Mathematical Analysis

Chapter 6
Sequences of Functions

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Overview

We now look at sequences of functions, which arise naturally in analysis and its applications.

In particular, we will

- discuss two types of convergence (pointwise and uniform), and
- prove some limit interchange theorems.

Outline

6.1 – Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ and $(f_n)_n$ be a **sequence of functions** $f_n : A \to \mathbb{R}$.

The sequence $(f_n(x))_n$ may converge for some $x \in A$ and diverge for others.

Let $A_0 = \{x \in A \mid (f_n(x))_n \text{ converges}\} \subseteq A$. For each $x \in A_0$, $(f_n(x))$ converges to a unique limit

$$f(x) = \lim_{n\to\infty} f(x),$$

the **pointwise limit** of (f_n), which we denote by $f_n \to f$ on A_0.

Examples:

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof. Let $\varepsilon > 0$ and $x \in \mathbb{R}$. According to the Archimedean Property, $\exists N_{\varepsilon,x} > \frac{|x|}{\varepsilon}$ so that

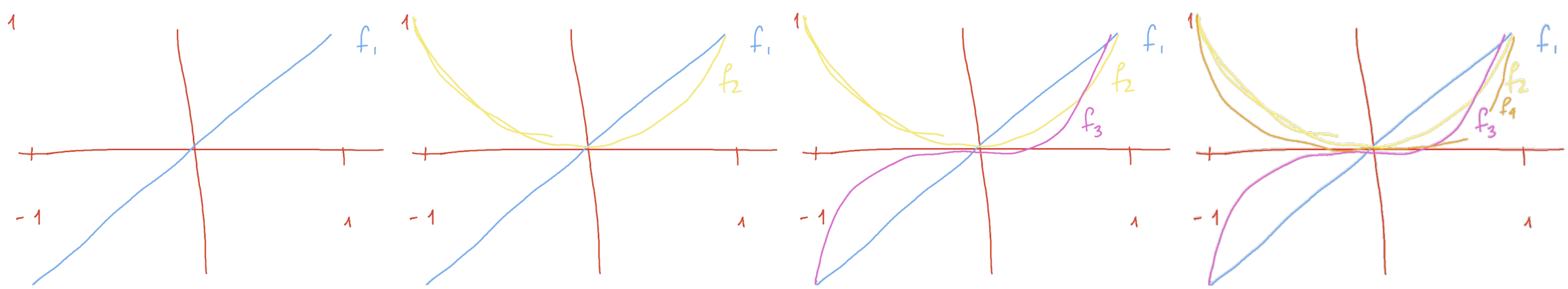
$$n > N_{\varepsilon,x} \implies \left|\frac{x}{n} - 0\right| < \frac{|x|}{n} < \frac{|x|}{N_{\varepsilon,x}} < \varepsilon,$$

   thus $f_n \to 0$ on $\mathbb{R}$. ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \to f$ on $(-1, 1]$.

Proof. Using various results seen in Chapters 3 and 4 and in the Exercises, we know that

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & x \in (-1,1) \\ 1 & x = 1 \\ \text{NA} & \text{otherwise} \end{cases}$$



Thus $f_n \to f$ on $(-1,1]$. Note that all f_n are continuous on $(1,1]$, but that f is not. ■

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^2+nx}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the identity function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof. As $f_n(x) = \frac{x^2}{n} + x \to f(x) = x$, $\forall x \in \mathbb{R}$, $f_n \to f$ on $\mathbb{R}$. ■

A sequence of functions $(f_n : A \to \mathbb{R})$ **converges uniformly on** $A_0 \subseteq A$ **to** $f : A_0 \to \mathbb{R}$, denoted by $f_n \rightrightarrows f$ on A_0, if the threshold $N_{\varepsilon,x} \in \mathbb{N}$ in the pointwise definition is in fact **independent** of $x \in A_0$:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \text{ and } x \in A_0 \implies |f_n(x) - f(x)| < \varepsilon.$$

The distinction between pointwise and uniform convergence is not unlike that between continuity and uniform continuity: convergence is uniform if the threshold is the same for all $x \in A_0$.

Clearly, if $f_n \rightrightarrows f$ on A_0, then $f_n \to f$ on A_0, but the converse is not necessarily true.

Examples:

1. Show that the sequence $f_n : [1,2] \to \mathbb{R}$ defined by $f_n(x) = \frac{\sin x}{nx}$ for $n \in \mathbb{N}$ converges uniformly to the zero function on $[1,2]$.

   Proof. Let $\varepsilon > 0$. According to the Archimedean Property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ so that

   $$n > N_\varepsilon \text{ and } x \in [1,2] \implies \left|\frac{\sin x}{nx} - 0\right| = \left|\frac{\sin x}{nx}\right| \leq \frac{1}{nx} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

   thus $f_n \rightrightarrows 0$ on $[1,2]$. ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \not\rightrightarrows f$ on $(-1, 1]$.

   Proof. We use the negation of the definition. Let $\varepsilon_0 = \frac{1}{4}$, and set $x_k = \frac{1}{2^{1/k}}$ and $(n_k) = (k)$. Then

$$|f_{n_k}(x_k) - f(x_k)| = \left|\tfrac{1}{2} - 0\right| = \frac{1}{2} \geq \varepsilon_0,$$

   which completes the proof. ■

A sequence of functions f_n does not converge uniformly to f on A_0 if

$\exists \varepsilon_0 > 0$ with $(f_{n_k}) \subseteq (f_n)$ and $(x_k) \subseteq A_0$ s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$, $\forall k \in \mathbb{N}$.

The definition of uniform convergence is only ever useful if a candidate for a uniform limit is available, a situation that we have encountered before.

Theorem 66. (Cauchy's Criterion for Sequences of Functions) *Let $f_n : A \to \mathbb{R}$, for all $n \in \mathbb{N}$. Then, $f_n \rightrightarrows f$ on $A_0 \subseteq A$ if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (indep. of $x \in A_0$) such that $|f_m(x) - f_n(x)| < \varepsilon$ whenever $m \geq n > N_\varepsilon \in \mathbb{N}$ and $x \in A_0$.*

Proof. Let $\varepsilon > 0$. If $f_n \rightrightarrows f$ on A_0, $\exists N_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ when $x \in A_0$ and $n > N_\varepsilon$. Hence,

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x)| \\ &\leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

whenever $x \in A_0$ and $m \geq n > N_\varepsilon$.

Conversely, let $\varepsilon > 0$ and assume that $\exists N_{\varepsilon/2} \in \mathbb{N}$ (independent of $x \in A_0$) such that

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies -\frac{\varepsilon}{2} < f_m(x) - f_n(x) < \frac{\varepsilon}{2}.$$

Since $x \in A_0$, we know that $f_m(x) \to f$ on A_0 when $m \to \infty$. Thus,

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies \lim_{m\to\infty} -\frac{\varepsilon}{2} \leq \lim_{m\to\infty} (f_m(x) - f_n(x)) \leq \lim_{m\to\infty} \frac{\varepsilon}{2},$$

or

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies -\varepsilon < -\frac{\varepsilon}{2} \leq f(x) - f_n(x) \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so $f_n \rightrightarrows f$ on A_0. ■

Example: Let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} nx, & x \in [0, 1/n] \\ 2 - nx, & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

for all $n \in \mathbb{N}$. Let $f : [0,1] \to \mathbb{R}$ be the zero function on $[0,1]$. Show that $f_n \to f$ on $[0,1]$ but $f_n \not\rightrightarrows f$ on $[0,1]$.

Proof. If $x = 0$, $f_n(0) = 0$ for all n so $(f_n(0))$ converges to 0.

If $x \in (0,1]$, $\exists N_x > 2/x$ by the Archimedean Property. Thus, for $n > N_x$, $f_n(x) = 0$ since $x > \frac{2}{N} > \frac{2}{n}$, so $f_n(x) \to 0$ on $(0,1]$

Combining these results, $f_n \to f$ on $[0,1]$.

Now, let $\varepsilon_0 = \frac{1}{2}$. Note that since $|f_n(\frac{1}{n}) - f(\frac{1}{n})| = 1$ for all $n \in \mathbb{N}$, we can never obtain

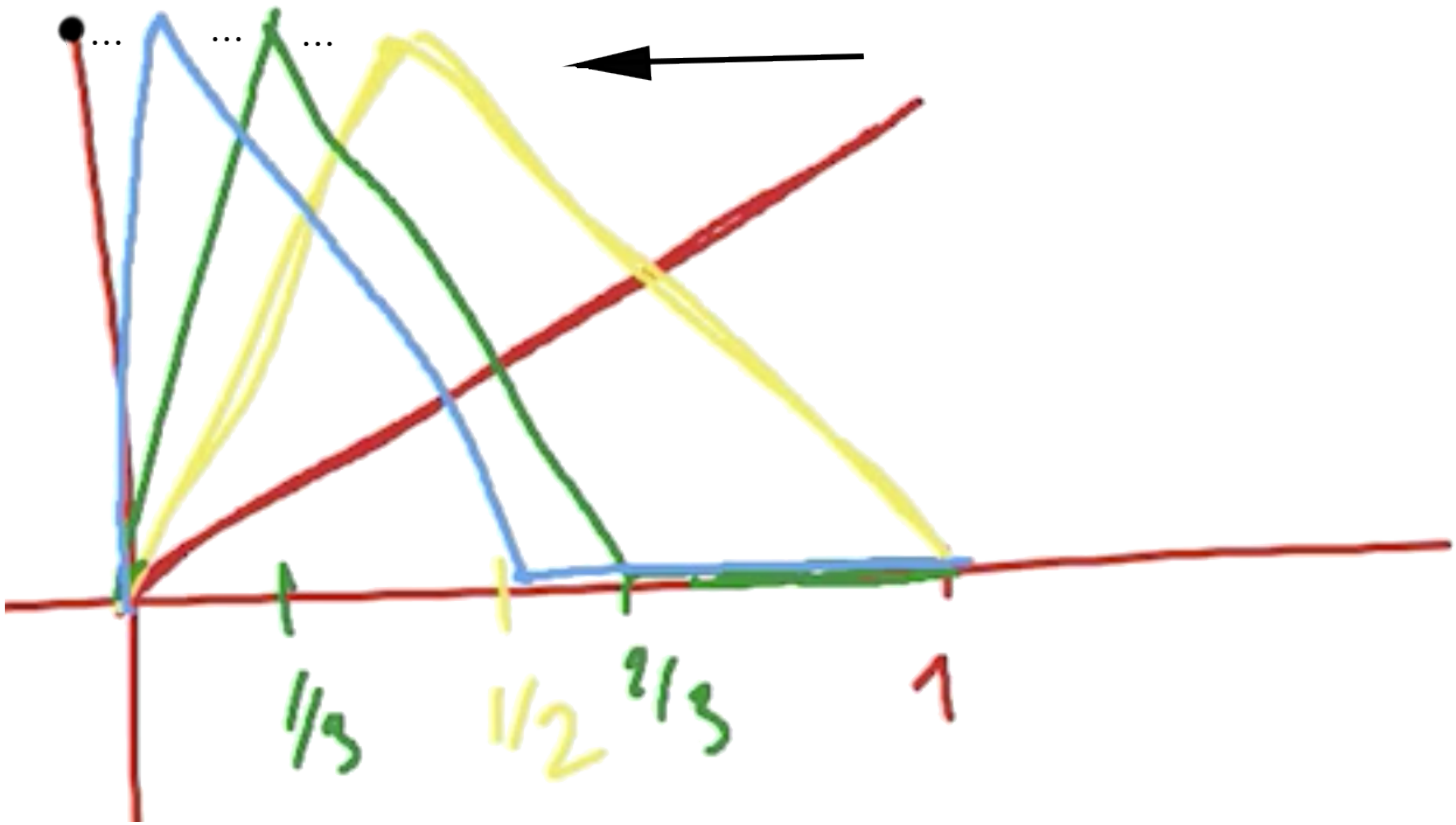
$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in [0,1]$, and so $f_n \not\rightrightarrows f$ on $[0,1]$. ■

The fact that we have to separate the proof for pointwise convergence into distinct argument depending on the value of x is a strong indication that the convergence cannot be uniform (although it could be that it was possible to do a one-pass proof and that the insight escaped us...)

Intuitively, we can think of the convergence process in the last example as being a flattening process: what happens to the tents' peak as $n \to \infty$?

The fact that we have to "break" the tents in order to get to the pointwise limit is another indication that the convergence cannot be uniform.


1/3
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6.2 – Limit Interchange Theorems

It is often necessary to know if the limit f of a sequence of functions (f_n) is continuous, differentiable, or Riemann-integrable. It is not always the case, even when the f_n are continuous, differentiable, or Riemann-integrable.

Examples:

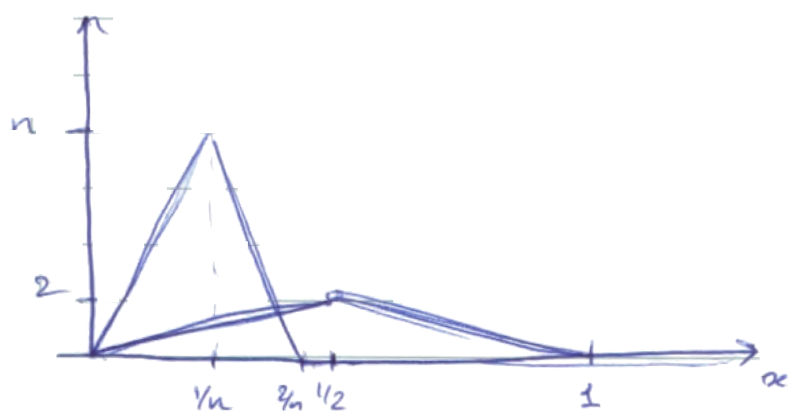
1. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ for $n \in \mathbb{N}$ and $f : [0,1] \to \mathbb{R}$ be the zero function except at $x = 1$ where $f(1) = 1$. Then f_n is continuous on $[0,1]$ for all $n \in \mathbb{N}$, but f is not.

2. The same functions f_n are differentiable on $[0,1]$ for all $n \in \mathbb{N}$, but f is not (as it is not continuous at $x = 1$).

3. Consider the functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x, & x \in [0, 1/n] \\ -n^2(x - 2/n), & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

for $n \geq 2$.



Since f_n is continuous on $[0,1]$ for all $n \geq 2$, f_n is Riemann-integrable on $[0,1]$ for all $n \geq 2$, with

$$\int_0^1 f_n = \frac{1}{2} \cdot \frac{2}{n} \cdot n = 1, \quad \text{for all } n \geq 2.$$

If $x = 0$, $f_n(0) = 0$ for all n so $(f_n(0))$ converges to 0.

If $x \in (0,1]$, $\exists N_x > 2/x$ by the Archimedean Property. Thus, for $n > N_x$, $f_n(x) = 0$ since $x > \frac{2}{N} > \frac{2}{n}$, so $f_n(x) \to 0$ on $(0,1]$

So $f_n \to f$ on $[0,1]$, but $\displaystyle\int_0^1 f = 0 \neq 1 = \lim_{n\to\infty} \int_0^1 f$. ■

Note that none of the "convergences" in the previous example are uniform on $[0,1]$. When the convergence $f_n \rightrightarrows f$ on A is uniform, then if the f_n are

- continuous on A, so is f;
- differentiable on A, so is f, with

$$f' = \frac{d}{dx}\Big[\lim_{n\to\infty} f_n\Big] = \lim_{n\to\infty}\left[\frac{d}{dx} f_n\right] = \lim_{n\to\infty} f_n';$$

- Riemann-integrable on A, then so is f, with

$$\int_A f = \int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n.$$

We finish this chapter by proving three **Limit Interchange Theorems**.

Theorem 67. *Let $f_n : A \to \mathbb{R}$ be continuous on A for all $n \in \mathbb{N}$. If $f_n \rightrightarrows f$ on A, then f is continuous on A.*

Proof. Let $\varepsilon > 0$. By definition, $\exists H_{\varepsilon/3} \in \mathbb{N}$ such that

$$n > H_{\varepsilon/3} \text{ and } x \in A \implies |f_n(x) - f(x)| < \tfrac{\varepsilon}{3}.$$

Let $c \in A$. According to the Triangle Inequality,

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_{H_{\varepsilon/3}}(x)| + |f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| + |f_{H_{\varepsilon/3}}(c) - f(c)| \\ &< \tfrac{\varepsilon}{3} + |f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| + \tfrac{\varepsilon}{3} \end{aligned}$$

whenever $n > H_{\varepsilon/3}$.

But $f_{H_{\varepsilon/3}}$ is continuous at c, so $\exists\delta_{\varepsilon/3} > 0$ such that $|f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| < \frac{\varepsilon}{3}$ when $x \in A$ and $|x - c| < \delta_{\varepsilon/3}$. Thus $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ and $|x - c| < \delta_{\varepsilon/3}$, so f is continuous at c. As $c \in A$ is arbitrary, f is continuous on A. ■

Theorem 68. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions on $[a, b]$ such that $\exists x_0 \in [a, b]$ with $f_n(x_0) \to z_0$, and $f_n'' \rightrightarrows g$ on $[a, b]$. Then $f_n \rightrightarrows f$ on $[a, b]$ for some function $f : [a, b] \to \mathbb{R}$ such that $f' = g$.*

Proof. Let $\varepsilon > 0$ and $x \in [a, b]$. Since $f_n' \rightrightarrows g$ on $[a, b]$, the sequence f_n' satisfies Cauchy's Criterion, and so $\exists N_1 \in \mathbb{N}$ such that

$$m \geq n > N_1 \text{ and } y \in [a, b] \implies |f_m'(y) - f_n'(y)| < \frac{\varepsilon}{2(b - a)}.$$

As $(f_n(x_0))$ converges it is also a Cauchy sequence, so $\exists N_2 \in \mathbb{N}$ such that

$$m \geq n > N_2 \implies |f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2}.$$

According to the Mean Value Theorem, $\exists y$ between x and x_0 such that

$$(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) = (f'_m(y) - f'_n(y))(x - x_0).$$

Hence,

$$\begin{aligned} |f_m(x) - f_n(x)| &\leq |f_m(x_0) - f_n(x_0)| + |f'_m(y) - f'_n(y)| \cdot |x - x_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon \end{aligned}$$

for all $m \geq n > \max\{N_1, N_2\}$.

Both N_1 and N_2 are independent of x, so $N_\varepsilon = \max\{N_1, N_2\}$ also is, and thus $(f_n)_n$ satisfies Cauchy's Criterion, which yields $f_n \rightrightarrows f$ on $[a, b]$.

It remains only to show that $f' = g$ on $[a, b]$. Let $\varepsilon > 0$ and $c \in [a, b]$. Since (f'_n) satisfies Cauchy's Criterion (as $f'_n \rightrightarrows g$), $\exists K_1 \in \mathbb{N}$ (independent of x) such that

$$m \geq n > K_1 \text{ and } y \in [a, b] \implies |f'_m(y) - f'_n(y)| < \frac{\varepsilon}{3}.$$

But $f' \rightrightarrows g'$, so $\exists K_2 \in \mathbb{N}$ (independent of c) such that

$$n \geq K_2 \text{ and } c \in [a, b] \implies |f'_n(c) - g(c)| < \frac{\varepsilon}{3}.$$

Set $K_\varepsilon > \max\{K_1, K_2\}$.

As $f'_{K_\varepsilon}(c)$ exists, $\exists \delta_\varepsilon > 0$ such that

$$0 < |x-c| < \delta_\varepsilon \text{ and } x \in [a,b] \implies \left| \frac{f_{K_\varepsilon}(x) - f_{K_\varepsilon}(c)}{x-c} - f'_{K_\varepsilon}(c) \right| < \frac{\varepsilon}{3}.$$

According to the Mean Value Theorem, $\exists y$ between x and c such that

$$(f_m(x) - f_n(x)) - (f_m(c) - f_n(c)) = (f'_m(y) - f'_n(y))(x-c).$$

If $x \neq c$, then $m \geq n > K_\varepsilon$ and $x \in [a,b] \implies$

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f'_m(y) - f'_n(y)| < \frac{\varepsilon}{3}.$$

Letting $m \to \infty$ (i.e. $f_m \to f$ on A), we get

$$n > K_\varepsilon \text{ and } x \in [a,b] \implies \left| \frac{f(x)-f(c)}{x-c} - \frac{f_m(c)-f_n(c)}{x-c} \right| \leq \frac{\varepsilon}{3}.$$

Combining all of these inequalities, for $0 < |x-c| < \delta_\varepsilon$, $x \in [a,b]$, and $k > K_\varepsilon$, we have

$$\left| \frac{f(x)-f(c)}{x-c} - g(c) \right| = \left| \frac{f(x)-f(c)}{x-c} - \frac{f_k(x)-f_k(c)}{x-c} \right.$$
$$\left. + \frac{f_k(x)-f_k(c)}{x-c} - f_k'(c) + f_k'(c) - g(c) \right|$$

$$\leq \left| \frac{f(x)-f(c)}{x-c} - \frac{f_k(x)-f_k(c)}{x-c} \right| + \left| \frac{f_k(x)-f_k(c)}{x-c} - f_k'(c) \right|$$
$$+ |f_k'(c) - g(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which is to say that $f'(c) = g(c)$. ■

Theorem 69. *Let $f_n : [a,b] \to \mathbb{R}$ be Riemann-integrable on $[a,b]$ for all $n \in \mathbb{N}$. If $f_n \rightrightarrows f$ on $[a,b]$, then f is Riemann-integrable on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$ on $[a,b]$, $\exists K_\varepsilon \in \mathbb{N}$ (independent of x) such that

$$n \geq K_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}.$$

Since f_{K_ε} is Riemann-integrable, $\exists P_\varepsilon = \{x_0, \ldots, x_n\}$ a partition of $[a,b]$ such that

$$U(P_\varepsilon; f_{K_\varepsilon}) - L(P_\varepsilon; f_{K_\varepsilon}) < \frac{\varepsilon}{2},$$

according to the Riemann Criterion.

For all $1 \leq i \leq n$, set

$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},\ m_i(f_{K_\varepsilon}) = \inf\{f_{K_\varepsilon}(x) \mid x \in [x_{i-1}, x_i]\},$$
$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},\ M_i(f_{K_\varepsilon}) = \sup\{f_{K_\varepsilon}(x) \mid x \in [x_{i-1}, x_i]\}.$$

Then according to the reverse triangle inequality, we have

$$|f(x)| < |f_{K_\varepsilon}(x)| + \frac{\varepsilon}{4(b-a)} \implies |f(x)| < M_i(f_{K_\varepsilon}) + \frac{\varepsilon}{4(b-a)} \text{ on } [x_{i-1}, x_i]$$
$$\implies M_i(f) < M_i(f_{K_\varepsilon}) + \frac{\varepsilon}{4(b-a)} \text{ on } [x_{i-1}, x_i].$$

Similarly, $m_i(f) \geq m_i(f_{K_\varepsilon}) - \frac{\varepsilon}{4(b-a)}$ on $[x_{i-1}, x_i]$. Thus,

$$U(P_\varepsilon; f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$
$$\leq \sum_{i=1}^{n} M_i(f_{K_\varepsilon})(x_i - x_{i-1}) + \frac{\varepsilon}{4(b-a)} \sum_{i=1}^{n}(x_i - x_{i-1}) = U(P_\varepsilon; f_{K_\varepsilon}) + \frac{\varepsilon}{4}.$$

Similarly, $L(P_\varepsilon; f) \geq L(P_\varepsilon; f_{K_\varepsilon}) - \frac{\varepsilon}{4}$. Hence

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) \leq U(P_\varepsilon; f_{K_\varepsilon}) - L(P_\varepsilon; f_{K_\varepsilon}) + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, according to the Riemann Criterion, f is Riemann-integrable.

Finally, let $\varepsilon > 0$. As $f_n \rightrightarrows f$ on $[a,b]$, $\exists \hat{K}_\varepsilon$ (indep. of x) such that

$$n > \hat{K}_\varepsilon \text{ and } x \in [a,b] \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}.$$

Consequently, $\int_a^b f_n \to \int_a^b f$, since $n > \hat{K}_\varepsilon \implies$

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| \leq \int_a^b \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} < \varepsilon. \quad \blacksquare$$

6.3 – Exercises

1. Show that $\lim\limits_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for all $x \in \mathbb{R}$.
2. Show that if $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on $\mathbb{R}$ but $f_n^2 \not\rightrightarrows g$ on $\mathbb{R}$ for any function g.
3. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Denote by f the pointwise limit of f_n on $[0,1]$. Does $f_n \rightrightarrows f$ on $[0,1]$?
4. Let (f_n) be the sequence of functions defined by $f_n(x) = \frac{x^n}{n}$, for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that (f_n) converges uniformly to a differentiable function $f : [0,1] \to \mathbb{R}$, and that the sequence (f_n') converges pointwise to a function $g : [0,1] \to \mathbb{R}$, but that $g(1) \neq f'(1)$.
5. Show that $\lim\limits_{n\to\infty} \displaystyle\int_1^2 e^{-nx^2}\, dx = 0$.

6. Show that $\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\, dx = 0$.
7. Show that if $f_n \rightrightarrows f$ on $[a, b]$, and each f_n is continuous, then the sequence of functions $(F_n)_n$ defined by

$$F_n(x) = \int_a^x f_n(t)\, dt$$

also converges uniformly on $[a, b]$.

Solutions

1. **Proof.** If $x = 0$, then $\frac{nx}{1+n^2x^2} = 0 \to 0$.

   If $x \neq 0$, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon|x|}$ (depending on x) s.t.

$$\left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{n|x|}{1+n^2x^2} < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|} < \frac{1}{N_\varepsilon|x|} < \varepsilon$$

   whenever $n > N_\varepsilon$, i.e. $\frac{nx}{1+n^2x^2} \to 0$ on $\mathbb{R}$. ∎

2. **Proof.** Let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ (independent of x) s.t.

$$|f_n(x) - f(x)| = \left| x + \frac{1}{n} - x \right| = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

whenever $n > N_\varepsilon$, i.e. $f_n \rightrightarrows 0$ on $\mathbb{R}$.

Now, $(f_n(x))^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \to x^2$ for all $x \in \mathbb{R}$. Hence, $f_n^2 \to g$ on $\mathbb{R}$, where $g(x) = x^2$. If f_n^2 converges uniformly to any function, it will have to do so to g. But let $\varepsilon_0 = 2$ and $x_n = n$. Then

$$\left| (f_n(x_n))^2 - g(x_n) \right| = \left| \frac{2x_n}{n} + \frac{1}{n^2} \right| = 2 + \frac{1}{n^2} \geq 2 = \varepsilon_0$$

for all $n \in \mathbb{N}$ (this is the negation of the definition of uniform convergence). Hence f_n^2 does not converge uniformly on $\mathbb{R}$. ■

3. **Proof.** First note that $1 \leq 1 + x$ on $[0, 1]$.

In particular, $\frac{1}{1+x} \leq 1$ on $[0, 1]$. If $x \in (0, 1]$, then $\frac{1}{(1+x)^n} \to 0$, according to one of the examples done in class.

If $x = 0$, $\frac{1}{(1+x)^n} = \frac{1}{1^n} = 1 \to 1$; i.e. $f_n \to f$ on $[0, 1]$, where

$$f(x) = \begin{cases} 0, & x \in (0, 1] \\ 1, & x = 0 \end{cases}.$$

However, $f_n \not\rightrightarrows f$ by theorem 67, since f_n is continuous on $[0, 1]$ for all $n \in \mathbb{N}$, but f is not. ∎

4. **Proof.** The sequence $f_n(x) = \frac{x^n}{n} \to f(x) \equiv 0$ on $[0,1]$.

Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ s.t.

$$\left| \frac{x^n}{n} - 0 \right| \leq \frac{|x|^n}{n} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

whenever $n > N_\varepsilon$. Note that f is differentiable and $f'(x) = 0$ for all $x \in [0,1]$. Furthermore, $f_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1} \to g(x)$ on $[0,1]$, where

$$g(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases},$$

by one of the examples I did in class. Then $g(1) = 1 \neq 0 = f'(1)$. ■

5. **Proof.** As $\left(e^{-nx^2}\right)' = -2nxe^{-nx^2} < 0$ on $[1,2]$ for all $n \in \mathbb{N}$, e^{-nx^2} is decreasing on $[1,2]$ for all n, that is

$$e^{-nx^2} \leq e^{-n(1)^2} = e^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Now,

$$f_n(x) = e^{-nx^2} \rightrightarrows f(x) \equiv 0 \quad \text{on } [1,2].$$

Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \ln \frac{1}{\varepsilon}$ (independent of x) s.t.

$$\left|e^{-nx^2} - 0\right| = e^{-nx^2} < e^{-Nx^2} \leq e^{-N} < \varepsilon$$

whenever $n > N_\varepsilon$. Then (and only because of this uniform convergence),

$$\lim_{n\to\infty}\int_1^2 e^{-nx^2}\,dx = \int_1^2 \lim_{n\to\infty} e^{-nx^2}\,dx = \int_1^2 0\,dx = 0,$$

by the Limit Interchange Theorem for Integrals. ∎

6. **Proof.** For $n \in \mathbb{N}$, define $f_n : [\pi/2, \pi] \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(nx)}{nx}.$$

Then each f_n is continuous. For all $n \in \mathbb{N}$, we have

$$\sup_{x\in[\pi/2,\pi]} \left\{ \left| \frac{\sin(nx)}{nx} \right| \right\} \leq \frac{2}{n\pi}.$$

Since $2/n\pi \to 0$ as $n \to \infty$, we have $f_n \rightrightarrows 0$ (why?). Then the limit interchange theorem for integrals applies, and we have

$$\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\, dx = \int_{\pi/2}^{\pi} 0\, dx = 0.$$

This completes the proof. ■

7. **Proof.** Define $F(x) = \int_a^x f(t)\,dt$. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, $\exists N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, x \in [a,b].$$

Then, for all $n \geq N$ and $x \in [a,b]$, we have

$$|F_n(x) - F(x)| = \left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| \leq \int_a^x |f_n(t) - f(t)|\,dt$$
$$\leq (x-a) \cdot \frac{\varepsilon}{b-a} \leq \varepsilon.$$

Thus $F_n \rightrightarrows F$ on $[a,b]$. ■