[redacted]

Theorem 4.2 *Let X be a set, let $A, B \subseteq X$, and let $\mathcal{P}_A$, $\mathcal{P}_B$ be the corresponding characteristic functions. The following holds (the prime denotes taking the complement):*

(i) $\neg\mathcal{P}_A = \mathcal{P}_{A'}$

(ii) $\mathcal{P}_A \wedge \mathcal{P}_B = \mathcal{P}_{A\cap B}$

(iii) $\mathcal{P}_A \vee \mathcal{P}_B = \mathcal{P}_{A\cup B}$

(iv) $\mathcal{P}_A \Rightarrow \mathcal{P}_B = \mathcal{P}_{(A\setminus B)'}$

(v) $\mathcal{P}_A \Leftrightarrow \mathcal{P}_B = \mathcal{P}_{(A\cap B)\cup(A\cup B)'}$.

PROOF. To prove (i) we note that the function $x \mapsto \neg\mathcal{P}_A(x)$ evaluates to TRUE if $x \notin A$ and to FALSE otherwise. However, from the definition of the complement of a set, we have $x \notin A \Leftrightarrow x \in A'$. Next we prove (iv). We'll prove instead

$$\neg(\mathcal{P}_A \Rightarrow \mathcal{P}_B) = \mathcal{P}_{(A\setminus B)}$$

which, together with (i), gives us (iv). Let $\mathcal{P}_A := (x \in A)$ and let $\mathcal{P}_B := (x \in B)$. From the truth table of the operator $\Rightarrow$, we find that $\neg(\mathcal{P}_A \Rightarrow \mathcal{P}_B)(x)$ is TRUE precisely when $\mathcal{P}_A(x)$ is TRUE and $\mathcal{P}_B(x)$ is FALSE. This means that

$$(x \in A) \wedge (x \notin B),$$

but this is just the definition of the characteristic function of the set $A \setminus B$, as desired.

The proof of (ii), (iii), (v) is left as an exercise. $\square$

Exercise

Exercise 1. Rewrite each symbolic sentence using the quantifier $\exists$.

1. $Y \not\supset X \quad \Leftrightarrow \quad \exists x \in X,\ x \notin Y$
2. $\#X > 1 \quad \Leftrightarrow \quad \exists x, y \in X,\ x \neq y$
3. $X \cap Y \neq \emptyset$
4. $z \in X \times Y$
5. $X \neq Y$
6. $\#(X \Delta Y) > 0$

Exercise 2. Write each sentence with symbols, using at least one quantifier.

1. The integer n is a cube. $\quad \Leftrightarrow \quad \exists k \in \mathbb{Z},\ n = k^3$
2. The equation $f(x) = 0$ has a rational solution.

$\Leftrightarrow \quad \exists x \in \mathbb{Q},\ f(x) = 0$

3. The fraction a/b is not reduced.
4. The unit circle has a rational point.
5. The polynomial $p(x)$ has no integer roots.

Exercise 3. The following expressions define sets; turn symbols into words.

1. $\{x \in \mathbb{N} : \forall m, n \in \mathbb{N},\ (x|mn) \Rightarrow (x|m \vee x|n)\}$

   The set of prime numbers, together with the integer 1.
2. $\{x \in \mathbb{N} : \forall m \in \mathbb{N},\ x|m^2 \Rightarrow x|m\}$
3. $\{x \in \mathbb{N} : \forall m \in \mathbb{N},\ x|m^3 \Rightarrow x|m^2\}$

Exercise 4. Decide if each sentence is true or false, hence write its negation with symbols. Then rewrite both with words.

1. $\forall n \in \mathbb{N},\ 1/n \notin \mathbb{N}$

(False). $\exists n \in \mathbb{N},\ 1/n \in \mathbb{N}$.

The reciprocal of a natural number is not a natural number.

There is a natural number whose reciprocal is also a natural number.

2. $\forall n \in \mathbb{N},\ \sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$
3. $\forall n \in \mathbb{Z},\ 2|n(n+1)$
4. $\forall n \in \mathbb{N},\ \exists r \in \mathbb{Q},\ n < r^2 < n+1$