Quiz-5 Solution (Point Processes)

Questions

1. (a) $\{Sn \leq t\}$ is the event that the n^{th} arrival occurs at some epoch $\tau \leq t$. This event implies that $N(\tau) = n$, and thus that $\{N(t) \geq n\}$. Similarly, $\{N(t) = m\}$ for some $m \geq n$ implies $\{S_m \leq t\}$, and thus that $\{S_n \leq t\}$.

   (b) for n=2:
$$f_{S_1,S_2}(s_1, s_2) = f_{X_1,S_2}(x_1, s_2) = f_{X_1}(x_1) f_{S_2|X_1}(s_2|x_1) = \lambda e^{-\lambda x_1} \times \lambda e^{-\lambda(s_2-x_1)} = \lambda^2 e^{-\lambda s_2} \tag{1}$$
   Then we suppose we have:
$$f_{S_1,S_2,...,S_n}(s_1, s_2, ..., s_n) = \lambda^n e^{-\lambda s_n},\ \textit{for some } n > 1 \tag{2}$$
   Then we can write:
$$\begin{aligned} &f_{S_1,...,S_n,S_{n+1}}(s_1, ..., s_n, s_{n+1}) = \\ &\qquad f_{S_1,...,S_n}(s_1, ..., s_n) f_{S_{n+1}|S_1,...,S_n}(s_{n+1}|s_1, ..., s_n) = \\ &\qquad \lambda^n e^{-\lambda s_n} f_{S_{n+1}|S_1,...,S_n}(s_{n+1}|s_1, ..., s_n) \end{aligned} \tag{3}$$
   We know that $S_{n+1} = S_n + X_{n+1}$. So:
$$f_{S_{n+1}|S_1,...,S_n}(s_{n+1}|s_1, ..., s_n) = \lambda e^{-\lambda(s_{n+1}-s_n)} \tag{4}$$
   Then
$$\begin{aligned} f_{S_1,...,S_n,S_{n+1}}(s_1, ..., s_n, s_{n+1}) &= \lambda^n e^{-\lambda s_n} \lambda e^{-\lambda(s_{n+1}-s_n)} \\ &= \lambda^{n+1} e^{-\lambda s_{n+1}} \end{aligned} \tag{5}$$

2. The likelihood function is the joint density function of all the points in the observed point pattern $(t_1, ..., t_n) \in [0, T)$, and can therefore be factorised into all the conditional densities of each points given all points before it. This yields
$$L = f^*(t_1) ... f^*(t_n)(1 - F^*(T)), \tag{6}$$
   where the last term $(1 - F^*(T))$ appears since the unobserved point $t_n + 1$ must appear after the end of the observation interval. So we can write:
$$\begin{aligned} L &= \left(\prod_{i=1}^{n} f^*(t_i)\right) \frac{f^*(T)}{\lambda^*(T)} \\ &= \left(\prod_{i=1}^{n} \lambda^*(t_i) \exp\left(-\int_{t_{i-1}}^{t_i} \lambda^*(s) ds\right)\right) \exp\left(-\int_{t_n}^{T} \lambda^*(s) ds\right) \\ &= \left(\prod_{i=1}^{n} \lambda^*(t_i)\right) \exp\left(-\int_0^T \lambda^*(s) ds\right) \end{aligned} \tag{7}$$

where t0 = 0.

3. (a) $Z \leq t$ if and only if $X_i \leq t$ for each i, $1 \leq n$, so

$$Pr\{Z \leq t\} = \prod_{i=1}^{n} Pr\{X_i \leq t\} = [1 - \exp(-\lambda t)]^n \tag{8}$$

   (b) You can view T_1 as the time of the first arrival out of n Poisson processes each of rate λ. Thus T_1 is exponential with parameter $n\lambda$. More directly yet, $T_1 > t$ if and only if $X_i > t$ for $1 \leq i \leq t$, so $Pr\{T_1 > t\} = [exp(-\lambda t)]^n = exp(-n\lambda t)$. The time T_2 is the remaining time until the next student out of the remaining n-1 finishes. Because of the memorylessness of the exponential distribution, each of these n-1 students has an exponential time to go, so $Pr\{T_2 > t_2\} = \exp(-(n-1)\lambda t_2)$. Each of these times-to-go are independent of T_1, so T_2 is independent of T_1. In the same way T_i is exponential wih parameter $(n - i + 1)\lambda$ and is independent of the earlier T_is.