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Tarea 2

CC3101: Discrete Mathematics for Computer Science

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Exercice 1

1.1) Give $D_{0,0}$, $D_{0,1}$ and show that $D_{0,n} = (n-1)(D_{0,n-1} + D_{0,n-2}) \forall n \geq 2$. If we consider $(n-1)$ as the number of possibilities where $\pi_1 \neq 1$, then it is easy to see that there is two possible types of output as permutations :

- First possibility : $\pi_1 = i$ and $\pi_i = 1$ where $i \in [\![2, n]\!]$
  . This implies that we are left with $D_{0,n-2}$ permutations.
- Second possibility : $\pi_1 = i$ and $\pi_i \neq 1$ where $i \in [\![2, n]\!]$
  . This case is exactly similar as having $D_{0,n-1}$ permutations left.

Moreover, we have $D_{0,0} = 1$, $D_{0,1} = 0$.

1.2) Demonstrate $P_n : \forall n \in \mathbb{N}^*, D_{0,n} = nD_{0,n-1} + (-1)^n$.

Proof. Case $n = 1$: We know $D_{0,1} = 0$ and by the formula $D_{0,1} = 1 \times D_{0,0} + (-1)^1 = 1 - 1 = 0$. The formula is true for $n = 1$.

Now let us suppose the property P_n true and let us prove P_{n+1}.

$$\begin{aligned} D_{0,n+1} &= n(D_{0,n} + D_{0,n-1}) \quad \text{Because of the first recurrent expression} \\ &= n(D_{0,n} + \frac{D_{0,n} - (-1)^n}{n}) \quad \text{Because of induction hypothesis} \\ &= nD_{0,n} + D_{0,n} + (-1)^{n+1} \\ &= (n+1)D_{0,n} + (-1)^{n+1} \end{aligned}$$

Thus the property P_n is true $\forall n \in \mathbb{N}^*$. □

1.3) Obtain a non-recurrent closed-form expression for $D_{0,n}$. From the previous result, provide an expression for $D_{k,n}$. We will need two notations to make the demonstration clearer. We define $G_n = \dfrac{D_{0,n}}{n!}$ and $\alpha_n = \dfrac{(-1)^n}{n!}$. As we have that $\forall n \in \mathbb{N}^*, D_{0,n} = nD_{0,n-1} + (-1)^n$ therefore $\forall n \in \mathbb{N}^*, \frac{D_{0,n}}{n!} = \frac{D_{0,n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$ and finally $G_n = G_{n-1} + \alpha_n$. Moreover we have $G_0 = 1$ and $\alpha_1 = 1$. Here is a short development to get the idea how we can obtain a value for $G_n, \forall n \in \mathbb{N}^*$.

$$\begin{aligned} G_1 &= G_0 + \alpha_1 \\ G_2 &= G_0 + \alpha_1 + \alpha_2 \\ G_3 &= G_0 + \alpha_1 + \alpha_2 + \alpha_3 \end{aligned}$$

If we continue to develop this way, we can easily guess that $G_n = G_0 + \displaystyle\sum_{i=1}^{n} \alpha_i$. As we know that $G_0 = 1 = \alpha_0$ hence finally $G_n = \displaystyle\sum_{i=0}^{n} \alpha_i \iff D_{0,n} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$. We should prove it by induction to be formal. However here the result is really obvious so we will not make the induction proof of this formula.

Now we will express $D_{k,n}$. The idea is that $D_{k,n}$ is equal to the number of permutations such that there are exactly k fixed points. This is equivalent to saying $D_{k,n}$ is the number of ways we can choose k elements, set them as fixed points and then having exactly zero fixed points in the permutations using the rest of the elements. The first part is equivalent to choosing k elements amongst n where the order of selection does not matter : $\binom{n}{k}$. The second part is simply $D_{0,n-k}$.

Finally, combinatorially, we showed that $D_{k,n} = \binom{n}{k} D_{0,n-k}$. Hence $D_{k,n} = \dfrac{n!}{k!} \displaystyle\sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$

1.4) Calculate the probability to get k fixed points and the esperance of the number of fixed points when n is set. Firstly, as we know that the number of permutations with k fixed points is $D_{k,n}$, the probability to obtain k fixed points, let us called it $p_{k,n}$, is simply $D_{k,n}$ over the total number of permutations. Hence, $p_{k,n} = \dfrac{D_{k,n}}{n!}$.

Let us define X a random variable that counts the number of fixed points in a permutation (length n set). We also define :

$X_i = \begin{cases} 1 \text{ , if } \pi_i = i \\ 0\text{, if } \pi_i \neq i \end{cases}$

We have $X = \displaystyle\sum_{i=1}^{n} X_i \implies E[X] = \sum_{i=1}^{n} E[X_i]$.

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} E[X_i] \\ &= \sum_{i=1}^{n} p(X_i = 1) \\ &= \sum_{i=1}^{n} \frac{1}{n} \\ &= 1 \end{aligned}$$

$p(X_i = 1) = \dfrac{1}{n}$ because it is just the chance that $\pi_i = i$, which is obviously just $\dfrac{1}{n}$.

1.5) What happens with the probability to obtain k fixed point when $n \to +\infty$? What about the esperance ? Calculate the esperance in a other way and show that $\displaystyle\sum_{k=0}^{n-1} \sum_{i=0}^{n-k-1} \frac{(-1)^i}{k!i!} = 1$. The probability to obtain k fixed points $p_{k,n} = \dfrac{D_{k,n}}{n!} = \dfrac{1}{k!} \displaystyle\sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$. Thus $p_{k,n} \to \dfrac{e^{-1}}{k!}$ as $n \to +\infty$. Moreover we know the esperance does not depend on n, thus $E[X] = 1$

still as $n \to +\infty$.

We can write the esperance in a other way :

$$\begin{aligned}
E[X] &= \sum_{k=0}^{n} kp(X=k) \\
&= \sum_{k=0}^{n-1} (k+1)p_{k+1,n} \\
&= \sum_{k=0}^{n-1} (k+1)\frac{1}{(k+1)!}\sum_{i=0}^{n-k-1}\frac{(-1)^i}{i!} \\
&= \sum_{k=0}^{n-1} \frac{1}{k!}\sum_{i=0}^{n-k-1}\frac{(-1)^i}{i!} \\
&= \sum_{k=0}^{n-1}\sum_{i=0}^{n-k-1}\frac{(-1)^i}{k!i!}
\end{aligned}$$

We know that $E[X] = 1$ thus we have that $\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-1}\frac{(-1)^i}{k!i!} = 1$.

1.6) Simulate permutation with k fixed and n fixed and study the probability $p_{k,n}$ We created functions to simulate permutations. We created a function that counts the number of fixed point in a given permutation. With all this tools we were able to compute empirical probabilities of having k fixed points in a sequence of length n.

Here is the graph of probability of having k fixed points in a permutation of length n fixed :

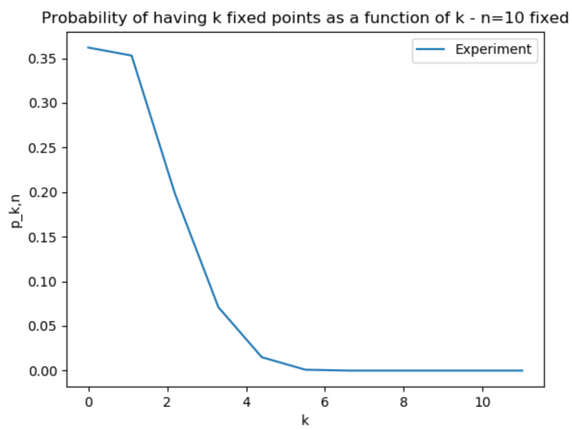

Figure 1: n fixed

We made 1000 permutation simulations to get the previous graph. The curve obtained makes sense since we know that $p_{k,n} = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!}$, thus $p_{k,n} \simeq \frac{e^{-1}}{k!}$. this shows why the curve converges extremely quickly towards zero as k is getting bigger. It is funny to notice that for $k = 0$ and $k = 1$ the probability is really close to $e^{-1} \simeq 0.3678$.

Here is the graph of probability of having $k = 2$ fixed points in a permutation of length n as n is getting bigger and bigger :

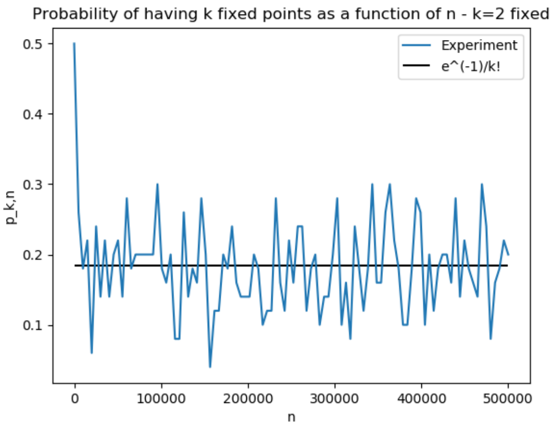

Figure 2: k fixed

This took quite a while to compute even though we used a step of 5000 for n (n going from k to $n_{max} = 500000$). We can clearly see that the probability oscillates. The curve makes sense since we have $p_{k,n} = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!} \simeq \frac{1}{k!}(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}...) \to \frac{e^{-1}}{k!}$. Empirically, the probabilities oscillate around the value $\frac{e^{-1}}{k!}$ (in the previous graph with $k = 2$). Eventually with n getting bigger and bigger the probability will converge to $\frac{e^{-1}}{2}$ (black horizontal line in the graph).

Exercice 2

2.1) Let a_n be the quantity of distinct EXP which length is equal to n. Provide a recurrent relation for a_n The idea is quite straightforward, for a given n, a_n counts the number of combinations of distinct expressions of length i and j such that $i + j + 3 = n$ for the symbols $*$ and $+$ and such that $i + j + 4 = n$ for the symbol $==$.

Seeing the problem this way, we can quickly give a recurrent formula for a_n.

$$a_n = \sum_{i=1}^{n-4} a_{n-i-3}a_i + \sum_{i=1}^{n-5} a_{n-i-4}a_i$$

However this formula is quite ugly and we can do much better. Firstly let us define $a_0 = 0$. Then we can gather the two sums and change the indices to make everything clearer :

$$a_n = \sum_{i=0}^{n-5} (2a_{i+1} + a_i)a_{n-4-i}$$

2.2) Create a function cantidad that computes a_n We have $a_0 = a_2 = a_3 = a_4 = 0$, $a_1 = |V \cup C| = 4$ and the recurrent formula for a_n. With all this information it is easy to make a recurrent function to compute a_n for a given n, there is nothing more to say about it.

2.3) Create a function esExp that verifies if a string is in EXP Here we used a recurrent approach like before nonetheless in this case the idea is not that straightforward.

The main idea is that we thought in a way that we suppose that the input expression is in EXP (it is easier to see it this way first). In that case, it necessarily needs to be decomposed as two substrings in EXP, each substring may be decomposed as two substrings in EXP and over and over again until we have a tree which leaves are simply elements in $V \cup C = \{a, b, 1, 2\}$.

So the idea is that, given a input string S, there are two cases : - Basic case : S has a length of 1 and $S \in \{a, b, 1, 2\}$ - Inductive case : both substrings of S are in EXP.

The idea seems great but if we do not consider the basic case which is easy to implement, the inductive case looks difficult : how do we extract the two substrings from S ?

The idea is the find the separation index i_{sep}. This i_{sep} can be computed be iterating through the string and counting the parenthesis ($\implies +1$ and) $\implies -1$. Of course we need to be careful and make some verification in the code to avoid obvious false cases like "((((((" for example which could make the code crashes.

Once we have the separation index, we need to be careful and verify the separation symbol (make sure it is in $\{*, +, =\}$). In the case of $=$, we also need to make sure that the next symbol is a $=$.

We did not describe everything here however this is quite the global idea. We encourage you to take a look at the code to get a better idea of how it works.

2.4) Provide an expresiones.py file that get an string input S and display esExp(S) and cantidad($|S|$) on a single line separated by a single space. This is simply a combination of the two previous functions.