Take-Home Quiz 3 (Point Processes) - Solution

1. Consider a homogeneous Poisson process with intensity λ.

   (a) Suppose that up to time t, exactly one arrival occured. Given this information, find the conditional distribution of the arrival time.

   (b) Suppose that exactly two arrivals occured. Compute the conditional expectations of both arrival times.

Solution:

a) For $0 \leq s \leq t$, we have:

$$\begin{aligned}
F_{S_1|N_t=1}(s) &= \mathbb{P}(S_1 \leq s \mid N_t = 1) = \\
&= \mathbb{P}(N_s \geq 1 \mid N_t = 1) = \\
&= \mathbb{P}(N_s = 1 \mid N_t = 1) = \\
&= \frac{\mathbb{P}(N_s = 1, N_t = 0)}{\mathbb{P}(N_t = 1)} = \\
&= \frac{\mathbb{P}(N_s = 1, N_t - N_s = 0)}{\mathbb{P}(N_t = 1)} = \\
&= \frac{\mathbb{P}(N_s = 1)\,\mathbb{P}(N_t - N_s = 0)}{\mathbb{P}(N_t = 1)} = \\
&= \frac{\lambda s\, e^{-\lambda s}\, e^{-\lambda(t-s)}}{\lambda t\, e^{-\lambda t}} = \\
&= \frac{s}{t}\,.
\end{aligned}$$

Thus, the desired conditional distribution is exactly the uniform distribution over $(0, t)$.

b) Similarly as before, for $0 \leq s \leq t$, compute:

$$\begin{aligned} F_{S_2|N_t=2}(s) &= \mathbb{P}(N_s \geq 2 \mid N_t = 2) = \\ &= \frac{\mathbb{P}(N_s = 2, N_t - N_s = 0)}{\mathbb{P}(N_t = 2)} = \\ &= \frac{s^2}{t^2}, \end{aligned}$$

so that $f_{S_2|N_t=2}(s) = \dfrac{2s}{t^2}$ and $\mathbb{E}(S_2 \mid N_t = 2) = \dfrac{2t}{3}$.

Next,

$$\begin{aligned} F_{S_1|N_t=2}(s) &= \mathbb{P}(N_s \geq 1 \mid N_t = 2) = \\ &= \frac{\mathbb{P}(N_s = 1, N_t - N_s = 1) + \mathbb{P}(N_s = 2, N_t - N_s = 0)}{\mathbb{P}(N_t = 2)} = \\ &= \frac{2st - s^2}{t^2}, \end{aligned}$$

so that $f_{S_1|N_t=2}(s) = \dfrac{2(t-s)}{t^2}$ and $\mathbb{E}(S_1 \mid N_t = 2) = \dfrac{t}{3}$.

2. Let $\rho : (0, \infty) \to [0, \infty)$ be a function. A Poisson process with intensity function ρ is a counting process characterized by the following two properties:

   - For $a \leq b$, $N_b - N_a \sim Poisson\left(\int_a^b \rho(t)dt\right)$. Consequently, $N_t \sim Poisson\left(\int_0^t \rho(s)ds\right)$.
   - Any restrictions of the process (regarded as a random subset of $(0, \infty)$) to disjoint intervals are independent.

   Consider a Poisson process with intensity function:

$$\rho(t) = \frac{1}{1+t} \tag{1}$$

   Find the distribution of the first two (inter)-arrival times T_1 and T_2.

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$$P(T_۱ > t) = P(N_t = ۰) = \exp\left(-\int_۰^t \frac{du}{۱+u}\right) = \frac{۱}{۱+t} = ۱ - F_{T_۱}(t) \qquad (۱)$$

با مشتق‌گیری از این عبارت، تابع چگالی احتمال به دست می‌آید:

$$f_{T_۱}(t) = \frac{۱}{(۱+t)^۲} \qquad (۲)$$

حال برای به دست آوردن تابع چگالی احتمال متغیر تصادفی $T_۲$، می‌دانیم متغیر تصادفی $N_{T_۱+s} - N_{T_۱}$ از توزیع پواسون با پارامتر زیر پیروی می‌کند:

$$\int_{T_۱}^{T_۱+s} \frac{۱}{۱+t} dt = \ln\left(\frac{۱+T_۱+s}{۱+T_۱}\right) \qquad (۳)$$

بنابراین داریم:

$$P(T_۲ > s|T_۱) = P(N_{T_۱+s} N_{T_۱} = ۰|T_۱) = \exp\left(-\ln\left(\frac{۱+T_۱+s}{۱+T_۱}\right)\right) = \frac{۱+T_۱}{۱+T_۱+s} \qquad (۴)$$

در گام بعد با انتگرال‌گیری، به احتمال غیر شرطی برای متغیر $T_۲$ می‌رسیم:

$$P(T_۲ > s) = \int_۰^\infty \frac{۱+t}{۱+t+s} f_{T_۱}(t) dt = \int_۰^\infty \frac{dt}{(۱+t)(۱+t+s)} = \frac{\ln(۱+s)}{s} \qquad (۵)$$

و در صورتی که مجددا مشتق بگیریم تابع چگالی احتمال به دست می‌آید:

$$f_{T_۲}(s) = \frac{s - (۱+s)\ln(۱+s)}{s^۲(۱+s)} \qquad (۶)$$

3. Let N be a random variable denoting the number of arrivals, ditributed by Poisson $Pois(\lambda)$. Each arrival is *successful* with probability p, independently of other arrivals, as well as of the number of arrivals. Denote by S the number of successful and by T the number of unsuccessful arrivals, that is, $T = N - S$.

   (a) Find the distribution of S and T.

   (b) Show that the random variables S and T are independent.

   (c) Show that under some other choice of the distribution of N, S and T are no longer necessarily independent.

a) Conditionally given N, we have $S \sim \mathrm{Bin}(N, p)$, i. e.:

$$\mathbb{P}(S = k \mid N = n) = \binom{n}{k} p^k (1-p)^{n-k}\,; \quad k = 0, 1, \ldots, n\,.$$

By the total probability theorem, we compute:

$$\begin{aligned}
\mathbb{P}(S = k) &= \sum_{n=k}^{\infty} \mathbb{P}(N = n)\, \mathbb{P}(S = k \mid N = n) = \\
&= \sum_{n=k}^{\infty} \frac{\lambda^n\, e^{-\lambda}}{n!} \binom{n}{k} p^k (1-p)^{n-k} = \\
&= \frac{\lambda^n p^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} = \\
&= \frac{(p\lambda)^k\, e^{-p\lambda}}{k!}\,.
\end{aligned}$$

Thus, $S \sim \mathrm{Pois}(p\lambda)$. Similarly, $T \sim \mathrm{Pois}((1-p)\lambda)$.

b) Observing that:

$$\begin{aligned}
\mathbb{P}(S = k, T = l) = \mathbb{P}(S = k, N = k + l) &= \mathbb{P}(N = k + l)\, \mathbb{P}(S = k \mid N = k + l) = \\
&= \frac{\lambda^{k+l}\, e^{-\lambda}}{(k+l)!} \binom{k+l}{k} p^k (1-p)^l = \frac{\lambda^{k+l}\, p^k (1-p)^l\, e^{-\lambda}}{k!\, l!} = \\
&= \mathbb{P}(S = k)\, \mathbb{P}(T = l)\,,
\end{aligned}$$

we find that S and T are indeed independent.

c) If $N = n$ is a constant S are T dependent, provided that $n \geq 1$ and $0 < p < 1$: in this case, S in T take at least two values (with positive probability), but $S = k$ implies $T = n - k$.