Name: Std. Number:

Quiz 5 (Point Processes)

Questions

1. An arrival process is a sequence of increasing rv s, $0 < S_1 < S_2 < ...$, where $S_i < S_{i+1}$ means that $S_{i+1} - S_i$ is a positive rv, and S_i is the time when i^{th} event occurs. Any arrival process $\{S_n;\ n \geq 1\}$ can also be specified by either of two alternative stochastic processes. The first alternative is the sequence of interarrival times, $\{X_i; i \geq 1\}$. The X_i here are positive rv s defined in terms of the arrival epochs by $X_1 = S_1$ and $X_i = S_i - S_{i-1}$ for $i > 1$. The second alternative is the counting process $\{N(t); t > 0\}$, where for each $t > 0$ the rv N(t) is the aggregate number of arrivals up to and including time t.

   (a) Show that $P(S_n \leq t) = P(N(t) \geq n)$

   (b) Suppose $X_1, X_2, ...$ are iid rv s from $f_X(x) = \lambda e^{-\lambda x}$. We define a new rv $S_n = X_1 + X_2 + ... + X_n$ for all $n \geq 1$. Show that

   $$f_{S_1,S_2,...,S_n}(s_1, s_2, ..., s_n) = \lambda^n e^{-\lambda s_n},\ \textit{for}\ 0 < s_1 < s_2 < ... < s_n,\ \textit{and}\ n > 1 \qquad (1)$$

2. Given an unmarked point pattern $(t_1, ..., t_n)$ on an observation interval [0, T), show that the likelihood function is as follows where $\lambda^*(t)$ is the conditional intensity function at time t.

   $$L = \left(\prod_{i=1}^{n} \lambda^*(t_i)\right) \exp\left(-\int_0^T \lambda^*(s)ds\right) \qquad (2)$$

3. A final exam is started at time 0 for a class of n students. Each student is allowed to work until completing the exam. It is known that each students time to complete the exam is exponentially distributed with density $f_X(x) = \lambda e^{-\lambda x}; x \geq 0$. The times $X_1, ..., X_n$ are IID.

   (a) Let Z be the time at which the last student finishes. Show that Z has a distribution function $F_Z(z)$ given by $[1 - e^{-\lambda z}]^n$.

   (b) Let T_1 be the time at which the first student leaves. Show that the probability density of T_1 is given by $n\lambda e^{-n\lambda t}$. For each i, $2 \leq i \leq n$, let T_i be the interval from the departure of the $i - 1^{\text{th}}$ student to that of the i^{th}. Show that the density of each T_i is exponential and find the parameter of that exponential density. Explain why T_is are independent.