Problèmes pour enfants de 5 à 15 ans

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Résumé

Cette brochure propose 77 exercices pour développer une culture de la pensée, sélectionnés ou élaborés par l'auteur. La plupart d'entre eux ne requièrent aucune connaissance particulière au-delà de la culture générale. Cependant, la résolution de certains exercices peut s'avérer difficile, même pour des enseignants. Cet ouvrage s'adresse aux écoliers, étudiants, enseignants, parents – à tous ceux qui considèrent la culture de la pensée comme un élément essentiel du développement personnel. nous.

Préface

J'ai écrit ces tâches à Paris au printemps 2004, lorsque des Parisiens russes m'ont demandé d'aider leurs jeunes enfants à acquérir une culture de pensée traditionnelle russe, mais bien supérieure à toutes les coutumes occidentales. Je suis profondément convaincu que cette culture est mieux nourrie par une réflexion indépendante et précoce sur des questions simples, mais non faciles, telles que celles ci-dessous (je recommande particulièrement les problèmes 1, 3, 13). Ma longue expérience m'a montré que les élèves médiocres les résolvent souvent mieux que les excellents, car dans leur « Kamtchatka », ils doivent réfléchir davantage pour survivre que pour « gouverner Séville et Grenade », comme le disait luimême Figaro, tandis que les excellents élèves ne comprennent pas « ce qui doit être multiplié par quoi » dans ces problèmes. J'ai même remarqué que les enfants de cinq ans résolvent ces problèmes mieux que les écoliers gâtés par les exercices, pour qui ils sont plus faciles que les étudiants qui ont bachoté à l'université, mais qui surpassent néanmoins leurs professeurs (les lauréats du prix Nobel et du prix Fields sont les plus maladroits a résoudre ces problèmes simples).

The problems

1. Il manquait sept kopecks à Macha pour acheter un livre, et il manquait un kopeck à Micha. Ils ont mis leur argent en commun pour l'acheter pour eux deux, mais il n'y avait toujours pas assez d'argent. Combien a coûté le livre ?

2. Une bouteille avec un bouchon coûte 10 kopecks, et la bouteille est 9 kopecks plus chère que le bouchon. Combien coûte une bouteille sans bouchon ?

3. Une brique pèse une livre et la demie du poids d'une brique. Combien de livres pèse une brique ?

4. Une cuillerée de vin a été versée d'un tonneau dans une verre (non plein) de thé. Puis la même cuillerée de mélange (hétérogène) du verre a été reversée dans le tonneau. Maintenant, le tonneau et le verre contiennent tous deux un certain volume de liquide étranger (vin dans le verre, thé dans le tonneau). Où le volume de liquide étranger est-il le plus grand : dans le verre ou dans le tonneau ?

5. Deux vieilles femmes sont parties de A à B et de B à A à l'aube (simultanément) pour se retrouver (sur la même route). Elles se sont rencontrées à midi, mais ne se sont pas arrêtées, et chacune a continué à marcher à la même vitesse. La première est arrivée (à B) à 16 heures, et la seconde (à A) à 21 heures. À quelle heure était l'aube ce jour-là ?

6. The hypotenuse of a right-angled triangle (in a standard American examination) is 10 inches, the altitude dropped onto it is 6 inches. Find the area of the triangle.

American school students had been coping successfully with this problem over a decade. But then Russian school students arrived from Moscow, and none of them was able to solve it as had their American peers (giving 30 square inches as the answer). Why ?

7. Vasya has 2 sisters more than he has brothers. How many daughters more than sons do Vasya's parents have ?

8. There is a round lake in South America. Every year, on June 1, a Victoria Regia flower appears at its centre (its stem rises from the bottom, and its petals lie on the water like those of a water lily). Every day the area of the flower doubles, and on July 1, it finally covers the entire lake, drops its petals, and its seeds sink to the bottom. On which date is the flower's area half the area of the lake ?

9. A peasant must take a wolf, a goat and a cabbage across a river in a boat. However the boat is so small that he is able to take only one of the three on board with him. How should he transport all three across the river ? (The wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage.)

10. During the daytime a snail climbs 3 cm up a post, and during the night, falling asleep, accidentally goes down by 2 cm. The post is 10 m high, and a delicious (for the snail) sweet is on its top. In how many days will the snail get the sweet ?

11. A ranger walked from his tent 10 km southwards, turned east, walked straight eastwards 10 km more, met his bear friend, turned north and after another 10 km found himself by his tent. What colour was the bear and where did all this happen ?

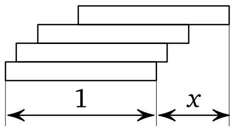
12. A tide was in today at 12 noon. What time will it be in (at the same place) tomorrow ?

13. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2 cm thick, and the cover – front and back each – is 2 mm. A bookworm has gnawed through (perpendicular to the pages) from the first page of volume 1 to the last page of volume 2. How long is the bookworm's track ? [This topological problem with an incredible answer 4 mm is absolutely impossible for academicians, but some preschoolers handle it with ease.]

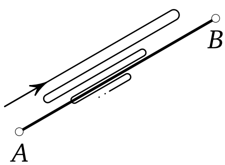
14. Find a body with views from the top and from the front as depicted (polytopes). Depict its side view (showing invisible edges of the polytope dashed).

15. How many ways are there to break up the number 64 into 10 positive integer summands, whose maximum is 12 ? [Ways that differ only by the order of their summands do not count as different.]

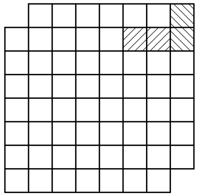
16. By putting a few similar bars one onto another (for example, domino pieces), one can make a length x overhang. What is the maximal attainable value of the overhang length x ?



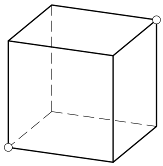
17. The distance between cities A and B is 40 km. Two cyclists leave respectively from A and B simultaneously towards one another, one with speed 10 km/h and the other with speed 15 km/h. A fly flies out with the first cyclist from A with the speed of 100 km/h, reaches the second, touches his forehead and flies back to the first, touches his forehead, returns to the second, and so on until the cyclists' foreheads collide and squash the fly. How many kilometres altogether has the fly flown ?



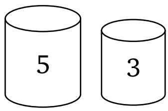
18. One domino piece covers two squares of a chessboard. Cover all the squares except for two opposite ones (on the same diagonal) with 31 pieces. [A chessboard consists of $8 \times 8 = 64$ squares.]



19. A caterpillar wants to slither from a corner of a cubic room (the left on the floor) to the opposite one (the right on the ceiling). Find the shortest route for such a journey along the walls of the room.

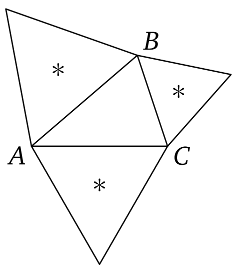


20. You have two vessels of volumes 5 litres and 3 litres. Measure out one litre (obtaining it in one of the vessels).

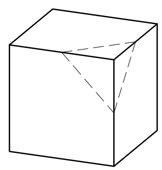


21. There are five heads and fourteen legs in a family. How many people and how many dogs are in the family ?

22. Equilateral triangles are constructed externally on sides AB, BC and CA of a triangle ABC. Prove that their centres ($*$) form an equilateral triangle.

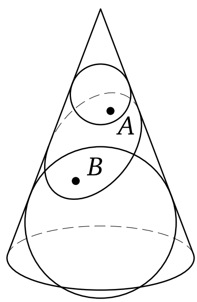


23. What polygons may be obtained as sections of a cube by a plane? Can we get a pentagon? A heptagon? A regular hexagon?

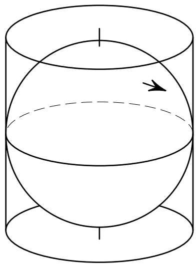


24. Draw a straight line through the centre of a cube so that the sum of squares of the distances to it from the eight vertices of the cube would be a) maximal, b) minimal (comparing with other such lines).

25. A right circular cone is cut by a plane along a closed curve. Two balls inscribed into the cone are tangent to the plane, one at point A and the other at point B. Find a point C on the cut line so that the sum of the distances $CA + CB$ would be a) maximal, b) minimal.

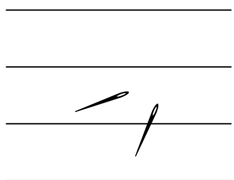


26. The Earth's surface is projected onto the cylinder formed by the lines tangent to the meridians at their equatorial points along the rays parallel to the equator and passing through the Earth's pole axis. Will the area of the projection of France be greater or smaller than the area of France itself?



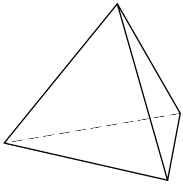
27. Prove that the remainder of division of the number 2^{p-1} by an odd prime p is 1. (Examples : $2^2 = 3a + 1$, $2^4 = 5b + 1$, $2^6 = 7c + 1$, $2^{10} - 1 = 1023 = 11 \cdot 93$.)

28. A needle 10 cm long is thrown randomly onto lined paper with the distance between neighbouring lines also 10 cm. This is repeated N (a million) times. How many times (approximately, up to a few per cent error) will the fallen needle intersect a line on the paper ?

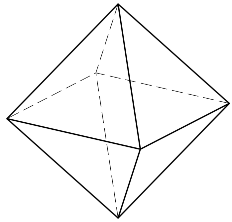


One can perform (as I did at the age of 10) this experiment with $N = 100$ instead of a million throws. [The answer to this problem is surprising : $\frac{2}{\pi}N$. Moreover even for a bent needle of length $a \cdot 10$ cm the number of intersections observed over N throws will be approximately $\frac{2a}{\pi}N$. The number $\pi \approx \frac{355}{113} \approx \frac{22}{7}$.]

29. Polyhedra with triangular faces are, for example, Platonic solids : tetrahedron (4 faces), octahedron (8 of them), icosahedron (20 – and all the faces are the same ; it is interesting to draw it, it has 12 vertices and 30 edges).



Tetrahedron (tetra = 4)



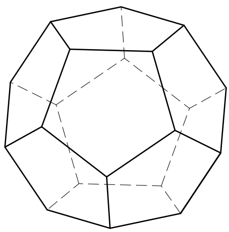
Octahedron (octo = 8)

?

Icosahedron

Is it true that for any such (bounded convex polyhedron with triangular faces) the number of faces is equal to twice the number of vertices minus four ?

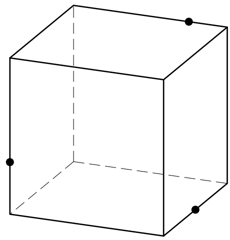
Yet another Platonic solid (there are 5 of them altogether) :



30. A dodecahedron is a convex polyhedron with twelve (regular) pentagonal faces, twenty vertices and thirty edges (its vertices are the centres of the faces of an icosahedron). Inscribe into a dodecahedron five cubes (the vertices of each cube are vertices of the dodecahedron) whose edges are diagonals of faces of the dodecahedron (a cube has 12 edges, one per face). [This was invented by Kepler for the sake of planets.]

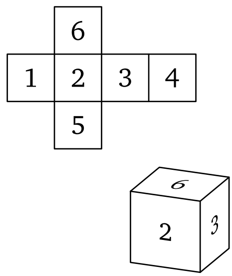
31. Find the intersection of two tetrahedra inscribed into a cube (so that the vertices of each are vertices of the cube, and the edges are diagonals of the faces). What fraction of the cube's volume is contained within the tetrahedra's intersection ?

31$^{\text{bis}}$. Construct the section of a cube by the plane passing through three given points on the edges. [Draw the polygon along which the planar section intersects the faces of the cube.]

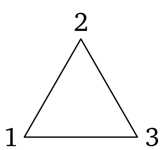


32. How many symmetries does a tetrahedron have? How many has a cube? octahedron? icosahedron? dodecahedron? A symmetry is a transformation preserving lengths. How many rotations are among them and how many reflections (in each of the five cases listed)?

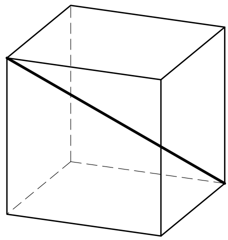
33. How many ways are there to paint 6 faces of similar cubes with six colours $(1, \ldots, 6)$ [one per face] so that no two of the coloured cubes obtained would be the same (could not be transformed one into another by a rotation)?



34. How many different ways are there to permute n objects? There are six of them for $n = 3$: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. What if the number of objectis is $n = 4$? $n = 5$? $n = 6$? $n = 10$?



35. A cube has 4 long diagonals. How many different permutations of these four objects are obtained by rotations of a cube?



36. The sum of the cubes of three integers is subtracted from the cube of the sum of these numbers. Is the difference always divisible by 3 ?

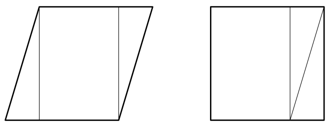
37. Same question for the fifth powers and divisibility by 5, and for the seventh powers and divisibility by 7.

38. Calculate the sum

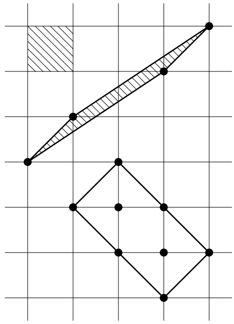
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$$

(with an error of not more than 1% of the answer).

39. If two polygons have equal areas, then they may be cut into a finite number of polygonal parts which may then be rearranged to obtain both the first and second polygons. Prove this! [For spatial solids this is not the case : the cube and tetrahedron of equal volumes cannot be cut this way!]



40. Four vertices of a parallelogram have been chosen at nodes of a piece of squared paper. It turns out that neither the parallelogram's sides nor its interior contain any other nodes of the squared paper. Prove that the area of such a parallelogram is equal to the area of one square of the paper.



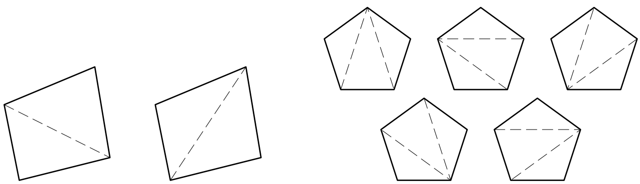
$a{=}2$, $b{=}2$

41. Under the conditions of question 40, a nodes have turned out to be in the interior and b on the sides of the parallelogram. Find its area.

42. Is the statement analogous to question 40 true for parallelepipeds in 3-space ?

43. The rabbit (or Fibonacci) numbers form a sequence $1, 2, 3, 5, 8, 13, 21, 34, \ldots$, in which $a_{n+2} = a_{n+1} + a_n$ for any $n = 1, 2, \ldots$ (a_n is the n-th number in the sequence). Find the greatest common divisor of the numbers a_{100} and a_{99}.

44. Find the (Catalan) number of ways to cut a convex n-gon into triangles by cutting along its non-intersecting diagonals. For example, $c(4) = 2$, $c(5) = 5$, $c(6) = 14$. How can one find $c(10)$?



45. A cup tournament has n participating teams, each losing team leaves, and the overall winner is decided after $n - 1$ games. The tournament schedule may be written symbolically as, for instance, $((a, (b, c)), d)$ meaning b plays c, the winner meets a, and the winner of those meets d]. What is the number of different schedules for 10 teams?

- For 2 teams, we have only (a, b), and the number is 1.
- For 3 teams, there are only $((a, b), c)$, or $((a, c), b)$, or $((b, c), a)$, and the number is 3.
- For 4 teams :

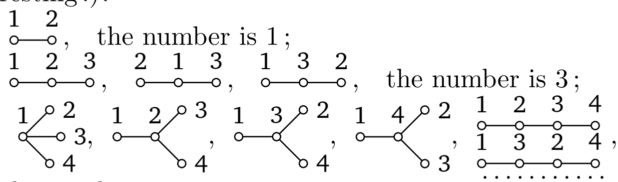
$$
\begin{array}{llll}
(((a,b),c),d) & (((a,c),b),d) & (((a,d),b),c) & (((b,c),a),d) \\
(((b,d),a),c) & (((c,d),a),b) & (((a,b),d),c) & (((a,c),d),b) \\
(((a,d),c),b) & (((b,c),d),a) & (((b,d),c),a) & (((c,d),b),a) \\
((a,b),(c,d)) & ((a,c),(b,d)) & ((a,d),(b,c)) &
\end{array}
$$

46. Join n points $1, 2, \dots, n$ by intervals ($n - 1$ of them) to obtain a tree. How many different trees may be obtained (the $n = 5$ case is already interesting!)?

$n = 2$: , the number is 1;

$n = 3$: , , , the number is 3;

$n = 4$:



the number is 16.

47. A permutation $(x_1, x_2, \dots, x_n)$ of numbers $\{1, 2, \dots, n\}$ is called a *snake* (of length n) if $x_1 < x_2 > x_3 < x_4 \cdots$.

EXAMPLE :

$$
\begin{array}{lll}
n = 2, \ \text{only} & 1 < 2, & \text{the number is } 1, \\
n = 3, & \left.\begin{array}{l} 1 < 3 > 2 \\ 2 < 3 > 1 \end{array}\right\}, & \text{the number is } 2, \\
n = 4, & \left.\begin{array}{l} 1 < 3 > 2 < 4 \\ 1 < 4 > 2 < 3 \\ 2 < 3 > 1 < 4 \\ 2 < 4 > 1 < 3 \\ 3 < 4 > 1 < 2 \end{array}\right\}, & \text{the number is } 5.
\end{array}
$$

Find the number of snakes of length 10.

48. Let s_n be the number of snakes of length n :

$$s_1 = 1, \quad s_2 = 1, \quad s_3 = 2, \quad s_4 = 5, \quad s_5 = 16, \quad s_6 = 61.$$

Prove that the Taylor series of the tangent is

$$\tan x = 1\,\frac{x^1}{1!} + 2\,\frac{x^3}{3!} + 16\,\frac{x^5}{5!} + \cdots = \sum_{k=1}^{\infty} s_{2k-1}\,\frac{x^{2k-1}}{(2k-1)!}.$$

49. Find the sum of the series

$$1 + 1\,\frac{x^2}{2!} + 5\,\frac{x^4}{4!} + 61\,\frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} s_{2k}\,\frac{x^{2k}}{(2k)!}.$$

50. For $s > 1$, prove the identity :

$$\prod_{p=2}^{\infty} \frac{1}{1-\frac{1}{p^s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

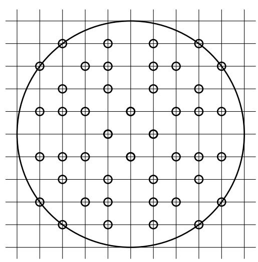
(The product is over all prime numbers p, and the summation over all natural numbers n.)

51. Find the sum of the series :

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[Prove that it is $\pi^2/6$, that is, approximately $3/2$.]

52. Find the probability of the irreducibility of a fraction p/q (this is defined as follows : in the disk $p^2 + q^2 \leqslant R^2$, we count the number N of vectors with integer p and q not having a common divisor greater than 1, after which the probability of the irreducibility is the limit of the ratio $N(R)/M(R)$, where $M(R)$ is the number of integer points in the disk ($M \sim \pi R^2$)).

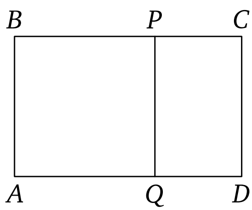


$M(5) = 81$, $N(5) = 44$, $N/M = 44/81$

53. For the sequence of Fibonacci numbers a_n from problem 43, find the limit of the ratio a_{n+1}/a_n when n tends to infinity :

$$\frac{a_{n+1}}{a_n} = 2,\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{34}{21}.$$

[The answer is « the golden ratio », $\frac{\sqrt{5}+1}{2} \approx 1.618$. This is the ratio of the sides of a card which stays similar to itself after cutting off the square whose side is the smaller side of the card, $\frac{AB}{BC} = \frac{PC}{CD}$. How is the golden ratio related to a regular pentagon and a five-pointed star ?]



54. Calculate the infinite continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

with $a_{2k} = 1$ and $a_{2k+1} = 2$ (that is, find the limit of the fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

for $n \to \infty$).

55. Find the polynomials

$$y = \cos 3(\arccos x),\ y = \cos 4(\arccos x),\ y = \cos n(\arccos x),$$

where $|x| \leqslant 1$.

56. Calculate the sum of the k-th powers of the n complex n-th roots of unity.

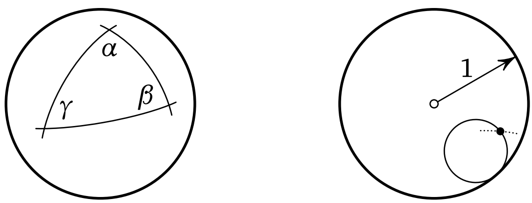
57. On the (x, y)-plane, draw the curves defined parametrically by :

$$\{x = \cos 2t, y = \sin 3t\}, \quad \{x = t^3 - 3t, y = t^4 - 2t^2\}.$$

58. Calculate $\int_0^{2\pi} \sin^{100} x \, dx$ (with an error of not more than 10% of the answer).

59. Calculate $\int_1^{10} x^x \, dx$ (with an error of not more than 10% of the answer).

60. Find the area of a triangle with angles (α, β, γ) on a radius 1 sphere, whose sides are great circles (sections of a sphere by planes passing through its centre).
ANSWER : $S = \alpha + \beta + \gamma - \pi$ (for example, for a triangle with three right angles, $S = \pi/2$, that is, one-eighth of the total area of the sphere).



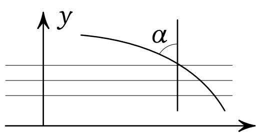
61. A circle of radius r rolls (without slipping) inside a circle of radius 1. Draw the whole trajectory of a point of the rolling circle (this trajectory is called a hypocycloid) for $r = 1/3$, for $r = 1/4$, for $r = 1/n$, and for $r = 1/2$.

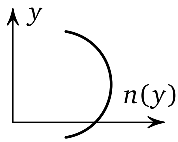
62. In a class of n pupils, estimate the probability of there being two pupils with the same birthdays. Is it high or is it low?
ANSWER : (very) high if the number of the pupils is (well) above n_0, (very) low if it is (well) below n_0, and what this n_0 actually is (when the probability $p \approx 1/2$) is to be found.

63. Snell's (Snellius') law states that the angle α made by a ray of light with the normal to layers of a stratified medium satisfies the equation

$$n(y)\sin\alpha = \text{const},$$

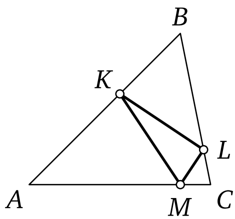
where $n(y)$ is the refractive index of the layer at height y (the quantity n is inversely proportional to the speed of light in the medium when taking its speed in vacuum as 1; in water $n = nicefrac43$).





Draw ray trajectories in the medium « air over a desert », where the index $n(y)$ has a maximum at a certain height. (A solution to this problem explains mirages in a desert to those understanding how trajectories of rays emanating from objects are related to the images.)

64. Inscribe into an acute-angled triangle ABC a triangle KLM of minimal perimeter (with its vertex K on AB, L on BC, M on CA).



HINT : The answer for non-acute-angled triangles is not similar to the beautiful answer for acute-angled ones.

65. Calculate the mean value of the function $1/r$ (where $r^2 = x^2 + y^2 + z^2$, r is the distance to the origin) on the radius R sphere centred at the point (X, Y, Z).
HINT : The problem is related to Newton's gravitation law and Coulomb's law of electricity theory. In the two-dimensional version of the problem, the function should be replaced by $\ln r$, and the sphere by the circle.

66. The fact $2^{10} = 1024 \approx 10^3$ implies $\log_{10} 2 \approx 0.3$. Estimate how much they differ, and calculate $\log_{10} 2$ to three decimal places.

67. Find $\log_{10} 4, \log_{10} 8, \log_{10} 5, \log_{10} 50, \log_{10} 32, \log_{10} 128, \log_{10} 125, \log_{10} 64$ with the same precision.

68. Using $7^2 \approx 50$, find an approximate value of $\log_{10} 7$.

69. Knowing $\log_{10} 64$ and $\log_{10} 7$, find $\log_{10} 9, \log_{10} 3, \log_{10} 27, \log_{10} 6, \log_{10} 12$.

70. Using $\ln(1+x) \approx x$ (ln is $\log_e$), find $\log_{10} e$ and $\ln 10$ from the relation[1]

$$\log_{10} a = \frac{\ln a}{\ln 10}$$

and from the values of $\log_{10} a$ calculated earlier (for example, for $a = 128/125, 1024/1000$ and so on).

[Solutions to problems 65–69 deliver after half an hour a table of four-digit logarithms of any numbers using products of numbers found already as basic data and the formula

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

for corrections.] (In this way Newton compiled a table of 40-digit logarithms!)

71. Consider the sequence of powers of two : 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, ... Among the first twelve numbers, four have their decimal expression starting with 1, and none has it starting with 7.

Prove that in the limit $n \to \infty$ the first digit of the numbers 2^m, $0 \leqslant m \leqslant n$, will be met with a certain frequency : $p_1 \approx 30\%, p_2 \approx 18\%, \dots, p_9 \approx 4\%$.

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1 The Euler number $e = 2.71828\dots$ is defined as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ for $n \to \infty$, and is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. It may also be defined via the quoted formula for $\ln(1+x)$: $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$.

72. Verify the behaviour of the first digits of powers of three : $1, 3, 9, 2, 8, 2, 7, \ldots$ Prove that, in the limit here, we also get certain frequencies and, moreover, the same as for the powers of two. Find an exact formula for $p_1, \ldots, p_9$.
HINT : The first digit of a number x is determined by the fractional part of the number $\log_{10} x$, therefore one has to consider the sequence of the fractional parts of the numbers $m\alpha$, where $\alpha = \log_{10} 2$.

Prove that these fractional parts are distributed over the interval from 0 to 1 uniformly : out of the n fractional parts of the numbers $m\alpha$, $0 \leqslant m < n$, a subinterval A will contain the quantity $k_n(A)$ such that, for $n \to \infty$, $\lim(k_n(A)/n) =$ (the length of the subinterval A).

73. Let $g\colon M \to M$ be a smooth map of a bounded domain M onto itself which is one-to-one and preserves areas (volumes in the multi-dimensional case) of domains.

Prove that in any neighbourhood U of any point of M and for any N there exists a point x such that $g^T x$ is also in U for a certain integer $T > N$ (« the recurrence theorem »).

74. Let M be the torus surface (with coordinates $\alpha \pmod{2\pi}$, $\beta \pmod{2\pi}$), and

$$g(\alpha, \beta) = (\alpha + 1, \beta + \sqrt{2}) \pmod{2\pi}.$$

Prove that the sequence of points $\{g^T(x)\}$, $T = 1, 2, \ldots$, is everywhere dense in the torus.

75. In the notation of problem 74, let

$$f(\alpha, \beta) = (2\alpha + \beta, \alpha + \beta) \pmod{2\pi}.$$

Prove that there is an everywhere dense subset of the torus consisting of periodic points x (that is, such that $f^{T(x)}x = x$ for certain integer $T > 0$).

76. In the notation of problem 74 prove that, for almost all points x of the torus, the sequence of points $\{g^T(x)\}$, $T = 1, 2, \ldots$, is everywhere dense in the torus (points x without this property constitute a set of measure zero).

77. In problems 74 and 76 prove that the sequence $\{g^T(x)\}$, $T = 1, 2, \ldots$, is distributed over the torus uniformly : if a domain A contains $k_n(A)$ points out of the n with $T = 1, 2, \ldots, n$, then

$$\lim_{n\to\infty} \frac{k_n(A)}{n} = \frac{\operatorname{mes} A}{\operatorname{mes} M}$$

(for example, for a Jordan measurable domain A of measure $\operatorname{mes} A$).

NOTE TO PROBLEM 13. I tried to illustrate with this problem the difference in approaches to tasks by mathematicians and physicists, in my invited paper in the journal « Physics – Uspekhi » for the 2000 Christmas jubilee. My success surpassed by far what I intended : the editors, unlike preschoolers, on whom I had been basing my experience, failed to solve the problem, and therefore altered it to match my 4 mm answer as follows : instead of « from the first page of volume 1 to the last page of volume 2 » they typeset « from the *last* page of volume 1 to the *first* page of volume 2 ».

This true story is so implausible that I am including it here : the proof is the editors' version published by the journal.

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