Solutions, TH-Quiz 6 (Point Processes)

Q1

Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1 + \lambda_2)$, independently of the time of the event.

Solution: We check that $N(t) = N_1(t) + N_2(t)$ satisfies Definition 1.
(i) $N(t) = 0$.
(ii) Note that $N_1(t)$ and N_2 have independent increments. Moreover, $N_1(t)$ and $N_2(t)$ are independent.
(iii) Indeed, for any $t, s > 0$,

$$\begin{aligned}
\mathbb{P}(N(t+s) - N(t) = n) &= \sum_{k=0}^{n} \mathbb{P}(N_1(t+s) - N_1(t) = n-k | N_2(t+s) - N_2(t) = k) \\
&\qquad \times \mathbb{P}(N_2(t+s) - N_2(t) = k) \\
&= \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k}}{(n-k)!} \exp\{-\lambda_1 s\} \frac{(\lambda_2 s)^k}{k!} \exp\{-\lambda_2 s\} \\
&= \exp\{-(\lambda_1 + \lambda_2)s\} \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k} (\lambda_2 s)^k}{(n-k)!k!} \\
&= \frac{((\lambda_1 + \lambda_2)s)^n}{n!} \exp\{-(\lambda_1 + \lambda_2)t\}.
\end{aligned}$$

Now to show that the probability of the first arrival is from $N_1(t)$. Let X be the first arrival time for $N(t)$, and X_1, X_2 the corresponding times for $N_1(t)$ and $N_2(t)$.

One way to do is, observing that, $X \sim \text{Exponential}(\lambda_1 + \lambda_2)$,

$$\begin{aligned}
\mathbb{P}(\text{first event from } N_1(t) | X = x) &= \lim_{\delta_x \to 0} \mathbb{P}(X_1 < X_2 | X \in [x, x + \delta_x]) \\
&= \lim_{\delta_x \to 0} \frac{\mathbb{P}(X_1 \in [x, x+\delta_x]) \mathbb{P}(X_2 > x) + o(\delta_x)}{\mathbb{P}(X \in [x, x+\delta_x])} \\
&= \frac{e^{-\lambda_1 x}(\lambda_1 \delta_x + o(\delta_x)) e^{-\lambda_2 x} + o(\delta_x)}{e^{-(\lambda_1+\lambda_2)x}((\lambda_1 + \lambda_2)\delta_x + o(\delta_x))} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

As required, this probability does not depend on the first event time for $N(t)$.

Q2

Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop the it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.
(a) Compute the expected time from when you arrive at the bus stop until you reach home.
(b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimizes by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.
(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.
Solution: **(a)** Let $E_s = \mathbb{E}(\text{journey time for strategy } s)$. The journey time is the function of the first arrival time of the rate λ Poisson process of bus arrivals. This has Exponential(λ) distribution (prop 2.2.1). So

$$E_s = \int_0^\infty \lambda e^{-\lambda t}[(t+R)\mathbf{1}(t \le s) + (s+W)\mathbf{1}(t > s)]\,dt$$

where $\mathbf{1}$ is the indicator function. Thus

$$\begin{aligned} E_s &= \int_0^s \lambda t e^{-\lambda t}\,dt + R\int_0^s \lambda e^{-\lambda t}\,dt + (s+W)\int_s^\infty \lambda e^{-\lambda t}\,dt \\ &= \frac{1-e^{-\lambda s}}{\lambda} + R(1-e^{-\lambda s}) + We^{-\lambda s} \end{aligned}$$

(b) Writing $E_s = (W - R - \frac{1}{\lambda})e^{-\lambda s} + \frac{1}{\lambda} + R$. We see that E_s is a decreasing function of s for $(W - R - 1/\lambda) > 0$, and increasing function for $(W - R - 1/\lambda) < 0$ and constant if $(W - R - 1/\lambda) = 0$.
(c) From the memoryless property of the exponential distribution , if it was worth waiting some time $s_0 > 0$ for a bus, and the bus has not arrived at s_0, then resetting time suggests that it must be worth waiting another s_0 time units. Thus, if the optimal s is positive, it must be infinite.