Introduction to $\LaTeX$

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Exercice 1

1) [1]

Exercice 2

1.1) Prove that all natural number can be expressed in factorial representation. First, we need the show this property :

Property 1. $P_n : \sum_{k=0}^{n}(k * k!) + 1 = (n+1)!$

We can prove this quite easily using induction.

Proof. Case $n = 0 :\ 0 * 0! + 1 = (0+1)! = 1$

Let us suppose that P_n is true. We have to prove that P_{n+1} is true.

$$\begin{aligned}\sum_{k=0}^{n+1}(k * k!) + 1 &= (n+1) * (n+1)! + \sum_{k=0}^{n}(k * k!) + 1 \\ &= (n+1) * (n+1)! + (n+1)! \\ &= (n+2)!\end{aligned}$$

Thus the property is true $\forall n \geq 0$. □

Now we are going to prove the following property which is the one we want to prove here.

Property 2. P_n : *The natural number n can be written using factorial representation. This means that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. The factorial representation of n is represented by $a_k ... a_1 a_0$ where k is the smallest natural number that satisfies the previous result.*

Proof. Case $n = 0$: $0 = a_0 * 0! = \sum_{i=0}^{k} a_i * i!$ with $k = 0$ and $a_0 = 0$.

Let us suppose that P_n is true. We have to prove that P_{n+1} is true.

As this is our induction hypothesis, we know that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. For convenience, let us take the smallest k such that $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. This way we do not work with useless zeros.

Let $i^* = min(i \in [\![0, k+1]\!] \mid a_i \neq i)$.

Here is a property we want to prove :

- If $i^* \leq k$, then $n+1 = \sum_{i=0}^{k'} a'_i * i!$ where $k' = k$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ a_i, \forall i > i^* \\ a_i + 1, \text{ if } i = i^* \end{cases}$

- If $i^* = k + 1$, then $n + 1 = 1 * (k + 1)! = \sum_{i=0}^{k'} a'_i * i!$ where $k' = k + 1$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

Proving this previous property implies that $n+1$ can be written in factorial representation.

Let us consider the first case where $i^* \leq k$. Then $\forall i \in [\![0, i^* - 1]\!]$, $a_i = i$ and $a_{i^*} \in [\![0, i^* - 1]\!]$.

$$\begin{aligned}
n + 1 &= \sum_{i=0}^{k} a_i * i! + 1 \\
&= \sum_{i=i^*+1}^{k} a_i * i! + a_{i^*} * i^*! + \sum_{i=0}^{i^*-1} (i * i!) + 1 \\
&= \sum_{i=i^*+1}^{k} a_i * i! + a_{i^*} * i^*! + i^*! \qquad \text{Because of the property 1} \\
&= \sum_{i=i^*+1}^{k} a_i * i! + (a_{i^*} + 1) * i^*! \\
&= \sum_{i=0}^{k'} a'_i * i!
\end{aligned}$$

Where $k' = k$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ a_i, \ \forall i > i^* \\ a_i + 1, \text{ if } i = i^* \end{cases}$

Let us now consider the second case where $i^* = k + 1$, which is a special case. Here $\forall i \in [\![0, k]\!]$, $a_i = i$.

$$\begin{aligned} n + 1 &= \sum_{i=0}^{k} i * i! + 1 \\ &= (k+1)! \qquad \text{Because of the property 1} \\ &= \sum_{i=0}^{k'} a'_i * i! \end{aligned}$$

Where $k' = k + 1$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

In both cases, $n + 1$ can be expressed in factorial representation.

To conclude, we have seen that the property is true when $n = 0$, we have proven that P_n implies that P_{n+1} is true. Thus the property stands $\forall n \geq 0$.

□

1.2) Prove that this expression is unique (there is not two ways to express a natural number in factorial representation).

Proof. Let us suppose there exists n, a natural number such that $n = \displaystyle\sum_{i=0}^{k} a_i * i!$ and $n = \displaystyle\sum_{i=0}^{k'} b_i * i!$ where there is at least a i such that $a_i \neq b_i$.

Without loss of generality, we can write that $k' = k$ (because in the case of $k' > k$ we can say that all the $a_i = 0$, $\forall i \in [\![k+1, k']\!]$).

Let $l = max(i \in [\![0, k]\!] \mid a_i \neq b_i)$, $A = (a_0, a_1, \ldots, a_{l-1})$ and $B = (b_0, b_1, \ldots, b_{l-1})$. We define $\phi_{A,l}$ as follows : $\phi_{A,l} = \dfrac{1}{l!} \displaystyle\sum_{i=0}^{l-1} a_i * i!$

Hence we have that :

$$\sum_{i=0}^{k} a_i * i! = \sum_{i=0}^{k} b_i * i! \Rightarrow \; a_l * l! + \sum_{i=0}^{l-1} a_i * i! = b_l * l! + \sum_{i=0}^{l-1} b_i * i!$$
$$\Rightarrow a_l + (\frac{1}{l!}\sum_{i=0}^{l-1} a_i * i!) = b_l + (\frac{1}{l!}\sum_{i=0}^{l-1} b_i * i!)$$
$$\Rightarrow \quad a_l + \phi_{A,l} = b_l + \phi_{B,l}$$
$$\Rightarrow \quad (a_l - b_l) = (\phi_{B,l} - \phi_{A,l})$$

By definition of a_l and b_l, we know that $(a_l - b_l) \in \mathbb{Z}\backslash\{0\}$. Moreover, by definition, we know that $\forall i \in [\![0, l-1]\!]$, $a_i \leq i$ and $b_i \leq i$. Thus :

Thus $0 \leq \phi_{A,l} \leq \frac{1}{l!}\sum_{i=0}^{l-1} i * i! \leq \frac{1}{l!}(\sum_{i=0}^{l-1}(i * i!) + 1 - 1) = \frac{l! - 1}{l!} = 1 - \frac{1}{l!} < 1.$ Then $-1 > -\phi_{A,l} \geq 0$. We can deduce the same results for $\phi_{B,l}$, and especially $0 \leq \phi_{B,l} < 1$. Given those two results, we have that :

$$-1 < \phi_{B,l} - \phi_{A,l} < 1$$

This result is absurd since we have that $(a_l - b_l) = (\phi_{B,l} - \phi_{A,l})$ and that $(a_l - b_l) \in \mathbb{Z}\backslash\{0\}$.

Hence we showed that supposing there exists a natural number that can be expressed in two different ways in factorial representation implies a absurd result. Therefore the factorial representation a all natural numbers is different.

□

Exercice 3

1.1) Prove that all natural number can be expressed in Fibonacci representation.

Property 3. P_n : *The natural number n can be written using Fibonacci representation. This means that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. The Fibonacci representation of n is represented by $a_k ... a_1 a_0$ where k is the smallest natural number that satisfies the previous result.*

The very first thing we have to consider is that the Fibonacci number associated to a_i is f_{i+2}, not f_i. The other important thing, is the constraint $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. This implies that we can not have two consecutive a_i equal to one. Now that we have a better understanding of P_n, we are going to prove it.

Proof. Case $n = 0$: $0 = 0 * 1 = \sum_{i=0}^{k} a_i * f_{i+2}$ where $k = 0$ and $a_0 = 0$.

Let us suppose P_n is true. We want to prove that P_{n+1} is true. As this is our induction hypothesis, we know that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. For convenience, let us take the smallest k such that $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. This way we do not work with useless zeros. We have $a_k = 1$ in all cases except if $n = 0$.

Let $i^* = min(i \in [\![0, k+1]\!] \mid a_i + a_{i+1} = 0)$. By this definition, we have that $a_{i^*} = 0$ and $a_{i^*+1} = 0$.

Here is a property we want to prove :

- If $i^* = 0$, then $n + 1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k = 0$ and $a'_0 = 1$.

- If $0 < i^* \leq k$, then $n + 1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ a_i, \forall i > i^* \\ 1, \text{ if } i = i^* \end{cases}$

- If $i^* = k + 1$, then $n + 1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k + 1$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

Moreover the property states that in all cases $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

Proving the previous property implies that $n+1$ can be written in Fibonacci representation. So let us prove it now :

The case $i^* = 0$ is trivial. This implies that $n = 0$, thus $n+1 = 1*1 = 1*f_2$. We obviously have $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

In the case $0 < i^* \leq k$, we can say that $\begin{cases} \forall i \in [\![0, i^*-1]\!], \ a_i a_{i+1} = 0 \\ \forall i \in [\![0, i^*-1]\!], \ a_i + a_{i+1} \neq 0 \\ a_{i^*-1} = 1 \end{cases}$

We can notice that i^* is necessarily different from k or $k - 1$.

In this case, n can be written in this way $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{i^*-1} a_i * f_{i+2}$. Here we can divide the problem in two parts : $i^* - 1$ is even and $i^* - 1$ is odd.

If $i^* - 1$ is even, $\forall i \in \mathbb{N}$ such that $2i \leq i^* - 1$, $a_{2i} = 1$ y $a_{2i-1} = 0$. Thus $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-1)/2} f_{2i+2}$. As $f_1 = 1$, and by the property of the Fibonacci sequence $f_{n+2} = f_{n+1} + f_n$, we can show that :

$$
\begin{aligned}
n+1 &= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-1)/2} f_{2i+2} + 1 \\
&= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \underbrace{\sum_{i=0}^{(i^*-1)/2} f_{2i+2} + f_1}_{=f_{i^*+2}} && \text{Property of the Fibonacci sequence} \\
&= \sum_{i=0}^{k'} a'_i * f_{i+2}
\end{aligned}
$$

If $i^* - 1$ is odd, $\forall i \in \mathbb{N}$ such that $2i + 1 \leq i^* - 1$, $a_{2i} = 0$ y $a_{2i+1} = 1$. Thus $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-2)/2} f_{2i+3}$. As $f_2 = 1$, and by the property of the Fibonacci sequence $f_{n+2} = f_{n+1} + f_n$, we can show that :

$$
\begin{aligned}
n+1 &= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-2)/2} f_{2i+3} + 1 \\
&= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \underbrace{\sum_{i=0}^{(i^*-2)/2} f_{2i+3} + f_2}_{=f_{i^*+2}} && \text{Property of the Fibonacci sequence} \\
&= \sum_{i=0}^{k'} a'_i * f_{i+2}
\end{aligned}
$$

In both cases ($i^* - 1$ even and odd), $k' = k$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ a_i, \forall i > i^* \\ 1, \text{if } i = i^* \end{cases}$

In both cases $\forall i > i^*$, $a'_i = a_i$ so as we have that $\forall i \in [\![i^* + 1, k - 1]\!]$, $a_i a_{i+1} = 0$ we have that $\forall i \in [\![i^* + 1, k' - 1]\!]$, $a'_i a'_{i+1} = 0$. Moreover, $\forall i < i^*$, $a'_i = 0$ so $\forall i \in [\![0, i^* - 1]\!]$, $a'_i a'_{i+1} = 0$. Finally, we know that $a'_{i^*} a'_{i^*+1} = 0$

because $a'_{i^*+1} = a_{i^*+1} = 0$ by definition of i^*. So $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$

If $i^* = k+1$, then if k is even $n+1 = \sum_{i=0}^{k/2} f_{2i+2} + 1 = \sum_{i=0}^{k/2} f_{2i+2} + f_1 = f_{k+3}$. If k is odd $n+1 = \sum_{i=0}^{(k-1)/2} f_{2i+3} + 1 = \sum_{i=0}^{(k-1)/2} f_{2i+3} + f_2 = f_{k+3}$.

So in both cases (k even and odd), $n+1 = \sum_{i=0}^{k'} f_{i+2}$ where $k' = k+1$ and
$a'_i = \begin{cases} 0, \forall i < i^* \\ 1, \text{if } i = i^* \end{cases}$
Then obviously, as only $a_{i^*} = 0$, $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

To conclude, we have seen that the property is true when $n = 0$, we have proven that P_n implies that P_{n+1} is true. Thus the property stands $\forall n \geq 0$.

□

1.2) Prove that this expression is unique (there is not two ways to express a natural number in Fibonacci representation). In order to proove this, we are going to use the same idea or method from the previous part.

Proof. Let us suppose there exists a natural number n such that $n = \sum_{i=0}^{k} a_i f_{i+2} = \sum_{i=0}^{k'} b_i f_{i+2}$ (like the previous exercice we can assume without loss of generality that $k' = k$) where $\forall i \in [\![0, k]\!]$, $a_i \in [\![0, 1]\!]$ and $b_i \in [\![0, 1]\!]$, where there exists at least a $i \in [\![0, k-1]\!]$ such that $a_i \neq b_i$. We have that $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$ and $\forall i \in [\![0, k-1]\!]$, $b_i b_{i+1} = 0$.

Let $l = max(i \in [\![0, k]\!] \mid a_i \neq b_i)$. We know that $|a_l - b_l| = 1$. Let $A = (a_0, a_1, \ldots, a_{l-1})$ and $B = (b_0, b_1, \ldots, b_{l-1})$. We define $\beta_{A,l}$ as follows : $\beta_{A,l} = \frac{1}{f_{l+2}} \sum_{i=0}^{l-1} a_i f_{i+2}$.

$$\begin{aligned}
\sum_{i=0}^{k} a_i f_{i+2} = \sum_{i=0}^{l-1} b_i f_{i+2} \Rightarrow & \quad a_l f_{l+2} + \sum_{i=0}^{k} a_i f_{i+2} = b_l f_{l+2} + \sum_{i=0}^{l-1} b_i f_{i+2} \\
& \Rightarrow a_l + (\frac{1}{f_{l+2}} \sum_{i=0}^{l-1} a_i f_{i+2}) = b_l + (\frac{1}{f_{l+2}} \sum_{i=0}^{l-1} b_i f_{i+2}) \\
& \Rightarrow \quad a_l + \beta_{A,l} = b_l + \beta_{B,l} \\
& \Rightarrow \quad (a_l - b_l) = (\beta_{B,l} - \beta_{A,l})
\end{aligned}$$

Because of the constraint that $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$, if $l-1$ is even :

$$0 \leq \beta_{A,l} = \frac{1}{f_{l+2}}(\sum_{i=0}^{l-1} a_i f_{i+2}) = \frac{1}{f_{l+2}}(f_{l+1}+f_{l-1}+\cdots+f_4+f_2+f_1-1) = \frac{1}{f_{l+2}}(f_{l+2}-1) = 1-\frac{1}{f_{l+2}} < 1$$

If $l-1$ is odd, we obtain the same result :

$$0 \leq \beta_{A,l} = \frac{1}{f_{l+2}}(\sum_{i=0}^{l-1} a_i f_{i+2}) = \frac{1}{f_{l+2}}(f_{l+1}+f_{l-1}+\cdots+f_5+f_3+f_2-1) = \frac{1}{f_{l+2}}(f_{l+2}-1) = 1-\frac{1}{f_{l+2}} < 1$$

Then $-1 > -\beta_{A,l} \geq 0$. We can deduce the same results for $\beta_{B,l}$, and especially $0 \leq \beta_{B,l} < 1$. Given those two results, we have that :

$$-1 < \beta_{B,l} - \beta_{A,l} < 1$$

Finally, we have that $1 = |a_l - b_l| = \beta_{B,l} - \beta_{A,l} < 1$ which is absurd. Thus the Fibonacci representation is unique for all $n \in \mathbb{N}$.

□

References

[1] J. F. C. Smith and J. Bourne, "The Pain of Having a Foolish Name," *Journal of Modern Fiction*, vol. 52, no. 1, p. 114, 2009.