WORKED EXAMPLES 4

1-1 MULTIVARIATE TRANSFORMATIONS

Given a collection of variables $(X_1, ...X_k)$ with range $\mathbb{X}^{(k)}$ and joint pdf $f_{X_1,...,X_k}$ we can construct the pdf of a transformed set of variables $(Y_1, ...Y_k)$ using the following steps:

1. Write down the set of transformation functions $g_1, ..., g_k$
$$\begin{aligned} Y_1 &= g_1 (X_1, ..., X_k) \\ &\vdots \\ Y_k &= g_k (X_1, ..., X_k) \end{aligned} \quad .$$
2. Write down the set of inverse transformation functions $g_1^{-1}, ..., g_k^{-1}$
$$\begin{aligned} X_1 &= g_1^{-1} (Y_1, ..., Y_k) \\ &\vdots \\ X_k &= g_k^{-1} (Y_1, ..., Y_k) \end{aligned} \quad .$$
3. Consider the joint range of the new variables, $\mathbb{Y}^{(k)}$.
4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives
$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_k} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & \cdots & \dfrac{\partial x_2}{\partial y_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_k}{\partial y_1} & \dfrac{\partial x_k}{\partial y_2} & \cdots & \dfrac{\partial x_k}{\partial y_k} \end{bmatrix}$$
where, for each (i, j)
$$\frac{\partial x_i}{\partial y_j} = \frac{\partial}{\partial y_j} \left\{ g_i^{-1} (y_1, ..., y_k) \right\}$$
and then set $|J (y_1, ..., y_k)| = |\det D_y|$

Note that
$$\det D_y = \det D_y^T$$
so that an alternative but equivalent Jacobian calculation can be carried out by forming D_y^T. *Note also that*
$$|J (y_1, ..., y_k)| = \frac{1}{|J (x_1, ..., x_k)|}$$
where $J (x_1, ..., x_k)$ *is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with* $(Y_1, ..., Y_k)$ *and transform to* $(X_1, ..., X_k)$*).*
5. Write down the joint pdf of $(Y_1, ...Y_k)$ as
$$f_{Y_1,...,Y_k} (y_1, ..., y_k) = f_{X_1,...,X_k} \left(g_1^{-1} (y_1, ..., y_k), ..., g_k^{-1} (y_1, ..., y_k)\right) \times |J (y_1, ..., y_k)|$$
for $(y_1, ..., y_k) \in \mathbb{Y}^{(k)}$.

EXAMPLE 1 Suppose that X_1 and X_2 have joint pdf

$$f_{X_1,X_2}(x_1,x_2) = 2, \qquad 0 < x_1 < x_2 < 1,$$

and zero otherwise. Compute the joint pdf of random variables

$$Y_1 = \frac{X_1}{X_2}, \qquad Y_2 = X_2.$$

SOLUTION

1. Given that $\mathbb{X}^{(2)} \equiv \{(x_1,x_2) : 0 < x_1 < x_2 < 1\}$ and

$$g_1(t_1,t_2) = \frac{t_1}{t_2}, \qquad g_2(t_1,t_2) = t_2.$$

2. Inverse transformations:

$$\left.\begin{array}{l} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} X_1 = Y_1Y_2 \\ X_2 = Y_2 \end{array}\right.$$

and thus

$$g_1^{-1}(t_1,t_2) = t_1t_2, \qquad g_2^{-1}(t_1,t_2) = t_2.$$

3. Range: to find $\mathbb{Y}^{(2)}$ consider point by point transformation from $\mathbb{X}^{(2)}$ to $\mathbb{Y}^{(2)}$ For a pair of points $(x_1,x_2) \in \mathbb{X}^{(2)}$ and $(y_1,y_2) \in \mathbb{Y}^{(2)}$ linked via the transformation, we have

$$0 < x_1 < x_2 < 1 \Leftrightarrow 0 < y_1y_2 < y_2 < 1$$

and hence we can extract the inequalities

$$0 < y_2 < 1 \text{ and } 0 < y_1 < 1 \qquad \therefore \qquad \mathbb{Y}^{(2)} \equiv (0,1) \times (0,1)\,.$$

4. The Jacobian for points $(y_1,y_2) \in \mathbb{Y}^{(2)}$ is found from

$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} \Rightarrow |J(y_1,y_2)| = |\det D_y| = |y_2| = y_2.$$

Note that for points $(x_1,x_2) \in \mathbb{X}^{(2)}$ is

$$D_x = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_2} & \dfrac{x_1}{x_2^2} \\ 0 & 1 \end{bmatrix} \Rightarrow |J(x_1,x_2)| = |\det, D_x| = \left|\frac{1}{x_2}\right| = \frac{1}{x_2},$$

verifying that

$$|J(y_1,y_2)| = \frac{1}{|J(x_1,x_2)|}.$$

5. Finally, we have

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2,y_2) \times y_2 = 2y_2, \qquad 0 < y_1 < 1,\, 0 < y_2 < 1,$$

and zero otherwise.

EXAMPLE 2 Suppose that X_1 and X_2 are **independent** and **identically distributed** random variables defined on $\mathbb{R}^+$ each with pdf of the form

$$f_X(x) = \sqrt{\frac{1}{2\pi x}} \exp\left\{-\frac{x}{2}\right\}, \qquad x > 0,$$

and zero otherwise. Compute the joint pdf of random variables $Y_1 = X_1$ and $Y_2 = X_1 + X_2$.

SOLUTION

1. Given that $\mathbb{X}^{(2)} \equiv \{(x_1, x_2) : 0 < x_1, 0 < x_2\}$ and

$$g_1(t_1, t_2) = t_1, \qquad g_2(t_1, t_2) = t_1 + t_2.$$

2. Inverse transformations:

$$\left.\begin{array}{l} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \end{array}\right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1, \qquad g_2^{-1}(t_1, t_2) = t_2 - t_1.$$

3. Range: to find $\mathbb{Y}^{(2)}$ consider point by point transformation from $\mathbb{X}^{(2)}$ to $\mathbb{Y}^{(2)}$ For a pair of points $(x_1, x_2) \in \mathbb{X}^{(2)}$ and $(y_1, y_2) \in \mathbb{Y}^{(2)}$ linked via the transformation, as both original variables are strictly positive, we can extract the inequalities

$$0 < y_1 < y_2 < \infty.$$

4. The Jacobian for points $(y_1, y_2) \in \mathbb{Y}^{(2)}$ is found from

$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_y| = |1| = 1.$$

Note, here, $J(x_1, x_2) = |\det D_x| = 1$ also so that again

$$|J(y_1, y_2)| = \frac{1}{|J(x_1, x_2)|}.$$

5. Finally, we have for $0 < y_1 < y_2 < \infty$,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(y_1, y_2 - y_1) \times 1 = f_{X_1}(y_1) \times f_{X_2}(y_2 - y_1) \qquad \text{by independence} \\ &= \sqrt{\frac{1}{2\pi y_1}} \exp\left\{-\frac{y_1}{2}\right\} \sqrt{\frac{1}{2\pi (y_2 - y_1)}} \exp\left\{-\frac{(y_2 - y_1)}{2}\right\} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{y_1 (y_2 - y_1)}} \exp\left\{-\frac{y_2}{2}\right\}, \end{aligned}$$

and zero otherwise.

Here, for $y_2 > 0$,

$$\begin{aligned}
f_{Y_2}(y_2) &= \int f_{Y_1,Y_2}(y_1,y_2)\,dy_1 = \int_0^{y_2} \frac{1}{2\pi}\frac{1}{\sqrt{y_1(y_2-y_1)}}\exp\left\{-\frac{y_2}{2}\right\}dy_1 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{y_2}\frac{1}{\sqrt{y_1(y_2-y_1)}}dy_1 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{1}\frac{1}{\sqrt{ty_2(y_2-ty_2)}}y_2dt \qquad \text{setting } y_1 = ty_2 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{1}\frac{1}{\sqrt{t(1-t)}}dt \\
&= \frac{1}{2}\exp\left\{-\frac{y_2}{2}\right\},
\end{aligned}$$

as (by MAPLE, or further transformations)

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}}dt = \pi.$$