Quiz-2 Solution (Gaussian Processes)

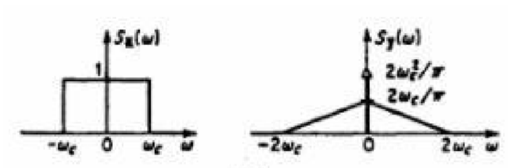
1. Solution:

   (a) According to the convolution theorem in frequency domain we have:

$$R_y(\tau) = E[X^2(t+\tau)X^2(t)] = E[X^2(t+\tau)]E[X^2(t)] + 2E^2[X(t+\tau)X(t)] = R_x^2(0) + 2R_x^2(\tau)$$

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + 2S_x(\omega) * S_x(\omega)$$

   (b) $S_y(\omega)$ would look like this:



2. Solution:

   (a) This can be done by standard (and quite tedious) manipulations, but if we first look at $t = 0$ and condition on a sample value of R, we are simply looking at $\cos(\theta)$, and since θ is uniform over $[0, 2\pi)$, it seems almost obvious that the mean should be zero. To capture this intuition, note that $\cos(\theta) = -\cos(\theta + \pi)$. Since θ is uniform between 0 and 2π, $E[\cos(\theta)] = E[\cos(\theta + \pi)]$, so that $E[\cos(\theta)] = 0$. The same argument works for any t, so the result follows.

   (b) Since θ and R are independent, we have:

$$E[X(t)X(t+\tau)] = E[R^2]E[\cos(2\pi ft + \theta)cos(2\pi f(t+\tau) + \theta)]$$

$$= E[R^2]\frac{1}{2}E[\cos(4\pi ft + 2\pi f\tau + 2\theta) + \cos(2\pi f\tau)]$$

$$= \frac{E[R^2]\cos(2\pi f\tau)}{2}$$

   (c) Let W_1, W_2 be iid normal Gaussian rv's. These can be expressed in polar coordinates as $W_1 = R\cos\theta$ and $W_2 = R\sin\theta$, where R is Rayleigh and θ is uniform. The rv $R\cos\theta$ is then $N(0, 1)$. Similarly, $X(t)$ is a linear combination of W_1 and W_2 for each t, so each set $X(t_1), X(t_2), ...X(t_k)$ of rv's is jointly Gaussian. It follows that the process is Gaussian.