Lab 3 1D Finite Differences

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1 Lab 3: 1D Finite Differences

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In this lab we explore boundary value problems (BVP). We start by implementing an implementation of the Matlab backslash operator called the Thomas algorithm. We then implement a finite difference solver to solve a BVP using both the backslash operator and the Thomas algorithm. The results are compared to each other and a convergence study is preformed on the approximated solution.

1.1 Thomas Algorithm

1.1.1 Code Deliverable

Write a function **thomas_solver** that takes as inputs vectors **a, b, c,** and **d**, and returns the solution **x**.

```
In [1]: #import external modules
        import numpy as np

        def thomas_solver(a, b, c, d):

            """
            The Thomas Algorithm is a simplified form of Gaussian elimination that
            can be used to solve tridiagonal systems of equations.

            Input:
                a, b, c, d - numpy vectors that make up tridiagonal matrix and solution d
            Output:
                x - solution numpy vector
            """

            #initialize number of equations
            #this will be the length of vector d
            n = len(d)

            #make a copy of each array
            #can use the map function to map an array function to each variable
            aa, bb, cc, dd = (x.astype(float) for x in (a, b, c, d))
```

```
    #loop for n iterations
    for i in range(1, n):

        #apply the Thomas Algorithm
        m = aa[i - 1] / bb[i - 1]
        bb[i] = bb[i] - m * cc[i - 1]
        dd[i] = dd[i] - m * dd[i - 1]

    #assign solution x to main diagonal b
    x = bb
    x[-1] = dd[-1] / bb[-1]

    for i in range(n - 2, -1, -1):
        x[i] = (dd[i] - cc[i] * x[i + 1]) / bb[i]

    #return solution
    return x
```

1.1.2 Thomas Algorithm Test

Test the **thomas_solver** on the 10x10 linear system where the main diagonal is **3**, the off diagonals are **-1**, and the solution is a 1x10 column vector **2, 1, ... ,1, 2.**

```
In [2]: #initialize arrays a b c and d
        a = np.full([9], -1)
        b = np.full([10], 3)
        c = np.full([9], -1)
        d = np.array([2,1,1,1,1,1,1,1,1,2])

        #call thomas_solver function with arrays
        result = thomas_solver(a, b, c, d)
        print(result)
```

```
[1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
```

```
In [3]: #compare against thomas solver
        A = np.array([[3, -1, 0, 0, 0, 0, 0, 0, 0, 0],
                      [-1, 3, -1, 0, 0, 0, 0, 0, 0, 0],
                      [0, -1, 3, -1, 0, 0, 0, 0, 0, 0],
                      [0, 0, -1, 3, -1, 0, 0, 0, 0, 0],
                      [0, 0, 0, -1, 3, -1, 0, 0, 0, 0],
                      [0, 0, 0, 0, -1, 3, -1, 0, 0, 0],
                      [0, 0, 0, 0, 0, -1, 3, -1, 0, 0],
                      [0, 0, 0, 0, 0, 0, -1, 3, -1, 0],
                      [0, 0, 0, 0, 0, 0, 0, -1, 3, -1],
                      [0, 0, 0, 0, 0, 0, 0, 0, -1, 3]])
```

```
d = np.array([2,1,1,1,1,1,1,1,1,2])
print(np.linalg.solve(A,d))
```

```
[1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
```

1.2 Finite Differences

1.2.1 Code Deliverable

Write a finite difference code using python's scipy built-in **spdiags** to generate $A_{i,j}$ in sparse form. Use **BOTH** python's numpy backslash operator and the **thomas_solver** to solve for **U**.

```
In [4]: #import external modules
        from scipy import sparse

        def finite_difference_sparse(f, a, b, ua, ub, N):

            """
            This function provides the approximate solution given the derivative of a function
            the bounded x values (a, b), the initial boundary conditions (ua, ub) and the numb

            Input:
                f - lambda function that is the derivative you are trying to approximate
                a - starting x value
                b - ending x value
                ua - starting boundary value condition
                ub - ending boundary value condition
                N - number of iterations to approx
            Output:
                U_backslash - approximations using python backslash operator
                U_thomas - approximations using thomas_solver function
                x - interior discretization points
            """

            #split the domian [a,b] into N+1 equally spaced nodes
            #x = a:b:h with h  = (b-a)/N+1
            h = (b - a) / (N + 1)

            #compute grid points
            x = np.arange(a, b, h)

            #compute right hand side and set boundary conditions
            f = f(x)*(-h**2)
            f[0], f[-1] = f[0] - ua, f[-1] - ub

            #define vectors for main diagonal and off diagonals
            main_diag = -2*np.ones((N+1, 1)).ravel()
            off_diag = 1*np.ones((N, 1)).ravel()
```

```
    #create sparse matrix
    diagonals = [main_diag, off_diag, off_diag]
    A = sparse.diags(diagonals, [0,-1,1], shape=(N+1,N+1)).toarray()

    #solve matrix system using backslash and thomas algorithm
    U_backslash = np.linalg.solve(A, f)
    U_thomas = thomas_solver(off_diag, main_diag, off_diag, f)

    #rreturn approximations and interior points
    return U_backslash, U_thomas, x
```

In [5]:
```
import matplotlib.pyplot as plt
%matplotlib inline

def generateApproxPlots(xf, x, UB, UT, exactF):

    """
    This function plots the approximate solutions
    versus the exact solution.

    Input:
        xf - vector of interior points for exact solution
        x - vector of interior points for approximate solutions
        UB - approximation for the backslash opperator
        UT - approximation for the thomas algorithm function
        exactF - lambda function that provides the exact solution
    Output:
        plot showing the approximate vs exact solution
    """

    fig, ax = plt.subplots(1, 2, figsize=(15,5))
    ax[0].plot(xf, exactF(xf), label='Exact')
    ax[0].plot(x, UB, 'o', label='Backslash')
    ax[0].plot(x, UT, 'x', label='Thomas')
    ax[0].set_title('Approximations vs Exact', fontsize=18)
    ax[0].legend()
    ax[1].plot(x, UB, 'o', label='Backslash')
    ax[1].plot(x, UT, '--', label='Thomas')
    ax[1].set_title('Difference between approximations', fontsize=18)
    ax[1].legend()
    plt.show()
```

1.2.2 Exercise

Given the exact solution,

$$u(x) = x^3 - sin(x) + 5$$

$$x = [0, 2\pi]$$

we can generate

$$-f(x) = u^{''}(x) = -(6x + sin(x)),$$

and the boundary conditions,

$$u_a = u(0) = 5,$$
$$u_b = u(2\pi) = 8\pi^3 + 5$$

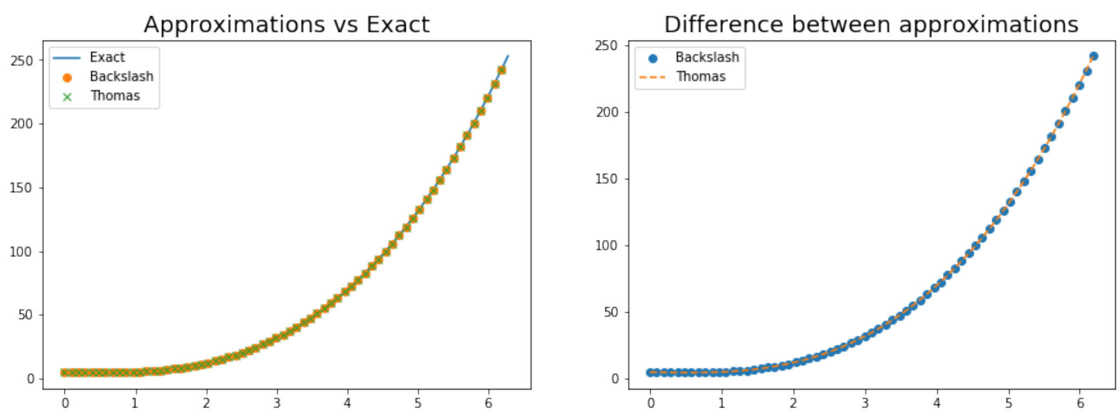
Appy the code to the $u^{''}$ when N = *64*. Plot **U_backslash and U_thomas** vs **x** against the exact solution at each x. Report the difference between **U_backslash and U_thomas**.

In [6]:
```
#initial conditions
N = 64
a = 0
b = np.pi * 2
ua = 5
ub = 8 * np.pi**3 + 5

#Define mesh and lambda functions for
#approximate and exact
xf = np.linspace(a, b, N)
exact = lambda x: x**3 - np.sin(x) + 5
func = lambda x: -6*x - np.sin(x)

#call finite difference function
UB, UT, x = finite_difference_sparse(func, a, b, ua, ub, N)

#plot approximations vs exact
generateApproxPlots(xf, x, UB, UT, exact)
```



1.2.3 Convergence study

Convergence study using *N = [4, 8, 16, 32, 64, 128]*, using the ℓ_2 norm and the ℓ_∞ norm. Provide a *loglog* plot with both norms and state the convergence order in your report. I am using **U_backslash** to show the order of convergence.

```
In [7]: def finiteDiffConvergence(func, a, b, ua, ub, h):

            """
            This function performs a convergence study
            """

            #array of different values of N
            N = [4, 8, 16, 32, 64, 128]

            #initialize empty array to hold error
            err = []

            #loop through N values in Narray
            for h in N:
                #call finite different functions
                UB, UT, x = finite_difference_sparse(func, a, b, ua, ub, h)
                #find error from approximate value and exact
                E = UB - exact(np.linspace(a, b, h+1))
                #append error to array
                err.append(E)

            #calculate the l2 and linf norm
            X = [np.linalg.norm(item, 2) for item in err]
            Y = [np.linalg.norm(item, np.inf) for item in err]

            #return norms
            return X/np.sqrt(N), Y, N
```

```
In [8]: def plotLoglog(X, Y, NN):
            """
            This function plots a convergence study on the approximated solutions

            Inputs:
                X - l2 norm of approximated solution
                Y - linf norm of the approximated solution
                NN - array of N input sizes
            """

            #calculate slope
            m1 = (np.log(2.18868523/4.36864701))/(np.log(128/64))
            m2 = (np.log(5.675683733054939/11.178104970785938))/(np.log(128/64))
            txt = "\nSlope of the log log plot for l_2 norm " + str(round(m1,6)) + "\nSlope of
```

```
    #plot convergence study
    fig, ax = plt.subplots(1, 2, figsize=(15,5))
    ax[0].loglog(NN, X)
    ax[0].set_xlabel('Step size N')
    ax[0].set_ylabel(r'$\ell_2$ norm of U_backslash')
    ax[0].set_title(r'Loglog plot for $\ell_2$ norm', fontsize=18)
    ax[1].loglog(NN, Y)
    ax[1].set_xlabel('Step size N')
    ax[1].set_ylabel(r'$\ell_{\infty}$ norm of U_backslash')
    ax[1].set_title(r'Loglog plot for $\ell_{\infty}$ norm', fontsize=18)
    fig.text(.5, .000005, txt, ha='center', fontsize=13)
    plt.show()
```

In [9]:
```
#perform and plot convergence study
X, Y, NN = finiteDiffConvergence(func, a, b, ua, ub, N)
plotLoglog(X, Y, NN)
```