Introduction to LaTeX

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Exercice 1

For the rest of the exercice 1, we assume that :

$p \in CNF \iff \exists k \in \mathbb{N}$ and $\forall i \in [\![1,k]\!], \exists J_i \in \mathbb{N}$ such that $p = \bigwedge\limits_{1\leq i\leq k} C_i = \bigwedge\limits_{1\leq i\leq k} (\bigvee\limits_{1\leq j\leq J_i} l_{i,j})$ where $l_{i,j}$ is a literal. C_i is the representation we will used to describe "clausulas".

1.1) Prove that all logical propositional formulas puden escribirse en CNF. In order to demonstrate this we will prove two properties first.

Property 1. *P_n : If $(\Phi, \Omega) \in CNF$ and are especially of the following form : $\Phi = \bigwedge\limits_{1\leq i\leq k} C_i$ and $\Omega = \bigwedge\limits_{1\leq i\leq k'} C'_i$ with $k, k' \leq n \implies (\Phi \bigvee \Omega) \in CNF$*

Proof. Case $n = 1$:

$$\begin{aligned}\Phi \bigvee \Omega &= \bigvee_{1\leq j\leq J_1^\Phi} l_{1,j}^\Phi \vee \bigvee_{1\leq j\leq J_1^\Omega} l_{1,j}^\Omega \\ &= \bigwedge_{1\leq i\leq k} (\bigvee_{1\leq j\leq J'_i} l'_{i,j})\end{aligned}$$

Where $k = 1$, $J'_1 = J_1^\Phi + J_1^\Omega$ and $l' = \begin{cases} l_{1,j}^\Phi \ , \forall j \in [\![1, J_1^\Phi]\!] \\ l_{1,j}^\Omega, \forall j \in [\![J_1^\Phi J_1^\Phi + J_1^\Omega]\!] \end{cases}$

□

Exercice 2

1.1) Prove that all natural number can be expressed in factorial representation. First, we need the show this property :

Property 2. *$P_n : \sum\limits_{k=0}^{n} (k * k!) + 1 = (n+1)!$*

We can prove this quite easily using induction.

Proof. Case $n = 0 : \ 0 * 0! + 1 = (0+1)! = 1$

Let us suppose that P_n is true. We have to prove that P_{n+1} is true.

$$\begin{aligned}\sum_{k=0}^{n+1}(k*k!)+1 &= (n+1)*(n+1)! + \sum_{k=0}^{n}(k*k!)+1 \\ &= (n+1)*(n+1)! + (n+1)! \\ &= (n+2)!\end{aligned}$$

Thus the property is true $\forall n \geq 0$. □

Now we are going to prove the following property which is the one we want to prove here.

Property 3. P_n : *The natural number n can be written using factorial representation. This means that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0,k]\!]$, $\exists a_i \in [\![0,i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. The factorial representation of n is represented by $a_k ... a_1 a_0$ where k is the smallest natural number that satisfies the previous result.*

Proof. Case $n = 0 : \ 0 = a_0 * 0! = \sum_{i=0}^{k} a_i * i!$ with $k = 0$ and $a_0 = 0$.

Let us suppose that P_n is true. We have to prove that P_{n+1} is true.

As this is our induction hypothesis, we know that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0,k]\!]$, $\exists a_i \in [\![0,i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. For convenience, let us take the smallest k such that $\forall i \in [\![0,k]\!]$, $\exists a_i \in [\![0,i]\!]$ such that $n = \sum_{i=0}^{k} a_i * i!$. This way we do not work with useless zeros.

Let $i^* = min(i \in [\![0,k+1]\!] \mid a_i \neq i)$.

Here is a property we want to prove :

- If $i^* \leq k$, then $n+1 = \sum_{i=0}^{k'} a'_i * i!$ where $k' = k$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ a_i, \ \forall i > i^* \\ a_i + 1, \text{ if } i = i^* \end{cases}$
- If $i^* = k+1$, then $n+1 = 1*(k+1)! = \sum_{i=0}^{k'} a'_i * i!$ where $k' = k+1$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

Proving this previous property implies that $n+1$ can be written in factorial representation.

Let us consider the first case where $i^* \leq k$. Then $\forall i \in [\![0, i^*-1]\!]$, $a_i = i$ and $a_{i^*} \in [\![0, i^*-1]\!]$.

$$
\begin{aligned}
n+1 &= \sum_{i=0}^{k} a_i * i! + 1 \\
&= \sum_{i=i^*+1}^{k} a_i * i! + a_{i^*} * i^*! + \sum_{i=0}^{i^*-1} (i * i!) + 1 \\
&= \sum_{i=i^*+1}^{k} a_i * i! + a_{i^*} * i^*! + i^*! \qquad \text{Because of the property 1} \\
&= \sum_{i=i^*+1}^{k} a_i * i! + (a_{i^*} + 1) * i^*! \\
&= \sum_{i=0}^{k'} a'_i * i!
\end{aligned}
$$

Where $k' = k$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ a_i, \forall i > i^* \\ a_i + 1, \text{ if } i = i^* \end{cases}$

Let us now consider the second case where $i^* = k+1$, which is a special case. Here $\forall i \in [\![0, k]\!]$, $a_i = i$.

$$
\begin{aligned}
n+1 &= \sum_{i=0}^{k} i * i! + 1 \\
&= (k+1)! \qquad \text{Because of the property 1} \\
&= \sum_{i=0}^{k'} a'_i * i!
\end{aligned}
$$

Where $k' = k+1$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

In both cases, $n+1$ can be expressed in factorial representation.

To conclude, we have seen that the property is true when $n = 0$, we have proven that P_n implies that P_{n+1} is true. Thus the property stands $\forall n \geq 0$.

$\square$

1.2) Prove that this expression is unique (there is not two ways to express a natural number in factorial representation).

Proof. Let us suppose there exists n, a natural number such that $n = \sum_{i=0}^{k} a_i * i!$ and $n = \sum_{i=0}^{k'} b_i * i!$ where there is at least a i such that $a_i \neq b_i$.

Without loss of generality, we can write that $k' = k$ (because in the case of $k' > k$ we can say that all the $a_i = 0$, $\forall i \in [\![k+1, k']\!]$).

Let $l = max(i \in [\![0, k]\!] \mid a_i \neq b_i)$, $A = (a_0, a_1, \ldots, a_{l-1})$ and $B = (b_0, b_1, \ldots, b_{l-1})$. We define $\phi_{A,l}$ as follows : $\phi_{A,l} = \frac{1}{l!} \sum_{i=0}^{l-1} a_i * i!$

Hence we have that :

$$\begin{aligned}
\sum_{i=0}^{k} a_i * i! = \sum_{i=0}^{k} b_i * i! \Rightarrow \ & a_l * l! + \sum_{i=0}^{l-1} a_i * i! = b_l * l! + \sum_{i=0}^{l-1} b_i * i! \\
\Rightarrow \ & a_l + (\frac{1}{l!} \sum_{i=0}^{l-1} a_i * i!) = b_l + (\frac{1}{l!} \sum_{i=0}^{l-1} b_i * i!) \\
\Rightarrow \ & \quad a_l + \phi_{A,l} = b_l + \phi_{B,l} \\
\Rightarrow \ & \quad (a_l - b_l) = (\phi_{B,l} - \phi_{A,l})
\end{aligned}$$

By definition of a_l and b_l, we know that $(a_l - b_l) \in \mathbb{Z} \backslash \{0\}$. Moreover, by definition, we know that $\forall i \in [\![0, l-1]\!]$, $a_i \leq i$ and $b_i \leq i$. Thus :

Thus $0 \leq \phi_{A,l} \leq \frac{1}{l!} \sum_{i=0}^{l-1} i * i! \leq \frac{1}{l!} (\sum_{i=0}^{l-1} (i * i!) + 1 - 1) = \frac{l! - 1}{l!} = 1 - \frac{1}{l!} < 1.$
Then $-1 > -\phi_{A,l} \geq 0$. We can deduce the same results for $\phi_{B,l}$, and especially $0 \leq \phi_{B,l} < 1$. Given those two results, we have that :

$$-1 < \phi_{B,l} - \phi_{A,l} < 1$$

This result is absurd since we have that $(a_l - b_l) = (\phi_{B,l} - \phi_{A,l})$ and that $(a_l - b_l) \in \mathbb{Z} \backslash \{0\}$.

Hence we showed that supposing there exists a natural number that can be expressed in two different ways in factorial representation implies a absurd result. Therefore the factorial representation a all natural numbers is different. $\square$

Exercice 3

1.1) Prove that all natural number can be expressed in Fibonacci representation.

Property 4. P_n : *The natural number n can be written using Fibonacci representation. This means that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. The Fibonacci representation of n is represented by $a_k ... a_1 a_0$ where k is the smallest natural number that satisfies the previous result.*

The very first thing we have to consider is that the Fibonacci number associated to a_i is f_{i+2}, not f_i. The other important thing, is the constraint $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. This implies that we can not have two consecutive a_i equal to one. Now that we have a better understanding of P_n, we are going to prove it.

Proof. Case $n = 0$: $0 = 0 * 1 = \sum_{i=0}^{k} a_i * f_{i+2}$ where $k = 0$ and $a_0 = 0$.

Let us suppose P_n is true. We want to prove that P_{n+1} is true. As this is our induction hypothesis, we know that $\exists k \in \mathbb{N}$ and $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. For convenience, let us take the smallest k such that $\forall i \in [\![0, k]\!]$, $\exists a_i \in [\![0, 1]\!]$ such that $n = \sum_{i=0}^{k} a_i * f_{i+2}$ and $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$. This way we do not work with useless zeros. We have $a_k = 1$ in all cases except if $n = 0$.

Let $i^* = min(i \in [\![0, k+1]\!] \mid a_i + a_{i+1} = 0)$. By this definition, we have that $a_{i^*} = 0$ and $a_{i^*+1} = 0$.

Here is a property we want to prove :

- If $i^* = 0$, then $n + 1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k = 0$ and $a'_0 = 1$.
- If $0 < i^* \leq k$, then $n + 1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ a_i, \forall i > i^* \\ 1, \text{if } i = i^* \end{cases}$

- If $i^* = k+1$, then $n+1 = \sum_{i=0}^{k'} a'_i * f_{i+2}$ where $k' = k+1$ and $a'_i = \begin{cases} 0, \forall i < i^* \\ 1, \text{if } i = i^* \end{cases}$

Moreover the property states that in all cases $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

Proving the previous property implies that $n+1$ can be written in Fibonacci representation. So let us prove it now :

The case $i^* = 0$ is trivial. This implies that $n = 0$, thus $n+1 = 1*1 = 1*f_2$. We obviously have $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

In the case $0 < i^* \leq k$, we can say that $\begin{cases} \forall i \in [\![0, i^*-1]\!],\, a_i a_{i+1} = 0 \\ \forall i \in [\![0, i^*-1]\!],\, a_i + a_{i+1} \neq 0 \\ a_{i^*-1} = 1 \end{cases}$
We can notice that i^* is necessarily different from k or $k-1$.

In this case, n can be written in this way $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{i^*-1} a_i * f_{i+2}$. Here we can divide the problem in two parts : i^*-1 is even and i^*-1 is odd.

If i^*-1 is even, $\forall i \in \mathbb{N}$ such that $2i \leq i^*-1$, $a_{2i} = 1$ y $a_{2i-1} = 0$. Thus $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-1)/2} f_{2i+2}$. As $f_1 = 1$, and by the property of the Fibonacci sequence $f_{n+2} = f_{n+1} + f_n$, we can show that :

$$
\begin{aligned}
n+1 &= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-1)/2} f_{2i+2} + 1 \\
&= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \underbrace{\sum_{i=0}^{(i^*-1)/2} f_{2i+2} + f_1}_{=f_{i^*+2}} && \text{Property of the Fibonacci sequence} \\
&= \sum_{i=0}^{k'} a'_i * f_{i+2}
\end{aligned}
$$

If i^*-1 is odd, $\forall i \in \mathbb{N}$ such that $2i+1 \leq i^*-1$, $a_{2i} = 0$ y $a_{2i+1} = 1$. Thus $n = \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-2)/2} f_{2i+3}$. As $f_2 = 1$, and by the property of the Fibonacci sequence $f_{n+2} = f_{n+1} + f_n$, we can show that :

$$
\begin{aligned}
n+1 &= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \sum_{i=0}^{(i^*-2)/2} f_{2i+3} + 1 \\
&= \sum_{i=i^*+2}^{k} a_i * f_{i+2} + \underbrace{\sum_{i=0}^{(i^*-2)/2} f_{2i+3} + f_2}_{=f_{i^*+2}} && \text{Property of the Fibonacci sequence} \\
&= \sum_{i=0}^{k'} a'_i * f_{i+2}
\end{aligned}
$$

In both cases (i^*-1 even and odd), $k'=k$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ a_i, \ \forall i > i^* \\ 1, \text{ if } i = i^* \end{cases}$

In both cases $\forall i > i^*$, $a'_i = a_i$ so as we have that $\forall i \in [\![i^*+1, k-1]\!]$, $a_i a_{i+1} = 0$ we have that $\forall i \in [\![i^*+1, k'-1]\!]$, $a'_i a'_{i+1} = 0$. Moreover, $\forall i < i^*$, $a'_i = 0$ so $\forall i \in [\![0, i^*-1]\!]$, $a'_i a'_{i+1} = 0$. Finally, we know that $a'_{i^*} a'_{i^*+1} = 0$ because $a'_{i^*+1} = a_{i^*+1} = 0$ by definition of i^*. So $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$

If $i^* = k+1$, then if k is even $n+1 = \sum_{i=0}^{k/2} f_{2i+2} + 1 = \sum_{i=0}^{k/2} f_{2i+2} + f_1 = f_{k+3}$. If k is odd $n+1 = \sum_{i=0}^{(k-1)/2} f_{2i+3} + 1 = \sum_{i=0}^{(k-1)/2} f_{2i+3} + f_2 = f_{k+3}$.

So in both cases (k even and odd), $n+1 = \sum_{i=0}^{k'} f_{i+2}$ where $k' = k+1$ and $a'_i = \begin{cases} 0, \ \forall i < i^* \\ 1, \text{ if } i = i^* \end{cases}$

Then obviously, as only $a_{i^*} = 0$, $\forall i \in [\![0, k'-1]\!]$, $a'_i a'_{i+1} = 0$.

To conclude, we have seen that the property is true when $n=0$, we have proven that P_n implies that P_{n+1} is true. Thus the property stands $\forall n \geq 0$.

□

1.2) Prove that this expression is unique (there is not two ways to express a natural number in Fibonacci representation). In order to proove this, we are going to use the same idea or method from the previous part.

Proof. Let us suppose there exists a natural number n such that $n = \sum_{i=0}^{k} a_i f_{i+2} =$

$\sum_{i=0}^{k'} b_i f_{i+2}$ (like the previous exercice we can assume without loss of generality that $k' = k$) where $\forall i \in [\![0, k]\!]$, $a_i \in [\![0, 1]\!]$ and $b_i \in [\![0, 1]\!]$, where there exists at least a $i \in [\![0, k-1]\!]$ such that $a_i \neq b_i$. We have that $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$ and $\forall i \in [\![0, k-1]\!]$, $b_i b_{i+1} = 0$.

Let $l = max(i \in [\![0, k]\!] \mid a_i \neq b_i)$. We know that $|a_l - b_l| = 1$. Let $A = (a_0, a_1, \ldots, a_{l-1})$ and $B = (b_0, b_1, \ldots, b_{l-1})$. We define $\beta_{A,l}$ as follows : $\beta_{A,l} = \frac{1}{f_{l+2}} \sum_{i=0}^{l-1} a_i f_{i+2}$.

$$
\begin{aligned}
\sum_{i=0}^{k} a_i f_{i+2} = \sum_{i=0}^{l-1} b_i f_{i+2} \Rightarrow \quad & a_l f_{l+2} + \sum_{i=0}^{k} a_i f_{i+2} = b_l f_{l+2} + \sum_{i=0}^{l-1} b_i f_{i+2} \\
\Rightarrow & \, a_l + \left(\frac{1}{f_{l+2}} \sum_{i=0}^{l-1} a_i f_{i+2}\right) = b_l + \left(\frac{1}{f_{l+2}} \sum_{i=0}^{l-1} b_i f_{i+2}\right) \\
\Rightarrow & \quad a_l + \beta_{A,l} = b_l + \beta_{B,l} \\
\Rightarrow & \quad (a_l - b_l) = (\beta_{B,l} - \beta_{A,l})
\end{aligned}
$$

Because of the constraint that $\forall i \in [\![0, k-1]\!]$, $a_i a_{i+1} = 0$, if $l-1$ is even :

$$0 \leq \beta_{A,l} = \frac{1}{f_{l+2}} (\sum_{i=0}^{l-1} a_i f_{i+2}) = \frac{1}{f_{l+2}} (f_{l+1} + f_{l-1} + \cdots + f_4 + f_2 + f_1 - 1) = \frac{1}{f_{l+2}} (f_{l+2} - 1) = 1 - \frac{1}{f_{l+2}} < 1$$

If $l-1$ is odd, we obtain the same result :

$$0 \leq \beta_{A,l} = \frac{1}{f_{l+2}} (\sum_{i=0}^{l-1} a_i f_{i+2}) = \frac{1}{f_{l+2}} (f_{l+1} + f_{l-1} + \cdots + f_5 + f_3 + f_2 - 1) = \frac{1}{f_{l+2}} (f_{l+2} - 1) = 1 - \frac{1}{f_{l+2}} < 1$$

Then $-1 > -\beta_{A,l} \geq 0$. We can deduce the same results for $\beta_{B,l}$, and especially $0 \leq \beta_{B,l} < 1$. Given those two results, we have that :

$$-1 < \beta_{B,l} - \beta_{A,l} < 1$$

Finally, we have that $1 = |a_l - b_l| = \beta_{B,l} - \beta_{A,l} < 1$ which is absurd. Thus the Fibonacci representation is unique for all $n \in \mathbb{N}$.

□

References

[1] J. F. C. Smith and J. Bourne, "The Pain of Having a Foolish Name," *Journal of Modern Fiction*, vol. 52, no. 1, p. 114, 2009.