Mathematical Analysis

Chapter 7
Series of Functions

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Overview

We discuss a specific type of sequence: the series.

In particular, we will discuss

- series of numbers,
- series of functions, and
- power series.

The latter is more naturally expressed using a complex analysis framework, but we will present it, and important theorems for regular series, in the real analysis framework.

Outline

7.1 – Series of Numbers

Let $(x_n) \subseteq \mathbb{R}$. The **series** associated with (x_n), denoted by

$$S_{(x_n)} = \sum_{n=1}^{\infty} x_n,$$

is the sequence (s_n), where

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\ldots \end{aligned}$$

If the sequence of partial sums s_n converges to S, we say the series $S_{(x_n)}$ **converges to the sum** S.

We start by producing a necessary condition for convergence.

Theorem 70. *If* $\sum_{n=1}^{\infty} x_n$ *converges, then* $x_n \to 0$.

Proof. Let S be the limit of the partial sums. Then

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0,$$

with the second equality being guaranteed by Theorem 14 and the convergence of the series. ∎

We can bypass the need to know the limit in order to prove convergence.

Theorem 71. (Cauchy Criterion for Series)

The series $\sum_{n=1}^{\infty} x_n$ *converges if and only if* $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ *such that*

$$m > n > N_\varepsilon \implies |x_{n+1} + \cdots + x_m| < \varepsilon.$$

Proof. Let (s_n) be the series of partial sums. If (s_n) converges, it is a Cauchy sequence, so that $\exists N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon \implies |s_m - s_n| < \varepsilon$. But $|s_m - s_n| = |x_m + \cdots + x_{n+1}|$, so the Cauchy Criterion holds.

Conversely, if the Cauchy Criterion holds, the sequence of partial terms is a Cauchy sequence, and so the series converges by completeness of $\mathbb{R}$. ■

But there are other tests that can be used to show the convergence of a series without knowing the limit.

Theorem 72. (Comparison Test)
Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ *be series whose terms are all non-negative. If* $\exists K \in \mathbb{N}$ *such that* $0 \leq x_n \leq y_n$ *when* $n > K$, *then*

1. $\sum_{n=1}^{\infty} y_n$ *converges* $\Longrightarrow$ $\sum_{n=1}^{\infty} x_n$ *converges.*

2. $\sum_{n=1}^{\infty} x_n$ *diverges* $\Longrightarrow$ $\sum_{n=1}^{\infty} y_n$ *diverges.*

Proof. We prove 1. The proof for 2. is simply the contrapositive. Let $\varepsilon > 0$. As $\sum_{n=1}^{\infty} y_n$ converges, $\exists N_\varepsilon \in \mathbb{N}$ such that $0 \leq \sum_{i=n+1}^{m} y_i < \varepsilon$ according to the Cauchy Criterion for series.

Hence, whenever $m \geq n > M_\varepsilon = \max\{N_\varepsilon, K\}$, then

$$0 \leq \sum_{i=n+1}^{m} x_i \leq \sum_{i=n+1}^{m} y_i < \varepsilon.$$

As such, $\sum_{n=1}^{\infty} x_n$ converges as it satisfies the Cauchy Criterion for series. ■

Examples: Discuss the convergence of 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

1. The limit of the partial sums converges to 1 as

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1 - 0 = 1. \quad \blacksquare$$

2. Since $n^2 \geq \frac{1}{2}(n^2+n) \geq 0$ for all $n \in \mathbb{N}$, then $\frac{2}{n(n+1)} \geq \frac{1}{n^2} \geq 0$ for all $n \in \mathbb{N}$, and

$$\infty > 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n^2},$$

thus the series converges according to the Comparison Theorem. $\blacksquare$

Theorem 73. (Alternating Series Test)
Let (a_n) be a sequence of non-negative numbers such that $a_n \searrow 0$ (i.e $a_n \to 0$ and $a_{n+1} \leq a_n$). Then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

Proof. Let (s_k) be the series of partial sums

$$s_k = \sum_{n=0}^{k} (-1)^n a_n.$$

The subsequence of even terms is $s_{2k} = s_{2k-2} - (a_{2k-1} - a_{2k})$; that of the odd terms is $s_{2k+1} = s_{2k-1} - (a_{2k} - a_{2k+1})$.

Since $a_n \searrow 0$, $a_{n+1} \leq a_n$ for all n. Thus $s_{2k} \leq s_{2k-2}$ and $s_{2k+1} \geq s_{2k-1}$ for all $k \in \mathbb{N}$.

But $s_{2k} \geq s_{2m+1}$ for all $k, m \in \mathbb{N}$ (left as an exercise), and so

$$a_0 = s_0 \geq s_2 \geq s_4 \geq \cdots \geq s_5 \geq s_3 \geq s_1 = a_0 - a_1.$$

Thus (s_{2k}) is a bounded decreasing sequence and (s_{2k-1}) is a bounded increasing sequence, and so $\lim_{k\to\infty} s_{2k}$ and $\lim_{k\to\infty} s_{2k-1}$ exist. According to Theorem 14, then, we have

$$\lim_{k\to\infty}(s_{2k} - s_{2k-1}) = \lim_{k\to\infty} a_{2k} = 0$$

since $a_n \searrow 0$, which implies that the alternating series converges:

$$\lim_{k\to\infty}\sum_{n=0}^{2k}(-1)^n a_n = \lim_{k\to\infty} s_{2k} = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty}\sum_{n=0}^{2k+1}(-1)^n a_n. \quad \blacksquare$$

Even though it was not part of the statement of the Alternating Series Test, the proof allows us to conclude that the value of a convergent alternating series lies between a_{2k} and a_{2m+1} for all $k, m \in \mathbb{N}$.

Example: The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Proof. Consider the sequence $(a_n) = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. As $\frac{1}{n} \to 0$ and $\frac{1}{n+1} \leq \frac{1}{n}$ for all n, then the corresponding alternating series converges.

Its value lies between $s_0 = 1$ and $s_1 = 1 - \frac{1}{2} = \frac{1}{2}$, between $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, between $s_2 = \frac{5}{6}$ and $s_3 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$, etc. ■

Two other convergence tests are often used in practice: the Ratio Test and the Root Test. We shall prove only the Ratio Test, the proof for the Root Test is similar.

Theorem 74. (RATIO TEST)
Let (a_n) be a sequence of positive real numbers.

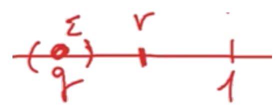
1. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

If $\frac{a_{n+1}}{a_n} \to 1$, then the series may converge or diverge, depending on the nature of the terms a_n.

Proof.

1. Assume $0 \leq \frac{a_{n+1}}{a_n} \to q < 1$. Let $r = \frac{q+1}{2}$. Thus $q < r < 1$ and there are only finitely many indices n for which $\frac{a_{n+1}}{a_n} > r$. Indeed, let $\varepsilon \in (0, \frac{1-q}{2})$.



Then, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \implies \frac{a_{n+1}}{a_n} - q < \varepsilon < \frac{1-q}{2} \implies \frac{a_{n+1}}{a_n} \leq \frac{q+1}{2} = r.$$

Then

$$n > N_\varepsilon \implies a_n = \frac{a_n}{a_{n-1}} \cdot \cdots \cdot \frac{a_{N+1}}{a_N} \cdot a_N \leq r^{n-N} a_N.$$

The tail of the original series converges, as

$$\sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} a_N r^{n-N} = \frac{a_N}{r^N} \sum_{n=N+1}^{\infty} r^n = \frac{a_N}{r^N} \left(\frac{r^{N+1}}{1-r} \right) < \infty,$$

where the last equality is left as an exercise.

But $a_0 + \cdots + a_N$ is also finite, so the full series converges.

2. Assume $\frac{a_{n+1}}{a_n} \to q > 1$. Using a similar argument as in part 1., we can show that $\exists r > 1$ and $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq r > 1$ for all $n \in \mathbb{N}$, so that $a_{n+1} > a_n$ for all $n \geq 1$.

   Thus $a_n \not\to 0$, and so $\sum_{n=0}^{\infty} a_n$ diverges, according to Theorem 70. ■

The key parts of the proof (namely, the convergence of the tail in the first case and the condition $a_n \not\to 0$ in the second) are also valid if the statement is relaxed to some extent.

Theorem 74. (Ratio Test – Reprise)
Let (a_n) be a sequence of real numbers with $a_n \neq 0$ for all n.

1. *If* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 75. (Root Test)
Let (a_n) be a sequence of positive real numbers.

1. *If* $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\liminf_{n\to\infty} \sqrt[n]{a_n} > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

If $\sqrt[n]{a_n} \to 1$, then the series may converge or diverge, depending on the nature of the terms a_n.

The proof of the Root Test follows the same general lines.

Examples: Discuss the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$; $\sum_{n=1}^{\infty} \frac{3^n}{n2^n}$; $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$.

1. The terms are all non-zero. We compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n}\right| = \frac{1}{2}\lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \frac{1}{2} < 1,$$

   so the series converges according to the Ratio Test. ■

2. The terms are all positive. We compute

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n2^n}} = \frac{3}{2}\lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{3}{2} > 1,$$

   so the series diverges according to the Root Test. ■

3. The terms are all positive. For all $p > 0$, we compute

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^p \to 1^p = 1.$$

Thus we cannot use the Ratio Test to determine if the series converges.

If $p = 1$, the harmonic series is bounded below by a divergent series

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=1/2} + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty,\end{aligned}$$

and so must itself be divergent. As $\frac{1}{n^p} > \frac{1}{n}$ for all n when $p < 1$, then the series diverges for all $0 < p \leq 1$ according to the Comparison Theorem.

If $p > 1$, the $p-$series is bounded above by a convergent series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} + \cdots \\
&\leq 1 + \underbrace{\frac{1}{2^p} + \frac{1}{2^p}}_{\text{2 times}} + \underbrace{\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}}_{\text{4 times}} + \frac{1}{8^p} + \cdots \\
&= 1 + 2^1 \cdot \frac{1}{(2^1)^p} + 2^2 \cdot \frac{1}{(2^2)^p} + \cdots = \sum_{k=0}^{\infty} 2^{k(1-p)} = \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k}.
\end{aligned}$$

But this series converges according to the Root Test.

Indeed, all the terms are positive, and, because $p > 1$,

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{(2^{p-1})^k}} = \lim_{k\to\infty} \frac{1}{2^{p-1}} < 1.$$

Thus the $p-$series diverges for $0 < 1 \leq p$ and converges for $p > 1$. ■

Theorem 76. (Absolute Convergence)

If the series $\displaystyle\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\displaystyle\sum_{n=0}^{\infty} a_n$ *(not an "iff" statement).*

Theorem 77. (Series Rearrangement)

If the series $\displaystyle\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\displaystyle\sum_{n=0}^{\infty} a_{\varphi(n)}$, $\varphi : \mathbb{N} \to \mathbb{N}$ *a bijection.*

7.2 – Series of Functions

Series of functions play the same role for sequences of functions that series played for sequences of numbers.

Let $I \subseteq \mathbb{R}$ and $f_n : I \to \mathbb{R}$, $\forall n \in \mathbb{N}$. If the sequence of partial sums

$$s_1(x) = f_1(x)$$
$$s_2(x) = f_1(x) + f_2(x)$$
$$\cdots$$

converges to some function $f : I \to \mathbb{R}$ for all $x \in I$, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges pointwise to f on I.**

Example: Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Does the sequence of partial sums $s_k(x)$ converge to some pointwise limit over some $A \subseteq \mathbb{R}$?

Solution. Formally, we have

$$(1 - x^{k+1}) = (1 - x)(1 + x + x^2 + \cdots + x^k) = (1 - x)s_k(x).$$

Thus

$$x \neq -1 \implies s_k(x) = \sum_{n=0}^{k} x^n = \frac{1 - x^{k+1}}{1 - x}.$$

Thus

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} s_k(x) = \frac{1}{1 - x}$$

when $x \in (-1, 1)$. ■

If the sequence of partial sums (s_n) converges uniformly to f on I, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges uniformly to f on I.**

If the convergence of the series of functions is uniform, the limit interchange theorems can be applied.

Theorem 78. (Cauchy Criterion for Series of Functions) *Let $f_n : I \to \mathbb{R}$ for all $n \in \mathbb{N}$. The series of functions with term f_n converges uniformly to some function $f : I \to \mathbb{R}$ if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (independent of $x \in I$) such that*

$$m > n > N_\varepsilon \implies \left| \sum_{i=n+1}^{m} f_i(x) \right| < \varepsilon.$$

Proof. The proof follows directly from Theorem 66 applied to the sequence of partial sums $s_m : I \to \mathbb{R}$. ■

The next result is a powerful tool to prove uniform convergence (and so to be able to use the Limit Interchange Theorems).

The simplicity of its proof belies its importance.

Theorem 79. (Weierstrass M–Test)
Let $f_n : I \to \mathbb{R}$ and $M_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $|f_n(x)| \leq M_n$ for all $x \in I$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} M_n \text{ converges} \implies \sum_{n=1}^{\infty} f_n \text{ converges uniformly on } I.$$

Proof. Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, its sequences of partial sums (s_k) is Cauchy and $\exists K_\varepsilon \in \mathbb{N}$ such that

$$m > n > K_\varepsilon \implies \sum_{i=n+1}^{m} M_i < \varepsilon.$$

But

$$m > n > K_\varepsilon \implies \left| \sum_{i=n+1}^{m} f_i(x) \right| \leq \sum_{i=n+1}^{m} |f_i(x)| \leq \sum_{i=n+1}^{m} M_i < \varepsilon;$$

since K_ε is independent of $x \in I$, $\sum_{n=1}^{\infty} f_n$ converges uniformly on I. ■

Example: Let $\varepsilon \in (0,1)$. Consider the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ defined by $g_n(x) = nx^{n-1}$ for each $n \in \mathbb{N}$. Does $\sigma_k(x) \rightrightarrows \sigma(x)$ on $I_\varepsilon = (-1+\varepsilon, 1-\varepsilon)$ for some σ? If so, find σ.

Solution. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$, and the corresponding sequence of partial sums $s_k(x)$ defined by $s_k(x) = 1 + x + \cdots + x^k$.

We have already shown that $s_k(x) \to \frac{1}{1-x}$ pointwise on $(-1+\varepsilon, 1-\varepsilon)$.

The partials sums s_k are differentiable on I_ε since

$$\sigma_k(x) = s_k'(x) = 1 + 2x + 3x^2 + \cdots + kx^{k-1}$$

are polynomials (in fact, σ_k is also continuous on I_ε).

Furthermore, note that the sequence of derivatives of partial sums $\sigma_k(x)$ converge uniformly on I_ε. To show this, note that

$$|g_n(x)| = |nx^{n-1}| \leq n|1-\varepsilon|^{n-1} = M_n \quad \forall x \in I_\varepsilon,\ \forall n \in \mathbb{N}.$$

But

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} n(1-\varepsilon)^{n-1}.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)(1-\varepsilon)^n}{n(1-\varepsilon)^{n-1}} = (1-\varepsilon)\lim_{n\to\infty} \frac{n+1}{n} = (1-\varepsilon) < 1,$$

then $\displaystyle\sum_{n=0}^{\infty} M_n$ converges according to the Ratio Test.

According to the Weierstrass $M-$Test, then, $\sigma_k(x) \rightrightarrows \sigma(x)$ on I_ε for some function $\sigma : I_\varepsilon \to \mathbb{R}$.

We can use the Limit Interchange Theorem 68 to identify σ:

$$\sigma(x) = \lim_{k\to\infty} \sigma_k(x) = \lim_{k\to\infty} \frac{d}{dx}[s_k(x)] = \frac{d}{dx}\Big[\lim_{k\to\infty} s_k(x)\Big] = \frac{d}{dx}\Big[\frac{1}{1-x}\Big],$$

which is to say $\sigma(x) = \frac{1}{(1-x)^2}$. ■

Incidentally, Theorem 68 also tells us that $s_k(x) \rightrightarrows \frac{1}{1-x}$ on I_ε, for all $0 < \varepsilon < 1$, and that for all $k \in \mathbb{N}$ and $x \in I_\varepsilon$, $\varepsilon \in (0,1)$, we have

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k}[x^n] = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} x^n = \frac{d^k}{dx^k}\Big(\frac{1}{1-x}\Big)$$

7.3 – Power Series

A **power series** around its **center** $x = x_0$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We have already seen an example of such a series, which converged uniformly on intervals containing $x_0 = 0$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{on } I_\varepsilon = (-1+\varepsilon, 1-\varepsilon),\ \forall \varepsilon \in (0,1)$$

(note, however, that the convergence is only pointwise on $(-1, 1)$).

Furthermore, the function $f : A \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$ is defined for all $x \neq 1$, yet the power series $1+x+x^2+\cdots$ does not converge to f outside of $(-1, 1)$.

Power series are commonly used as a formal guessing procedure to solve differential equations, but this is not a topic we will tackle at the moment.

A natural question to ask is: for which functions $f : A \to \mathbb{R}$ (and which A) can we find a sequence of coefficients (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n, \quad \forall x \in A?$$

Questions of this ilk are more naturally answered in $\mathbb{C}$; a more complete treatment will be provided in a complex analysis course.

Examples: Where do the following power series converge?

$$1.\ \sum_{n=0}^{\infty} x^n, \quad 2.\ \sum_{n=1}^{\infty} (nx)^n, \quad 3.\ \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n.$$

1. We have seen that the series converges only on $(-1, 1)$. ■

2. The power series obviously converges when $x = 0$. To show that it fails to converge on $\mathbb{R} \setminus \{0\}$, note that if $|x| > 0$, then by the Archimedean Property, $\exists N \in \mathbb{N}$ such that $N > \frac{2}{|x|}$. Thus,

$$n > N \implies |(nx)^n| = n^n |x|^n > 2^n$$

and the sequence $(nx)^n$ is unbounded, which means that the terms do not go to 0, and so the series diverges. ■

3. Let $x \in \mathbb{R}$. By the Archimedean Property, $\exists N \in \mathbb{N}$ s.t. $N > 2|x|$. Thus,

$$n > N \implies \left|\left(\frac{x}{n}\right)^n\right| = \frac{|x|^n}{n^n} < \frac{1}{2^n}.$$

According to the Weierstrass $M-$Test and Theorem 76, the series thus converges uniformly on $\mathbb{R}$. ■

The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

If the limit exists, we can replace $\limsup$ by $\lim$. Intuitively, this says that for all large enough n,

$$-R^{-n} \leq -|a_n| \leq a_n \leq |a_n| \leq R^{-n},$$

so that

$$-\sum_{n>N} \left(\frac{x-x_0}{R}\right)^n \leq \sum_{n>N} a_n(x-x_0)^n \leq \sum_{n>N} \left(\frac{x-x_0}{R}\right)^n.$$

The bounds are geometric series, and they converge when $|x-x_0| < R$.

We would expect the original power series to converge on the **interval of convergence** $|x-x_0| < R$.

Theorem 80. *Let R be the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Then, if

- $R=0$, *the power series converges for $x=x_0$ and diverges for $x \neq x_0$;*
- $R=\infty$, *the power series converges absolutely on $\mathbb{R}$, and*
- $0<R<\infty$, *the power series converges absolutely on $|x-x_0|<R$, diverges on $|x-x_0|>R$; the extremities must be analyzed separately.*

Proof. Follows immediately from the Root Test. ■

Theorem 81. *The power series of Theorem 80 converges uniformly on any compact sub-interval*

$$[a,b] \subseteq (x_0 - R, x_0 + R).$$

Proof. Let $\ell = \max\{|a-x_0|, |b-x_0|\} < R$. For every $n \in \mathbb{N}$, set $M_n = \ell^n |a_n| \geq 0$ and $\varepsilon = \frac{1}{4}(R-\ell)$.

Since $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$, $\exists N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies |a_n| \leq (\frac{1}{R-\varepsilon})^n$. Thus, for all $n > N_\varepsilon$, we have

$$0 \leq M_n = \ell^n |a_n| = (R-4\varepsilon)^n |a_n| \leq \left(\frac{R-4\varepsilon}{R-\varepsilon}\right)^n = \Big(1 - \frac{3\varepsilon}{\underbrace{R-\varepsilon}_{>0}}\Big)^n,$$

so that

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n>N_\varepsilon} M_n \leq \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n>N_\varepsilon} \left(1 - \frac{3\varepsilon}{R-\varepsilon}\right)^n$$

$$\leq \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n=0}^{\infty} \left(1 - \frac{3\varepsilon}{R-\varepsilon}\right)^n = \underbrace{\sum_{n=0}^{N_\varepsilon} M_n}_{\text{finite}} + \frac{R-\varepsilon}{3\varepsilon} < \infty.$$

But for all $x \in [a,b]$, we have

$$|a_n(x-x_0)^n| \leq |a_n|\ell^n = M_n, \quad \text{for all } n \in \mathbb{N}.$$

According to the Weierstrass $M-$Test, the power series converges uniformly on $[a,b]$. ∎

In what follows, we let $f:(x_0-R,x_0+R)\to\mathbb{R}$ be the function defined by

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n, \quad \text{and} \quad s_N(x)=\sum_{n=0}^{N}a_n(x-x_0)^n.$$

Theorem 82. *The function f is continuous on any closed bounded interval $[a,b]\subseteq(x_0-R,x_0+R)$.*

Proof. The functions $a_n(x-x_0)^n$ are continuous on $[a,b]$ for all n, and

$$s_N(x)=\sum_{n=0}^{N}a_n(x-x_0)^n \rightrightarrows f(x) \text{ on } [a,b] \quad \text{when } N\to\infty.$$

According to Theorem 67, f is continuous on $[a,b]$. ■

Theorem 83. *Let $x \in (x_0 - R, x_0 + R)$. Then f is Riemann-integrable between x_0 and x and*

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

Proof. Without loss of generality, assume $x > x_0$. As in the proof of Theorem 82, $s_N(x) \rightrightarrows f(x)$ on $[x_0, x]$ when $N \to \infty$. Thus, according to Limit Interchange Theorem 69, we have

$$\begin{aligned}\int_{x_0}^{x} f(t)\,dt &= \lim_{N\to\infty} \int_{x_0}^{x} s_N(t)\,dt = \lim_{N\to\infty} \int_{x_0}^{x} \sum_{n=0}^{N} a_n(t - x_0)^n\,dt \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_{x_0}^{x} a_n(t - x_0)^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}. \quad \blacksquare\end{aligned}$$

Theorem 84. *The function f is differentiable on $(x_0 - R, x_0 + R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Proof. As $n^{1/n} \to 1$,

$$\limsup_{n\to\infty} (n|a_n|)^{1/n} = \limsup_{n\to\infty} n^{1/n} \cdot \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R},$$

so the radius of convergence of both power series is identical, and so, in particular, $s_N'(x)$ converges uniformly on any closed bounded interval $[a, b] \subseteq (x_0 - R, x_0 + R)$.

Thus, according to Limit Interchange Theorem 68, we have

$$\frac{d}{dx}[f(x)] = \lim_{N\to\infty}\frac{d}{dx}[s_N(x)] = \lim_{N\to\infty}\frac{d}{dx}\sum_{n=0}^{N}[a_n(x-x_0)^n]$$
$$= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{d}{dx}[a_n(x-x_0)^n] = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}. \quad \blacksquare$$

How do we compute the power series coefficients a_n? Combining Theorems 82 and 84, we see that f is **smooth** in its interval of convergence (i.e. all of its derivatives are continuous).

Theorem 85. *If $R > 0$, then*

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Proof. If $x = x_0$, then $f(x_0) = a_0$, which corresponds to the case $n = 0$.

When $n = k > 0$, then repeated application of Theorem 84 yields

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k} \quad \text{on } (x_0 - R, x_0 - R).$$

If we evaluate at $x = x_0$, we get $f^{(k)}(x_0) = k! a_k$, thus $a_k = \frac{f^{(k)}(x_0)}{k!}$. ■

Corollary. If $\exists r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

and $f(x) = g(x)$ for all $x \in (x_0 - r, x_0 + r)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

Attempts to strengthen this uniqueness result must fail.

Example: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f does not have a power series expansion.

Proof. For all $n \in \mathbb{N}$, it can be shown that

$$f^{(n)}(x) = \begin{cases} \frac{d^n}{dx^n}\left[\exp(-1/x^2)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous and that $f^{(n)}(0) = 0$.

According to the Corollary to Theorem 85, if f is equal to its power series on some interval $(-r,r)$, then all of the coefficients a_n would be 0, and so $f \equiv 0$, but $f \not\equiv 0$, so f cannot be equal to its power series expansion. ■

Thus, we cannot always assume that a function is equal to its power series.

There are other ways to expand a function as infinite series, most notable being **Laurent Series** and **Fourier Series**. These topics are covered in courses in complex analysis and partial differential equations, respectively.

7.4 – Exercises

1. Answer the following questions about series.
   (a) If $\sum_{k=1}^{\infty}(a_k+b_k)$ converges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?
   (b) If $\sum_{k=1}^{\infty}(a_k+b_k)$ diverges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?
   (c) If $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$ converges, what about $\sum_{k=1}^{\infty} a_k$?
   (d) If $\sum_{k=1}^{\infty} a_k$ converges, what about $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$?
2. Show that
$$\frac{1}{r-1}=\frac{1}{r+1}+\frac{2}{r^2+1}+\frac{4}{r^4+1}+\frac{8}{r^8+1}+\cdots$$
   for all $r>1$.

3. Using Riemann integration, find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (compare with the approach used in the notes).

4. Which of the following series converge?

   (a) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$

   (b) $\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{2^n-1+\cos^2 n^3}$

   (d) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

   (e) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+3n^3}{7n^3+2}\right)^n$

5. Give an example of a power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence $[-\sqrt{2}, \sqrt{2})$.

6. Find the values of x for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} (nx)^n$;

(b) $\displaystyle\sum_{n=1}^{\infty} x^n$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

7. If the power series $\sum a_k x^k$ has radius of convergence R, what is the radius of convergence of the series $\sum a_k x^{2k}$?

8. Obtain power series expansions for the following functions.

(a) $\dfrac{x}{1+x^2}$;

(b) $\dfrac{x}{(1+x^2)^2}$;

(c) $\dfrac{x}{1+x^3}$;

(d) $\dfrac{x^2}{1+x^3}$;

(e) $f(x) = \displaystyle\int_0^1 \frac{1-e^{-sx}}{s}\, ds$, about $x = 0$.

Solutions

1. **Proof.**

   (a) They might both diverge. Consider $a_k = -k$ and $b_k = k$. However, if one converges, then so does the other, by the arithmetic of limits/series.

   (b) At least one of them diverges because if they both converged, then the series of sums would converge as well (according to a proposition seen in class).

   (c) Nothing. Consider $a_{2k} = k$, $a_{2k+1} = -k$, for which $\sum_{k=1}^{\infty} a_k$ diverges, but $a_{2k} = \frac{1}{k^2}$, $a_{2k+1} = 0$, for which $\sum_{k=1}^{\infty} a_k$ converges.

(d) It also converges. The sequence of partial sums of the second series is

$$(a_1 + a_2, a_1 + a_2 + a_3 + a_4, , a_1 + a_2 + a_3 + a_4 + a_5 + a_6, \ldots)$$

is a subsequence of the sequence of partial sums of the first series

$$(a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots).$$

If the first series sequence of partial sums converges, so does the subsequence's series. ■

2. **Proof.** From the hint, we see that

$$\frac{1}{\ell+1}=\frac{1}{\ell-1}-\frac{2}{\ell^2-1}.$$

Thus, for all $k\in\mathbb{N}$, if $\ell=2^k$, we have

$$\begin{aligned}\frac{1}{r^{2^k}+1}&=\frac{1}{r^{2^k}-1}-\frac{2}{r^{2^{k+1}}-1}\\ \Longrightarrow\frac{2^k}{r^{2^k}+1}&=\frac{2^k}{r^{2^k}-1}-\frac{2^{k+1}}{r^{2^{k+1}}-1}.\end{aligned}$$

Therefore, we have a telescoping sum

$$\sum_{k=1}^{\infty}\frac{2^k}{r^{2^k}+1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^k}{r^{2^k}+1}=\lim_{n\to\infty}\left(\frac{1}{r-1}-\frac{2^n}{r^{2^n}-1}\right)=\frac{1}{r-1},$$

where the last equality follows from the fact that, for $r > 1$, we have

$$\lim_{m\to\infty} \frac{m}{r^m} = 0.$$

This completes the proof. ■

3. **Proof.** If $p \leq 0$, then $\frac{1}{n^p} \not\to 0$ so the series diverges. In what follows, then, let $p > 0$.

For $k \in \mathbb{N}$, consider the function $f_{k;p} : [1,k] \to \mathbb{R}$ defined by $f_{k;p}(x) = \frac{1}{x^p}$. Since $f'_{k;p}(x) = -\frac{p}{x^{p+1}} < 0$ for all $x \geq 1$, $f_{k;p}$ is strictly decreasing on $[1,k]$. Thus $f_{k;p}$ is Riemann-integrable on $[1,k]$.

Consider the partition $P_k = \{1,2,\ldots,k,k+1\}$ of $[1,k+1]$. Since $f_{k;p}$ is Riemann-integrable,

$$L(f_{k;p};P_k) \leq \int_1^{k+1} f_{k;p} \leq U(f_{k;p};P_k).$$

As $f_{k;p}$ is decreasing on the sub-interval $[\mu,\nu]$, $f_{k;p}$ reaches its maximum at μ and its minimum at ν;

Hence

$$U(f_{k;p};P_k)=\sum_{n=1}^{k}f_{k;p}(n)(n+1-n)=\sum_{n=1}^{k}\frac{1}{n^p},\quad \text{and}$$

$$L(f_{k;p};P_k)=\sum_{n=2}^{k+1}f_{k;p}(n+1)(n+1-n)=\sum_{n=2}^{k+1}\frac{1}{n^p}.$$

But

$$\sum_{n=2}^{k+1}\frac{1}{n^p}=\frac{1}{(k+1)^p}-1+\sum_{n=1}^{k}\frac{1}{n^p}.$$

Thus

$$\frac{1}{(k+1)^p}-1+\sum_{n=1}^{k}\frac{1}{n^p}\leq\int_1^{k+1}f_{k;p}\leq\sum_{n=1}^{k}\frac{1}{n^p}.$$

Write $s_{k;p}$ for the partial sum and note that

$$\int_1^{k+1} f_{k;p} = \int_1^{k+1} \frac{dx}{x^p} = \begin{cases} \ln(k+1), & \text{when } p = 1 \\ \frac{1}{1-p}(k^{1-p} - 1), & \text{when } p \neq 1 \end{cases}$$

If $p = 1$, then $\ln(k+1) \leq s_{k;1}$ for all k. Since the sequence $\{\ln(k+1)\}_k$ is unbounded, so must $\{s_{k;1}\}_k$ be unbounded, which means that the corresponding series cannot converge.

If $p > 1$, then

$$\lim_{k\to\infty} \left(\frac{1}{1-p}(k^{1-p} - 1) + 1 - \frac{1}{(k+1)^p} \right) = \frac{p}{p-1}.$$

Since $s_{k;p}$ is monotone (as every additional $\frac{1}{n^p}$ added to the partial sum is positive) and since $s_{k;p}$ is bounded above by the convergent sequence

$$\left\{\frac{1}{1-p}(k^{1-p}-1)+1-\frac{1}{(k+1)^p}\right\}_k,$$

$s_{k;p}$ is a convergent sequence.

If $p<1$, then

$$\left\{\frac{1}{1-p}(k^{1-p}-1)\right\}_k$$

is unbounded. As $s_{k;p}\geq\frac{1}{1-p}(k^{1-p}-1)$ for all k, $\{s_{k;p}\}$ is also unbounded, which means that the corresponding series cannot converge.

Thus, the series converges if and only if $p>1$. ■

4. **Proof.** We use the various tests at our disposal.

(a) Since
$$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)^2} = 1 \neq 0,$$
the series diverges .

(b) Since $-1 \leq \sin^3(n+1) \leq 1$, we have
$$0 \leq \frac{2+\sin^3(n+1)}{2^n+n^2} \leq \frac{1}{2^n+n^2} \leq \frac{1}{2^n}.$$
Thus the given series converges by comparison with the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(c) If a_n denotes the n-th term of the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{2^n - 1 + \cos^2 n^3}{2^{n+1} - 1 + \cos^2(n+1)^3} \to \frac{1}{2} < 1.$$

Thus the series converges by the ratio test.

(d) We have

$$\frac{n+1}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

Thus the series diverges by comparison with the harmonic series.

(e) We have

$$0 \leq \frac{n+1}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

(f) For $n \geq 2$, we have

$$0 \leq \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3 \cdot 4 \cdots n}{n^{n-2}} \leq \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

(g) If a_n denotes the n-th term in the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} = \frac{n+1}{5} \to \infty.$$

Thus the series diverges by the ratio test.

(h) We have

$$\left(\frac{n^n}{3^{1+2n}}\right)^{1/n} = \frac{n}{3^{2+1/n}} \to \infty.$$

Thus the series diverges by the root test.

(i) We have

$$\left(\left(\frac{5n+3n^3}{7n^3+2}\right)^n\right)^{1/n} = \frac{5n+3n^3}{7n^3+2} \to \frac{3}{7} < 1.$$

Thus the series converges by the root test. ■

5. **Proof.** Consider the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

We have

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt[k]{k}} = |x|.$$

Therefore, by the root test, the series converges when $|x| < 1$ and diverges for $|x| > 1$.

For $x = 1$, the series is the harmonic series, which diverges. For $x = -1$, it is the alternating harmonic series, which converges.

Thus, the series converges precisely on the interval $[-1, 1)$.

Now, replace x by $x/\sqrt{2}$. The corresponding power series is thus

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{2}^k k} x^k.$$

We have

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{\sqrt{2}^k k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt{2}\sqrt[k]{k}} = \frac{|x|}{\sqrt{2}}.$$

The series converges on $\frac{|x|}{\sqrt{2}} < 1$ and diverges on $\frac{|x|}{\sqrt{2}} > 1$. For $x = \sqrt{2}$, the series is the harmonic series, which diverges. For $x = -\sqrt{2}$, it is the alternating harmonic series, which converges.

Thus, the series converges precisely on the interval $[-\sqrt{2}, \sqrt{2})$. ■

6. **Proof.**

(a) The series diverges whenever $x \neq 0$ since the terms $(nx)^n$ do not tend to zero when $n \to \infty$. (For large enough n, we have $n|x| \geq 1$.) Thus, this power series converges *only* at its center.

(b) The geometric series converges precisely on the interval $(-1, 1)$, and the series takes on the value $\frac{1}{1-x}$ there.

(c) For $|x| \leq 1$, we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2},$$

and thus the series converges for these values of x. If $|x| > 1$, the terms $|x^n/n^2| \to \infty$, and so the series diverges. Hence the series converges precisely on the interval $[-1, 1]$.

(d) Let $x \in \mathbb{R}$. Using the ratio test we have

$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Thus the series converges for all $x \in \mathbb{R}$ (and takes on the value e^x). $\blacksquare$

7. **Proof.** The new series can be written as $\sum_{k=0}^{\infty} b_k x^k$, where $b_k = a_{k/2}$ if k is even and $b_k = 0$ if k is odd. Thus

$$\limsup_{k\to\infty} \sqrt[k]{|b_k|} = \lim_{k\to\infty} \sqrt[k]{|a_{k/2}|} = \lim_{k\to\infty} \sqrt[2k]{|a_k|} = \lim_{k\to\infty} \left(\sqrt[k]{|a_k|}\right)^{1/2}$$

$$= \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right)^{1/2} = R^{1/2}.$$

Therefore, the radius of convergence of the new series is $\sqrt{R}$. ■

8. **Proof.**

(a) Since
$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k,$$
we have
$$\frac{x}{1+x^2}=x\sum_{k=0}^{\infty}(-x^2)^k=\sum_{k=0}^{\infty}(-1)^k x^{2k+1}.$$

(b) We know that, for $x\in(-1,1)$, $\displaystyle\frac{1}{1-x}=\sum_{k=1}^{\infty}x^k$.

For any $-1<a<b<1$, the series $\displaystyle\sum_{k=1}^{\infty}kx^{k-1}$ converges uniformly on $[a,b]$.

Indeed, let $c = \max\{|a|, |b|\} < 1$. Then, for all $x \in [a, b]$, we have

$$|kx^{k-1}| \leq kc^{k-1}.$$

Now,

$$\frac{(k+1)c^k}{kc^{k-1}} = \frac{k+1}{k}c \to c \quad \text{as } k \to \infty.$$

Since $c < 1$, the ratio test tells us that $\sum_{k=1}^{\infty} kc^{k-1}$ converges.

Thus, $\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly by the Weierstrass M-test.

Consequently, we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

and so for any $x \in [a,b] \subseteq (-1,1)$:

$$\frac{x}{(1+x^2)^2} = x\sum_{k=1}^{\infty} k(-x^2)^{k-1} = \sum_{k=1}^{\infty}(-1)^{k-1}kx^{2k-1}.$$

(c) Using the geometric series, we have

$$\frac{x}{1+x^3} = x\sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k+1}.$$

(d) Using the geometric series, we have

$$\frac{x^2}{1+x^3} = x^2 \sum_{k=0}^{\infty} (-x^3)^k = \sum_{k=0}^{\infty} (-1)^k x^{3k+2}.$$

(e) Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we have

$$\frac{1-e^{-sx}}{s} = -\frac{1}{s} \sum_{k=1}^{\infty} \frac{(-sx)^k}{k!} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{s^{k-1} x^k}{k!}.$$

This series converges absolutely for all s and all x (use the ratio test or compare it to the series for e^x). Therefore, viewing it as a power series in s (for some fixed x), its interval of convergence is ∞, and its centre is 0. Thus the series can be integrated term by term:

$$\begin{aligned}
\int_0^1 \frac{1-e^{-sx}}{s}\,ds &= \int_0^1 \sum_{k=1}^{\infty}(-1)^{k+1}\frac{s^{k-1}x^k}{k!}\,ds \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\int_0^1 s^{k-1}\,ds\right)\frac{x^k}{k!} \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{s^k}{k}\right]_{s=0}^{s=1}\frac{x^k}{(k!)} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k(k!)}.
\end{aligned}$$

This completes the exercises for the course. ■