Quiz 2 (Gaussian Processes) Solutions

1 -a)

$$E[X(t)] = E\left[\int_0^t W(\tau)\, d\tau\right] = \int_0^t E[W(\tau)]\, d\tau = \int_0^t 0\ d\tau = 0$$

$$R_X(t, t+\tau) = E[X(t)X(t+\tau)] = E\left[\int_0^t \int_0^{t+\tau} W(u)W(v)\, dv\, du\right]$$

$$= \int_0^t \int_0^{t+\tau} E[W(u)W(v)]\, dv\, du$$

$$= \int_0^t \int_0^{t+\tau} \alpha\delta(u-v)\, dv\, du$$

Case 1: $\tau \geq 0$

$$R_X(t, t+\tau) = \int_0^t \alpha\, du = \alpha t$$

Case 2: $\tau < 0$

$$R_X(t, t+\tau) = \int_0^{t+\tau} \int_0^t \alpha\delta(u-v)\, du\, dv = \int_0^{t+\tau} \alpha\, dv = \alpha(t+\tau)$$

$$\Rightarrow\ R_X(t, t+\tau) = \alpha \min\{t, t+\tau\}$$

1-b)

Since $Var\big(X(t)\big) = \alpha.t$ is dependent on t, it cannot be stationary.

2)

Let $W = [W(t_1),\ W(t_2),\ \ldots, W(t_n)]^T$ denote a vector of samples of a Brownian motion process. To prove that W(t) is a Gaussian process, we must show that W is a Gaussian random vector. To do so, let

$$X = [X_1, X_2, \ldots, X_n]^T = [W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})]^T$$

denote the vector of increments. By the definition of Brownian motion, $X_1, X_2, \ldots, X_n$is a sequence of independent Gaussian random variables. Thus, X is a Gaussian random vector. Finally,

$$W = \begin{bmatrix} X_1 \\ X_1 + X_2 \\ \dots \\ X_1 + X_2 + \dots + X_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \dots & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix} . X$$

Since X is a Gaussian random vector and $W = AX$, where A is a rank n matrix, so W is also a Gaussian random vector.