Take-Home Quiz 1 (Gaussian Process)

1 Problem 1

Assume X, Y are joint Gaussian with zero mean and their variances are σ_X^2, σ_Y^2 respectively. let normalized covariance matrix be ρ.

1.1 A

let $V = Y^3$ find joint p.d.f. of $f_{X|V}(x|v)$.

1.2 B

let $U = Y^2$ find joint p.d.f. of $f_{X|U}(x|u)$.

1.3 Solution A

$$\rho = \frac{E[XY]}{\sigma_X \sigma_Y}$$

$$f_{X|V}(x|v) = \frac{f(X=x, V=v)}{f(V=v)} = \frac{f(X=x, Y^3=v)}{f(Y^3=v)} = \frac{f(X=x, Y=\sqrt[3]{v})}{f(Y=\sqrt[3]{v})}$$

$$= \frac{\frac{1}{2\pi}\frac{1}{|\Sigma|^{\frac{1}{2}}} exp\{-\frac{1}{2}[x\,\sqrt[3]{v}]\Sigma^{-1}[x\,\sqrt[3]{v}]^T\}}{\frac{1}{2\pi}\frac{1}{\sigma_y} exp\{-\frac{v^{\frac{2}{3}}}{2\sigma_y^2}\}}$$

We know that:

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \rightarrow det(\Sigma) = \sigma_x^2\sigma_y^2(1-\rho^2)$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|}\begin{pmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_x^2(1-\rho^2)} & \frac{\rho}{\sigma_x\sigma_y(1-\rho^2)} \\ -\frac{\rho}{\sigma_x\sigma_y(1-\rho^2)} & \frac{1}{\sigma_y^2(1-\rho^2)} \end{pmatrix}$$

$$\Rightarrow [x\,\sqrt[3]{v}]\Sigma^{-1}[x\,\sqrt[3]{v}]^T = \frac{1}{1-\rho^2}[\frac{x}{\sigma_x^2} + \frac{\sqrt[3]{v^2}}{\sigma_Y} - \frac{2\rho x\sqrt[3]{v}}{\sigma_X\sigma_Y}]$$

Try to derive last equation your self!

So we can write the p.d.f.:

$$f_{X|V}(x|v) = \frac{1}{2\pi}\frac{1}{\sigma_x\sqrt{1-\rho^2}} exp\{-\frac{1}{2}[\frac{x}{\sigma_x^2} + \frac{\sqrt[3]{v^2}}{\sigma_Y} - \frac{2\rho x\sqrt[3]{v}}{\sigma_X\sigma_Y}]\frac{1}{1-\rho^2} + \frac{v^{\frac{2}{3}}}{2\sigma_y^2}\}$$

Simplify the above expression and you'll se that:

$$f_{X|V}(x|v) \sim \mathcal{N}(\rho\frac{\sigma_x}{\sigma_y}v^{\frac{1}{3}}, \sigma_x^2(1-\rho^2))$$

1.4 Solution B

The overall path to solve this part is like previous one except:

$$
\begin{aligned}
f_{X|U}(x|u) &= \frac{f(X=x,U=u)}{f(U=u)} = \frac{f(X=x,Y^2=u)}{f(Y^2=u)} = \frac{f(X=x,Y=\pm\sqrt{u})}{f(Y=\pm\sqrt{u})} \\
&= \frac{f(X=x,Y=\sqrt{u})+f(X=x,Y=-\sqrt{u})}{f(Y=\sqrt{u})+f(Y=-\sqrt{u})}
\end{aligned}
$$

Again by forming numerator and denumerator terms we have:

$$
f_{X|U}(x|u) = \frac{1}{2}[\mathcal{N}(\rho\frac{\sigma_X}{\sigma_Y}\sqrt{u}, \sigma_x^2(1-\rho^2)) + \mathcal{N}(-\rho\frac{\sigma_X}{\sigma_Y}\sqrt{u}, \sigma_x^2(1-\rho^2))]
$$

2 Problem 2

2.1 A

Assume $X_1 \sim \mathcal{N}(0,\sigma_1^2)$, $X_2 \sim \mathcal{N}(0,\sigma_2^2)$ are independent. show $X_1 + X_2 \sim \mathcal{N}(0,\sigma_1^2+\sigma_2^2)$.

2.2 B

Assume W_1, W_2 are normalized iid gaussian random variables. show $\alpha_1 W_1 + \alpha_2 W_2 \sim \mathcal{N}(0,\alpha_1^2+\alpha_2^2)$

2.3 C

Show that any linear combination of normalized iid gaussian random variables is gaussian random variable.

2.4 Solution A

We show this by using moment generating function. First if X, Y are independent random variables then $\phi_{x+y}(t) = \phi_x(t)\phi_y(t)$.

$$
E[e^{t(x+y)}] = E[e^{tx}e^{ty}] \overset{independence}{\rightarrow} E[e^{tx}]E[e^{ty}]
$$

Now lets calculate MGF for normal distribution:

$$
\begin{aligned}
&\phi_x(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}exp(\frac{-x^2}{2\sigma^2})dx \\
&\Rightarrow \frac{-x^2}{2\sigma^2} + tx = -\frac{(x-\mu)^2}{2\sigma^2} + z \\
&\mu = t\sigma^2, z = \frac{t^2}{2}\sigma^2 \\
&\Rightarrow \phi_x(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{(x-t\sigma^2)^2}{2\sigma^2})exp(\frac{t^2}{2}\sigma^2)dx = exp(\frac{t^2}{2}\sigma^2)
\end{aligned}
$$

So from these two statements we have:

$$
\phi_x(t)\phi_y(t) = exp(\frac{t^2}{2}\sigma_1^2)exp(\frac{t^2}{2}\sigma_2^2) = exp(\frac{t^2}{2}(\sigma_1^2+\sigma_2^2)) = \phi_{x+y}(t)
$$

2.5 Solution B

You know that:

$$E[\alpha x] = \alpha E[x]$$
$$Var(\alpha x) = \alpha^2 Var(x)$$

$$W_1 \sim \mathcal{N}(0,1) \rightarrow \alpha_1 W_1 \sim \mathcal{N}(0, \alpha_1^2)$$
$$W_2 \sim \mathcal{N}(0,1) \rightarrow \alpha_2 W_2 \sim \mathcal{N}(0, \alpha_2^2)$$

The apply part A:)

2.6 Solution C

Prove by induction.

Base has been proven by part b.

We know that $Y_k = \sum_{i=1}^{k} \alpha_i W_i \sim \mathcal{N}(0, \sum_{i=1}^{k} \alpha_i^2)$ and want to show $Y_{k+1} \sim \mathcal{N}(0, \sum_{i=1}^{k+1} \alpha_i^2)$

$$Y_{K=1} = Y_k + \alpha_{k+1} W_{k+1}$$

If random variable X is independent of random variables $a_1, a_2, ..., a_n$ then it is also independent of a $g(a_1, a_2, ..., a_n)$. By using this statement and part b the proof is complete.

3 Problem 3

Let X, Y be iid normalized gaussian random variables. let $Z = |Y|Sgn(x)$ and $Sgn(x)$ is sign function where is 1 for $x \geq 0$ and -1 otherwise. Show that X, Z are each gaussian but are not jointly gaussian.

3.1 Solution

Note that Z has the magnitude of Y but sign of X. so X, Z are both positive or both negative. i.e., their joint density is non-zero only in the first and third quadrant of X-Z plane.

Conditional on a given X, conditional density of Z is twice conditional density of Y since both Y and -Y map to the same Z.