Flervariabelanalys, 10HP 2013

Föreläsningsanteckningar

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I detta dokument är föreläsningsanteckningar till kursen komplex analys, som gavs av Jörgen Östensson på Uppsala Universitet 2017. Samtliga figurer är ritade med vektorgrafik direkt i LaTeX, så om något inte syns tydligt nog är det bara att zooma in utan att det blir grynigt (fantastiskt, eller hur?).

January 17, 2017

Uppskattar du att all info för kursen finns i detta dokument, så att du (kanske) slipper köpa kurslitteraturen? Känner du att du vill öka min livskvalitet litegrann som tack för arbetet jag lagt ner? Swisha valfri summa (typ 20-30kr) till 070-422 40 81

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Introduction

Definition 1. A complex number is a number on the form $x + iy$, where $x, y \in \mathbb{R}$. Two omplex numbers $x_1 + iy_1$, $x_2 + iy_2$ are said to be equal iff. $x_1 = x_2$ and $y_1 = y_2$. The number x is called the real part, and the number y is called the imaginary part.

We write $x = \mathfrak{Re}(x + iy)$, $y = \mathfrak{Im}(x + iy)$. The set of complex numbers is denoted $\mathbb{C}$.

We define addition and multiplication as follows:

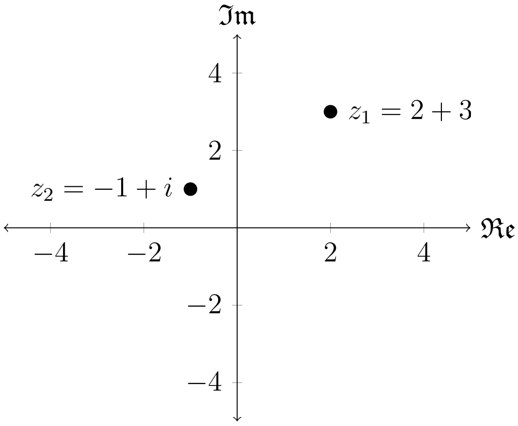
$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

complex numbers are often denoted by z or w.

Definition 2. The complex conjugate of $z = x + iy$ is denoted $\overline{z}$ and is defined by $\overline{z} = x - iy$. It holds that

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}.$$

Note also that $\mathfrak{Re}(z) = \dfrac{z + \overline{z}}{2}$, $\mathfrak{Im} = \dfrac{z - \overline{z}}{2i}$.
It is natural to represent a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Thus geometric representation is called the complex/Argand plane.



Definition 3. The absolute value of a complex number $z = x + iy$ is denoted $|z|$, and is defined by $|z| = \sqrt{x^2 + y^2}$. It holds that

$$|z|^2 = z \cdot \overline{z}$$
$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

Note also that every $z \in \mathbb{C}$, $z \neq 0$ has a multiplicative inverse $\dfrac{1}{z}$ given by $\frac{1}{z} = \dfrac{\overline{z}}{|z|^2}$

Theorem 1 (The Triangle Inequality)**.** *For z_1, $z_2 \in \mathbb{C}$ it holds that*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Corollary 1. *For z_1, $z_2 \in \mathbb{C}$, it holds that*

$$||z_1| - |z_2|| \leq |z_1| - |z_2|$$

Proof.

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$
$$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$$

Now let $z_1 \longleftrightarrow z_2$ $\square$

Polar form

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, so $\exists\theta$ s.t.

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta).$$

Therefore $z = x + iy$ can be written as follows:

$$z = |z|(cos(\theta) + i\sin(\theta))$$

Note that $r = |z|$ is uniquely determined by z, but θ is **not**. θ is only unique up to integer multiples of 2π, i.e. if a particular θ suffices, then so does $\theta + 2\pi n$, $n \in \mathbb{Z}$. We let all these numbers be denoted by $\arg(z)$

It is practical to have a notation for one of these values of $\arg(z)$. The so called **principal value** of $\arg(z)$, denoted $\mathrm{Arg}(z)$ is specified as the value of $\arg(z)$ which belongs to the interval $(-\pi, \pi]$.

Example 1.

$$\begin{aligned} \arg(1+i) &= \{\frac{\pi}{4} + 2\pi n \mid n \in \mathbb{Z}\} \\ \mathrm{Arg}(1+i) &= \frac{\pi}{4} \end{aligned}$$

Remark 1. *One calls* $\mathrm{Arg}(z)$ *a **branch** of* $\arg(z)$. *Note that* $\mathrm{Arg}(z)$ *is 'discontinuous' along the negative real axis, which is called the branch cut of this function*

Suppose $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$, $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) \\ &+ i(\sin(\theta_1)cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Correctly interpreted, then, $|z_1 z_2| = |z_1||z_2|$, and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
Definition 4 (The exponential function)**.** For $z = x + iy$, let $e^z := e^x(\cos(y) + i\sin(y))$

Remark 2. *Note that* e^z *agrees with the 'usual' exponential function if* $z \in \mathbb{R}$, *i.e. the above definition extends the 'usual' exponential function to all of* $\mathbb{C}$.

Note in particular that $e^{iy} = \cos(y) + i\sin(y)$, $y \in \mathbb{R}$ is called Euler's formula. In polar form, z can be written as $z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$. Moreover, if $z_1 = r_1 r^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i\theta_1} e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

From this it follows that $e^{z_1} e^{z_2} ? e^{z_1 + z_2}$. Note also that $(e^{i\theta})^n = e^{i\theta} \cdot \ldots \cdot e^{i\theta} = e^{in\theta}$, i.e. $(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta))$ [de Moivre's formula].

The Logarithm Function

In real analysis, one defines the logarithm $\ln(x)$ as the inverse of the exponential function e^x, but the problem here is that e^z is not an injective function (and has no inverse).

Given $z \in \mathbb{C} \setminus \{0\}$ one chooses the define $\log(z)$ as the set of all $w \in \mathbb{C}$ whose image is z under the exponential function, i,e, $w = \log(z) \iff e^w = z$ (So $\log(z)$ is a multivalued function).

Write $z = re^{i\theta}$, $w = u + iv$. Then

$$\begin{aligned} e^w = z &\iff re^{i\theta} = e^r e^{iv} \\ &\iff u = \log(r) = \log(|z|) \end{aligned}$$

and

$$v = \theta + 2\pi k,\ k \in \mathbb{Z} = \arg(z)$$

The explicit definition is:

Definition 5. For $z \neq 0$ we define $\log(z)$ as

$$\begin{aligned} \log(z) &= \ln(|z|) + i \arg(z) \\ &= \ln(|z|) + i\mathrm{Arg}(z + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$

Example 2. Compute $\log(1 + i)$

$$\begin{aligned} \log 1 + i &= \ln(|1 + i|) + i \arg(1 + i) \\ &= \ln(\sqrt{2}) + i(\frac{\pi}{4} + 2\pi k),\ k \in \mathbb{Z} \\ &= \frac{1}{2} \ln 2 + i(\frac{\pi}{4} + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$