ (You should show this).

4. It is enough to show $\frac{1}{y_n} \to \frac{1}{y}$ under the hypotheses above; then the result will hold by part 3. Since $y \neq 0$, $\frac{|y|}{2} > 0$. Hence, as $y_n \to y$, $\exists N_{|y|/2} \in \mathbb{N}$ such that $|y_n - y| < \frac{|y|}{2}$, whenever $n > N_{\frac{|y|}{2}}$. According to Theorem 6,

$$|y| - |y_n| < |y - y_n| < \frac{|y|}{2}, \quad \text{and so}$$

$$\frac{|y|}{2} < |y_n| \quad \text{or} \quad \frac{1}{|y_n|} < \frac{2}{|y|} \tag{3}$$

whenever $n > N_{|y|/2}$ (these expressions make sense as neither y_n nor y is 0 for all n).

Let $\varepsilon > 0$. Then $|y|^2\frac{\varepsilon}{2} > 0$. As $y_n \to y$, $\exists N_{|y|^2\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$|y_n - y| < |y|^2\frac{\varepsilon}{2} \tag{4}$$

whenever $n > N_{|y|^2\frac{\varepsilon}{2}}$. Set $N_\varepsilon = \max\left\{N_{\frac{|y|}{2}}, N_{|y|^2\frac{\varepsilon}{2}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned}
\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y-y_n}{y_ny}\right| &= \frac{|y-y_n|}{|y_ny|} \\
\boxed{\text{by (3)}} &< \frac{2|y-y_n|}{|y|^2} \\
\boxed{\text{by (4)}} &< \frac{2}{|y|^2}\cdot|y|^2\frac{\varepsilon}{2} = \varepsilon, \quad \text{i.e. } \frac{1}{y_n} \to \frac{1}{y}. \qquad \blacksquare
\end{aligned}$$

Can the limit of a sequence whose terms are all near 2 be -19? 0? 1? 2?

Theorem 15. (Comparison Theorem for Sequences)
Let $(x_n), (y_n)$ be convergent sequences of real numbers with $x_n \to x$, $y_n \to y$, and $x_n \leq y_n\ \forall n \in \mathbb{N}$. Then $x \leq y$.

Proof. Suppose that it is not the case, namely, that $x > y$. Then $x - y > 0$. Set $\varepsilon = \frac{x-y}{2} > 0$. Since $x_n \to x$ and $y_n \to y$, $\exists N_\varepsilon^x, N_\varepsilon^y \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \text{ whenever } n > N_\varepsilon^x \quad \text{and} \quad |y_n - y| < \varepsilon \text{ whenever } n > N_\varepsilon^y.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^y\}$. Then, if $n > N_\varepsilon$, we have

$$y_n < y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} = x - \frac{x-y}{2} = x - \varepsilon < x_n.$$

But this contradicts the assumption that $x_n \leq y_n$ for all n, $\therefore x \leq y$. ■

⚠ The "$\leq$"s in the statement of Theorem 15 cannot be replaced by "$<$"s throughout. For instance, if $(x_n) = (\frac{1}{n+1})$ and $(y_n) = (\frac{1}{n})$, then $x_n < y_n$ for all $n \in \mathbb{N}$, but $x_n \to x = 0$, $y_n \to y = 0$, and $0 = x \not< y = 0$.

Theorem 16. (Squeeze Theorem for Sequences)
Let $(x_n), (y_n), (z_n) \subseteq \mathbb{R}$ be such that $x_n, z_n \to \alpha$ and $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$. Then $y_n \to \alpha$.

Proof. Let $\varepsilon > 0$. By convergence of $(x_n), (z_n)$ to α, $\exists N_\varepsilon^x, N_\varepsilon^z \in \mathbb{N}$ s.t.

$$|x_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^x \quad \text{and} \quad |z_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^z.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^z\}$. When $n > N_\varepsilon$, $\alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon$, which is to say, that $|y_n - \alpha| < \varepsilon$. Consequently, $y_n \to \alpha$. ■

We can use these various results to compute the following limits.

Examples:

1. Compute $\lim_{n\to\infty} \dfrac{3n+1}{n}$, if the limit exists.

   Solution. Note that $\frac{3n+1}{n} = 3 + \frac{1}{n}$. According to Theorem 14, **if the limit exists** we must have

$$\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty} \left(3 + \frac{1}{n}\right) = \lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n} = 3 + 0 + 3.$$

   Reading the string of equality backwards, we see that the original limit must exist and be equal to 3. ■

2. Compute $\lim_{n\to\infty} \dfrac{\sin(n^2+212)}{n}$, if the limit exists.

Solution. We cannot use Theorem 14 since neither the numerator nor the denominator limit exists. This does not necessarily mean that the limit of the quotient does not exist. In order to determine if it does, we need to use another approach.

By definition of the $\sin$ function (which we take for granted for now), we have $-1 \leq \sin x \leq 1,\ \forall x \in \mathbb{R}$. Thus

$$-1 \leq \sin(n^2+212) \leq 1,\ \forall n \implies -\frac{1}{n} \leq \frac{\sin(n^2+212)}{n} \leq \frac{1}{n},\ \forall n.$$

As $\pm\frac{1}{n} \to 0$, we can use the Squeeze Theorem to conclude that

$$\lim_{n\to\infty} \frac{\sin(n^2+212)}{n} = 0. \qquad \blacksquare$$

3. Compute $\lim_{n\to\infty} \frac{2n-1}{n+7}$, if the limit exists.

Solution. We cannot apply Theorem 14 directly since neither the numerator nor the denominator limits exist.

However,

$$\frac{2n-1}{n+7} = \frac{1/n \cdot (2n-1)}{1/n \cdot (n+7)} = \frac{2-1/n}{1+7/n} \quad \text{when } n \neq 0.$$

Because each of the constituent parts converge (and because the denominator is never equal to 0, either in the limit or in the sequence), repeated applications of Theorem 14 yield

$$\lim_{n\to\infty}\frac{2n-1}{n+7}=\frac{\lim_{n\to\infty}(2-1/n)}{\lim_{n\to\infty}(1+7/n)}=\frac{2-\lim_{n\to\infty}1/n}{1+7\cdot\lim_{n\to\infty}1/n}=\frac{2-0}{1+7\cdot 0}=2.$$

This is basically a calculus argument. ■

4. Let (x_n) be such that $|x_n|\to 0$. Show that $x_n\to 0$.

Proof. Since $-|x_n|\leq x_n\leq |x_n|$ for all $n\in\mathbb{N}$ according to Theorem 6, and since $-|x_n|,|x_n|\to 0$ by assumption, then $x_n\to 0$ according to the Squeeze Theorem. ■

Note, however that if $|x_n| \to \alpha \neq 0$, we cannot necessarily conclude that $x_n \to \alpha$. Consider, for instance, the sequence $(x_n) = (-1)^n$.

5. Let $|q| < 1$. Compute $\lim_{n\to\infty} q^n$, if the limit exists.

Proof. If $q = 0$, then $q^n = 0 \to 0$.

If $q \neq 0$, then $\frac{1}{|q|} > 1$. Thus, $\exists t > 0$ such that $\frac{1}{|q|} = 1 + t$.

From Bernoulli's Inequality, we have

$$\left(\frac{1}{|q|}\right)^n = (1+t)^n \geq 1 + nt, \ \forall n \in \mathbb{N},$$

so that $0 \leq |q^n| \leq |q|^n \leq \frac{1}{1+nt}$.

But $\frac{1}{1+nt} = 0$ when $n \to \infty$ (does this need to be proven?); thus $|q^n| \to 0$ according to the Squeeze Theorem, and so $q^n \to 0$ by the previous example. ■

6. Let $|q| < 1$. Compute $\lim_{n\to\infty} nq^n$, if the limit exists.

Solution. The proof that $nq^n \to 0$ is left as an exercise; it is similar to the proof of part of the previous example, but uses an extension of Bernoulli's Inequality:

$$(1+t)^n \geq 1 + nt + \frac{n(n-1)}{2}t^2, \text{ for } t > 0, n \geq 1,$$

which can be proven by induction. ■

7. Show that $\sqrt[n]{n} \to 1$.

Proof. Let $\varepsilon > 0$. Then $1+\varepsilon > 1$ and $0 < \frac{1}{1+\varepsilon} < 1$.

Claim: $n\left(\frac{1}{1+\varepsilon}\right)^n \to 0$ when $n \to \infty$ (use previous example with $q = \frac{1}{1+\varepsilon}$.

Hence, $\exists M_1 \in \mathbb{N}$ such that

$$\left|\frac{n}{(1+\varepsilon)^n} - 0\right| < 1 \text{ when } n > M_1 \implies 1 \leq n < (1+\varepsilon)^n \text{ when } n > M_1.$$

Set $N_\varepsilon = M_1$. Then $1-\varepsilon < 1 \leq n^{1/n} < 1+\varepsilon$ when $n > N_\varepsilon$. But this is precisely the same as $|n^{1/n} - 1| < \varepsilon$ when $n > N_\varepsilon$; thus $n^{1/n} \to 1$. ■

8. Compute $\lim_{n\to\infty} \frac{n!}{n^n}$, if the limit exists.

Solution. Since

$$0 \leq \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1}{n \cdot n \cdot \cdots \cdot n \cdot n} \leq \frac{1}{n}, \ \forall n \in \mathbb{N},$$

and $\frac{1}{n} \to 0$, the squeeze theorem implies $\frac{n!}{n^n} \to 0$. ■

9. Let $a > 0$. Compute $\lim_{n\to\infty} a^{1/n}$, if the limit exists.

Solution. Since $a > 0$, we have $\frac{1}{a} > 0$. According to the Archimedean Property, $\exists N_a \geq \max\{a, \frac{1}{a}\}$. For every $n \geq N_a$, we then have $\frac{1}{n} \leq a \leq n$. Thus $\frac{1}{\sqrt[n]{n}} \leq \sqrt[n]{a} \leq \sqrt[n]{n}$ for all $n \geq N_a$. But $\sqrt[n]{n} \to 1$ by a previous example, so $\sqrt[n]{a} \to 1$ by the Squeeze Theorem. ■

10. Compute $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$, if the limit exists.

Solution. Since

$$5^n \leq 3^n + 5^n \leq 5^n + 5^n = 2 \cdot 5^n \leq n \cdot 5^n, \ \forall n \geq 2,$$

then

$$5 \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{n} \cdot 5, \ \forall n \geq 2.$$

But we have seen previously that $\sqrt[n]{n} \to 1$.

The Squeeze Theorem can then be applied to the above chain of inequalities to conclude $\sqrt[n]{3^n + 5^n} \to 5$. ■

Theorem 17. *Let* $y_n \to y$. *If* $y_n \geq 0\ \forall n \in \mathbb{N}$, *then* $\sqrt{y_n} \to \sqrt{y}$.

Proof. According to Theorem 15, $y \geq 0$. There are 2 cases:

- If $y = 0$, let $\varepsilon > 0$. Then $\varepsilon^2 > 0$. Since $y_n \to 0$, $\exists M_{\varepsilon^2} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon^2}$, we must have $|y_n - 0| = y_n < \varepsilon^2$. Now, set $N_\varepsilon = M_{\varepsilon^2}$.

  Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - 0| = \sqrt{y_n} < \sqrt{\varepsilon^2} = \varepsilon$.

- If $y > 0$, let $\varepsilon > 0$. Then $\varepsilon\sqrt{y} > 0$. Since $y_n \to y$, $\exists M_{\varepsilon\sqrt{y}} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon\sqrt{y}}$, $|y_n - y| < \varepsilon\sqrt{y}$. Now, set $N_\varepsilon = M_{\varepsilon\sqrt{y}}$.

  Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - \sqrt{y}| = \frac{|y_n - y|}{\sqrt{y_n} + \sqrt{y}} \leq \frac{|y_n - y|}{\sqrt{y}} < \frac{\varepsilon\sqrt{y}}{\sqrt{y}} = \varepsilon$.

In both cases, we have $\sqrt{y_n} \to \sqrt{y}$. ■

3.4 – Bounded Monotone Convergence Theorem

A sequence (x_n) is **increasing** if

$$x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots, \quad \forall n \in \mathbb{N}$$

and it is **decreasing** if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \cdots, \quad \forall n \in \mathbb{N}.$$

If (x_n) is either increasing or decreasing, we say that it is **monotone**. If it is both increasing and decreasing, it is **constant**.

When the inequalities are strict, then the sequence is **strictly increasing** or **strictly decreasing**, depending, and thus **strictly monotone**.

Theorem 18. (Bounded Monotone Convergence)
Let (x_n) be an increasing sequence, bounded above. Then (x_n) converges to $\sup\{x_n \mid n \in \mathbb{N}\}$.

Proof. Since the sequence (x_n) is bounded above, so it its range $\{x_n\}$. By completeness of $\mathbb{R}$, $x^* = \sup\{x_n\}$ exists. It remains only to show $x_n \to x^*$. Let $\varepsilon > 0$. By definition, $x^* - \varepsilon$ is not an upper bound for $\{x_n\}$. Then $\exists N_\varepsilon \in \mathbb{N}$ such that

$$x^* - \varepsilon < x_{N_\varepsilon} \leq x^* < x^* + \varepsilon.$$

But (x_n) is increasing; in particular, $x_{N_\varepsilon} \leq x_n$ when $n > N_\varepsilon$. Thus

$$n > N_\varepsilon \implies x^* - \varepsilon < x_n < x^* + \varepsilon,$$

so $x_n \to x^*$. ■

A similar result holds for decreasing sequences that are bounded below.

Examples:

- Does the sequence $(x_n) = (1 - \frac{1}{n})$ converge? If so, what is its limit?

  Solution. As $\frac{1}{n} \geq \frac{1}{n+1}$ for all $n \in \mathbb{N}$,

$$x_n - 1 - \frac{1}{n} \leq 1 - \frac{1}{n+1} \leq x_{n+1},$$

  and so (x_n) is increasing. Furthermore, $x_n \leq 1$ for all $n \in \mathbb{N}$. Then (x_n) converges by the Bounded Monotone Convergence Theorem, and

$$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}}\{x_n\} = \sup_{n\in\mathbb{N}}\{1-1/n\} = 1+\sup_{n\in\mathbb{N}}\{-1/n\} = 1-\inf_{n\in\mathbb{N}}\{1/n\} = 1.$$

- Let (x_n) be defined by $x_n = \sqrt{2x_{n-1}}$ when $n \geq 2$, with $x_1 = 1$. Does (x_n) converge? If so, to what limit?

  Solution. We first show, by induction, that (x_n) is increasing.

  – **Base Case:** $x_2 = \sqrt{2} \geq 1 = x_1$.
  – **Induction Step:** Suppose $x_k \geq x_{k-1}$. Then

$$2x_k \geq 2x_{k-1} \implies \sqrt{2x_k} \geq \sqrt{2x_{k-1}} \implies x_{k+1} \geq x_k.$$

  Thus $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

  Next we show, again by induction, that (x_n) is bounded above by 2.

- **Base Case:** $1 \leq x_1 = 1 \leq 2$.
- **Induction Step:** Suppose $1 \leq x_k \leq 2$. Then

$$2 \leq 2x_k \leq 2{\cdot}2 = 4 \implies 1 \leq \sqrt{2} \leq \sqrt{2x_k} \leq \sqrt{4} = 2 \implies 1 \leq x_{k+1} \leq 2.$$

Thus $x_n \leq 2$ for all $n \in \mathbb{N}$ (why did we include the lower bound 1?).

Thus, according to the Bounded Monotone Convergence Theorem, $x_n \to x = \sup\{x_n \mid n \in \mathbb{N}\}$. But

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} = \sqrt{2 \lim_{n\to\infty} x_n} = \sqrt{2x},$$

whence $x^2 = 2x$. So either $x = 0$ or $x = 2$. But $x_n \geq 1$ for all $n \in \mathbb{N}$, so $x \geq 1$ according to Theorem 15. Thus $x_n \to 2$. ■

3.5 – Bolzano-Weierstrass Theorem

The main result of this section is a corner stone of analysis, concerning bounded sequences and their subsequences.

Let $(x_n) \subseteq \mathbb{R}$ be a sequence and $n_1 < n_2 < \cdots$ be an increasing string of positive integers. The sequence $(x_{n_k})_k = (x_{n_1}, x_{n_2}, \ldots)$ is a **subsequence** of (x_n), denoted by $(x_{n_k}) \subseteq (x_n)$. Note that $n_k \geq k$ for all $k \in \mathbb{N}$.

Examples:

- Let $(x_n) = (\frac{1}{n})$. Both $(\frac{1}{2k}) = (\frac{1}{2}, \frac{1}{4}, \ldots)$ and $(1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \ldots)$ are subsequences of (x_n) as they sample the original sequence while preserving the order in which the terms appear. But $(1, \frac{1}{3}, \frac{1}{2}, \frac{1}{8}, \ldots)$ is not a subsequence of (x_n) as $\frac{1}{3} = x_3$ appears before $\frac{1}{2} = x_2$.

- The sequence $(x_{3n}) = (x_3, x_6, x_9, \ldots)$ is a subsequence of (x_n) for any sequence (x_n).

- Every sequence (x_n) is a (**non-proper**) subsequence of itself.

- If $(y_k) = (x_{n_k})$ is a subsequence of (x_n) and $(z_j) = (y_{k_j})$ is a subsequence of (y_k), then $(z_j) = (x_{n_{k_j}})$ is a subsequence of (x_n).

Theorem 19. *Let $x_n \to x$. If $(x_{n_k}) \subseteq (x_n)$, then $x_{n_k} \to x$ as well.*

Proof. Let $\varepsilon > 0$. Since $x_n \to x$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n > N_\varepsilon$. But (x_{n_k}) is a subsequence of (x_n), so $n_k \geq k$ for all $k \in \mathbb{N}$.

Then $|x_{n_k} - x| < \varepsilon$ whenever $n_k \geq k > N_\varepsilon$, so $x_{n_k} \to x$ when $k \to \infty$. ■

The converse of this theorem is false: can you find a divergent sqeuence with convergent subsequences?

The next result is surprising, deep and useful.

Theorem 20. (Bolzano-Weierstrass)
If $(x_n) \subseteq \mathbb{R}$ is bounded, it has (at least) one convergent subsequence.

Proof. We build a subsequence as follows: as (x_n) is bounded, there is an interval $I_1 = [a, b]$ s.t. $(x_n) \subseteq I_1$. Let $n_1 = 1$. Then $x_{n_1} = x_1 \in I_1$ and

$$\mathrm{length}(I_1) = b - a = \frac{b-a}{2^0}.$$

Set $I_1' = [a, \frac{a+b}{2}]$ and $I_1'' = [\frac{a+b}{2}, b]$,

$A_1 = \{n \in \mathbb{N} \mid n > n_1$ and $x_n \in I_1'\}$, $\quad B_1 = \{n \in \mathbb{N} \mid n > n_1$ and $x_n \in I_1''\}$.

At least one of A_1, B_1 must be infinite as $A_1 \cup B_1 = \{n \in \mathbb{N} \mid n > n_1\}$:

- If A_1 is infinite, set $I_2 = I_1'$. Since A_1 is an infinite set of integers, it is not empty. By the Well-Ordering Principle, A_1 contains a smallest element, say n_2.

- If A_1 is finite, set $I_2 = I_1''$. Since B_1 is an infinite set of integers, it is not empty. By the Well-Ordering Principle, B_1 contains a smallest element, say n_2.

Either way, there is an integer $n_2 > n_1$ such that $x_{n_2} \in I_2$, $I_1 \supseteq I_2$ and

$$\text{length}(I_2) = \frac{b-a}{2^1}.$$

Now, suppose that $I_{k-1} \supseteq I_k$ are intervals with

$$\text{length}(I_{k-1}) = \frac{b-a}{2^{k-2}} \quad \text{and} \quad \text{length}(I_k) = \frac{b-a}{2^{k-1}},$$

that $\exists n_{k-1}, n_k \in \mathbb{N}$ such that $n_{k-1} < n_k$, $x_{n_{j-1}} \in I_{k-1}$, $x_{n_k} \in I_k$, and that at least one of the corresponding sets A_{k-1}, B_{k-1} is infinite.

Write $I_k = [\alpha, \beta]$. Set $I_k' = [\alpha, \frac{\alpha+\beta}{2}]$ and $I_k'' = [\frac{\alpha+\beta}{2}, \beta]$,

$$A_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k'\}, \quad B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k''\}.$$

One of A_k, B_k must be infinite as $A_k \cup B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k\}$ is infinite.

- If A_k is infinite, set $I_{k+1} = I'_k$. Since A_k is an infinite set of integers, it is not empty. By the Well-Ordering Principle, A_k contains a smallest element, say n_{k+1}.

- If A_k is finite, set $I_{k+1} = I''_k$. Since B_k is an infinite set of integers, it is not empty. By the Well-Ordering Principle, B_k contains a smallest element, say n_{k+1}.

Either way, there is an integer $n_{k+1} > n_k$ s.t. $x_{n_{k+1}} \in I_{k+1}$, $I_k \supseteq I_{k+1}$ and

$$\text{length}(I_{k+1}) = \frac{b-a}{2^k}.$$

By induction, we have

1. $I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq I_{k+1} \supseteq \cdots$;
2. for each $k \in \mathbb{N}$, $\text{length}(I_k) = \frac{b-a}{2^{k-1}}$;
3. for each $k \in \mathbb{N}$, $x_{n_k} \in I_k$, and
4. $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$.

Furthermore, $\frac{b-a}{2^k} \to 0$ (since it is a subsequence of $\frac{b-a}{n} \to 0$). According to the Nested Intervals Theorem, then, $\exists \xi \in [a,b]$ such that

$$\bigcap_{k \geq 1} I_k = \{\xi\}.$$

It remains to show that $x_{n_k} \to \xi$.

Let $\varepsilon > 0$. By the Archimedean Property, $\exists K_\varepsilon \in \mathbb{N}$ such that $2^{K_\varepsilon - 1} > \frac{b-a}{\varepsilon}$. Hence

$$k > K_\varepsilon \implies 2^{K_\varepsilon - 1} < 2^{k-1} \implies 0 \leq \frac{b-a}{2^{k-1}} < \frac{b-a}{2^{K_\varepsilon - 1}} < \varepsilon.$$

Since $\xi \in I_k$ for all $k \in \mathbb{N}$, then

$$k > K_\varepsilon \implies |x_{n_k} - \xi| \leq \frac{b-a}{2^{k-1}} < \frac{b-a}{2^{K_\varepsilon - 1}} < \varepsilon,$$

which is to say $x_{n_k} \to x$. ■

We have mentioned that a sequence (x_n) which diverges is one for which

$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L.$$

If (x_n) does not converge to L, it is easy to construct a subsequence (x_{n_k}) that also fails to converge to L:

- let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - L| \geq \varepsilon_L$;
- let $n_2 \in \mathbb{N}$ be such that $n_2 \geq n_1$ and $|x_{n_2} - L| \geq \varepsilon_L$;
- etc.

Note that there might be some subsequences of (x_n) that do converge to some L, however: $x_n = (-1)^n$ diverges, but $x_{2n} = (-1)^{2n} = 1 \to 1$.

Theorem 21. *Let $(x_n) \subseteq \mathbb{R}$ be a bounded sequence such that every one of its proper converging subsequence converges to the same $x \in \mathbb{R}$. Then $x_n \to x$.*

Proof. Let $M > 0$ be a bound for (x_n). Then $|x_n| \leq M$ for all $n \in \mathbb{N}$.

If (x_n) does not converge to x, then $\exists (x_{n_k}) \subseteq (x_n)$ and an $\varepsilon_0 > 0$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

But (x_{n_k}) is also a bounded sequence, and so, by the Bolzano-Weierstrass Theorem, there is convergent subsequence $(x_{n_{k_j}}) \subseteq (x_{n_k}) \subseteq (x_n)$.

But all subsequences of (x_n) converge to x, by assumption, so $x_{n_{k_j}} \to x$. That is to say, for $\varepsilon_0 > 0$, $\exists N_{\varepsilon_0} \in \mathbb{N}$ such that $|x_{n_{k_j}} - x| < \varepsilon_0$ whenever $k_j > j > N_{\varepsilon_0}$, which contradicts the above property. Hence $x_n \to x$. ■

3.6 – Cauchy Sequences

A main challenge with the definition of a limit of a sequence is that we need to know what the limit is **before** we can show what it is, in which case we do not need to show what it is...

A sequence (x_n) is a **Cauchy sequence** if

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } m, n > N_\varepsilon \implies |x_m - x_n| < \varepsilon.$$

Incidentally, (x_n) is not a Cauchy sequence if

$$\exists \varepsilon_0 > 0,\ \forall N \in \mathbb{N},\ \exists m_N, n_N > N \text{ such that } |x_{m_N} - x_{n_N}| \geq \varepsilon_0.$$

Examples:

1. Show that $(x_n) = (\frac{1}{n})$ is a Cauchy sequence.

   Proof. Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$. Thus

   $$m, n > N_\varepsilon \implies \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \frac{1}{N_\varepsilon} + \frac{1}{N_\varepsilon} = \frac{2}{N_\varepsilon} < \varepsilon.$$

   Thus (x_n) is Cauchy. ■

2. Show that $(x_n) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ is not a Cauchy sequence.

Proof. Let $m > n$. Then $\frac{1}{n} \geq \frac{1}{n+1} \geq \cdots \geq \frac{1}{m}$ and

$$|x_m - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{m} \geq \underbrace{\frac{1}{m} + \cdots + \frac{1}{m}}_{m-n \text{ terms}} = \frac{(m-n)}{m} = 1 - \frac{n}{m}.$$

In particular, if $m = 2n$, then $|x_m - x_n| \geq \frac{1}{2}$ for every $n \in \mathbb{N}$, and so (x_n) is not a Cauchy sequence. ■

In essence, a Cauchy sequence is a sequence for which the terms can get as close to one another as one wishes, after a threshold (which depends on the desired distance).

The next result shows that Cauchy sequences behave like convergent sequences in $\mathbb{R}$ – we will soon see that the similarity is in fact not pure happenstance.

Theorem 22. *If (x_n) is a Cauchy sequence, then it is bounded.*

Proof. Let $1 > \varepsilon > 0$. Since (x_n) is a Cauchy sequence, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $m, n > N_\varepsilon$.

Set $m^* = N_\varepsilon + 1$. If $n > N_\varepsilon$, then

$$|x_n| = |x_{m^*} + (x_n - x_{m^*})| \leq |x_{m^*}| + |x_n - x_{m^*}| < |x_{m^*}| + \varepsilon.$$

Set $M = \max\{|x_1| + 1, \ldots, |x_{N_\varepsilon}| + 1, |x_{m^*}| + 1\}$.

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

We could also show that the sum of two Cauchy sequences is a Cauchy sequence, that every bounded Cauchy sequence admits at least one convergent subsequence, and so forth.

Cauchy sequences in $\mathbb{R}$ behave like convergent sequences in $\mathbb{R}$ because ...

Theorem 23. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. Let (x_n) be the sequence under consideration.

Suppose that $x_n \to x$, say. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$ and $\exists M_{\varepsilon/2}$ such that

$$n > M_{\varepsilon/2} \implies |x_n - x| < \frac{\varepsilon}{2}.$$

Set $N_\varepsilon = M_{\varepsilon/2}$. When $n, m > N_\varepsilon$, we have

$$|x_m - x_n| \leq |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is to say that (x_n) is Cauchy.

Now suppose that (x_n) is Cauchy. According to Theorem 22, it is a bounded sequence, and so must admit a convergent subsequence $(x_{n_k}) \subseteq (x_n)$ by the Bolzano-Weierstrass Theorem, with $x_{n_k} \to x$, say.

Let $\varepsilon > 0$. Since (x_n) is Cauchy, $\exists M_{\varepsilon/2} \in \mathbb{N}$ such that

$$n, m > M_{\varepsilon/2} \implies |x_m - x_n| < \frac{\varepsilon}{2}.$$

Since (x_{n_k}) converges to x, $\exists N > M_{\varepsilon/2}$ such that $|x_N - x| < \frac{\varepsilon}{2}$. Set $N_\varepsilon = M_{\varepsilon/2}$. Then

$$n > N_\varepsilon \implies |x_n - x| = |x_n - x_N + x_N - x| \leq |x_n - x_N| + |x_N - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so (x_n) is convergent. ■

Examples:

1. As the sequence $(x_n) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ is not a Cauchy sequence, it does not converge.

2. Compute the limit of the sequence defined by $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $n > 2$, with $x_1 = 1$ and $x_2 = 2$.

   Solution. We cannot use the Bounded Monotone Convergence Theorem as (x_n) is not monotone. However, (x_n) is a Cauchy sequence. Indeed,

$$
\begin{aligned}
|x_{n+1} - x_n| &= \left|\tfrac{1}{2}(x_{n-1} + x_n) - x_n\right| = \tfrac{1}{2}|x_n - x_{n-1}| = \tfrac{1}{2^2}|x_{n-1} - x_{n-2}| \\
&= \tfrac{1}{2^3}|x_{n-2} - x_{n-3}| = \cdots = \tfrac{1}{2^{n-1}}|x_2 - x_1| = \tfrac{1}{2^{n-1}}.
\end{aligned}
$$

Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{2^{N_\varepsilon - 2}} < \varepsilon$. Then, whenever $m \geq n > N_\varepsilon$,

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &= \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} < \frac{1}{2^{N_\varepsilon - 2}} < \varepsilon. \end{aligned}$$

Being a Cauchy sequence, (x_n) is convergent according to Theorem 23. Let $x_n \to x$. From Theorem 19, we must have $x_{2n+1} \to x$ as well.

It is left as an induction exercise to show that

$$x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} = 1 + \frac{3}{4}\left(1 - \frac{1}{4^n}\right).$$

Then $x_{2n+1} \to 1 + \frac{2}{3} = \frac{5}{3} = x$. ∎

Cauchy sequences illustrate the fundamental difference between $\mathbb{R}$ and $\mathbb{Q}$. A sequence is Cauchy if the points of the sequence "accumulate" on top of one another. In $\mathbb{R}$, every Cauchy sequence is convergent, and $vice\text{-}versa$.

In $\mathbb{Q}$, the converging sequences are Cauchy, but there are Cauchy sequences that do not converge: it is possible that the points of such a sequence "accumulate" around of the (uncountably infinitely) many holes of $\mathbb{Q}$.

For instance, the sequence $(1, 1.4, 1.41, 1.414, \ldots)$ is Cauchy in $\mathbb{Q}$, but does not converge in $\mathbb{Q}$.

This leads to one of the ways of building $\mathbb{R}$: we take all Cauchy sequences in $\mathbb{Q}$ and add whatever point the sequences "accoumulates" around to $\mathbb{R}$ (there is more to it than that, but that is the main idea). In this example, we would get to add $\sqrt{2}$ to $\mathbb{R}$.

3.7 – Exercises

1. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) $(5, 7, 9, 11, \ldots)$;

   (b) $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right)$;

   (c) $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$;

   (d) $(1, 4, 9, 16, \ldots)$.
2. Use the definition of the limit of a sequence to establish the following limits.
   (a) $\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$;
   (b) $\lim_{n\to\infty} \left(\frac{2n}{n+1}\right) = 2$;

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$, and

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$.

3. Show that

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n+7}}\right)=0$;

(b) $\displaystyle\lim_{n\to\infty}\left(\frac{2n}{n+2}\right)=2$;

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1}\right)=0$, and

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{(-1)^n n}{n^2+1}\right)=0$.

4. Show that $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0$.

5. Find the limit of the following sequences:
   (a) $\lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)^2\right)$;
   (b) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n+2}\right)$;
   (c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, and
   (d) $\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt{n}}\right)$.
6. Let $y_n=\sqrt{n+1}-\sqrt{n}$. Show that (y_n) and $(\sqrt{n}y_n)$ converge.
7. Let (x_n) be a sequence of positive real numbers such that $x_n^{1/n}\to L<1$. Show $\exists r\in(0,1)$ such that $0<x_n<r^n$ for all sufficiently large $n\in\mathbb{N}$. Use this result to show that $x_n\to 0$.
8. Give an example of a convergent (resp. divergent) sequence (x_n) of positive real numbers with $x_n^{1/n}\to 1$.
9. Let $x_1=1$, $x_{n+1}=\sqrt{2+x_n}$ for $n\in\mathbb{N}$. Show that (x_n) converges; find the limit.

10. Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$. Show that (x_n) is increasing and bounded above.
11. Show that $c^{1/n} \to 1$ if $0 < c < 1$.
12. Let (x_n) be a bounded sequence.
    For each $n \in \mathbb{N}$, let $s_n = \sup\{x_k : k \geq n\}$. If $S = \inf\{s_n\}$, show that there is a subsequence of (x_n) that converges to S.
13. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $((-1)^n x_n)$ converges. Show that (x_n) converges.
14. Show that if (x_n) is unbounded, there exists a subsequence (x_{n_k}) with $1/x_{n_k} \to 0$.
15. If $x_n = \frac{(-1)^n}{n}$, find the convergent subsequence in the proof of the Bolzano-Weierstrass theorem, with $I_1 = [-1, 1]$.
16. Show directly that a bounded increasing sequence is a Cauchy sequence.
17. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is Cauchy.
18. If $x_1 < x_2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \in \mathbb{N}$, show that (x_n) is convergent and compute its limit.
19. Suppose that (a_n) is a bounded sequence and $b_n \to 0$. Show that $a_n b_n \to 0$.

20. Consider the sequence given by the recursion $a_{n+1} = \frac{1}{2}(a_n + a_n^{-1})$, with some initial condition $a_1 \in (-\infty, 0) \cup (0, \infty)$. Find and prove the limit, if it exists.
21. Let (a_n) be a sequence with no convergent subsequences. Show that $|a_n| \to \infty$.
22. We define the **limit inferior** and the **limit superior** of a sequence as follows:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}.$$

    Let (a_n) be bounded. Show that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ exist and are in $\mathbb{R}$.
23. Let (a_n) be unbounded. Show that $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$.
24. Let (a_n), (b_n) be two sequences. Show that

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

    Furthermore, find a pair of sequences for which the second inequality is strict.

Solutions

1. **Proof.** There is no general method. This question is a wee bit on the easy side...

   (a) Odd integers ≥ 5: $x_n = 2n + 3$ for all $n \geq 1$;

   (b) Alternating powers of $\frac{1}{2}$: $x_n = (-1)^{n+1}\frac{1}{2^n}$ for all $n \geq 1$;

   (c) Fractions where the denominator is one more than the numerator: $x_n = \frac{n}{n+1}$ for all $n \geq 1$;

   (d) Perfect squares ≥ 1: $x_n = n^2$ for all $n \geq 1$. ■

2. **Proof.**

(a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{2}{\varepsilon}$. Then

$$\left|\frac{2n}{n+1} - 2\right| = \left|-\frac{2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{13}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|-\frac{13}{2(2n+5)}\right| = \frac{13}{2} \cdot \frac{1}{2n+5} < \frac{13}{2} \cdot \frac{1}{2n} = \frac{13}{4} \cdot \frac{1}{n} < \frac{13}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$.

(d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{5}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|-\frac{5}{2(2n^2+3)}\right| = \frac{5}{2} \cdot \frac{1}{2n^2+3} < \frac{5}{2} \cdot \frac{1}{2n^2} \leq \frac{5}{4} \cdot \frac{1}{n} < \frac{5}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$. ■

3. **Proof.**

(a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{4}{\varepsilon}$. Then

$$\left|\frac{2n}{n+2} - 2\right| = \left|-\frac{4}{n+2}\right| = \frac{4}{n+2} < \frac{4}{n} < \frac{4}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$. ■

4. **Proof.** Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\sqrt{\varepsilon}}$.

Then

$$\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| = \frac{1}{n(1+n)} < \frac{1}{n^2} < \frac{1}{N_\varepsilon^2} < \varepsilon,$$

whenever $n > N_\varepsilon$. ■

5. **Proof.** We can only use the definition if we have a candidate. Throughout, we will assume that it is known that $\frac{1}{n} \to 0$.

(a) Note that $(2+\frac{1}{n})^2 = 4 + \frac{2}{n} + \frac{1}{n^2}$. Then, by theorem 14 (operations on sequences and limits),

$$\frac{2}{n} = 2 \cdot \frac{1}{n} \to 2 \cdot 0 = 0 \quad \text{and} \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} \to 0 \cdot 0 = 0,$$

so that $4 + \frac{2}{n} + \frac{1}{n^2} \to 4 + 0 + 0 = 4$.

(b) Clearly,

$$\frac{-1}{n+2} \leq \frac{(-1)^n}{n+2} \leq \frac{1}{n+2}, \quad \forall n \in \mathbb{N}.$$

Note that $n+2 \geq n$ for all n so that

$$0 \leq \frac{1}{n+2} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N};$$

as a result, $\frac{1}{n+2} \to 0$ by the squeeze theorem. Then $-\frac{1}{n+2} \to -0 = 0$ by theorem 14, so that $\frac{(-1)^n}{n+2} \to 0$ by the squeeze theorem.

(c) Re-write $\frac{\sqrt{n}-1}{\sqrt{n}+1} = 1 - \frac{2}{\sqrt{n}+1}$. Note that

$$0 \leq \frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}, \quad \forall n \in \mathbb{N}.$$

We have seen that $\frac{1}{\sqrt{n}} \to 0$; as a result of the squeeze theorem, $\frac{1}{\sqrt{n}+1} \to 0$. Then $1 - \frac{2}{\sqrt{n}+1} \to 1 - 2 \cdot 0 = 1$, by theorem 14.

(d) Note that $n \leq n\sqrt{n} \leq n^2$ for all $n \in \mathbb{N}$ so

$$\frac{1}{n^2} \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

But $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{\sqrt{n}} \to 0$ (see previous problems) so that $\frac{1}{n\sqrt{n}} \to 0$ by the squeeze theorem. Furthermore,

$$\frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}} \to 0 + 0 = 0,$$

by theorem 14. ■

6. **Proof.** As

$$0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad \forall n \in \mathbb{N},$$

and $\frac{1}{\sqrt{n}} \to 0$, then $\sqrt{n+1} - \sqrt{n} \to 0$ by the squeeze theorem.

Note that $\sqrt{n}y_n = \sqrt{n(n+1)} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$ for all $n \in \mathbb{N}$. Then, according to theorem 14,

$$\lim_{n\to\infty} \sqrt{n}y_n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{2},$$

since $\sqrt{1+\frac{1}{n}}+1 > 2$ for all $n \in \mathbb{N}$. ■

7. **Proof.** Since $L < 1$, $\exists \varepsilon_0 > 0$ such that $L < L + \varepsilon_0 < 1$. Then, $\exists N_0 \in \mathbb{N}$ such that

$$|x_n^{1/n} - L| < \varepsilon_0 \quad \text{whenever } n > N_0.$$

Hence $L - \varepsilon_0 < x_n^{1/n} < L + \varepsilon_0$ for all $n > N_0$. Set $r = L + \varepsilon_0$. Then $r \in (0,1)$ and

$$0 < x_n < r^n, \quad \forall n > N_0.$$

Let $\varepsilon > 0$. $r^n \to 0$ (do you know how to show this?), $\exists N_\varepsilon \geq N_0$ such that $r^n < \varepsilon$ whenever $n > N_\varepsilon$, hence

$$|x_n - 0| = x_n < r^n < \varepsilon$$

whenever $n > N_\varepsilon$. ■

8. **Proof.** The sequences $(x_n) = \frac{1}{n}$ and $(x_n) = (n)$ do the trick. You should fill in the details or ask for hints if you're not sure how to show this – there are other solutions. ■

9. **Proof.** We show (x_n) is increasing and bounded by induction; by a theorem seen in class, (x_n) converges.

A quick computation shows $x_2 = \sqrt{3}$.

Initial case: Clearly, $1 \leq x_1 \leq x_2 \leq 2$.

Induction hypothesis: Suppose $1 \leq x_k \leq x_{k+1} \leq 2$. Then

$$3 \leq x_k + 2 \leq x_{k+1} + 2 \leq 4$$

and so

$$1 \leq \sqrt{3} \leq \sqrt{x_k + 2} \leq \sqrt{x_{k+1} + 2} \leq \sqrt{4} = 2,$$

i.e. $1 \leq x_{k+1} \leq x_{k+2} = 2$.

Hence (x_n) is increasing and bounded above by 2; as such $x_n \to x$ for some $x \in \mathbb{R}$. exists.

But

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{2 + \lim_{n\to\infty} x_n} = \sqrt{2 + x},$$

that is, $x^2 = 2 + x$. The only solutions are $x = 2$ or $x = -1$, but $x = -1$ must be rejected since $1 \leq x_n$ for all n.

Thus, $x_n \to 2$. ∎

10. **Proof.** As $\frac{1}{(n+1)^2} > 0$ for all $n \in \mathbb{N}$, we have

$$x_n = \frac{1}{1^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = x_{n+1}.$$

Furthermore, for any $k \geq 2 \in \mathbb{N}$, we have $\frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k}$. Then

$$\begin{aligned} x_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + 0 + \cdots + 0 - \frac{1}{n} < 2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence (x_n) is increasing and bounded above by 2. ■

11. **Proof.** Let $x_n = c^{1/n}$ for all $n \in \mathbb{N}$.

Since $x_{n+1} = c^{1/(n+1)} > c^{1/n} = x_n$ for all $n \in \mathbb{N}$ (as $c < 1$), then (x_n) is increasing. Furthermore, $0 < c^{1/n} < 1^{1/n} = 1$ for all $n \in \mathbb{N}$, so (x_n) is bounded above.

Hence (x_n) converges, and $x_n \to x$, for some $x \in \mathbb{R}$. As all subsequences of a convergent sequence converge to the same limit as the convergent sequence, $x_{2n} = c^{1/2n} \to x$. As such,

$$x = \lim_{n\to\infty} c^{1/2n} = \lim_{n\to\infty} \sqrt{c^{1/n}} = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n} = \sqrt{x},$$

and so either $x = 0$ or $x = 1$. But as x_n increases to 1, there comes a point after which all x_n are "far" from 0 (you should mathematicize this statement...), so $x_n \to 1$. ■

12. **Proof.** As (x_n) is bounded, $\exists M > 0$ such that $-M < x_n < M$ for all $n \in \mathbb{N}$. By definition, $s_1 \geq s_2 \geq \cdots$ and $s_n \geq x_k$ for all $n \in \mathbb{N}$, $k \geq n$.

    Hence $s_n > -M$ for all n and (s_n) is bounded below and decreasing, i.e. (s_n) is convergent.

    Furthermore, for each $n \in \mathbb{N}$, as $s_n = \sup\{x_k : k \geq n\}$, $\exists k_n \in \mathbb{N}$ s.t.

$$s_n - \frac{1}{n} \leq x_{k_n} < s_n$$

    (otherwise s_n is not the supremum).

    The sequence (x_{k_n}) might not necessarily be a subsequence of (x_n), but by deleting the terms that are out of order, the resulting sequence, which we will also denote by (x_{k_n}) is a subsequence of (x_n).

Then

$$0 \leq |x_{k_n} - s_n| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

By the squeeze theorem,

$$0 \leq \lim_{n\to\infty} |x_{k_n} - s_n| \leq 0, \quad \text{so} \lim_{n\to\infty} |x_{k_n} - s_n| = 0.$$

From a theorem seen in class, this means that

$$\lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} s_n = S,$$

where the last equality comes from the theorem on bounded increasing/decreasing sequences. ■

13. **Proof.** Let $(-1)^n x_n \to \alpha$.

Consider its subsequences

$$\left((-1)^{2n} x_{2n}\right) = (x_{2n}) \quad \text{and} \quad \left((-1)^{2n+1} x_{2n+1}\right) = (-x_{2n+1}) \,.$$

Then $x_{2n} \to \alpha$ and $(-x_{2n+1}) \to \alpha$.

But $x_{2n} \geq 0 \ \forall n \in \mathbb{N}$ so $\alpha \geq 0$. Similarly, $-x_{2n+1} \leq 0 \ \forall n \in \mathbb{N}$ so $\alpha \leq 0$.

Since $0 \leq \alpha \leq 0$, $\alpha = 0$.

By Theorem 14 (operations on limits),

$$\lim_{n\to\infty} |(-1)^n x_n| = |0| = 0.$$

But $|(-1)^n x_n| = x_n \ \forall n$, so $x_n \to 0$. ■

14. **Proof.** As (x_n) is unbounded, $\exists n_1 \in \mathbb{N}$ such that $|x_{n_1}| \geq 1$.

Moreover, $\forall k \geq 2$, $\exists n_k \in \mathbb{N}$ such that $|x_{n_k}| \geq k$ and $n_{k+1} > n_k$ (otherwise the sequence would be bounded).

Let $\varepsilon > 0$. By the Archimedean property, $\exists K_\varepsilon \in \mathbb{N}$ such that $K_\varepsilon > \frac{1}{\varepsilon}$ and

$$\left|\frac{1}{x_{n_k}} - 0\right| = \frac{1}{|x_{n_k}|} \leq \frac{1}{k} < \frac{1}{K_\varepsilon} < \varepsilon$$

whenever $k > K_\varepsilon$.

Thus, $1/x_{n_k} \to 0$. ■

15. **Proof.** We must first note that (x_n) is bounded by -1 and 1, so the question makes sense.

Let $n_1 = 1$. Then $x_{n_1} = x_1 = -1$ and length$(I_1) = 2$. Set $I_1' = [-1, 0]$ and $I_1'' = [0, 1]$.

We have

$$A_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1'\} = \{3, 5, 7, 9, 11, \ldots\}$$

and

$$B_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1''\} = \{2, 4, 6, 8, 10, \ldots\}.$$

Since A_1 is infinite (why?), set $I_2 = I_1' = [-1, 0]$ and $n_2 = \min A_1 = 3$, so that $x_{n_2} = -1/3$. Note that $n_2 > n_1$, $I_2 \subseteq I_1$, and length$(I_2) = 1$.

Set $I_2' = [-1, -1/2]$ and $I_2'' = [-1/2, 0]$.

We have

$$A_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2'\} = \varnothing$$

and

$$B_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2''\} = \{5, 7, 9, 11, 13, \ldots\}.$$

Since A_2 is finite, set $I_3 = I_2'' = [-1/2, 0]$ and $n_3 = \min B_2 = 5$, so that $x_{n_3} = -1/5$.

Note that $n_3 > n_2 > n_1$, $I_3 \subseteq I_2 \subseteq I_1$, and length$(I_3) = 1/2$.

For $k \geq 3$, we set $I_k' = [-1/2^{k-2}, -1/2^{k-1}]$ and $I_k'' = [-1/2^{k-1}, 0]$. Then

$$A_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k'\} = \varnothing$$

and

$$B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k''\} = \{2k+1, 2k+3, 2k+5, \ldots\}.$$

A_k is finite, so set $I_{k+1} = I_k'' = [-1/2^{k-1}, 0]$.

Furthermore, $n_{k+1} = \min B_k = 2k+1$ so that $x_{n_k} = \frac{-1}{2k+1}$.

Note that $n_{k+1} > n_k > \cdots > n_2 > n_1$, $I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_2 \subseteq I_1$ and length$(I_{k+1}) = 1/2^{k-2}$.

The convergent subsequence is thus is $-1, -1/3, -1/5, \ldots \to 0$. ■

16. **Proof.** Let $\varepsilon > 0$.

By completeness of $\mathbb{R}$, $x^* = \sup\{x_n \mid n \in \mathbb{N}\}$ exists as $\{x_n \mid n \in \mathbb{N}\}$ is bounded and non-empty.

In particular, $\exists M_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$x^* - \frac{\varepsilon}{2} < x_{M_{\frac{\varepsilon}{2}}} \leq x^*.$$

But $x^* \geq x_n > x_{M_{\frac{\varepsilon}{2}}}$ whenever $n > M_{\frac{\varepsilon}{2}}$.

Let $N_\varepsilon = M_{\frac{\varepsilon}{2}}$. Then

$$|x_m - x_n| = |x_m - x^* + x^* - x_n| \leq |x^* - x_m| + |x^* - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m, n > N_\varepsilon$. ■

17. **Proof.** Let $\varepsilon > 0$.

By the Archimedean property, $\exists N_\varepsilon > \log_r(\varepsilon(1-r)) + 1$, i.e. $r^{N_\varepsilon - 1} < \varepsilon$.

Then

$$|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$
$$< r^{m-1} + \cdots + r^n < \frac{r^{n-1}}{1-r} < \frac{r^{N_\varepsilon - 1}}{1-r} < \varepsilon$$

whenever $m > n > N_\varepsilon$.

(The third last inequality holds since $r^{m-1} + \cdots + r^n$ is a geometric progression.) ■

18. **Proof.** We start by showing that (x_n) is a Cauchy sequence. let $L = x_2 - x_1$. Then

$$|x_n - x_{n-1}| \leq \frac{L}{2^{n-2}}$$

by induction (show this!).

Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $\frac{L}{2^{N_\varepsilon - 2}} < \varepsilon$. Then

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \frac{L}{2^{m-2}} + \cdots \frac{L}{2^{n-1}} \leq \frac{L}{2^{n-2}} < \frac{L}{2^{N_\varepsilon - 2}} < \varepsilon \end{aligned}$$

whenever $m > n > N_\varepsilon$. Hence (x_n) is a Cauchy sequence, and so it converges.

Let $x_n \to x$. We can show, by induction (do this!), that

$$x_{2n+1} = x_1 + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}$$

for all $n \in \mathbb{N}$. In particular,

$$\begin{aligned}
x = \lim_{n\to\infty} x_{2n+1} &= x_1 + \lim_{n\to\infty}\left(\frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}\right) \\
&= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{2n-2}}\right) \\
&= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(\frac{1-(1/2^2)^n}{1-(1/2^2)}\right) = x_1 + \frac{2}{3}L = \frac{1}{3}(x_1 + 2x_2).
\end{aligned}$$

For instance, when $x_1 = 1$ and $x_2 = 2$, $x_n \to 5/3$. ∎

19. **Proof.** Since (a_n) is bounded, there exists some $0 \leq M < \infty$ so that $\sup_n |a_n| \leq M$.

Next, we will check that $a_n b_n \to 0$.

Fix some $\varepsilon > 0$. Since $b_n \to 0$, there exists some N_ε so that for all $n > N_\varepsilon$, $|b_n| \leq \frac{\varepsilon}{M}$. Thus, for all $n > N_\varepsilon$,

$$|a_n b_n| \leq M|b_n| \leq M \frac{\varepsilon}{M} = \varepsilon.$$

Thus, $a_n b_n \to 0$. ■

20. **Proof.** Consider a value of n for which $a_n \geq 1$. For this value,

$$a_{n+1} = \frac{1}{2}(a_n + a_n^{-1}) \leq \frac{1}{2}(a_n + 1).$$

On the other hand, consider the function $f(x) = \frac{1}{2}(x + x^{-1})$ with domain $x \in (0, \infty)$. We recognize (e.g. from completing the square) that, on this domain, the function is minimized at $x = 1$. In particular, $f(x) \geq 1$ for all $x \in (0, \infty)$. Thus,

$$a_{n+1} = \frac{1}{2}(a_n + a_n^{-1}) \geq 1.$$

Putting together the two displayed equations, for $a_n \geq 1$ we have

$$1 \leq a_{n+1} \leq \frac{1}{2}(a_n + 1).$$

We note that, by this bound, $a_n \geq 1$ for all $n \geq 2$ for *any* value of $a_1 \in (0, \infty)$. Iterating the upper and lower bounds, we have

$$1 \leq a_{n+1} \leq \frac{1}{2}(a_n + 1) \leq \frac{1}{2}\left(\frac{1}{2}(a_{n-1} + 1) + 1\right) = \frac{1}{4}a_{n-1} + \frac{3}{4}.$$

Continuing to iterate, we find

$$1 \leq a_{n+1} \leq 2^{-n+1}a_2 + (1 - 2^{-n+1}).$$

Applying the Squeeze Theorem, we calculate

$$1 \leq \lim_{n\to\infty} a_{n+1} \leq \lim_{n\to\infty} (2^{-n+1}a_2 + (1 - 2^{-n+1})) = 1.$$

This completes the proof. ■

21. **Proof.** We prove this by contradiction. Assume that $|a_n|$ does not diverge to infinity. Then there exists some $M < \infty$ such that the set $\{n \in \mathbb{N} \mid |a_n| < M\}$ is infinite.

Let $1 \leq m_1 \leq m_2 \leq m_3 \leq \ldots$ be the indices satisfying $|a_{m_n}| < M$.

Set $b_n = a_{m_n}$. Then $\{b_n\}$ is a bounded sequence and so has a convergent subsequence $\{b_{k_n}\}_n$ by the Bolzano-Weierstrass Theorem.

But $\{a_{m_{k_n}}\}_n = \{b_{k_n}\}_n$ is in fact a convergent subsequence of (a_n), contradicting the information given in the question. We conclude that our assumption was false, and so that $|a_n|$ diverges to infinity. ■

22. **Proof.** Define the sequence of sets $B_n = \{a_k \mid k \geq n\}$ and the sequence of numbers $b_n = \sup(B_n)$, so that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

We note that $B_1 \supset B_2 \supset \ldots$, which implies $\sup(B_1) \leq \sup(B_2) \leq \ldots$, which means that $\{b_n\}$ is monotone decreasing.

Furthermore, since (a_n) is bounded, there exists some $-\infty < M < \infty$ so that $a_n \geq M$ for all $n \in \mathbb{N}$.

But this M is a lower bound for (a_n), which means it must be a lower bound for B_n for all $n \in \mathbb{N}$, which means $b_n = \sup(B_n) \geq M$ for all $n \in \mathbb{N}$ as well.

Thus, we have shown that $\{b_n\}$ is a monotone decreasing sequence that is bounded from below. Hence, by the monotone convergence theorem, it has a limit and so

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

exists.

The proof for the $\liminf$ statement follows a similar path. $\blacksquare$

23. **Proof.** Since (a_n) is unbounded, for all $0 < M < \infty$ there exists $n = n(M)$ satisfying $|a_n| > M$.

Define the subsequence $\{b_k\}$ by setting $b_k = a_{n(k)}$, so that $|b_k| > k$ for all $k \in \mathbb{N}$. Since this is an infinite sequence, we have by the Pigeonhole Principle that at least one of the two sets $I_+ = \{k \in \mathbb{N} \mid b_k \geq 0\}$, $I_- = \{k \in \mathbb{N} \mid b_k \leq 0\}$ is infinite.

In the case that I_+ is infinite, write the elements $i_1 < i_2 < i_3 < \ldots$ in order and define the subsequence $\{c_\ell\}$ of $\{b_n\}$ by the formula $c_\ell = b_{i_\ell} = a_{n(i_\ell)}$. But then for all n, we have

$$\begin{aligned}\sup\{a_k \mid k \geq n\} &\geq \sup\{a_{n(i_\ell)} \mid \ell \geq n\} \\ &= \sup\{c_k \mid k \geq n\} \geq \sup\{k \mid k \geq n\} = \infty.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} a_n = \infty.$$

The case that I_- is infinite is essentially the same, with the conclusion

$$\liminf_{n\to\infty} a_n = -\infty.$$

This completes the proof. ■

As an aside, if I_-, I_+ are both infinite, then we have

$$\limsup_{n\to\infty} a_n = \infty, \quad \liminf_{n\to\infty} a_n = -\infty,$$

which you can check holds for sequences such as $a_n = (-n)^n$, say.

24. **Proof.** Fix $\varepsilon > 0$. Then there exists some $N_\varepsilon \in \mathbb{N}$ such that, for all $m > N_\varepsilon$, the following inequalities all hold:

$$\frac{\varepsilon}{2} + \limsup_{n\to\infty} a_n \geq a_m \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} a_n$$
$$\frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n \geq b_m \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} b_n.$$

Adding the left-hand sided inequalities, we get:

$$a_m + b_m \leq \varepsilon + \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

We conclude with our first desired inequality,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

To obtain the reverse inequality, again fix $\varepsilon > 0$. Then there exists a sequence $\{k_n\}$ so that

$$b_{k_m} \geq -\frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n \quad \text{for all } m.$$

Chopping off the finitely-many terms in the sequence occurring before the threshold N_ε and applying the above inequalities, we have, for all $m \in \mathbb{N}$:

$$a_{k_m} + b_{k_m} \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} a_n - \frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n.$$

We conclude with the desired reverse inequality,

$$\limsup_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

For the second question, consider the sequences

$$a_n = (-1)^n,\ b_n = (-1)^{n+1}.$$

It is clear that $a_n + b_n = 0$ for all n, so $\limsup_{n\to\infty}(a_n + b_n) = 0$. However, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. ■