Notes

March 01^{st}**, 2011**

Exercise 2.10 pag 90 in [1]

Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop then it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to time s, and if a bus has not yet arrived by that time then you walk home.

a) Compute the expected time from when you arrive at the bus stop until you reach home.

b) Show that if $W < 1/\lambda + R$ then the expected time of part a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimized by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.

c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.

Solution

Let X be the arrival time of the bus, and T_s be the total time to get home under the strategy "wait for bus till time s", then we have

$$T_s = (s + W)\,1\{X > s\} + (X + R)\,1\{X \le s\}\ .$$

a) Computing the expectation we have that

$$\begin{aligned}
\mathbb{E}[T_s] &= \mathbb{E}[(s+W)\,1\{X > s\} + (X+R)\,1\{X \le s\}] \\
&= (s+W)\Pr(X > s) + \int_0^s (x+R) f_X(x)dx \\
&= (s+W)e^{-s\,\lambda} - s\,e^{-s\,\lambda} + (1/\lambda + R)(1 - e^{-s\,\lambda}) \\
&= \Big(W - (1/\lambda + R)\Big)e^{-s\,\lambda} + (1/\lambda + R)
\end{aligned}$$

b) From the relation above we see that if $W - (1/\lambda + R)$ is positive we minimize the expectation by letting $e^{-s\,\lambda} \to 0$, i.e. $s = \infty$. If $W - (1/\lambda + R)$ is negative, the expectation is minimized by the largest value of $e^{-s\,\lambda}$ that we have for $s = 0$. If $W = (1/\lambda + R)$ then $\mathbb{E}[T_s] = (1/\lambda + R)$ independently of the choice of s.

c) An intuitive explanation of the above result is the following. If $W < (1/\lambda + R)$, it is obvious that on average it is better to walk home since going home by bus will take an average time of $(1/\lambda + R)$. Assume now the contrary, i.e. $W > (1/\lambda + R)$, hence it is better to wait at least for a while for the bus. But as soon as we wait, for the memoryless property of the Poisson process, we are back in the initial situation and taking again the decision if to wait or not it will be better to wait for a while, it follows that $s \to \infty$.

Take-home exercise

Try to answer to the questions a)–c) above assuming that the arrival process for the buses is a renewal process with inter-arrival times distributed according to a distribution Erlang$(2, \lambda)$.

References

[1] S.M. Ross (1996): *Stochastic processes*, 2ed. *John Wiley & Sons*.