Accelerated Failure Time Model

Estimation and Inference

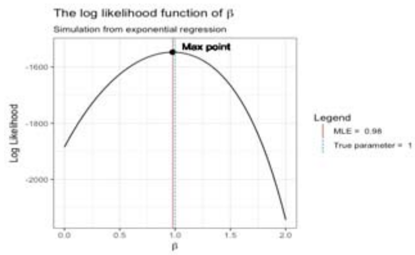
Parameter Estimation

- The AFT model is of the form that
  $Y = \log(T) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p + \sigma\varepsilon.$
- Parameters to be estimated
  - $\boldsymbol{\beta}$=(β_0, β_1, …, β_p): regression coefficients.
  - σ: standard deviation of error term.
- Data consists of independent observations from n individuals,
  $$(t_i, \delta_i, \boldsymbol{x}_i)_{i=1}^n.$$

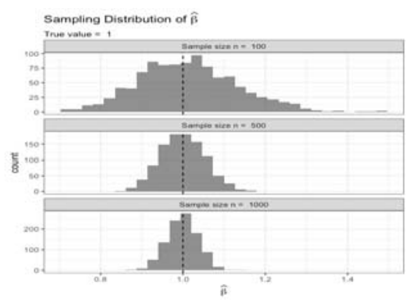
Maximum likelihood estimation

- The likelihood function is
  $$\mathrm{L}(\boldsymbol{\beta}, \sigma | (t_i, \delta_i, \boldsymbol{x}_i)_{i=1}^n) = \prod_i [f(t_i; \boldsymbol{\beta}, \sigma)]^{\delta_i} [S(t_i; \boldsymbol{\beta}, \sigma)]^{1-\delta_i}$$
  - $f(t; \boldsymbol{\beta}, \sigma)$ and $S(t; \boldsymbol{\beta}, \sigma)$ are density and survival functions of true survival time T_i.
- Estimate ($\boldsymbol{\beta}$, σ) by maximizing the likelihood function or log likelihood function.
  $$(\widehat{\boldsymbol{\beta}}, \widehat{\sigma}) = \text{argmax } \mathrm{L}(\boldsymbol{\beta}, \sigma | (t_i, \delta_i, \boldsymbol{x}_i)_{i=1}^n)$$
  $$= \text{argmax } \log \mathrm{L}(\boldsymbol{\beta}, \sigma | (t_i, \delta_i, \boldsymbol{x}_i)_{i=1}^n)$$

An Example



An Example



Newton-Raphson algorithm

- Generally speaking, the likelihood function is a nonlinear function of parameters.
- We must find the MLE numerically such as the Newton-Raphson algorithm.
- We will illustrate estimation of AFT model using SAS.

Hypothesis Testing

- It is often of interest to test hypothesis on the parameters in the AFT model
  $Y = \log(T) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p + \sigma\varepsilon$
- Example 1: to test if covariate x_j has a significant effect on survival time, consider
  $H_0: \beta_j = 0$ vs. $H_1: \beta_j \neq 0$
- Example 2: to test Exponential regression model v.s. Weibull regression model, consider
  $H_0: \sigma = 0$ vs. $H_1: \sigma \neq 0$

Likelihood based testing methods

- Wald test
  - Calculated using quadratic forms of parameter estimates and their estimated variances and covariances.
- Score test
  - Based on quadratic forms of first order derivatives of log-likelihood function and its estimated variances and covariances.
- Likelihood ratio test
  - Maximizing the likelihood function twice: under the null and under the alternative, then twice the positive difference in the two log-likelihood functions.

Likelihood based testing methods

- The three tests are asymptotically equivalent, meaning that their approximate large sample distributions are identical.
- Three test statistics approximately follow the chi-square distribution with the same number of degrees of freedom.
- Some research show that the LR test has better finite sample performance. But in general, Wald and score tests are easier to calculate.

Wald Test

- Wald test for $H_0: \beta_j = 0$ vs. $H_1: \beta_j \neq 0$

$$\chi^2_{Wald} = \left\{\frac{\hat{\beta}_j}{SE(\hat{\beta}_j)}\right\}^2 \sim \chi^2_1$$

Analysis of Maximum Likelihood Parameter Estimates

Parameter		DF	Estimate	Standard Error	95% Confidence Limits		Chi-Square	Wald test Pr > ChiSq
Intercept		1	0.6834	0.6408	-0.5726	1.9394	1.14	0.2862
Allo	allo	1	0.1333	0.4607	-0.7697	1.0363	0.08	0.7723
Allo	auto	0	0.0000	.	.	.	.	.
hodgkins	2	1	-1.3185	0.5275	-2.3524	-0.2845	6.25	0.0124
hodgkins	Non-Hodgkins	0	0.0000	.	.	.	.	.
Kscore		1	0.0758	0.0094	0.0574	0.0942	64.90	<.0001
Wtime		1	0.0093	0.0072	-0.0049	0.0235	1.66	0.1975
Scale		0	1.0000	0.0000	1.0000	1.0000		
Weibull Shape		0	1.0000	0.0000	1.0000	1.0000		

Wald test
$0.08 = (\frac{0.1333}{0.4607})^2$

Likelihood ratio test

- Suppose we want to compare survival of cancer patients with four different stages. In particular we want to test if stage 2 patients have same survival time as phrase 3 cancer patients
  $H_0: \beta_2 = \beta_3$ vs. $H_1: \beta_2 \neq \beta_3$
- Fit two different Weilbull regressions
  - Full model
    $Y = \log(T) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \sigma\varepsilon$
  - Null model
    $Y = \log(T) = \beta_1 x_1 + \beta^*(x_2 + x_3) + \beta_4 x_4 + \sigma\varepsilon$

Likelihood ratio test

Model Information

Data Set	WORK.CAN
Dependent Variable	Log(time)
Censoring Variable	status
Censoring Value(s)	0
Number of Observations	143
Noncensored Values	109
Right Censored Values	34
Left Censored Values	0
Interval Censored Values	0
Number of Parameters	5
Name of Distribution	Weibull
Log Likelihood	-136.890528

Full model

Model Information

Data Set	WORK.CAN
Dependent Variable	Log(time)
Censoring Variable	status
Censoring Value(s)	0
Number of Observations	143
Noncensored Values	109
Right Censored Values	34
Left Censored Values	0
Interval Censored Values	0
Number of Parameters	4
Name of Distribution	Weibull
Log Likelihood	-144.624315

Null model

LR test: 2(144.62-136.89)=15.46
Compare with χ^2_1,p-value=0.000

Summary

- Parameters are estimated by MLE.
- Hypothesis testing methods are developed based on likelihood functions.
- Different type of censoring can be easily incorporated into likelihood function.
- PROC LIFEREG is the only procedure can handle different types of censoring.