In-Class Quiz 3 (Point Processes) - Solution

1. Explain what is an Evolutionary Point Process? What does the history $\mathcal{H}_t$ consist of in this type of point process?
Solution:
Usually we think of time as having an evolutionary character: what happens now may depend on what happened in the past, but not on what is going to happen in the future. This order of time is also a natural starting point for defining practically useful temporal point processes. Roughly speaking, we can define a point process by specifying a stochastic model for the time of the next event given we know all the times of previous events. The term evolutionary point process is used for processes defined in this way.
If we consider the time t, then the history $\mathcal{H}_t$ is the list of times of events $(..., t_1, t_2, ..., t_n)$ up to but not including time t.

2. As you know, $f^*(t) = f(t|\mathcal{H}_t)$ is the conditional density function of the time of the next event given the history of previous events. The conditional intensity function is also defined by

$$\lambda^*(t) = \frac{f^*(t)}{1 - F^*(t)} \tag{1}$$

(a) Prove the following equation where dt is an infinitisemal interval around t:

$$\lambda^*(t)\mathrm{d}t = \mathbb{E}[N(\mathrm{d}t)|\mathcal{H}_t] \tag{2}$$

Solution:

$$\begin{aligned}
\lambda^*(t)\mathrm{d}t &= \frac{f^*(t)\mathrm{d}t}{1 - F^*(t)} \\
&= \frac{\mathbb{P}(\text{point in dt}|\mathcal{H}_t)}{\mathbb{P}(\text{point not before t}|\mathcal{H}_t)} \\
&= \frac{\mathbb{P}(\text{point in dt, point not before t}|\mathcal{H}_t)}{\mathbb{P}(\text{point not before t}|\mathcal{H}_t)} \\
&= \mathbb{P}(\text{point in dt}|\text{point not before t}, \mathcal{H}_t) \\
&= \mathbb{P}(\text{point in dt}|\mathcal{H}_t) \\
&= \mathbb{E}[N(\mathrm{d}t)|\mathcal{H}_t].
\end{aligned}$$

(b) Considering the previous equation, interpret the conditional intensity function.
Solution:
$N(A)$ denotes the number of points falling in an interval, and the last equality follows from the assumption that no points coincide, so that there is either zero or one point in an infinitisemal interval. In other words, the conditional intensity function specifies the mean number of events in a region.