a. $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

$a = 3$

$b = 3$

$f(n) = \sqrt{n}$

$\Rightarrow n^{\log_b^a} = n^{\log_3^3} = \Theta(n)$

$\xrightarrow{\text{حالت ۱}} f(n) = n^{\frac{1}{2}} \in O(n^{\log_3^3 - \varepsilon}), 0 < \varepsilon < \frac{1}{2} \Rightarrow f(n) = O(n^{\log_3^3 - \varepsilon}) . \varepsilon = \frac{1}{3}$

پس طبق شرط حالت اول master theorem ، $T(n) = \Theta(n)$

b. $T(n) = 3T(n/4) + n\log n$

$f(n) = n\log n$

$a = 3$

$b = 4$

$n^{\log_b a} = n^{\log_4^3} = n^{\frac{1}{2}\log_2^3} = n^{\frac{1.2}{2}} \approx n^{0.79}$

$n\log n \in O(n^{.79 - \varepsilon})$ ✗

$n\log n \in \Theta(n^{0.79 - \varepsilon})$ ✗

$f(n) = n\log n \in \Omega(n^{0.79 + \varepsilon})$ $\varepsilon > 0$

$\varepsilon = 0.2$

Regularity condition : $af(\frac{n}{b}) \leqslant cf(n) \Rightarrow 3(\frac{n}{4})\log\frac{n}{4} \bigcirc c.n\log n$

$\Rightarrow c = \frac{3}{4} \Rightarrow \frac{3n}{4}\log\frac{n}{4} \leqslant \frac{3n}{4}\log n$ ✓

$T(n) = \Theta(n\log n)$

c.

$a = 4$

$b = 2$

$f(n) = n^2$

$n^{\log_b^a} = n^2$

$\xrightarrow{\text{حالت ۲}} f(n) = \Theta(n^{\log_2 4}) = \Theta(n^2) \Rightarrow T(n) = \Theta(n^2 \lg n)$

در حالت ۲ master theorem صدق می‌کند .

d. $T(n) = 4T(n/2) + n^2\log^2 n$

در حالت ۲ master theorem نتیجه می‌گیریم :

If $f(n) = \Theta(n^{\log_b^a}\log^k n)$ with $k \geq 0$ then $T(n) = \Theta(n^{\log_b^a}\log^{k+1} n)$.

$a = 4$

$b = 2$

$n^{\log_b^a} = n^2$

$f(n) = n^2\log^2 n$

$\Rightarrow f(n) = \Theta(n^2\log^2 n) \Rightarrow T(n) = \Theta(n^2\log^3 n)$

e. $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n} \Rightarrow T(n) = 2\left(2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}\right) + \frac{n}{\log n} \Rightarrow$

$T(n) = 2\left(2\left(2T(\frac{n}{8}) + \frac{\frac{n}{4}}{\log \frac{n}{4}}\right)\right) + \frac{n}{\log \frac{n}{2}} + \frac{n}{\log n} \Rightarrow \dots \Rightarrow T(n) = 2^k T(\frac{n}{2^k}) + n \times \sum_{i=0}^{k-1} \frac{1}{\log \frac{n}{2^i}}$

$\Rightarrow T(n) = 2^k T(\frac{n}{2^k}) + n \times \sum_{i=0}^{k-1} \frac{1}{\log n - \log 2^i} \Rightarrow T(n) = 2^k T(\frac{n}{2^k}) + n \times \sum_{i=0}^{k-1} \frac{1}{\log n - i}$

$\simeq \lg i$

$$\xrightarrow[\substack{n = 2^k \\ k = \lg n}]{T(\frac{n}{2^k}) = T(1)} T(n) = nT(1) + n\left(\frac{1}{k} + \frac{1}{k-1} + \dots + \frac{1}{1}\right) \Rightarrow T(n) = n + nH_k = n(1 + \log k)$$

$\Rightarrow T(n) \cong \Theta(n) + n \times H_{\lg n} \simeq \Theta(n) + n\ln(\lg n) \Rightarrow T(n) \in \Theta(n\ln(\lg(n)))$

f. $T(n) = T(\sqrt{n}) + 1 \Rightarrow T(n) = T(n^{\frac{1}{4}}) + 1 + 1 \Rightarrow \dots \Rightarrow T(n) = T(n^{\frac{1}{2^k}}) + \underbrace{1 + 1 + \dots + 1}_{k}$

$\Rightarrow T(n) = T(n^{\frac{1}{2^k}}) + k$

$n^{\frac{1}{2^k}} = 1 \Rightarrow \frac{1}{2^k} = \log n$ ✗ از بین مقدارهای که رابطه بازگشتی در آن تکراری شود 2 بهتر از ۱ است. $\Rightarrow$

$\Rightarrow n^{\frac{1}{2^k}} = 2 \Rightarrow n^1 = 2^{2^k} \Rightarrow \log_2(\log_2(n)) = k$

$\Rightarrow T(n) = T(1) + \lg(\lg n) \Rightarrow T(n) = \Theta(\lg(\lg n))$