Hazard function for discrete random variables

Hazard function: discrete case

Suppose T is a discrete random variable.

- **Probability mass function** (*unconditional failure probability*)

$$f(t_j) = P(T = t_j), for\ j = 1,2,\cdots$$

- **Hazard function**(conditional failure probability)

$$\mathrm{h}(t_j) = P(T = t_j | T \geq t_j) = \frac{f(t_j)}{S(t_{j-1})}$$

A discrete example: uniform

- Suppose 10 patients whose event times are uniformly distributed on 1,2,3,4,5,6,7,8,9,10

t	0	1	2	3	4	5	6	7	8	9	10
f(t)	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
S(t)	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
h(t)	0	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1

- Hazard goes up because the risk set dwindles.

Hazard function

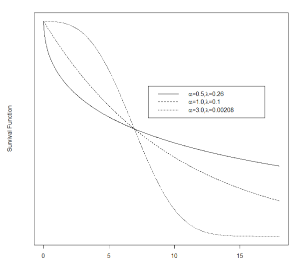
- The hazard function not only depends on the number of events, but also the size of risk set at time t.
- For discrete r.v.

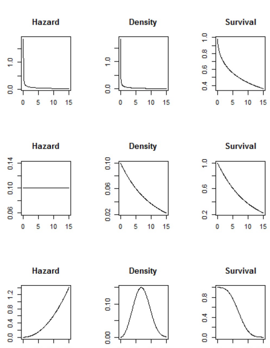
$$h(t_j) = \frac{f(t_j)}{S(t_{j-1})}$$

- For continuous r.v.

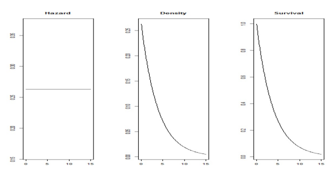
$$h(t) = \frac{f(t)}{S(t)}$$

Three Weibull Distributions



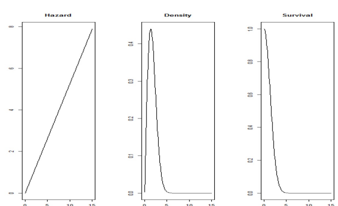


Typical shapes of hazard functions



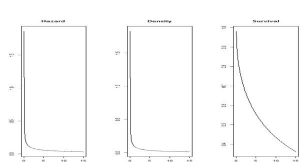
Constant hazard: appropriate for lifetime of light bulbs

Typical shapes of hazard functions



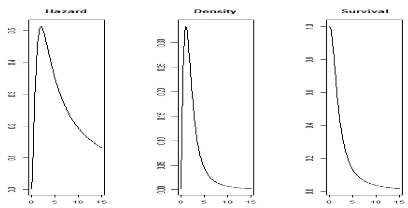
Increasing hazard: typical to model aging effect.

Typical shapes of hazard functions



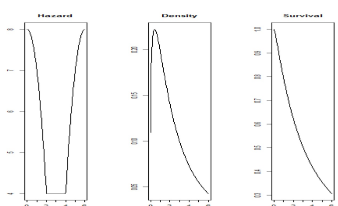
Decreasing hazard: survival following surgery

Typical shapes of hazard functions



Increasing then decreasing hazard: survival after tuberculosis infection

Typical shapes of hazard functions



Bathtub hazard: lifespan of animals

Cumulative hazard function

- Let T be a nonnegative random variable.
- Cumulative hazard function

  $H(t) = \int_0^t h(u)du$.
- It measures the total amount of risk that has been cumulated up to time t.

Summary: continuous case

If T is a continuous non-negative random variable,

- Cumulative distribution function
$$F(t) = P(T \le t)$$
- Probability density function
$$\mathrm{f}(t) = \frac{dF(t)}{dt} = \lim_{\Delta t \to 0} \frac{P(t \le T < t + \Delta t)}{\Delta t}$$
- Survival function
$$\mathrm{S}(t) = P(T > t)$$
- Hazard function
$$\mathrm{h}(t) = \lim_{\Delta t \to 0} \frac{P(t \le T < t + \Delta t | T \ge t)}{\Delta t}$$
- Cumulative hazard function
$$H(t) = \int_0^t h(u)du$$

Relationship

- There is one-to-one relationship between the quantities.
- If we know any one of the functions, we can derive the other functions.
- For example
$$\mathrm{f}(t) = -\frac{dS(t)}{dt} \qquad h(t) = \frac{f(t)}{S(t)}$$
$$H(t) = -\log\big(S(t)\big) \quad S(t) = \exp(-H(t))$$