Complex Analysis, 10C 2017

Lecture Notes

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In this document are lecture notes for the course complex analysis, which was given by Jörgen Östensson at Uppsala University in 2017. All figures are drawn with vector graphics directly in LaTeX, so that you can zoom in if anything is unclear (fantastic, right?).

May 29, 2017

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1 Introduction

Definition 1. A complex number is a number on the form $x + iy$, where $x, y \in \mathbb{R}$. Two omplex numbers $x_1 + iy_1$, $x_2 + iy_2$ are said to be equal iff. $x_1 = x_2$ and $y_1 = y_2$. The number x is called the real part, and the number y is called the imaginary part.

We write $x = \mathfrak{Re}(x + iy)$, $y = \mathfrak{Im}(x + iy)$. The set of complex numbers is denoted $\mathbb{C}$.

We define addition and multiplication as follows:

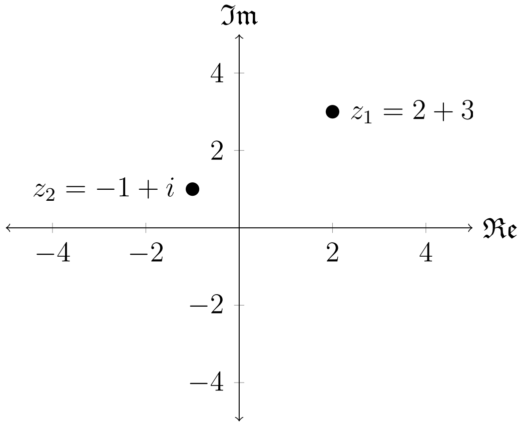
$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2)\\ (x_1 + iy_1) \cdot (x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\end{aligned}$$

complex numbers are often denoted by z or w.

Definition 2. The complex conjugate of $z = x + iy$ is denoted $\overline{z}$ and is defined by $\overline{z} = x - iy$. It holds that

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}\\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2}.\end{aligned}$$

Note also that $\mathfrak{Re}(z) = \dfrac{z + \overline{z}}{2}$, $\mathfrak{Im} = \dfrac{z - \overline{z}}{2i}$. It is natural to represent a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Thus geometric representation is called the complex/Argand plane.



Definition 3. The absolute value of a complex number $z = x + iy$ is denoted $|z|$, and is defined by $|z| = \sqrt{x^2 + y^2}$. It holds that

$$\begin{aligned}|z|^2 &= z \cdot \overline{z}\\ |z_1 \cdot z_2| &= |z_1| \cdot |z_2|\end{aligned}$$

Note also that every $z \in \mathbb{C}$, $z \neq 0$ has a multiplicative inverse $\dfrac{1}{z}$ given by $\frac{1}{z} = \dfrac{\overline{z}}{|z|^2}$

Theorem 1 (The Triangle Inequality)**.** *For z_1, $z_2 \in \mathbb{C}$ it holds that*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Corollary 1. *For z_1, $z_2 \in \mathbb{C}$, it holds that*

$$||z_1| - |z_2|| \leq |z_1| - |z_2|$$

Proof.

$$\begin{aligned}|z_1| = |(z_1 - z_2) + z_2| &\leq |z_1 - z_2| + |z_2|\\ \Rightarrow |z_1| - |z_2| &\leq |z_1 - z_2|\end{aligned}$$

Now let $z_1 \longleftrightarrow z_2$ $\square$

Polar form

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, so $\exists\theta$ s.t.

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta).$$

Therefore $z = x + iy$ can be written as follows:

$$z = |z|(cos(\theta) + i\sin(\theta))$$

Note that $r = |z|$ is uniquely determined by z, but θ is **not**. θ is only unique up to integer multiples of 2π, i.e. if a particular θ suffices, then so does $\theta + 2\pi n$, $n \in \mathbb{Z}$. We let all these numbers be denoted by $\arg(z)$

It is practical to have a notation for one of these values of $\arg(z)$. The so called **principal value** of $\arg(z)$, denoted $\text{Arg}(z)$ is specified as the value of $\arg(z)$ which belongs to the interval $(-\pi, \pi]$.

Example 1.

$$\begin{aligned}\arg(1+i) &= \{\frac{\pi}{4} + 2\pi n \mid n \in \mathbb{Z}\}\\ \text{Arg}(1+i) &= \frac{\pi}{4}\end{aligned}$$

Remark 1. *One calls* $\text{Arg}(z)$ *a* ***branch*** *of* $\arg(z)$*. Note that* $\text{Arg}(z)$ *is 'discontinuous' along the negative real axis, which is called the branch cut of this function*

Suppose $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$, $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then

$$\begin{aligned}z_1 z_2 &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))\\ &= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))\\ &+ i(\sin(\theta_1)cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))]\\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]\end{aligned}$$

Correctly interpreted, then, $|z_1 z_2| = |z_1||z_2|$, and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
Definition 4 (The exponential function)**.** For $z = x + iy$, let $e^z := e^x(\cos(y) + i\sin(y))$

Remark 2. *Note that* e^z *agrees with the 'usual' exponential function if* $z \in \mathbb{R}$*, i.e. the above definition extends the 'usual' exponential function to all of* $\mathbb{C}$*.*

Note in particular that $e^{iy} = \cos(y) + i\sin(y)$, $y \in \mathbb{R}$ is called Euler's formula. In polar form, z can be written as $z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$. Moreover, if $z_1 = r_1 r^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

$$\begin{aligned}z_1 z_2 &= r_1 r_2 e^{i\theta_1} e^{i\theta_2}\\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}\end{aligned}$$

From this it follows that $e^{z_1}e^{z_2}?e^{z_1+z_2}$. Note also that $(e^{i\theta})^n = e^{i\theta} \cdot \ldots \cdot e^{i\theta} = e^{in\theta}$, i.e. $(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta))$ [de Moivre's formula].

The Logarithm Function

In real analysis, one defines the logarithm $\ln(x)$ as the inverse of the exponential function e^x, but the problem here is that e^z is not an injective function (and has no inverse).

Given $z \in \mathbb{C} \setminus \{0\}$ one chooses the define $\log(z)$ as the set of all $w \in \mathbb{C}$ whose image is z under the exponential function, i,e, $w = \log(z) \iff e^w = z$ (So $\log(z)$ is a multivalued function).

Write $z = re^{i\theta}$, $w = u + iv$. Then

$$\begin{aligned} e^w = z &\iff re^{i\theta} = e^r e^{iv} \\ &\iff u = \log(r) = \log(|z|) \end{aligned}$$

and

$$v = \theta + 2\pi k,\ k \in \mathbb{Z} = \arg(z)$$

The explicit definition is:

Definition 5. For $z \neq 0$ we define $\log(z)$ as

$$\begin{aligned} \log(z) &= \ln(|z|) + i\arg(z) \\ &= \ln(|z|) + i\mathrm{Arg}(z + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$

Example 2. Compute $\log(1 + i)$

$$\begin{aligned} \log 1 + i &= \ln(|1 + i|) + i\arg(1 + i) \\ &= \ln(\sqrt{2}) + i(\frac{\pi}{4} + 2\pi k),\ k \in \mathbb{Z} \\ &= \frac{1}{2}\ln 2 + i(\frac{\pi}{4} + 2\pi k),\ k \in \mathbb{Z} \end{aligned}$$

2 Branches of the complex logarithm and complex powers

Branches of the logarithm

If we replace $\arg z$ by $\mathrm{Arg}(z)$ in our definition of the logarithm from the previous section, we obtain a single-valued function; a so called 'branch' of $\log(z)$

$$\mathrm{Log}(z) := \ln(|z|) + i\mathrm{Arg}(z),\ z \in \mathbb{C} \setminus \{0\}.$$

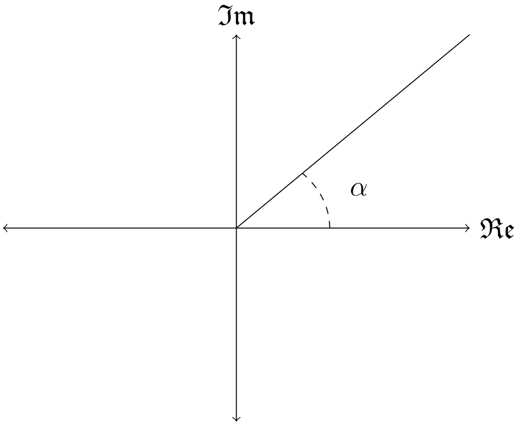
This particular logarithm is called the principal logarithm. Note that $\mathrm{Log}(z)$ extends the 'usual' logarithm (defined on $(0, \infty)$) to $\mathbb{C} \setminus \{0\}$.Note also that $\mathrm{Log}(z)$ is 'discontinuous' along the negative real line; its so called 'branch cut'.

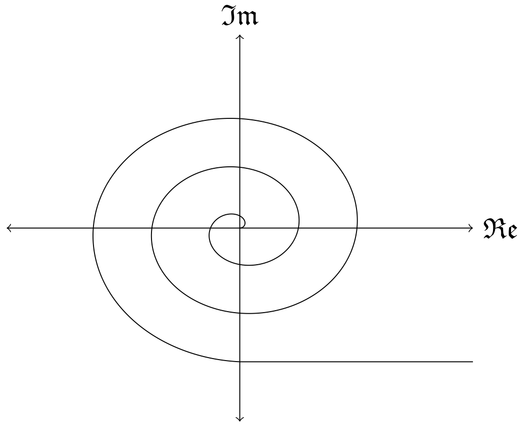
We shall see that $\mathrm{Log}(z)$ is differentiable in $\mathbb{C} \setminus (-\infty, 0]$.

There are other branches of the logarithm $\log(z)$, e.g. a branch cut at the angle α (left figure), resulting in the logarithm being defined as

$$\log(z) := \ln(|z|) + i\arg_\alpha(z),\ \arg_\alpha(z) \in (\alpha, \alpha + 2\pi]$$

We could also have a spiraling branch cut, like in the right right figure. This is not very useful, but that's not the point. The point is that we can define our branch cut however we want.





To understand/visualize a function $f : \mathbb{C} \to \mathbb{C}$ one can check how it maps various regions. One usually writes $f(z) = f(x+iy)?w = u+iv$ and then draw two copies of the complex plane $\mathbb{C}$; a z-plane for the domain, and a $w-$plane for the range

Complex Powers

Given a complex number $z \in \mathbb{C}$, consider the equation $w^n = z$ (*). The set of all solutions w of (*) is denoted by $x^{\frac{1}{n}}$, and is called the n^{th} root of z. If $z = 0$, then $w = 0$. Suppose $z \neq 0$ and write $w = |w|e^{i\alpha}$, $z = |z|e^{i\theta}$ with help from de Moivres formula. We then have that $|w|^n e^{in\alpha} = |z|e^{i\theta}$, and clearly

$$\begin{cases} |w| = \sqrt[n]{|z|} \\ n\alpha = \theta + 2k\pi,\ k \in \mathbb{Z} \end{cases} \iff \begin{cases} |w| = \sqrt[n]{|z|} \\ \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}. \end{cases}$$

Every $k \in \mathbb{Z}$ gives a solution of (*), but only $k \in \{0, 1, \ldots, n-1\}$ solutions are unique; namely:

$$z^{\frac{1}{n}} = \sqrt[n]{|z|}e^{i(\frac{\theta}{n} + k\frac{2\pi}{n})},\ k \in \{0, 1, \ldots, n-1\}.$$

Suppose again that $z \neq 0$. For $n \in \mathbb{Z}$, it holds that $z^n = e^{n\log(z)}$ for every value of $\log(z)$. It is also true that $z^{\frac{1}{n}} = e^{\frac{1}{n}\log(z)}$, i.e. the right-hand-side

$$e^{n(\ln|z| + i\arg(z))} = e^{n\ln|z|}e^{ni\arg(z)} = |z|^n(e^{i\arg(z)})^n.$$

Definition 6. For $\alpha \in \mathbb{C}$, we let $z^\alpha = e^{\alpha\log(z)},\ z \in \mathbb{C} \setminus \{0\}$.

This makes z^α (in general) a multi-valued function.

Example 3. Compute i^{-2i}.

Solution:

$$i^{-2i} = e^{-2i\log(i)} = e^{-2i \cdot i(\frac{\pi}{2}+k2\pi)} = e^{(4k+1)\pi},\ k \in \mathbb{Z}.$$

By choosing a branch of $\log(z)$ (i.e. of $\arg(z)$) in the definition if z^α, one obtains branches of z^α, for example, the principal branch of z^α is defined as follows: $z^\alpha = e^{\alpha \log(z)}$.
Example 4. The principal branch of $z^{\frac{1}{3}}$ is given as follows:

$$e^{\frac{1}{3}\mathrm{Log}(z)} = e^{\frac{1}{3}(\ln|z|+i\arg(z))} = |z|^{\frac{1}{3}} e^{i\frac{\mathrm{Arg(z)}}{3}}.$$

Trigonometric/Hyperbolic Functions

We have

$$\begin{cases} e^{iy} & = \cos(y) + i\sin(y) \\ e^{-iy} & = \cos(y) - i\sin(y) \end{cases} \forall y \in \mathbb{R}$$

$$\Rightarrow \cos(y) = \frac{e^{iy}+e^{-iy}}{2}, \quad \sin(y) = \frac{e^{iy}-e^{-iy}}{2i}, \quad \forall y \in \mathbb{R}.$$

Definition 7. For $z \in \mathbb{C}$ we define $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. We also define tan, cot analogously to the real cases. We define the hyperbolic functions as $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$ and $\sinh(z) = \frac{e^{z}-e^{-z}}{2i}$.

Example 5. Solve $\sin(z) = 2$.

Solution:

$$\begin{aligned} \sin(z) = 2 \iff & \frac{e^{iz}-e^{-iz}}{2i} = 2 \iff e^{iz} - e^{-iz} = 4i \\ \iff & (e^{iz})^2 - 4e^{iz} - 1 = 0 \iff e^{iz} = 2i \pm \sqrt{-4+1} \\ \iff & e^{iz} = 2i \pm i\sqrt{3} \\ iz = \log(i(2\pm\sqrt{3})) = & \ln|2\pm\sqrt{3}| + i\arg(\frac{\pi}{2}+2k\pi), k \in \mathbb{Z} \\ \iff & z = \frac{\pi}{2} + 2k\pi - i\ln(2\pm\sqrt{3}). \end{aligned}$$

3 Topology of the Complex Plane

The set $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is called the open disk with center z_0 and radius r. A subset $\mathcal{M}$ of $\mathbb{C}$ is called open is for every $z_0 \in \mathcal{M}$ there exists an $r > 0$ s.th. $D_r(z_0) \subseteq M$.

Example 6. The disk $D_r(z_0)$ is open (hence the name 'open disk').

A subset $\mathcal{L}$ of $\mathbb{C}$ is called closed if its complement $\mathcal{L}^c$ is open. $\mathcal{L}^c = \mathbb{C} \setminus \mathcal{L}$.
Example 7. $\{z \in \mathbb{C} : |z - z_0| \leq r\}$.

A point z_0 is called an interior point of $\mathcal{M}$ if there is an $r > 0$ s.th. $D_r(z_0) \subseteq \mathcal{M}$. A point $z_0 \in \mathbb{C}$ is called a boundary point of $\mathcal{M}$ is $\forall r > 0$ is holds that $D_r(z_0) \cap \mathcal{M} \not\emptyset$ and $D_r(z_0) \cap \mathcal{M}^c \neq \emptyset$. The set of all interior points is denoted $\mathrm{Int}(M)$ or M°. The set of all boundary points is denoted ∂M.

It holds that

- $\mathcal{M}$ is closed $\iff \partial M \subseteq \mathcal{M}$.
- $\mathcal{M}$ is open $\iff \partial \mathcal{M} \subseteq \mathcal{M}^c$.

An open set $\mathcal{M}$ is called path connected if every pair of points can be connected by a polygonal path contained in $\mathcal{M}$.

Remark 3. *one can assume that the polygonal paths have segments parallel to the coordinate axes.*

Definition 8. An open connected set is called a domain.

Theorem 2. *Suppose that $u(x, y)$ is a real-valued function in a domain $D \in \mathbb{R}^2$. Suppose also that*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

in all of D. Then u is constant in D.

A domain D is called simply connected if every closed curve in D can be, within D, continuously deformed to a point (this essentially means that D has no 'holes').

Limits and Continuity

Definition 9. A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to have the limit z_0 or to converge to z_0 and we write $\lim_{n\to\infty} z_n = z_0$ if for every $\varepsilon > 0$, $\exists n \geq N$ s.th. $|z - z_0| < \varepsilon \, \forall n \in \mathbb{N}$.

Remark 4.

$$z_n \to z_0 \iff \begin{cases} \Re(z_n) \to \Re(z_0) \\ \Im(z_n) \to \Im(z_0). \end{cases}$$

Definition 10. Let f be a function defined in some punctured neighbourhood of z_0. We say that f has the limit w_0 as $z \to z_0$ and we write

$$\lim_{z \to z_0} f(z) = w_0$$

if for every $\varepsilon > 0\, \exists \delta > 0$ s.th. $0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

Theorem 3. *For $z = x + iy$, let $u(x, y) = \Re(f(z))$, $v(x, y) = \Im(f(z))$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then,*

$$\lim_{z \to z_0} f(z) = w_0 \iff \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \end{cases}$$

Proof. exercise. □

Definition 11. Let f be a function defined in a neighbourhood of z_0. Then f is said to be continuous at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$ and a function is said to be continuous on the (open) set $\mathcal{M}$ is it is continuous at every point of $\mathcal{M}$.

Theorem 4. *If $\lim\limits_{z \to z_0} f(z) = A$ and $\lim\limits_{z \to z_0} g(z) = B$ then*

- $\lim\limits_{z \to z_0} f(z) + g(z) = A + b$
- $\lim\limits_{z \to z_0} f(z)g(z) = AB$ $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$

Corollary 2. *If f and g are continuous at z_0, then so are $f + g$, fg, and $\frac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$.*

Example 8. It's easy to show that $f(z) =$ const. and $g(z) = z$ are continuous in the entire complex plane $\mathbb{C}$. It follows that polynomials are continuous in $\mathbb{C}$. Rational functions are continuous everywhere except where the denominator is zero.

Example 9. $f(z) = e^z = e^x(\cos(y) + i\sin(y)$ is a continuous function in $\mathbb{C}$.

The Complex Derivative, Analytic Functions

Definition 12. Let f be a continuous function defined in a neighbourhood of a point z_0. We say that f is differentiable at z_0 if the following limit exists

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

The limit is called the derivative of f at z_0 and is denoted $f'(z_0)$ of $\frac{\partial f}{\partial z}(z_0)$.

Remark 5. *Δz is a complex number, so it can approach 0 'in many different ways'. In order for the derivative to exist, the result must be independent of how Δz gets to 0.*

Example 10. $f(z) = \bar{z}$ is nowhere differentiable.

Proof.

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta_z)} - \overline{z_0}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

Now let $\Delta z \to 0$ in two different ways. First let $\Delta z = \Delta x$; the limit will approach 0. Second, let $\Delta z = \Delta y$; the limit will approach -1 □

Example 11. Let $n \in \mathbb{N}_{>0}$. The $\frac{d}{dz} z^n = nz^{n-1}$ since by the binomial theorem,

$$\frac{(z + \Delta z)^n - z^n}{\Delta z} = \frac{nz^{n-1}\Delta z + \binom{n}{k} z^{n-2}\Delta z^2 + \dots}{\Delta z} \to nz^{n-1}.$$

Theorem 5. *If f and g are differentiable at z, then*

- $(f + g)'(z) = f'(z) + g'(z)$
- $(cf)'(z) = cf'(z)$
- $(f \cdot g)'(z) = (f'g)(z) + (fg')(z)$
- $\left(\frac{f}{g}\right)'(z) = \frac{f'g - fg'}{g^2}(z)$.

Also, the Chain-rule holds, i.e. if g is differentiable at z and f is differentiable at $g(z)$, then $(f \circ g)'(z) = f'(g(z))g'(z)$.

Definition 13. A complex-valued function f is said to be analytic in an open set G if it is differentiable at every point of G. We say that f is analytic at the point z_0 if it is differentiable in a neighbourhood if z_0. If f is analytic in all of $\mathbb{C}$, we call f an 'entire' function.

Remark 6. *The book also assumes $f'(z)$ to be continuous in G in order for f to be analytic in G. It turns out that the continuity of f is automatic.*

4 Cauchy-Riemann Equations, Harmonic Functions

Suppose that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(x_0 + \Delta x + i(y_0 + \Delta y)) - f(x_0 + iy_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z} \end{aligned}$$

1: Consider $\Delta z = \Delta x$, i.e. $\Delta y = 0$

$$\begin{aligned}\Rightarrow f'(z_0) &= \lim_{\Delta x \to 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0).\end{aligned}$$

2: Now let $\Delta z = \Delta y$, i.e. $\Delta x = 0$

$$\begin{aligned}\Rightarrow f'(z_0) &= \lim_{\Delta y \to 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0).\end{aligned}$$

It must therefore hold that $u_x + v_y = -iu_y + v_y$ which is equivalent to

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \tag{Cauchy-Riemann Equations}$$

Theorem 6. *A necessary condition for $f = u + iv$ to be differentiable at $z_0 = x_0 + iy_0$ is that the Cauchy-Riemann Equations be satisfied by f at z_0.*

Remark 7. *We also saw the following: If f is differentiable at z_0, then the derivative is given by $f'(z_0) = u_x(x_0, y_0) + iv(x_0, y_0)$.*

The following provides a sufficient condition for differentiability:

Theorem 7. *Suppose that f is defined in an open set G containing $z_0 = x_0 + iy_0$. Suppose also that u_x, u_y, v_x, v_y exist in 'G', are continuous at z_0, and satisfy the Cauchy-Riemann Equations. Then f is differentiable at z_0.*

Proof. In view of the continuity of the first partial derivatives at (x_0, y_0), it follows that:

$$\begin{aligned}u(x_0, y_0) + u(x_0 + \Delta x, y_0 + \Delta y) &= u_x(x_0, y_0)\Delta x + u_u(x_0, y_0)\Delta y \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_1(\Delta x, \Delta y)\end{aligned}$$

and

$$\begin{aligned}v(x_0, y_0) + v(x_0 + \Delta x, y_0 + \Delta y) &= v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_2(\Delta x, \Delta y)\end{aligned}$$

[See Calculus; C^1 implies differentiability]

$$\begin{aligned}\Rightarrow f(z_0 + \Delta z) - f(z_0) &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y \\ &\quad + i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y) \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}(\rho_1(\Delta x, \Delta y) + \rho_2(\Delta x, \Delta y)) \\ &= u_x(x_0, y_0)\Delta z + iv_x(x_0, y_0)\Delta z \\ &\quad + |\Delta z|(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)).\end{aligned}$$

$\rho_1, \rho_2 \to 0$ as $\Delta z \to 0$, which implies that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, and is equal to

$$= u_x(x_0 + y_0) + iv_x(x_0, y_0).$$

$\square$

Example 12. Show that e^z is entire and find its derivative.

Solution: $e^z = e^x(\cos(y) + i\sin(y))$

$$\begin{cases} u(x,y) = e^x \cos(y) \\ v(x,y) = e^x \sin(y) \end{cases} \Rightarrow \begin{cases} u_x = e^x \cos(y) = v_y \\ u_y = -e^x \sin(y) = -v_x \end{cases}$$

thus, $u, v \in C^1$ and satisfy the Cauchy-Riemann Equations, which implies that e^z is entire. Moreover, $\frac{d}{dz} e^z = u_x + iv_x = u + iv = e^z$.

Remark 8. *By the uniqueness of so called 'analytic continuation', the above definition of e^z is the only one which makes e^z entire. (See uniqueness principle in book, p.156).*

Example 13. Since e^{iz} and e^{-iz} are both entire, so are of course $\cos(z)$ and $\sin(z)$. Moreover,

$$\frac{d}{dz}\sin(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos(z). \qquad \text{(Chain rule)}$$

Inverse Mappings

Suppose f is analytic (with $f'(z)$ continuous) in a domain D. Consider a mapping

$$(x,y) \to \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

as a mapping of $D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. Its Jacobian matrix is given by

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

It's determinant is given by

$$\det(J_f) = u_x v_y - u_y v_x = (u_x)^2 + (v_x)^2 = |f'(z)|^2$$

The inverse function theorem leads to the following:

Theorem 8 (Inverse Function Theorem)**.** *Suppose that $f(z)$ is analytic on a domain D (with $f'(z)$ continuous), and also that $f'(z_0) \neq 0$. Then there is a neighbourhood U of z_0 and a neighbourhood V of $f(z_0)$ s.th. $f : U \to V$ is a bijection and the inverse mapping $f^{-1} : V \to U$ is analytic with derivative*

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)}, \qquad w = f(z)$$

Example 14. $f(z) = e^z$ maps the region $\{z : \Im(z) \in (-\pi, \pi)\}$ to $\mathbb{C} \setminus (-\infty, 0]$. $f'(z) \neq 0$ and $f^{-1}(w) = \mathrm{Log}(w)$ is defined in $\mathbb{C} \setminus (-\infty, 0]$. Then $\mathrm{Log}(w)$ must be analytic in $\mathbb{C} \setminus (-\infty, 0]$. Moreover

$$\frac{d}{dw}\mathrm{Log}(w) = \frac{1}{f'(z)} = \frac{1}{e^z} = \frac{1}{w}.$$

Clearly, $\frac{d}{dw}\mathrm{Log}(w) = \frac{1}{w}$ for *any* branch of $\log(w)$ away from the cut.
Example 15.

$$\begin{aligned} f(z) &= z^\alpha = e^{\alpha \log(z)};\ \alpha \in \mathbb{C} \\ \Rightarrow f'(z) &= e^{\alpha \log(z)} \frac{d}{dz}(\alpha \log(z)) \\ &= e^{\alpha \log(z)} \frac{\alpha}{z} = z^\alpha \frac{\alpha}{z} = \alpha z^{\alpha - 1} \end{aligned}$$

Harmonic Functions

Definition 14. A real-valued finction $\varphi(x, y)$ is said to be harmonic in a domain D if $\varphi \in C^2(D)$ and φ satisfies the so called Laplace-equation:

$$\Delta\varphi = \nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0 \qquad \text{in } D$$

Theorem 9. *Suppose $f = u + iv$ is analytic in a domain D. Then u and v are harmonic in D*

Proof. We later prove that $u, v \in C^\infty$. (see theorem on page 114 in book).

$$\begin{cases} u_x = v_y \Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x \Rightarrow u_{yy} = -v_{xy} \end{cases}$$

since $v_{yx} = v_{xy}$, the result follows. Similarly, $v_{xx} + v_{yy} = 0$. □

Definition 15. If u is harmonic in a domain D and v is a harmonic function in D s.th. $f = u + iv$ is analytic in D, then we say that v is a harmonic conjugate of u in D.

Example 16. Construct an analytic function whose real-part is $u(x,y) = y^3 - 3x^2y$.

Solution: Note that u is harmonic:

$$\Delta u = u_{xx} + u_{yy} = -6y + 6y = 0.$$

If $f = u + iv$ is analytic, then by the Cauchy-Riemann Equations, we get that

$$v_y = -u_x = -6xy \tag{1}$$

$$v_x = -u_y = -3y^2 + 3x^2 \tag{2}$$

Now, integrate (1) w.r.t y, which gives us $v(x,y) = -3xy^2 + \Phi(x)$ and insert into (2)

$$-3y^2 + \Phi'(x) = -3y^2 + 3x^2$$

$$\Phi'(x) = 3x^2 \Rightarrow \Phi(x) = x^3 + c$$

Thus $v = -3xy^2 + x^3 + c$, so $f = u + iv = y^3 - 3x^2y + i(-3xy^2 + x^3 + c)$.

Note: $f(x + i0) = i(x^3 + c)$. Now put $g(z) = i(z^3 + c)$. g is an entire function, and agrees with f on $\mathbb{R}$. By the uniqueness principle, $f \equiv g$, so $f(z) = i(z^3 + c)$, $c \in \mathbb{R}$

5 Conformal Mappings and Stereographic Projections

Conformal Mappings

Let D be a domain in $\mathbb{C}$, $z_0 \in D$. Suppose $f : D \to \mathbb{C}$ is analytic with $f'(z_0) \neq 0$. Let $\gamma(t) = x(t) + iy(t)$ be a C^1-curve in D that passes through $z_0 = z(0)$ with $f'(0) \neq 0$. Then $(f \circ gamma)(t) = f(\gamma(t))$ is a C^1-curve through $(f \circ \gamma)(0) = f(z_0)$. Moreover,

$$\begin{aligned}(f \circ \gamma)(0) &= \frac{d}{dt}(f(\gamma(t)))|_{t=0} = \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \\ &= \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \frac{\gamma(t) - \gamma(0)}{t} \\ &= f'(z_0)\gamma'(0).\end{aligned}$$

i.e. $(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$ is a tangent vector to $(f \circ gamma)$ at $f(z_0)$. Note that $\arg(f \circ \gamma)'(0) = \arg(f'(z_0)) + \arg(\gamma'(0))$.

Remark 9. *If γ_1, γ_2 are two C^1-curves which intersect at z_0, then the angle from $(f \circ \gamma_1)'(0)$ to $(f \circ \gamma_2)'(0)$ are the same as the angles from $\gamma_1'(0)$ to $\gamma_2'(0)$.*

Definition 16. A C^1-mapping $f : D \to \mathbb{C}$ is said to be conformal at z_0 if it satisfies Remark 9 (angles are preserved). If f maps the domain D Bijectively onto V and if f is conformal at every point of D, we call $f : D \to V$ a conformal mapping.

We have proven: if f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.

Example 17.

1. $f(z) = e^z$ is a conformal mapping at every point $z \in \mathbb{C}$, since $f'(z) = e^z \neq 0$

2. $f(z) = z^2$ is conformal at every point $z \in \mathbb{C} \setminus \{0\}$, since $f'(z) = 2z \neq 0$ iff. $z \neq 0$.

Stereographic Projections

Consider the unit sphere S in $\mathbb{R}^3$ (the Riemann-sphere).

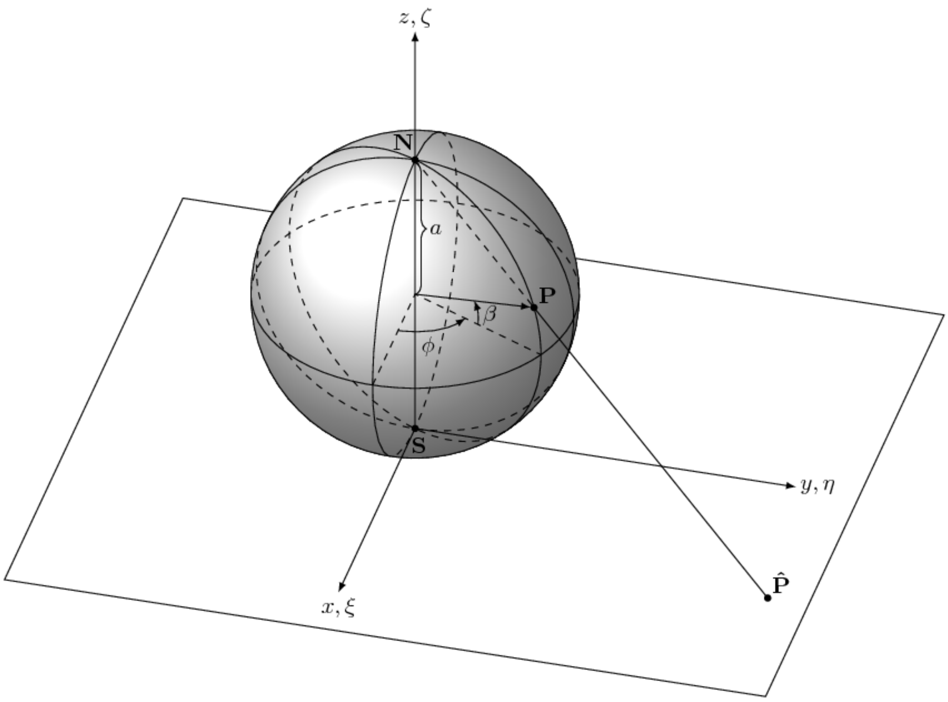


Figure 2: Stereographic projection

Given any point $P = (x, y, x) \in S$ other than N, we draw the line through N and P. We define the stereographic projection of P to be the point $\hat{P} = (\xi, \eta, 0)$ where the line intersects the plane at $\zeta = 0$.

Clearly, $(\xi, \eta, 0) = (0, 0, 1) + t[(x, y, z) - (0, 0, 1)]$., where t is given by the fact that $1 + t(z - 1) = 0$. Solve for t:

$$t = \frac{1}{1-z}, \text{ i.e. } \hat{P} = \xi + i\eta = \frac{x + iy}{1 - z}. \tag{2}$$

Conversely, given $\hat{P} = \xi + i\eta \in \mathbb{C} \sim (\xi, \eta, 0)$, then the line through N and $\hat{P}$ is given by $(x, y, z) = (0, 0, 1) + t[(\xi, \eta, 0) - (0, 0, 1)]$, $t \in \mathbb{R}$. It intersects S when

$$\begin{aligned} \xi^2 + \eta^2 + \zeta^2 = 1 &\iff (t\xi)^2 + (t\eta)^2 + (1 - t)^2 = 1 \\ &\iff t^2(\xi^2 + \eta^2 + \zeta^2) = 2t \\ &\iff t = 0 \quad \vee \quad t = \frac{2}{\xi^2 + \eta^2 + 1} = \frac{2}{|\zeta|^2 + 1} \end{aligned}$$

This corresponds to

$$P = \left(\frac{2\xi}{|\zeta|^2 + 1}, \frac{2\eta}{|\zeta|^2 + 1}, \frac{|\zeta|^2 + 1}{|\zeta|^2 + 1} \right).$$

Thus, the stereographic projection $s : S \setminus \{0\} \to \mathbb{C}$ defines a bijection. Letting $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane and define $s(N) = \infty$, then s becomes a bijective map $s : S \to \hat{\mathbb{C}}$.

Theorem 10. *Under stereographic projections, circles on S correspond to circles and lines on $\mathbb{C}$.*

Remark 10. *We refer to circles as lines in $\mathbb{C}$ as 'Circles' in $\hat{\mathbb{C}}$, where lines are considered as circles through '∞'.*

Proof. The general equation for a circle or line in the plane $\hat{P} = \xi + i\eta$ is as follows:

$$A(\xi^2 + \eta^2) + B\xi + C\eta + D = 0$$

Using (2), we get the following:

$$\begin{aligned}
&A\left(\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2\right) + B\left(\frac{x}{1-z}\right) + C\left(\frac{y}{1-z}\right) + D = 0 \\
\iff &A(x^2 + y^2) + Bx(1-z) + Cy(1-z) + D(1-z)^2 = 0 \\
\iff &A(1-z)^2 + Bx(1-z) + Cy(1-z) + D(1-z)^2 = 0 \\
\iff &A(1-z) + Bx + Cy + D(1-z) = 0 \\
\iff &Bx + Cy + (A-D)z + D + A = 0.
\end{aligned}$$

This is the equation for a plane in $\mathbb{R}^3$, which intersects the Riemann-Sphere in a circle. $\square$

Möbius Mappings/Transformations

Definition 17. A Möbuis transformation is a mapping on the form

$$T(z) = \frac{az + b}{cz + d}, \; a, b, c, d \in \mathbb{C}$$

where $ad - bc \neq 0$ (so that T is not constant).

If $c = 0$ we let $T(\infty) = \infty$. Then $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is bijective. If $c \neq 0$, then $T : \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$ is a bijection. Letting $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$ we extend T to a bijective mapping $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

1. Note that

$$\begin{aligned}
T'(z) &= \frac{d}{dz}\left(\frac{az + b}{cz + d}\right) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} \\
&= \frac{ad - bc}{(cz + d)^2}
\end{aligned}$$

   thus $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a conformal mapping.

2. If

$$T(z) = \frac{ax+b}{cz+d}, \quad S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

$$\begin{aligned}\Rightarrow (S\circ T)(z) = S(T(z)) = \frac{\alpha T(z)+\beta}{\gamma T(z)+\delta} &= \frac{\alpha\left(\dfrac{az+b}{cz+d}\right)+\beta}{\gamma\left(\dfrac{az+b}{cz+d}\right)+\delta} \\ &= \frac{(\alpha a+\beta c)z+(ab+\beta d)}{(\gamma a+\delta c)z+(\gamma b+\delta d)}\end{aligned}$$

so the composition of Möbius transformations are Möbius transformations (so they form a group under the operation of composition). Note the following:

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \begin{pmatrix}\alpha a+\beta c & \alpha b+\beta d \\ \gamma a+\delta c & \gamma b+\delta d\end{pmatrix}$$

Meaning that the composing transformations is 'the same' as multiplying matrices containing the coefficients of the transformations (by 'the same', we mean that there is an isomorphism between these two).

The inverse, $T^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, is given by solving the following:

$$z = T^{-1}(w) = \begin{cases} -\dfrac{dw+b}{cw+a}, & w \neq \dfrac{a}{c} \\ \infty, & w = \dfrac{a}{c} \\ -\dfrac{d}{c}, & w = \infty \end{cases}$$

Lemma 1. *If a Möbius Transformation T has more than 2 fixed points in $\hat{\mathbb{C}}$ (i.e. z s.th. $T(z) = z$) then it is the identity mapping.*

Proof. If $c = 0$ then

$$\begin{aligned} T(z) &= \frac{az+b}{d} \\ T(z) = z &\iff \frac{az+b}{c} = z \iff (a-d)z+b = 0 \end{aligned}$$

so T has at most one fixed point in $\mathbb{C}$ unless $a = d$ and $b = 0$, which is equivalent to $T(z)$ being the identity mapping. This implies that T has at most two fixed points in $\hat{\mathbb{C}}$ unless T is identically 0.

If $c \neq 0$ then

$$T(z) = z \iff \frac{az+b}{cz+d} = z \iff az^2 + (d-a)z - b = 0.$$

So T has at most 2 fixed points in $\mathbb{C}$ unless $c = 0$ and $a = d$, $b = 0$. But $c \neq 0$, which is a contradiction and the result follows. $\square$

Theorem 11. *If S, T are Möbius Transformation s.th. $S(z_i) = T(z_i)$ for three different z_i, then $S \equiv T$.*

Proof. $T^{-1} \circ S^{-1}$ is a Möbius Transformation s.th. $(T^{-1} \circ S)(z_i) = z_i$, $i = 1, 2, 3$. This implies that $(T^{-1} \circ S)(z) = z$, $\forall z$ by lemma. So $S \equiv T$. □

6 More on Möbius Transformations

Recall that a Möbius transformation is a mapping on the form

$$T(z) = \frac{az + b}{cz + d},\ a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

Some particular cases:

1. $T(z) = z + b$; a translation
2. $T(z) = az = |a|e^{i\arg(z)}$; a rotation & magnification
3. $T(z) = \frac{1}{z}$; an inversion

Note that if $c \neq 0$,

$$T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2}\frac{1}{z + \frac{d}{c}}$$

so every Möbius transformation is a composition of Möbius transformation of the type 1, 2, or 3.

Theorem 12. *Every Möbius transformation maps 'circles' onto 'circles'*

Remark 11. *Recall that a 'circle' in $\hat{\mathbb{C}}$ is a circle or a line in $\mathbb{C}$. A line in $\mathbb{C}$ is a 'circle' through ∞ in $\hat{\mathbb{C}}$.*

Proof. It is easy to see that mappings on the form of 1 or 2 map circles onto circles and lines onto lines. It is enough to prove that inversion

$$T(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = u + iv$$

maps 'circles' onto 'circles'. [Note that $u^2 + v^2 = \frac{1}{x^2+y^2}$].

A 'circle' in $\hat{\mathbb{C}}$ has equation

$$\begin{aligned} A(x^2 + y^2) + Cx + Dy + E = 0 \iff & A + \frac{Cx}{x^2 + y^2} + \frac{Dy}{x^2 + y^2} + \frac{E}{x^2 + y^2} = 0 \\ \iff & A + Cu + Dv + E(u^2 + v^2); \end{aligned}$$

Which is precisely the equation of a 'circle' in $\hat{\mathbb{C}}$. □

Example 18. Determine the image of the disk $|z-2|=2$ under the transformation $T(z)=\dfrac{z}{2z-8}$.

Solution: First determine the image of $c:|z-2|=2$.

$$\begin{aligned}&T(4)=\infty \Rightarrow T(z) \text{ must be a line}\\ &T(0)=0, \quad T(2+2i)=\frac{2+2i}{2(2+2i)-8}=\frac{2+2i}{-4+2i}=-\frac{1}{2}\frac{1+i}{1-i}=-\frac{i}{2}.\end{aligned}$$

Since $T(2)=-\frac{1}{2}$, clearly $\{z:|z-2|<2\}$ maps to the left half-plane.

Given a 'circle' C_z in the z-plane, and a 'circle' C_w in the w-plane, can one find a Möbius transformation T s.th. $T(C_z)=C_w$? Yes!

The Cross-Ratio

Definition 18. Let $z_1,z_2,z_3\in\hat{\mathbb{C}}$ be distinct points. Put

$$(z,z_1,z_2,z_3):=\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}\in\hat{\mathbb{C}}.$$

If some of the z_i is infinity, then the right-hand side should be interpreted as

$$(z,z_1,z_2,z_3)=\begin{cases}\dfrac{\frac{z_2-z_3}{z-z_3}}{} & z_1=\infty\\ \dfrac{z-z_1}{z-z_3} & z_2=\infty\\ \dfrac{z-z_1}{z_2-z_1} & z_3=\infty\end{cases}$$

(z,z_1,z_2,z_3) is called the cross-ratio of the points. Note that $S(z):=(z,z_1,z_2,z_3)$ is a Möbius transformation s.th. $S(z_1)=0,\ S(z_2)=1,\ S(z_3)=\infty$.

By an earlier proposition, it is the unique Möbius transformation that maps z_1,z_2,z_3 to 0,1,∞.

Theorem 13. *Given a triple $z_1,z_2,z_3\in\hat{\mathbb{C}}$ of distinct points, and another triple $w_1,w_2,w_3\in\hat{\mathbb{C}}$ of distinct points, there is a unique Möbius transformation T s.th. $T(z_i)=w_i$, $i=1,2,3$.*

Proof. By our earlier proposition, there is at most one such T. We now prove that there is a unique such T by constructing a Möbius transformation T s.th. $T(z_i)=w_i$. Put $S(z)=(z,z_1,z_2,z_3)$, $U(w)=(w,w_1,w_2,w_3)$.

$$\begin{aligned}S(z_1)&\longrightarrow 0\longleftarrow U(w_1)\\ S(z_2)&\longrightarrow 1\longleftarrow U(w_2)\\ S(z_3)&\longrightarrow \infty\longleftarrow U(w_3)\end{aligned}$$

So the transformation $T(z) := (U^{-1} \circ S)(z) = U^{-1}(S(z))$. Furthermore

$$\begin{aligned} w = T(z) &\iff U^{-1}(S(z)) = w \iff U(w) = S(z) \\ &\iff (z, z_1, z_2, z_3) = (w, w_1, w_2, w_3). \end{aligned}$$

□

Let z_1, z_2, z_3 be three distinct points on a 'circle' C_z in $\hat{\mathbb{C}}$. Note that C_z is *oriented* by the order of z_1, z_2, z_3. that is, C_z is acquiring an orientation by proceeding through z_1, z_2, z_3 in succession.

Since Möbius transformations are conformal, it maps the ragion to the left of C_z oriented by z_1, z_2, z_3 onto the region to the left of $C_w = T(C_z)$, if oriented by w_1, w_2, w_3.

Example 19. Find a Möbius transformation that maps the region $|z| > 1$ onto $\Re(w) < 0$

Solution: Let $z_1 = 1$, $z_2 = -i$, $z_3 = -1$, and $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

$$\begin{aligned} \frac{w - w_1}{w - \infty} \frac{i - \infty}{i - 0} &= \frac{z - 1}{z + 1} \frac{-i + 1}{-i - 1} \\ \iff w = (-i) \frac{z - 1}{z + 1} \frac{1 - i}{1 + i} &= \frac{1 - z}{1 + z}. \end{aligned}$$

Symmetry preserving property

Two points z_1, z_2 are said to be symmetric w.r.t. a line L if L is the perpendicular bisector of the line segment from z_1 to z_2. This means that every circle or line through the points z_1 and z_2 intersect L orthogonally. This motivates the following definition:

Definition 19. Two points are said to be symmetric w.r.t. a circle if every circle or line through z_1, z_2 intersects the circle orthogonally. In particular, the center of the circle and ∞ are symmetric w.r.t the circle.

Theorem 14 (Symmetry Principle)**.** *Let C_z be a circle or line in the z-plane and $w = T(z)$ be a Möbius transformation. Then the two points z_1, z_2 are symmetric w.r.t C_z iff.*

$$w_1 = T(z_1) \textit{ and } w_2 = T(z_2) \textit{ are symmetric w.r.t. } C_w = T(C_z).$$

Proof. Follows since Möbius transformations preserve the class of circles and lines as well as orthogonality.

□

Example 20. Let C be a proper circle in $\mathbb{C}$ with center a and radius R. Given a point $\alpha \in \hat{\mathbb{C}}$ there is a unique point α^* s.th. α and α^* are symmetric w.r.t C. Indeed it holds that

$$(\alpha^* - \alpha)\overline{(\alpha - \alpha^*)} = R^2$$

or

$$\alpha^* = \alpha + \frac{R^2}{\overline{(\alpha - a)}} = a + \frac{R^2}{|\alpha - a|^2}(\alpha - a)$$

Note: $\arg(\alpha^* - \alpha) = \arg(\alpha - a)$. $|\alpha^* - a||\alpha - a| = R^2$.

7 Dirichlet's Problem

Many applications involve solving Dirichlet's problem: Find a function $\varphi(x, y)$ defined on $D \cup \partial D$, $\varphi \in C^2$ in D s.th.

1. $\Delta\varphi = \varphi_{xx} + \varphi_{yy} = 0$ in D
2. $\varphi =$ some given function on ∂D.

This problem is easily solved in some special cases.

1. If the region we are interested in is the area between two points on the $x-$axis, and the temperature at the endpoints are given as A and B, then

$$\begin{cases} \Delta\varphi = 0 \text{ in } D \\ \varphi(a, y) = A \\ \varphi(x, b) = B \end{cases}$$

   Let $\varphi(x, y) = \alpha x + \beta$, choose

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \alpha = \dfrac{B - A}{b - a} \\ \beta = \dfrac{Ab - Ba}{b - a} \end{cases}$$

   So $\varphi(x, y) = \dfrac{A - B}{b - a}x + \dfrac{Ab - Ba}{b - a}$.

2. If the region we are interested in is say, the right half-plane, and the temperature at some point on the positive imaginary axis is given as A, and the temperature at some point on the negative imaginary axis is given as B, then

$$\varphi(x, y) = \alpha \mathrm{Arg}(z) + \beta. \text{ choose } \alpha,\ \beta \text{ s.th.}$$

$$\begin{cases} \alpha \cdot \frac{\pi}{2} + \beta = A \\ \alpha \cdot -\frac{\pi}{2} + \beta = B \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \alpha = \dfrac{A - B}{\pi} \\ \beta = \dfrac{A + B}{2}. \end{cases}$$

$$\varphi(x, y) = \frac{A - B}{\pi}\mathrm{Arg}(z) + \frac{A + B}{2}.$$

3. If the region we are interested in lies between the positive real axis and a ray centered at the origin with argument α relative to the positive real axis
$$\varphi(x, y) = \frac{1}{\alpha}\text{Arg}(z).$$
4. If the region is the upper half-plane, and given a set of n ordered points x_i on the real axis, the temperature between the points x_i, x_{i+1} is given by a_i. Then the temperature distribution is given by
$$\varphi(x, y) = a_n + \frac{1}{\pi}\sum_{i=1}^{n}(a_{i-1} - a_i)\text{Arg}(z - x_i)$$
5. If the region is an annulus with inner radius r_1 and outer radius r_2, and the temperature at these are A and B respectively, then the temperature distribution is given by
$$\varphi(x, y) = \alpha \ln |z] + \beta, \quad \alpha = \frac{B - A}{\ln(r_r) - \ln(r_1)}, \quad \beta = \frac{A\ln(r_2) - B\ln(r_1)}{\ln(r_2) - \ln(r_1)}.$$

How about more complicated Dirichlet problems? The idea is to map the more complicated problem to an easier problem using a conformal mapping, e.g. Möbius transformations.

Theorem 15. *Suppose $f : D \to D'$ is analytic, $f = u + iv$, where D, D' are domains. If $\psi(u, v)$ is harmonic in D', then $\phi(x, y) := \phi(u(x, y), v(x, y))$ (*) is harmonic.*

Proof. Take $z_0 \in D$, then $w_0 = f(z_0) \in D'$, and since D' is open, there exists a disk $V \ni w_0$ s.th. $v \subseteq D'$. Since f is continuous we know that there is a disk $U \in D$ s.th. $f(U) \subseteq V$. Since ψ is harmonic in D' which is simply connected, there is an analytic functions in V s.th. $\psi = \Re(g)$. But then $g \circ f$ is analytic in U and moreover $\Re(g \circ f)(z) = \psi(u(x, y), v(x, y)) = \phi(x, y)$. Hence ϕ is harmonic in U. Since z_0 was chosen arbitrarily, this implies that *phi* is harmonic in D. □

Suppose now that the function $f_D \to D'$ maps D bijectively on D' and that ∂D maps bijectively onto $\partial D'$. . Suppose also that the boundary conditions for ψ in D' correspond to the boundary conditions for ϕ in D, i.e. (*) holds for $(x, y) \in \partial D$.

If we can solve the Dirichlet problem for $\psi \in D'$, we can also solve it for ϕ.

Example 21. Find a function ϕ which is harmonic inside the unit disk $|z| < 1$ s.th. $\phi(x, y) = 1$ on the upper half-circle and -1 on the lower half-circle.

Solution: The Möbius transformation taking
$$\begin{cases} -1 & \to 0 \\ 0 & \to 1 \\ 1 & \to \infty \end{cases} \quad \text{is given by} \quad \frac{w - 0}{w - \infty}\frac{1 - \infty}{1 - 0} = \frac{z + 1}{z - 1}\frac{0 - 1}{0 + 1}$$
$$w = -\frac{z + 1}{z - 1} = \frac{1 + z}{1 - z}.$$

$$\Rightarrow \psi(u,v) = \frac{2}{\pi}\text{Arg}(w)$$
$$\Rightarrow \phi(x,y) = \frac{2}{\pi}\text{Arg}\left(\frac{1+z}{1-z}\right) = \frac{2}{\pi}\text{Arg}\left(\frac{1+x+iy}{1-x-iy}\right) = \text{Arg}\left(\frac{2y}{1-x^2-y^2}\right)$$

Theorem 16 (The Riemann Mapping Theorem)**.** *Every simply connected domain $D \neq \mathbb{C}$ can be conformally mapped onto the open unit disk.*

Theorem 17 (Poisson Integral Formula for the disk)**.** *Let f be a continuous function defined on the unit circle $|z| = 1$. Then*

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(e^{it})dt$$

Proof. will follow from the Cauchy Integral Formula. □

8 Complex Integration

We shall now study so-called contour integral (or line integrals) of complex-valued functions. This theory of integration will teach more about the properties of analytic functions. There are infinitely many parameterizations for a given curve, e.g. change the 'direction'. Obviously, there are two natural directions of a smooth curve. A smooth curve with a specified directions is called a directed smooth curve.

Example 22. Give parameterizations of the following directed smooth curves

(a) The line from z_1 to z_2.

(b) The circle if radius r and z_0

(a) $z(t) = z_1 + t(z_2 - z_1)$, $0 \leq t \leq 1$

(b) $z(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$

Definition 20. A contour γ is either a single point or a finite sequence $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ of oriented smooth curves s.th. the terminal point of the k^{th} curve is the starting point of the $k+1^{\text{th}}$ curve. The contour γ is said to be a closed contour if the terminal and initial points coincide. If these are the only self-intersections, we call γ a simple closed curve.

If γ is a smooth curve, the length of γ is given by

$$l(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \left|\frac{dz}{dt}\right| dt$$

Contour Integrals

We shall now see how to define the contour integral of a complex-valued function over a contour γ

$$\int_\gamma f(z)dz$$

where γ is a smooth curve.

Definition 21 (Properties of Contour Integrals)**.**

- $\displaystyle\int_\gamma (f+g)(z)dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$
- $\displaystyle\int_\gamma cf(z) == c\int_\gamma f(z)dz$
- $\displaystyle\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$

Theorem 18. *If f in continuous along γ, then f is integrable.*

How do we compute it? First consider a segment of $\mathbb{R}$

$$\int_a^b f(t)dt, \quad f = u + iv$$

and u, v continuous on $[a,b]$. Let $F(t)$ be an anti-derivative of $f(t)$, i.e. $F(t) = U(t) + iV(t)$, where $U'(t) = u(t)$ and $V'(t) = v(t)$, so

$$\int_a^b f(t)dt = \int_a^b u(t) + iv(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt = [U(t)]_a^b + i\,[V(t)]_a^b = F(b) - F(a).$$

Theorem 19. *If f is continuous on $[a,b]$ and $F'(t) = f(t)$ for $t \in [a,b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

The integral along any directed smooth curve can be reduced to an integral on the above form by using a parameterization of the directed smooth curve $z(t,\ a \le t \le b)$.

Let $z_0 = z(t_0),\ z_1 = z(t_1), \ldots, z_n = z(t_n)$, where $a = t_0 < t_1 < t_2 < \ldots < t_n = b$. then

$$\sum_{k=1}^{n} f(z_k)\Delta z_k = \sum_{k=1}^{n} f(z(t_k))(z(t_k) - z(t_{k-1})) \qquad \text{Riemann sum for } f(z(t))z'(t)$$
$$= \sum_{k=1}^{n} f(z(t_k))z'(t_k)\Delta t_k$$

This suggests the following theorem:

Theorem 20. *Let f be a continuous function on a directed smooth curve having admissible parameterization $z(t)$, $a \leq t \leq b$. Then*

$$\int_{\gamma} f(z)dx = \int_{a}^{b} f(z(t))z'(t)dt.$$

Example 23. compute

$$\int_{C_r} (z - z_0)^n dz,\ n \in \mathbb{Z}$$

over $C_r = \{z : |z - z_0| = r\}$, oriented counter-clockwise.

Solution: Let $z(t) = re^{it}$, $0 \leq t \leq 2\pi$. then $z'(t) = ire^{it}$, so

$$\int_{C_r} (z - z_0)^n dz = \int_{0}^{2\pi} r^n e^{it} ire^{it} dt = ir^{n+1} \int_{0}^{2\pi} e^{i(n+1)t} dt$$
$$= \begin{cases} ir^{n+1} \left[\dfrac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi}, & n \neq -1 \\ i2\pi, & n = -1 \end{cases}$$

Definition 22. Suppose $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$ and let f be continuous on γ, then we let

$$\int_{\gamma} f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz \qquad \text{and if } \gamma = z_0 \text{ the integral vanishes identically.}$$

ML-Inequality

Suppose $|f(z) \leq M\ \forall Z \in \gamma$. Then

$$\left| \sum_{k=1}^{n} f(z_k)\Delta z_k \right| \leq \sum_{k=1}^{n} |f(z_k)\Delta z_k| \leq M \sum_{k=1}^{n} |\Delta z_k| \leq Ml(\gamma).$$

Letting $\mu(P_n) \to 0$ leaves us with

$$\left| \int_\gamma f(z)dz \right| \leq ML, \; L = l(\gamma)$$

Theorem 21. *Suppose f is continuous on γ and that $|f(z)| \leq M \, \forall z \in \gamma$. Then*

$$\left| \int_\gamma f(z)dz \right| \leq ML, \; L = l(\gamma)$$

9 Independence of Path, Cauchy's Integral Theorem

Independence of Path

Theorem 22. *Suppose the f is continuous in a domain D, and that f has an anti-derivative F in D, i.e. $F'(z) = f(z) \, \forall z \in D$. Let γ be a contour in D with initial point z_i and terminal points z_t. Then*

$$\int_\gamma f(z)dz = F(z_t) - F(z_i)$$

Proof.

$$\int_\gamma f(z)dz = \sum_k \int_{\gamma_k} f(z)dz = \sum_k \int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt$$

where $z(t)$, $\tau_{k-1} \leq t \leq \tau_k$ is a parameterization of γ_k. Now, $\frac{d}{dt}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t)$. So by Theorem 18 from the previous chapter,

$$\int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt = [F(z(t))]_{\tau_{k-1}}^{\tau_k} = F(z(\tau_k)) - F(z(\tau_{k-1})).$$

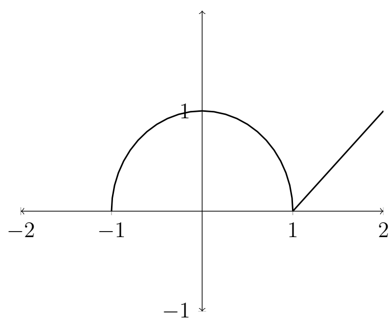
Summing over k yields our result.

□

Example 24. Compute

$$\int_\gamma \cos(z)dz$$

with γ as in figure.



Solution:

$$\int_\gamma \cos(z)dz = [\sin(z)]_{-1}^{2+i} = \sin(2+i) - \sin(-1).$$

Corollary 3. *If f is continuous in a domain D and has an anti-derivative in D, then*

$$\int_\gamma f(z)dz = 0$$

for any closed contour $\gamma \in D$.

Example 25. Let $f(z) = z^n$, $n \neq 1$, and let γ be any closed contour not passing through the origin. Then

$$\int_\gamma z^n dz = 0$$

since z^n has an anti-derivative $\frac{z^{n+1}}{n+1}$ in $\mathbb{C} \setminus \{0\}$.

Theorem 23. *Let f be continuous in a domain D. Then the following are equivalent:*

(i) f has an anti-derivative in D.

(ii) $\int_\gamma f(z)dz = 0$ for every closed contour $\gamma \in D$.

(iii) Contour integrals are independent of path in D. (i.e. if γ_1 and γ_2 are two contours with the same initial and terminal points, then the integrals give the same value).

Proof.

$(i) \Rightarrow (ii)$ Shown above.

$(i) \Rightarrow (iii)$ Shown above.

$(ii) \Rightarrow (iii)$ Given γ_1 and γ_2, let $\gamma := \gamma_1 + (-\gamma_2) \Rightarrow$ the integral is identically zero.

$(iii) \Rightarrow (i)$ Fix $z_0 \in D$. This implies that for any $z \in D$ there exists a polygonal path γ_z from z_0 to z. Define $F(z) = \int_{\gamma_z} f(\zeta)d\zeta$. $F(z)$ is well-defined, i.e. independent of choice of γ_z, by (iii).

$$\begin{aligned}\Rightarrow F(z+\Delta z) - F(z) &= \int_L f(\zeta)d\zeta = \int_L f(z)d\zeta + \int_L (f(\zeta) - f(z))d\zeta \\ &= f(z)\Delta z + \int_L (f(\zeta) - f(z))d\zeta\end{aligned}$$

i.e.

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z}\int\limits_L (f(\zeta)-f(z))d\zeta$$

But by the ML-Inequality

$$\left|\frac{1}{\Delta z}\int\limits_L (f(\zeta)-f(z))d\zeta\right| \le \frac{1}{\Delta z}\max_{\zeta\in L}|f(\zeta)-f(z)||\Delta z| \to 0$$

as $\Delta z \to 0$ by the continuity of f. This implies that $F'(z) = f(z)$.

$\square$

Cauchy's Integral Theorem

Let γ be a simple closed contour in $\mathbb{C}$ parameterized by $z(t)$, $a \le t \le b$.

$$\begin{aligned}
\Rightarrow \int\limits_\gamma f(z)dz &= \int\limits_a^b f(z(t))z'(t)dt = \int\limits_a^b [u(x(t),y(t)) + iv(x(t),y(t))]\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt \\
&= \int\limits_a^b \left[u(x(t),y(t))\left(\frac{dx}{dt}\right) - v(x(t),y(t))\left(\frac{dy}{dt}\right)\right]dt \\
&\quad + i\int\limits_a^b \left[v(x(t),y(t))\left(\frac{dx}{dt}\right) - u(x(t),y(t))\left(\frac{dy}{dt}\right)\right]dt
\end{aligned}$$

i.e.

$$\int\limits_\gamma f(z)dz = \int\limits_\gamma udx - vdy + i\int\limits_\gamma vdx + udy.$$

Theorem 24 (Green's Theorem)**.** *Let $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$ be a C^1 vector field defined on a simply connected domain D, and let γ be a positively oriented simple closed contour in D. Then*

$$\int\limits_\gamma (F_1dx + F_2dy) = \iint\limits_\Omega \left(\frac{\partial F_2}{\partial dx} - \frac{\partial F_1}{\partial dy}\right)dxdy$$

Where $\gamma = \partial\Omega$.

Let us use this expression for the integral of $f(z)$ over γ:

$$\int\limits_\gamma f(z)dz = \iint\limits_\Omega \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint\limits_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\right)dxdy.$$

If we assume $u, v \in C^1$. If we moreover assume that f is analytic in D, the entire integral vanishes identically in view of the Cauchy-Riemann Equations.

Theorem 25 (Cauchy's Integral Theorem)**.** *Suppose that f is analytic in a simply connected domain D and let γ be any closed contour in D. Then*

$$\int_\gamma f(z)dz = 0$$

Remark 12. *The theorem generalizes the discussion thus far in two ways:*

(i) γ can be any closed contour, i.e. need not be simple.

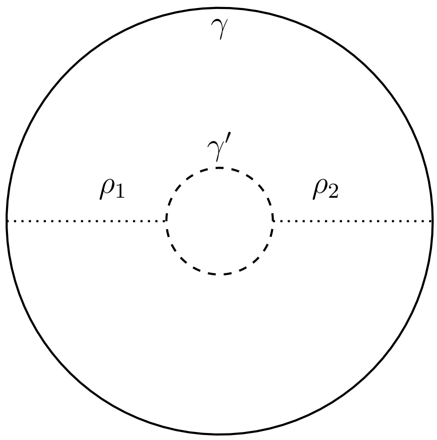
(ii) The assumption that $u, v \in C^1$ has been dropped.

The fact that the second assumption can be dropped was proven by Edward Goursat. The Theorem is therefore often called the Cauchy-Goursat Theorem.

Remark 13. *The Theorem implies that if f is analytic inside and on a simple closed contour, then the integral of f along said contour is identically zero.*

Theorem 26. *Suppose f is analytic in a simply connected domain. Then f has an anti-derivative, contour integrals are path-independent, and integrals over closed contours vanish identically.*

Example 26. Consider the contours γ (black) and γ' (dashed) in the figure. Suppose that f is analytic in a domain containing γ and γ', and the region between these two contours. (In the equations below, γ_1, γ_1' denote the parts of the contours 'above' ρ_1, ρ_2 and γ_2, γ_2' denote the parts of the contours 'below' ρ_1, ρ_2). By Cauchy's Integral Theorem



$$\int_{\gamma_1} + \int_{\rho_1} + \int_{-\gamma_1'} + \int_{\rho_2} = 0 \qquad \int_{\gamma_2} + \int_{\rho_2} + \int_{-\gamma_2'} + \int_{\rho_1} = 0$$

$$\Rightarrow \int_{\gamma_1} + \int_{\gamma_2} = \int_{-\gamma_1'} + \int_{-\gamma_2'}$$
$$\int_\gamma = \int_{\gamma'} .$$

So we can 'deform' the contour γ into γ' without affecting the integral. This illustrates the 'Deformation Theorem'.

10 Cauchy's Integral Formula and Applications

Theorem 27 (Cauchy's Integral Formula)**.** *Suppose f is analytic in a simply connected domain D. Let γ be a simple closed positively oriented contour in D. Moreover, let $z_0 \in \gamma$. then*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z)}{z - z_0} dz.$$

Proof. $\dfrac{f(z)}{z - z_0}$ is analytic in $D \setminus \{z_0\}$. As in the last example from the previous chapter, we see that

$$\begin{aligned} \int\limits_{\gamma} \frac{f(z)}{z - z_0} dz = \int\limits_{C_r} \frac{f(z)}{z - z_0} dz &= \int\limits_{C_r} \frac{f(z_0)}{z - z_0} dz + \int\limits_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \\ &= 2\pi i f(z_0) + \int\limits_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz. \end{aligned}$$

Note: Clearly, the second term above is independent of r. So it is enough to show that

$$\lim_{r \to 0} \int\limits_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

Let $M_r := \max\limits_{z \in C_r} |f(z) - f(z_0)|$. f is continuous, which implies that $M_r \to 0$ as $r \to 0^+$. So, by the ML-Inequality:

$$\left| \int\limits_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{M_r}{r} 2\pi r \to 0.$$

$\square$

Remark 14. *$f(z_0)$ is determined by $f(z)$, $z \in \gamma$. so, Cauchy's formula says that*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad z \text{ interior to } \gamma.$$

By differentiability under the integral sign, it seems plausible that:

Theorem 28 (Cauchy's Generalized Integral Formula)**.** *If f is analytic inside and on a simple closed positively oriented contour γ, and z is a point interior to γ, then*

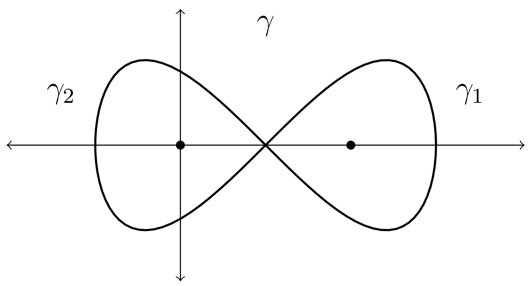
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Proof. The proof is by induction by using the definition of the derivative.. ☐

Example 27. Compute

$$\int_\gamma \frac{2z+1}{z(z-1)^2}dz$$

where γ is as in the figure.



Note that $\gamma = \gamma_1 + \gamma_2$, and γ_1, γ_2 are simple closed contours.

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{\gamma_1} \frac{\frac{2z+1}{z}}{(z-1)^2}dz + \int_{\gamma_2} \frac{\frac{2z+1}{(z-1)^2}}{z}dz \\ &= 2\pi i \frac{d}{dz}\left(\frac{2z+1}{z}\right)\Bigg|_{z=1} - 2\pi i \left(\frac{2z+1}{(z-1)^2}\right)\Bigg|_{z=0} \\ &= -4\pi i.\end{aligned}$$

Theorem 29. *If f is analytic in a domain D, then <u>all</u> derivatives f, f', f'', $\ldots$ exist in D and are analytic in D.*

Proof. Apply Cauchy's Generalized Integral Formula in a positively oriented contour (circle) around an arbitrary point $z_0 \in D$. ☐

So, in particular, if f is analytic, then f' is analytic as well.

Suppose f is analytic. Then $f'(z) = u_x + iv_x = v_y - iu_y$ by Cauchy-Riemann equations. Since f' is analytic, hence continuous, it follows that $u, v \in C^1$. Since f'' exists and is continuous; and moreover $f''(z) = u_{zz} + iv_{xx} = v_{xy} - iu_{xy}$. It follows that all second order derivatives are continuous, etc.

Theorem 30. *If $f = u + iv$ is analytic, then $u, v \in C^\infty(D)$.*

Remark 15. *This completes the proof that u, v are harmonic functions. Suppose f is continuous in D and that*

$$\int_\gamma f(z)dz = 0$$

for all closed contours $\gamma \in D$. Independence of path theorem implies that f has an anti-derivative in D, i.e. $F' = f$ in D. Since F is analytic, this implies that $F' = f$ is analytic.

Theorem 31 (Morera). *If f is continuous in a domain d, and*

$$\int_{\gamma} f(z)dz = 0$$

for all closed contours $\gamma \in D$, then f is analytic in D

Consequences of Cauchy's Generalized Integral Formula

Theorem 32 (Cauchy Estimate). *Let f be analytic inside and on a circle C_R of radius R centered at z_0. Suppose $|f(z)| \leq M \, \forall z \in C_R$. Then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n} \qquad \forall n \in \mathbb{N}$$

Proof. Give C_r a positive orientation. Apply Cauchy's Generalized Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \qquad \zeta \in C_r$$

$$\left|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \leq \frac{M}{R^{n+1}}$$

so, by the ML-Inequality

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi i} \frac{M}{r^{n+1}} 2\pi R = \frac{n!M}{R^n}.$$

□

Suppose f is an entire function, $|f(z)| \leq M \, \forall z \in \mathbb{C}$. Then a Cauchy-estimate implies

$$|f'(z_0)| \leq \frac{M}{R} \, \forall R \Rightarrow |f'(z_0)| = 0$$

i.e. $f'(z_0) = 0$. Since z_0 was arbitrary $f'(z) = 0 \, \forall z \in \mathbb{C} \Rightarrow f$ constant in $\mathbb{C}$.

Theorem 33 (Liouville's Theorem). *The only bounded entire functions are the constant functions.*

11 The Fundamental Theorem of Algebra and Applications to Harmonic Functions

Theorem 34 (Fundamental Theorem of Algebra). *Every non-constant polynomial with complex coefficients has at least one (1) zero.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,\ a_n \neq 0$. Suppose $p(z)$ has no zeros. Put $f(z) = (p(z))^{-1}$. Then f is entire. We next show that $f(z)$ is bounded.

1.

$$\begin{aligned} p(z) &= z^n \left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right), \quad p(z) \to a_n \text{ as } z \to \infty \\ &\Rightarrow \exists \varphi : |z| \geq \varphi \Rightarrow \left| \frac{p(z)}{z^n} \right| \geq \frac{|a_n]}{z^n} \\ &\Rightarrow |f(z)| = \frac{1}{|p(z)|} \leq \frac{2}{|z|^n |a_n|} \leq \frac{2}{\varphi^n |a_n|},\ |z| \geq \varphi \end{aligned}$$

2. For $|z| \leq \varphi$ the function $|f(z)|$ is a continuous functions on a compact set, which means it must have an upper bound said set ($|z| \leq \varphi$). This implies that $\frac{1}{p(z)}$ is a bounded entire function, hence constant by Liouville's Theorem. This in turn implies $p(z)$ is constant, meaning $n = 0$.

In other words: The only polynomials without zeros are the constant polynomials. □

The Mean-Value Property and Maximum Modulus Principle

Suppose f is analytic inside and one a circle C_R of radius R centered at z_0. By Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} dz.$$

Parameterize $C_R : z(t) = z_0 + Re^{it},\ t \in [0, 2\pi]$.

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it} dt = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + Re^{it}) dt. \qquad (*)$$

$(*)$ is called the 'mean value property'.

Lemma 2. *Suppose f is analytic in a disk centered at z_0, sat $D(z_0)$. Suppose also that*

$$\max_{z \in D(z_0)} |f(z)| = |f(z_0)|$$

Then

$$|f(z)| = |f(z_0)| \quad \forall z \in D(z_0)$$

Proof. Suppose $|f(z)|$ is <u>not</u> constant. Then $\exists z_1 \in D(z_0)$ s.th. $|f(z_1)| < |f(z_0)|$. Let C_R be the circle with center in z_0 passing through z_1. By assumption, $|f(z)| \leq |f(z_0)|\, \forall z \in$

C_R. Since f is continuous, there exists a segment of C_R containing z_1 in which $|f(z)| < |f(z_0)|$. Say that $|f(z)| < |f(z_0)| - 2\pi\varepsilon$ on a segment of length δ.

$$\begin{aligned}
\Rightarrow |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) dt \right| && \text{By the mean-value-property} \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})| dt \\
&< \frac{1}{2\pi} \left(|f(z_0)|(2\pi - \delta) + (|f(z_0)| - 2\pi\varepsilon)\delta \right) \\
&= |f(z_0)| - \varepsilon\delta.
\end{aligned}$$

Which is a contradiction. □

Theorem 35 (The Maximum Modulus Principle)**.** *If f is analytic in a domain D and $|f(z)|$ attains its maximum value at a point $z_0 \in D$, then f is constant in D.*

Proof. We show that $|f|$ is constant in D. The result then follows from the Cauchy-Riemann Equations. Suppose that $|f(z)|$ is not constant. This implies

$$\exists z_1 \in D : |f(z_1)| < |f(z_0)|.$$

Let γ be a polygonal path from z_0 to z. We consider now the values of $|f(z)|$ for z on γ, starting at z_0. Then $\exists w \in \gamma$ s.th.

(i) $|f(z)| = |f(z_0)|$, $\forall z$ preceding $z \in \gamma$.

(ii) $\exists$ points $z \in \gamma$ arbitrarily close to $w \in \gamma$ s.th. $|f(z)| < |f(z_0)|$

follows from the supremum property of $\mathbb{R}$. Parameterize γ s.th. $z = z(t)$, $t|in[0,1]$, $z(0) = z_0$, $z(1) = z_1$. Let M be the set

$$M = \{z \in [0,1] : |f(z(t))| < |f(z)|\}.$$

We know that $M \neq \emptyset$ since $1 \in M$, and M is bounded below by 0. Let $\alpha = \inf(M)$ (greatest lower bound). Put $w = f(z(\alpha))$.

(i) α is a lower bound implies that for all $t \in M$ it holds that $\alpha \leq t$

$$\begin{aligned}
&\Rightarrow t < \alpha \Rightarrow t \notin M, \text{ i.e. } |f(z(t))| = |f(z)| \\
&\Rightarrow (i) \text{ holds true.}
\end{aligned}$$

(ii) α is the greatest lower bound

$$\Rightarrow \forall \beta > \alpha \; \exists t \in M : t < \beta,$$

i.e. there are points t with $|f(z(0))| < |f(z_0)|$. So (ii) holds.

Since f is continuous, (i) implies $|f(w)| = |f(z_0)|$. There exists $D(w)$ contained in D. By lemma, $|f(z)|$ is constant in $D(w)$, but this contradicts (ii). Thus the assumption the $|f|$ is not constant is false. □

Now, suppose f is analytic in a bounded domain D, and that f is continuous up to ∂D. This implies that $|f|$ attains a maximum value in $\bar{D}$.

Theorem 36. *A function which is analytic in a bounded domain D and continuous up to ∂D attains its maximum and minimum modulus on the boundary*

Recall that if ϕ is a harmonic function in a simply connected domain D, then there exists a function f such that $\Re(f) = \phi$. Now, we prove the above theorem:

Proof. If there is such an f, say $f = \phi + i\psi$, then $f' = \phi_x + i\psi_y = \phi_x - i\phi_y$. So, put $g(z) = \phi_x - i\phi_y$. Then

$$\begin{cases} (\phi_x)_x = (-\phi_y)_y; & \phi \text{ harmonic} \\ (\phi_x)_y = -(-\phi_y)_x; & \phi \in C^2 \end{cases} \Rightarrow \text{Cauchy-Riemann Equations satisfied}$$

Also, g^{s} real and imaginary parts are C^1 since $\phi \in C^2$. This implies the following:

- $g(z)$ is analytic in a simply connected domain.
- g has an anti-derivative $G \in D$, $G = u + iv$, since $G' = g$.
- $u_x - iu_y = \phi_x - i\phi_y$
- $(\phi - u)_x = (\phi - u)_y = 0 \Rightarrow (\phi - u)$ is some constant $c \in \mathbb{R}$

Now put $f(z) = g(z) + c$ and the result follows. $\square$

Theorem 37 (Maximum Principle for Harmonic Functions)**.** *If ϕ is harmonic in a simply connected domain D and attains its maximum or minimum at some point inside the domain, then ϕ is constant.*

Proof. Let ϕ be harmonic in a simply connected domain D and let $f = \phi + i\psi$ be analytic in D. Then

$$|e^f| = |e^{\phi + i\psi}| = |e^\phi|.$$

The exponential function is monotonically increasing, so if ϕ has maximum in D, then so will $|e^f|$. This implies that e^f is constant by the Maximum Modulus Principle. $\square$

Theorem 38. *If ϕ is harmonic in a bounded simply connected domain D, and continuous up to ∂D, then ϕ attains its maximum and minimum on ∂D.*

Proof. Since ϕ attains its minimum precisely when $-\phi$ attains its maximum, this implies the corresponding minimum principle. $\square$

12 More on Analytic Functions and Notions of Convergence

We begin by making a remark on Theorems 37 and 38 from the last chapter. In the formulation of the two theorems, the assumption of simple connectedness is not necessary (the first theorem gives an analog of a Lemma from the same chapter, then argue similarly as in the proof of the Maximum Modulus Principle).

Recall now the Dirichlet problem; the problem of finding a function $\phi(x, y)$ which is harmonic in a domain D, and continuous up to ∂D, with given values on ∂D. An interesting question to pose is: 'Is there a solution to Dirichlet's Problem, and if so: is it unique?'. The above implies uniqueness for bounded domains.

Theorem 39. *Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be harmonic in a bounded domain D, and continuous up to ∂D, s.th. $\phi_1 = \phi_2$ on ∂D. Then $\phi_1 \equiv \phi_2$ in D.*

Proof. Let $\phi = \phi_1 - \phi_2$. By Theorem 38, ϕ is harmonic and attains both maximum and minimum on ∂D. But $\phi = 0$ on ∂D, so $\phi_1 = \phi_2$. $\square$

An explicit solution can be found if e.g. D is a disk.

Poisson's Integral Formula

Suppose $f = \phi + i\psi$ is analytic inside and on the circle C_R centered at the origin and with radius R. then, by Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint\limits_{C_R} \frac{f(\zeta)}{\zeta - z} dz.$$

Let z^*, z be symmetric w.r.t. C_R, i.e.

$$(z^* - 0)\overline{(z - 0)} = R^2 \iff z^* = \frac{R^2}{\bar{z}}.$$

Then $z^* > R$, so by Cauchy's theorem

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint\limits_{C_R} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z^*} \right) dz = \frac{1}{2\pi i} \oint\limits_{C_R} \frac{R^2 - |z|^2}{(\zeta - z)(R^2 - \bar{z}\zeta)} f(\zeta) d\zeta \\ &= \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - \bar{z}Re^{it})} f(Re^{it}) iRe^{it} dt \\ &= \frac{R^2 - |z|^2}{2\pi} \int\limits_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2} dt \end{aligned}$$

Taking the real part and identifying (x, y) with re^{it} yields

$$\phi(re^{it}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(Re^{it})}{R^2 - 2R\cos(z - \theta) + r^2} dt$$

Sequences and Series of Functions, Uniform Convergence

In Section 3 we defined what is meant by convergence of a sequence of complex numbers. Recall that the sequence $\{a_n\}_{n=0}^{\infty}$ has a limit a if for $\varepsilon > 0\, \exists N : n > N \Rightarrow |a_n - a| < \varepsilon$.

Definition 23. A series if a formal expression of the form

$$c_0 + c_1 + c_2 + \ldots + c_j + \ldots \iff \sum_{j=0}^{\infty} c_j \qquad c_j \in \mathbb{C}$$

The n^{th} partial sum of the series, s_n, is the sum

$$\sum_{j=0}^{n} c_j$$

of the first $n + 1$ terms. If $\{s_n\}_{n=0}^{\infty}$ has a limit s, then the series is said to converge to s. In this case, we say that

$$s = \sum_{j=0}^{\infty} c_j.$$

If the series does not converge, it is said to diverge.

Example 28. The following holds:

$$\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}, \quad |c| < 1.$$

Proof.

$$\begin{aligned}(1-c)(1 + c + \ldots + c^{n-1} + c^n) &= 1 + c + \ldots + c^{n-1} + c^n - (c + c^2 + \ldots + c^n + c^{n+1}) \\ &= 1 - c^{n+1}\end{aligned}$$

i.e.

$$\frac{1}{1-c} - (1 + c + \ldots + c^{n-1} + c^n) = \frac{c^{n+1}}{1-c}.$$

The result follows, since $\frac{c^{n+1}}{1-c} \to 0$ as $n \to \infty$ if $|c| < 1$ $\square$

Theorem 40 (Comparison Theorem)**.** *Suppose that $|c_j| \leq M_j$, $\forall j \in \mathbb{N}$. Then*

$$\sum_{j=0}^{\infty} M_j \text{ convergent } \Rightarrow \sum_{j=0}^{\infty} c_j \text{ convergent.}$$

The proof of the above theorem is rather straight forward if one is familiar with the Cauchy Convergence Criterion.

Definition 24 (Cauchy Convergence Criterion)**.**

$$\{a_n\}_{n=0}^{\infty} \text{ converges } \iff \{a_n\}_{n=0}^{\infty} \text{ is Cauchy.}$$
$$\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N} : n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$$

Definition 25 (Absolute Convergence)**.** The series

$$\sum_{j=0}^{\infty} c_j$$

is said to be absolutely convergent if

$$\sum_{j=0}^{\infty} |c_j|$$

is convergent. An absolutely convergent series is convergent by a trivial use of the comparison test.

Theorem 41 (Ratio Test)**.** *Suppose that the terms of the series*

$$\sum_{j=0}^{\infty} c_j$$

have the following property: $\dfrac{|c_{j+1}|}{|c_j|} \to L$ *as* $j \to \infty$*. Then the given series converge if* $L < 1$ *and diverges if* $L > 1$.

Example 29. Show that $\displaystyle\sum_{j=0}^{\infty} \frac{4^j}{j!}$ converges.

Solution: Put $c_j = \frac{4^j}{j!}$. Then

$$\frac{|c_{j+1}|}{|c_j|} = \frac{4^{j+1}}{(j+1)!} \frac{j!}{4^j} = \frac{4}{j+1} \to 0 \text{ as } j \to \infty.$$

Convergence follows from the ratio test.

Definition 26 (Pointwise Convergence). Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions defined on the set $E \subseteq \mathbb{C}$. Then we say that the sequence converges pointwise to f on E if for each $z \in E$ the sequence of numbers $\{f_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$, i.e. for every $z|inE$ and for every $\varepsilon > 0$:

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon.$$

Definition 27 (Uniform convergence). Let $\{f_n\}_{n=0}^{\infty}$ be defined on $E \subseteq \mathbb{C}$. We say that the sequence converges uniformly to f on E if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} : n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon\, \forall z \in E$$

Remark 16. *The above definition is equivalent to* $\sup\limits_{z \in E} |f_n(z) - f(z)| \to 0$ *is* $n \to \infty$*.*

Why care about uniform convergence? Because it preserves properties of functions under the limit operation.

Theorem 42. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of continuous functions on e set $E \subseteq \mathbb{C}$, and suppose it converges uniformly to f on E. Then f is continuous on E.*

Proof. Take $z_0 \in E$ and let $\varepsilon > 0$ be given. We want to show that

$$z \in E,\ |z - z_0| < \delta \Rightarrow |f_n(z_0) - f(z)| < \varepsilon.$$

First choose N so large that $|f_n(z) - f(z)| < \varepsilon/3\, \forall z \in E$. This is possible since the sequence is uniformly convergent. Since f_N is continuous at z_0, there exists δ s.th. $|f_N(z_0) - f_N(z)| < \varepsilon/3$ if $|z - z_0| < \delta$. For such z is then holds that

$$\begin{aligned} |f(z_0) - f(z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z_0)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

13 Power Series and Taylor Series

Definition 28 (Power Series). A series of the form

$$\sum_{j=0}^{\infty} aj(z - z_o)^j$$

is called a 'power series'. The constants a_j are called the coefficients of the power series.

Theorem 43. *For any power series, there is an $R \in [0, +\infty)$ (dependent on a_j) s.th*

(i) The series converges absolutely for $|z - z_0| < R$.

(ii) The series converges uniformly in any closed subdisk $|z - z_0 \leq r < R|$.

(iii) The series diverges for $|z - z_0| > R$.

The number R is called the 'radius of convergence' of the series.

The proof depends on the following lemma:

Lemma 3. *Suppose that*

$$\sum_{j=0}^{\infty} a_z w^j \tag{$*$}$$

converges at some points having modulus $r_0 > 0$. Then $(*)$ converges absolutely and uniformly in any subdisk $|w7 \leq r < r_0$.

Proof. Suppose that $(*)$ converges at some w_0 with $|w_0| = r_0$. This implies

$$|a_j w^j| \to 0 \text{ as } j \to \infty \Rightarrow \exists M : |a_j w^j| = |a_j| r_0^j \leq M \,\forall j.$$

For any $|w| \leq r$ it follows that

$$|a_j w^j| = |a_j| r_0^j \left(\frac{|w|}{r_0}\right)^j \leq M \left(\frac{r}{r_0}\right)^j.$$

Letting $M_j = M\left(\frac{r}{r_0}\right)^j$ the result follows by Weierstrass M-test. ☐

Proof (theorem). Let $w = z - z_0$. If the series $(*)$ converges only for $w = 0$, or if it converges for all w we are done (by Lemma above). Otherwise, let r be the 'greatest' r s.th. $(*)$ converges for some w with $|w| = r$. More precisely, let $M = \{r > 0 : (*) \text{ converges for some } |w| = r\}$. then $M \neq \emptyset$ and upper-bounded. Let $R = \sup(M)$. For any $r < R$ there exists an $r < r_1 \leq R$ s.th. $r_1 \leq M$. By the lemma, $(*)$ converges absolutely and uniformly on $|w| \leq r$.

If $|w| > R$, then $(*)$ diverges. ☐

Example 30. the geometric series

$$\sum_{j=0}^{\infty} z^j$$

has radius of convergence 1. The series diverge if $|z| = 1$.

Example 31. The power series

$$\sum_{j=0}^{\infty} \frac{z^j}{j^2}$$

converges uniformly for $|z| < 1$. This follows by Weierstrass M-test with $M_j = j^{-2}$. The series diverges if $|z| > 1$.

Theorem 44. *Consider a power series*

$$\sum_{j=0}^{\infty} a_j(z-z_0)^j$$

(i) Ratio test: If the limit $L = \lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|}$ *exists, then* $R = 1/L$.

(ii) Root test: If the limit $L = \lim_{j\to\infty} \sqrt[j]{|a_j|}$ *exists, then* $R = 1/L$.

Remark 17.

(i) In both cases, it is understood that $R = +\infty$ *of* $L = 0$ *and that* $R = 0$ *if* $L = +\infty$.

(ii) In fact, the following formula, due to Hadamard, is true for any power series:

$$R = \frac{1}{\lim_{j\to\infty} \sup(\sqrt[j]{|a_j|})}.$$

Proof. Follows directly form the standard ratio/root test, e.g. let

$$c_j(z) = a_j(z-z_0)^j \Rightarrow \left|\frac{c_{j+1}(z)}{c_j(z)}\right| = \left|\frac{a_{j+1}}{a_j}\right| |z-z_0| \to L|z-z_0|,\ j\to\infty.$$

By the ratio test, if $L|z-z_0| < 1$, the series converges. If $L|z-z_0| > 1$, the series diverges. So $R = 1/L$. □

Example 32. (i) The series $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ has $R = +\infty$ since

$$a_j = \frac{1}{j!} \Rightarrow \frac{a_{j+1}}{a_j} = \frac{1}{j+1} \to 0 \text{ as } j\to\infty.$$

(ii) The series $\sum_{j=0}^{\infty} j!z^j$ has $R = 0$ since

$$a_j = j! \Rightarrow \frac{a_{j+1}}{a_j} = j+1 \to +\infty \text{ as } j\to\infty.$$

Since the partial sums of power series are analytic and converge uniformly in each closed subdisk of the disk of convergence, $\{|z-z_0| < R\}$, the convergence theorems from previous chapters immediately give the following theorem:

Theorem 45. *Suppose*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

has radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is analytic in the same disk. The derivatives of $f(z)$ can be obtained by termwise differentiation of the series:

$$\begin{aligned} f'(z) &= \sum_{k=0}^{\infty} k a_k (z - z_0)^{k-1} \\ f''(z) &= \sum_{k=0}^{\infty} k(k-1) a_k (z - z_0)^{k-2} \\ &\vdots \end{aligned}$$

In particular, $a_k = \dfrac{f^{(n)}(z_0)}{k!}$.

Example 33. We have that, for $|z| < 1$

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \Rightarrow \frac{1}{(1-z)^2} = \sum_{j=0}^{\infty} j z^{j-1}$$

by termwise differentiation. One can also integrate termwise:

$$\int_0^w \frac{1}{1-z} dz = \sum_{k=0}^{\infty} \int_0^w z^k dz = \sum_{k=0}^{\infty} \left[\frac{z^{k+1}}{k+1} \right]_0^w = \sum_{k=0}^{\infty} \frac{w^{k+1}}{k+1} = \sum_{n=1}^{\infty} \frac{w^n}{n}.$$

But $\displaystyle\int_0^w \frac{1}{1-z} dz = [-\mathrm{Log}(1-z)]_0^w = -\mathrm{Log}(1-w)$ for $|w| < 1$. This all implies

$$\begin{aligned} \mathrm{Log}(1-w) &= -\sum_{n=1}^{\infty} \frac{w^n}{n}, \ |w| < 1 \\ \Rightarrow \mathrm{Log}(1+z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \ |z| < 1. \end{aligned}$$

Example 34. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Since $R = +\infty$, this is an entire function.

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = f(z), \text{ also } f(0) = 1. \\ \Rightarrow \frac{d}{dz}\left(f(z)e^{-z}\right) &= f'(z)e^{-z} - f(z)e^{-z} = 0 \\ \Rightarrow f(z)e^{-z} &= c \in \mathbb{C}. \quad z = 0 \Rightarrow c = 0 \\ \Rightarrow e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

Theorem 46 (Taylor's Theorem). *Suppose that $f(z)$ is analytic in some disk $|z - z_0 < r|$. Then*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(n)}(z_0)}{k!}(z - z_0)^k, \ |z - z_0| < R.$$

Proof Sketch. Let $r < R$ and fix z s.th. $|z - z_0| < r$. By Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z)} d\zeta.$$

Now

$$\begin{aligned}
\frac{f(\zeta)}{(\zeta - z)} &= \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{f(\zeta)}{(\zeta - z_0)} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{f(\zeta)(z - z_0)^k}{(\zeta - z_0)^{k+1}}
\end{aligned}$$

where the convergence is uniform w.r.t. $\zeta \in C_r$. All in all, this implies

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)(z - z_0)^k}{(\zeta - z_0)^{k+1}} d\zeta = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right) \cdot (z - z_0)^k \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k
\end{aligned}$$

□

14 Laurent Series, Zeros and Singularities

Theorem 47 (Laurent's Theorem). *Suppose that f is analytic in $r < |z - z_0| < R$. then f can be expressed as the sum of two series*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}. \tag{1}$$

Where both series converge absolutely in $r < |z - z_0| < R$, and they both converge uniformly in any sub-annulus

$$r - \rho_1 < |z - z_0| < \rho_2 < R.$$

The coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta, \quad j \in \mathbb{Z} \tag{2}$$

where C is any positively oriented circle $|z-z_0| = \rho$, with $r < \rho < R$. Any pointwise convergent expansion of f in $r < |z-z_0| < R$ of the form (1) agrees with that above, i.e. the coefficients a_j agree with (2), and therefore the expansion is unique.

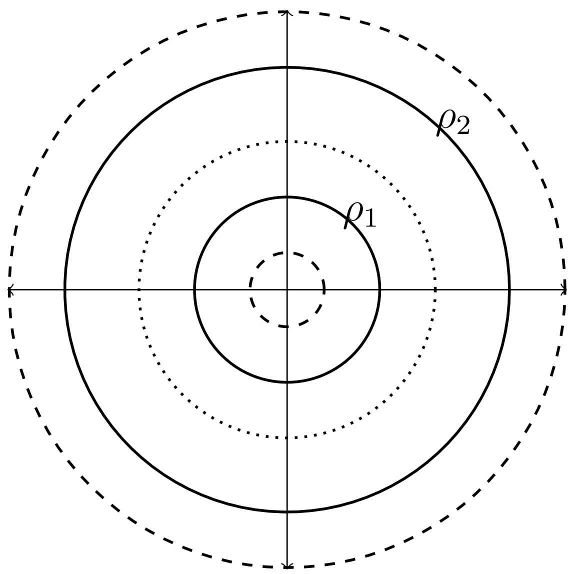
Remark 18. *1. We allow that $r = 0$ and $R = +\infty$.*

2. The above expansion of f is called the 'Laurent Series' of f in $r < |z-z_0| < R$. It is often written shortly

$$f(x) = \sum_{j=-\infty}^{\infty} a_j (a-a_0)^j$$

3. If f is analytic in $|z-z_0| < R$, then $a_j = 0$ for $j < 0$ by Cauchy's Integral Theorem, and the other terms reproduce the Taylor series of f (According to Cauchy's Generalized Integral Formula).

Proof Sketch. We start by proving existence. Fix $z \in \{\rho_1 \leq |z-z_0| \leq \rho_2\} = A_{\rho_1,\rho_2}$. Let C_1, C_2 be two positively oriented circles of radii $\tilde{\rho}_1$, $\tilde{\rho}_2$ (dashed in figure), where $\tilde{\rho}_1 < \rho_1 < \rho_2 < \tilde{\rho}_2$.



By Cauchy's Integral Formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z)}d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)}d\zeta \qquad (3)$$

As in the proof of Taylor's Theorem, for $\zeta \in C_2$:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0}\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \frac{f(\zeta)}{(\zeta - z_0)}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^j = \sum_{j=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{j+1}}(z-z_0)^j$$

with uniform convergence in the variable $\zeta \in C_2$.

$$\Rightarrow \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta - z)}d\zeta = \sum_{j=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{j+1}}d\zeta\right](z-z_0)^j = \sum_{j=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_2} d\zeta\right](z-z_0)^j$$
$$= \sum_{j=0}^{\infty} a_j (z-z_0)^j,$$

with a_j as in (2).

In fact, the series converges pointwise got any C inside C_2, hence it does converge absolutely and uniformly on any smaller disk (and so on A_{ρ_1,ρ_2}, since $\tilde{\rho}_2$ is arbitrarily smaller than R, the first term in (3) is analytic on $|z-z_0|<R$).

Similarly for $\zeta \in C_1$:

$$-\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{z-z_0}\frac{1}{1-\frac{\zeta-z_0}{z-z_0}} = \frac{f(\zeta)}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^j = \sum_{j=1}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{j+1}}(z-z_0)^{-j}$$

with uniform convergence w.r.t. $\zeta \in C_1$.

$$\Rightarrow -\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{\zeta-z}d\zeta = \sum_{j=1}^{\infty}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-j+1}}d\zeta\right](z-z_0)^{-j} = \sum_{j=1}^{\infty}\left[\frac{1}{2\pi i}\int_{C_1}d\zeta\right](z-z_0)^{-j}$$
$$= \sum_{j=0}^{\infty}a_{-j}(z-z_0)^{-j}.$$

The latter series in fact converges pointwise for any t outside C_1. An argument similar to that in the power series lemma shoes that the series converges absolutely and uniformly outside any circle strictly larger than C_1, and so on this annulus A_{ρ_1,ρ_2} (since in fact $\tilde{\rho}_1$ was arbitrarily chosen, the second term in (3)) is analytic on $|z-z_0|>r$.

This proves the existence. Suppose now that we have an expansion of f

$$f(z) = \sum_{j=0}^{\infty}a_j(z-z_0)^j$$

which converges pointwise in the given annulus. Then convergence is uniform on C, so

$$\int_C \frac{f(\zeta)}{(\zeta-z)^{k+1}}d\zeta = \sum_{j=0}^{\infty}a_j\int_C (z-z_0)^{j-k-1}dz$$

the integral evaluates to $2\pi i$ if $j=k$ and 0 otherwise, so:

$$= 2\pi i a_k.$$

the coefficients are uniquely defined by f and given by (2). $\square$

Example 35. Determine the Laurent series for

$$f(z) = \frac{z^2-2z+3}{z-2}, \quad |z-1|>1.$$

Solution:

1. f is analytic in $\mathbb{C}\setminus\{2\}$
2. $z_0 = 1,\ r = 1,\ R = +\infty$.

$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{1-\frac{1}{z-1}} = \frac{1}{z-1}\sum_{j=0}^{\infty}(z-z_0)^{-j} = \sum_{j=1}^{\infty}\frac{1}{(z-1)^j}.$$

Now

$$z^2-2z+3 = (z-1)^2+2 \Rightarrow \frac{z^2-2z+3}{z-2} = [z^2-2z+3]\left(\frac{1}{z-1}+\frac{1}{(z-1)^2}+\dots\right)$$
$$= (z-1)+1+\sum_{j=1}^{\infty}\frac{3}{(z-1)^j}, \quad |z-1|>1.$$

Example 36. Determine the Laurent Series for $e^{\frac{1}{z}}$, around $z=0$.

Solution: We know that

$$e^w = \sum_{j=0}^{\infty}\frac{w^j}{j!} \Rightarrow e^{\frac{1}{z}} = \sum_{j=0}^{\infty}\frac{1}{j!z^j}, \quad z \in \mathbb{C}\setminus\{0\}.$$

Zeros and Singularities

Definition 29. Suppose that f is analytic at z_0. Then z_0 is called a zero of order m for out function f if the following holds:

$$f^{(j)}(z_0) = 0,\ j = 0, 1, \dots,\ m-2,\ m-1, \quad f^{(m)}(z_0) \neq 0$$

A zero of order 1 is called a simple zero.

Theorem 48. *Suppose f is analytic at z_0. Then f has a zero of order m at z_0 iff. it can be written as follows:*

$$f(z) = (z-z_0)^m g(z),$$

where g is analytic at z_0 and

$$g(z_0) \neq 0.$$

Proof Sketch. By Taylor's Theorem (Theorem 46),

$$\begin{aligned} f(z) &= a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots \\ &= (z-z_0)^m[a_m + a_{m+1}(z-z_0) + \dots] \end{aligned}$$

$$a_m = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

Let $g(z) = [a_m + a_{m+1}(z-z_0) + \dots]$ which is analytic near z_0 and $g(z_0) = a_m \neq 0$. Taylor expanding g we find that f has power series representation

$$f(z) = g(z_0)(z-z_0)^m + g'(z_0)(z-z_0)^{m-1} + \dots$$

The coefficients are $\dfrac{f^{(j)}(z_0)}{m!}$, hence

$$\frac{f^{(j)}(z_0)}{j!} = 0, \quad j = 0, 1, \ldots, m-1$$
$$\frac{f^{(m)}(z_0)}{m!} = g(z_0) \neq 0.$$

$\square$

15 Zeros and Singularities

Recall that if f is analytic at z_0, we call z_0 a zero of order m for f if

$$f^{(j)}(z_0) = 0,\ j = 0, 1, \ldots,\ m-2,\ m-1, \quad f^{(m)}(z_0) \neq 0.$$

We also proved Theorem 48.

Corollary 4. *Suppose f is analytic at z_0 and that $f(z_0) = 0$. Then, either f is identically zero in a neighbourhood of z_0, or there is a punctured disk about z_0 in which f is non-vanishing.*

Proof. Let $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ be the Taylor expansion of f about z_0. If $a_j = 0\,\forall j > 0$, then f must be identically zero in a neighbourhood if z_0. Otherwise, let $m = \min\{j : a_j \neq 0\}$. Clearly then, f has a zero of order m at z_0, so $f(z)?(z-z_0)^m g(z)$ where g is analytic at z_0, and $g(z_0) \neq 0$. Then g is continuous at z_0 and $g(z_0) \neq 0$, so there is a disk of radius δ in which $g(z_0) \neq 0$. Then $f(z) \neq 0$ in $0 < |z-z_0| < \delta$. $\square$

If f is analytic in some domain D and vanishes identically in some disk in D, it in fact vanishes identically in all of D. This can be proven by an argument similar to that in the proof of the Maximum Modulus Principle (Theorem 35).

We therefore have:

Theorem 49 (The Uniqueness Principle)**.** *If f and g are analytic on a domain D and if they agree ($f = g$) for z belonging to a set that has a non-isolated point, then $f = g$ on all of D.*

Definition 30. A point z_0 is called an isolated singularity of f if f is analytic in some punctured disk $0 < |z-z_0| < R$ about z_0, but not analytic at z_0.

Let z_0 be an isolated singularity of f. Then f has a Laurent series expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j, \quad 0 < |z-z_0| < R \qquad (*)$$

Definition 31. Let z_0 be an isolated singularity of f, and let $(*)$ be its Laurent series expansion on $0 < |z - z_0| < R$.

(i) If $a_j = 0\, \forall j < 0$, we say that z_0 is a removable singularity of f.

(ii) If $a_{-m} \neq 0$ for some positive integer m, but $a_j = 0$ for all $j < -m$, we say that z_0 is a pole of order m for f.

(iii) If $a_j \neq 0$ for infinitely many $j < 0$, we say that z_0 is an essential singularity of f.

If f has a removable singularity at z_0, then its Laurent series takes the following form:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j, \quad 0 < |z - z_0| < R.$$

By putting $f(z_0) = a_0$, f becomes analytic in $|z - z_0| < R$, and hence bounded in a neighbourhood of z_0.

Conversely, we have Riemann's theorem on removable singularities

Theorem 50 (Riemann's Theorem of Removable Singularities)**.** *Let z_0 be an isolated singularity of f. If f is bounded in a punctured neighbourhood of z_0, then f has a removable singularity at z_0.*

Proof. From Laurent's theorem (Theorem 47),

$$a_j = \frac{1}{2\pi i} \oint\limits_{|z-z_0|=\rho} \frac{f(\zeta)}{(\zeta - z)^{j+1}} d\zeta \quad 0 < \rho < R.$$

So if $|f(\zeta)| \leq M$ for $|\zeta - z_0| = \rho$, then $a_j \leq \frac{1}{2\pi} \frac{M}{\rho^{j+1}} \cdot 2\pi\rho = \frac{M}{\rho^j}$ by the ML-Inequality. So if $j < 0$, letting $\rho \to 0$, we see tat $a_j = 0$. $\square$

If f has a pole of order m at z_0, the Laurent series $(*)$ takes the following form:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + \ldots, \quad a_{-m} \neq 0.$$

One easily obtains the following:

Theorem 51. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of order m iff. f in a punctured neighbourhood of z_0 can be represented as follows:*

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is analytic and $g(z_0) \neq 0$.

Proof. $\Rightarrow$ Let $g(z) = a_{-m} + a_{-(m-1)}(z - z_0) + \ldots$

$\Leftarrow$ Taylor expand g near z_0. $\square$

By the above characterization of zeros and poles of order m, we easily prove the following result (exercise!)

Theorem 52. *If f has a zero of order m at z_0, then $\frac{1}{f}$ has a pole of order m at z_0. Conversely, if f has a pole of order m at z_0, then $\frac{1}{f}$ has a removable singularity at z_0, and if we define $\frac{1}{f(z_0)}$ to be 0, then $\frac{1}{f}$ is analytic at z_0 and has a zero of order m at z_0.*

Theorem 53. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of f iff. $|f(z)| \to +\infty$ as $z \to z_0$.*

Proof. If z_0 is a pole of order m, then form $f = \frac{g}{(z-z_0)^m}$ with g analytic at z_0, $g(z_0) \neq 0$. Clearly, $|f(z)| = |z - z_0|^{-m}|g(z)| \to +\infty$. Conversely, if $|f(z) \to +\infty|$, $z \to z_0$. Clearly $f(z) \neq 0$ in a punctured disk about z_0, i.e. $h(z) = \frac{1}{f(z)}$ is analytic in a punctured disk about z_0. Further, $h(z) \to 0$ as $z \to z_0$. So by Riemann's theorem above, h has a removable singularity at z_0. So h extends to an analytic function near z_0 with $h(z_0) = 0$ if m denotes the finite order of the zero of h at z_0 and we are done. □

Definition 32. A function f is said to be meromorphic in a domain D if at every point of D, f is either analytic or a pole.

Theorem 54 (Picard)**.** *A function with and essential singularity assumes every complex number, with one possible exception, as a value in any neighbourhood of this singularity.*

Proof. See book. □

The Residue Theorem

Let Γ be a simple closed positively oriented contour in $\mathbb{C}$. Suppose f is analytic inside and on Γ except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside Γ. Then

$$\oint_{\Gamma} f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz.$$

in a punctured neighbourhood z_j, f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^j.$$

According to Laurent's theorem, $a_{-1} = \dfrac{1}{2\pi i} \displaystyle\int_{C_j} f(z)dz.$

Definition 33. If f has an isolated singularity at z_0, then the coefficient a_{-1} in the Laurent series expansion for f around z_0 is called the 'residue' of f at z_0 and is denoted $\mathrm{Res}(f, z_0)$.

Theorem 55 (Residue Theorem)**.** *Let Γ be a simple closed positively oriented contour, and let f be analytic inside and on Γ, except at finitely many isolated singularities $z_1, z_2, \ldots, z_n \in \Gamma$. Then*

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j).$$

16 Integral Calculations Using Residue Calculus

Recall The Residue Theorem (Theorem 55).

Example 37. Compute

$$\oint_{|z|=4} ze^{\frac{3}{z}} dz.$$

Solution:

$$e^w = \sum_{j=0}^{\infty} \frac{w^j}{j!},\ w \in \mathbb{C} \quad \Rightarrow \quad e^{\frac{3}{z}} = \sum_{j=0}^{\infty} \left(\frac{3}{z}\right)^j,\ z \neq 0.$$

$$\Rightarrow ze^{\frac{3}{z}} = z\left(1 + \frac{3}{z} + \frac{3^2}{2z^2} + \ldots\right) = z + 3 + \frac{9}{2z} + \ldots, \quad z \neq 0$$

$$\Rightarrow \mathrm{Res}(ze^{\frac{3}{z}}, 0) = \frac{9}{2} \Rightarrow \oint_{|z|=4} ze^{\frac{3}{z}} dz = 2\pi i \frac{9}{2} = 9\pi i.$$

The Residue Calculus

1. If f has a removable singularity at z_0, $\mathrm{Res}(f, z_0) = 0$.
2. Suppose that f has a simple pole, i.e. a pole of order one at z_0. then f has a Laurent series expansion

$$f(z) = \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + \ldots$$

   in a punctured neighbourhood about z_0. Hence

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Example 38. $f(z) = \dfrac{e^z}{z(z+1)}$ has simple poles at $z = 0,\ 1$.

$$\mathrm{Res}(f, 0) = \lim_{z \to 0} \frac{e^z}{z+1} = 1$$

$$\mathrm{Res}(f, 1) = \lim_{z \to -1} \frac{e^2}{z} = -\frac{1}{e}$$

Example 39. Suppose $f(z) = \frac{g(z)}{h(z)}$ where g, h are analytic at z_0 and suppose that h has a simple zero at z_0, while g is non-zero at that point. Clearly, f has a simple pole at z_0, and

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\frac{h(z)-h(z_0)}{(z-z_0)}} = \frac{g(z)}{h'(z)}.$$

Thus, in such a case, we have $\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$.

3. Suppose f has a pole of order m at z_0. Then f has Lauren series expansion

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \ldots + a_0 + a_1(z - z_0) + \ldots$$

in a punctured neighbourhood about z_0. It follows that

$$(z - z_0)^m f(z) = a_{-m} + a_{-(m-1)}(z - z_0) + \ldots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \ldots$$

and so

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right] = (m - 1)! a_{-1}(z - z_0)^m + m! a_0 (z - z_0)^m + \ldots$$

Taking the limit as $z \to z_0$ we get:

Theorem 56. *If f has a pole of order m at z_0, then*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{m!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0 =^m f(z)\right].$$

Calculations of Integrals

Trigonometric integrals can sometimes be integrated using residues calculus:

Example 40. Suppose you want to compute $I = \int\limits_0^{2\pi} \frac{\sin^2(\theta)}{5 + 4\cos(\theta)} d\theta$.

Solution: Put $z = e^{i\theta}$. Then, clearly $e^{-i\theta} = \frac{1}{z}$, so

$$\begin{cases} \cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} & = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) \\ \sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} & = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right) \end{cases}, \quad dz = ie^{i\theta} d\theta \iff d\theta = \frac{dz}{iz}$$

$$I = \int\limits_0^{2\pi} \frac{\sin^2(\theta)}{5 + 4\cos(\theta)} d\theta = \int\limits_{|z|=1} \frac{\frac{1}{2i}\left(z - \frac{1}{z}\right)^2}{5 + \frac{4}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = -\frac{1}{4i} \int\limits_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)} dz$$
$$= -\frac{1}{4i} \int\limits_{|z|=1} \frac{(z^2 - 1)^2}{z^2\left(z + \frac{1}{2}\right)(z + 2)} dz$$

f has simple poles at $z = -\frac{1}{2}, 2$, and a double pole at $z = 0$. Only $z = 0, -\frac{1}{2}$ contribute ti the integral, so:

$$\Rightarrow I = 2\pi i \left[\text{Res}\left(f, -\frac{1}{2}\right) + \text{Res}(f, 0)\right] \cdot -\frac{1}{4i}.$$

By the residue Theorem:

$$\begin{aligned}
\text{Res}\left(f, -\frac{1}{2}\right) &= \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \frac{(z^2-1)^2}{z^2(z+2)} = \frac{3}{4} \\
\text{Res}\,(f, 0) &= \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left[z^2 f(z)\right] = \lim_{z \to 0} \frac{d}{dz} \left[\frac{(z^2-1)^2}{2z^2+5z+2}\right] = \\
&= \left.\frac{2(z^2-1)2z(2z^2+5z+2) - (z^2-1)^2(4z+5)}{(2z^2+5z+2)^2}\right|_{z=0} \\
&= -\frac{5}{4}.
\end{aligned}$$

$$\Rightarrow I = -\frac{1}{4i} \frac{2\pi i}{2} \left(\frac{3}{4} - \frac{5}{4}\right) = -\frac{\pi}{2} \cdot -\frac{2}{4} = \frac{\pi}{4}.$$

Integrals over $\mathbb{R}$ an often be computed by considering a contour in, say, the upper half-plane.

Example 41. Compute the integral $I = \int\limits_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$.

Solution: Let $f(z) = \frac{z^2}{(z^2+1)^2}$. Also, let $\Gamma_R = [-R, R] \cup C_R^+$, where C_R^+ is a half-circle in the upper half-plane of radius $R > 1$. By t he residue theorem, we then have:

$$\int\limits_{\Gamma_R} f(z) dz = \int\limits_{-R}^{R} f(x) dx + \int\limits_{C_R^+} f(z) dz = 2\pi i \text{Res}(f, i).$$

Since $z = i$ is a pole of order 2,

$$\begin{aligned}
\text{Res}(f, i) &= \lim_{z \to i} \frac{1}{1!} \frac{d}{dz} [(z-i)^2 f(z)] = \left.\frac{d}{dz} \frac{z^2}{(z+i)^2}\right|_{z=i} = \left.\frac{2z(z+i)^2 - z^2 2(z+i)}{(z+i)^4}\right|_{z=i} \\
&= \frac{2i \cdot 2i - (-1) \cdot 2}{-8i} = \frac{1}{4i}
\end{aligned}$$

Thus the integral over Γ_R is equal to $\frac{\pi}{2}$. But

$$\left|\int\limits_{C_R^+} f(z)dz\right| \le \max_{z\in C_R^+} |f(z)| \cdot l(C_R^+) \text{ and } |f(z)| = \left|\frac{z^2}{(z^2+1)^2}\right| = \frac{|z^2|}{|z^2+1|^2}.$$

$$\frac{|z^2|}{|z^2+1|^2} \le \frac{|z^2|}{||z^2|-|1||^2} = \frac{R^2}{(R^1-1)^2},\ z\in C_R^+$$

$$\Rightarrow \left|\int\limits_{C_R^+} f(z)dz\right| \le \frac{R^2}{(R^2-1)^2}\pi R \to 0,\ R\to\infty.$$

So, letting $R\to\infty$, we obtain

$$\int\limits_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int\limits_{-R}^{R} f(x)dx = \frac{\pi}{2}$$

Remark 19. *The same method can be used to calculate any integral on the form*

$$\int\limits_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx,\ \deg(Q) \le \deg P + 2,\ Q\neq 0 \text{ on } \mathbb{R}.$$

Example 42. Compute $I = \int\limits_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+1}dx$

Solution: It is tempting to consider $\frac{\cos(3z)}{z^2+1}$, but it holds that $\cos(3z) = \frac{e^{i3z}+e^{-i3z}}{2}$. Since $|e^{i3z}| = e^{-3y}$ and $|e^{-i3z}| = e^{3y}$, we <u>cannot</u> proceed as in the previous example. We can, however, use that

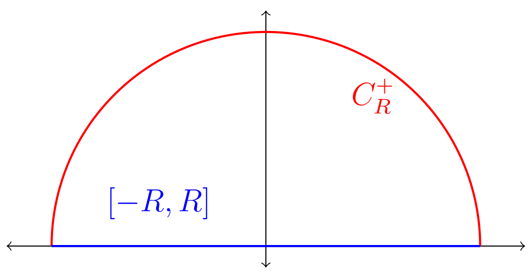
$$I = \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{e^{i3x}}{x^2+1}dx + \frac{1}{2}\int\limits_{\infty}^{\infty}\frac{e^{-i3x}}{x^2+1}dx := I_1 + I_2.$$

One can compute I_1 by considering

$$\int\limits_{\Gamma_R}\frac{e^{i3z}}{z^2+1}dz$$

and similarly for I_2. It is easier to note that

$$I = \Re\int\limits_{-\infty}^{\infty}\frac{e^{i3x}}{x^2+1}dx.$$



Therefore, let $f(z) = \frac{e^{i3z}}{z^2+1}$ and consider the integral of $f(z)$ over Γ_R (See Figure). Arguing as before:

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = 2\pi i \mathrm{Res}(f(z), i) = 2\pi i \frac{e^{i3\cdot i}}{2z}\bigg|_{z=i} = \frac{\pi}{e^3}.$$

Taking real part, we get that $I = \Re \pi e^{-3} = \pi e^{-3}$.

Remark 20. *Integrals of the form*

$$\int_{\mathbb{R}} f(x)e^{-i\omega x}dx$$

are important in Fourier analysis.

17 More About Integral Calculations

Recall the last example from the previous chapter, and suppose now that we would like to compute

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+1}dx = \Im \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}dx$$

using the previous method. Then we want to show that the following integral

$$\lim_{R\to\infty}\int_{C_R^+} \frac{ze^{iz}}{z^2+1}dz = 0.$$

The previous rough estimate using $|e^{iz}| \le e^{-y} < l$ only shows that

$$\left|\int_{C_R^+} \frac{ze^{iz}}{z^2+1}dz\right| \le \frac{R}{R^2-1}\pi R \le c \in \mathbb{R}.$$

A more accurate estimate shows the following result:

Theorem 57 (Jordan's Lemma)**.** *Suppose $m > 0$, and that P, Q are polynomials s.th.* $\deg(Q) \ge \deg(P) + 1$. *Then*

$$\lim_{R\to\infty}\int_{C_R^+} \frac{e^{imz}P(z)}{Q(z)}dz = 0$$

where C_R^+ is the upper half of a circle with radius R centered at the origin.

Proof. Parameterize C_R^+ by $z(t) = Re^{it},\ t \in [0, \pi]$

$$\left|\int\limits_{C_R^+} e^{imz}\frac{P(z)}{Q(z)}dz\right| \leq \left|\int\limits_0^\pi e^{imRe^{it}}\frac{P(Re^{it})}{Q(Re^{it})}iRe^{it}dt\right| \leq \int\limits_0^\pi \left|e^{imR(\cos(t)+i\sin(t))}\frac{P(Re^{it})}{Q(Re^{it})}iRe^{it}dt\right|$$

$$\leq \left\{\left|\frac{P(Re^{it})}{Q(Re^{it})}\right| \leq \frac{K}{R} \text{ for some } k \text{ if } R \text{ is large}\right\}$$

$$\leq \frac{k}{R}\int\limits_0^\pi e^{-mR\sin(t)}Rdt = k\int\limits_0^\pi e^{-imR\sin(t)}dt.$$

$$\Rightarrow \quad 2\int\limits_0^{\frac{\pi}{2}} e^{-imR\sin(t)}dt \leq 2\int\limits_0^{\frac{\pi}{2}} e^{-imR\frac{\pi}{2}t}dt = 2\left[-\frac{\pi}{2mR}e^{-mR\frac{\pi}{2}}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \leq \frac{\pi}{mR}.$$

$$\Rightarrow \left|\int\limits_{C_R^+} e^{imz}\frac{P(z)}{Q(z)}dz\right| \leq \frac{k\pi}{mR} \to 0,\ R \to \infty.$$

□

Remark 21. *The inequality*

$$\int\limits_0^\pi e^{-R\sin(\theta)}d\theta \leq \frac{\pi}{R}$$

is usually called the 'Jordan inequality'.

Example 43. Compute the integral

$$I = \int\limits_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+1}dx.$$

Solution: Let $I = \Im \int\limits_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1} dx$. Put $f(z) = \frac{ze^{iz}}{z^2+1}$ and consider

$\int\limits_{\Gamma_R} f(z)dz$ over the same Γ_R as before.

$$\int\limits_{\Gamma_R} f(z)dz = \int\limits_{-R}^{R} f(x)dx + \int\limits_{C_R^+} f(z)dz = 2\pi i \mathrm{Res}(f, i)$$

$$= 2\pi i \frac{ie^{i \cdot i}}{2i} = \frac{i\pi}{e}.$$

$$I = \Im \left(\frac{i\pi}{e} \right) = \frac{\pi}{e}.$$

Principal Values

Recall from calculus the following definition:
Definition 34. Suppose that f is continuous on $\mathbb{R}$. We say that the improper integral

$$\int\limits_{-\infty}^{\infty} f(x)dx \text{ is convergent if } \begin{cases} \lim\limits_{a \to -\infty} \int\limits_{a}^{0} f(x)dx \\ \lim\limits_{b \to \infty} \int\limits_{0}^{b} f(x)dx \end{cases} \text{ exist.}$$

$$\text{Then } \int\limits_{-\infty}^{\infty} f(x)dx = \lim_{a \to -\infty} \int\limits_{a}^{0} f(x)dx + \lim_{b \to \infty} \int\limits_{0}^{b} f(x)dx = \lim_{R \to \infty} \int\limits_{-R}^{R} f(x)dx.$$

Definition 35. Suppose that f is continuous on $\mathbb{R}$ as before. The principal value of

$$\int\limits_{-\infty}^{\infty} f(x)dx \text{ is defined as } PV \int\limits_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int\limits_{-R}^{R} f(x)dx.$$

Provided that the limit exists.

Remark 22. *1. PV may exist, even if the integral is not convergent.*

2. One can show that

$$\int\limits_{-\infty}^{\infty} \frac{e^{imx}P(x)}{Q(x)} dz$$

is convergent for $m \neq 0$ provided that $\deg(Q) \geq \deg(P) + 1$, $Q(x) \neq 0$ *on* $\mathbb{R}$.

Definition 36. Let $a \leq x_0 < b$ and suppose that f is continuous for $a \leq x \leq x_0$ and for $x_0 < x \leq b$ (i.e. continuous on $[a, b] \setminus \{x_0\}$). Then the principal value of

$$\int_a^b f(x)dx = PV \int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \left(\int_a^{x_0-\varepsilon} f(x)dx + \int_{x_0+\varepsilon}^b f(x)dx \right).$$

Example 44. We want to compute

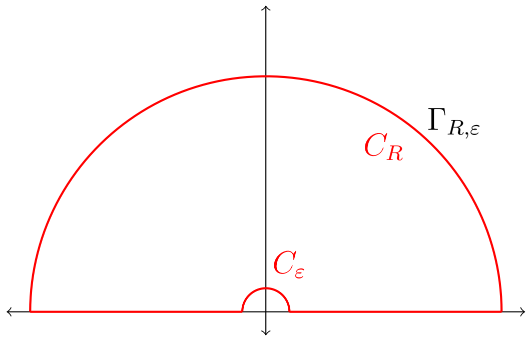
$$I = PV \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \lim_{R\to\infty} \lim_{\varepsilon \to 0^+} \left(\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx \right).$$

It seems natural to let $f(z) = \frac{e^{iz}}{z}$ and consider

$$\int_{\Gamma_{R,\varepsilon}} f(z)dz$$

with $\Gamma_{R,\varepsilon}$ as in figure. We then want to compute the limit

$$\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} f(z)dz$$



We then want to compute the limit

$$\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} f(z)dz.$$

Theorem 58 (The Fractional Residue Theorem)**.** *Suppose z_0 is a simple pole of $f(z)$ and that C_ε is the circular arc $C_\varepsilon := z_0 + \varepsilon e^{i\theta}$, $\theta \in [\theta_1, \theta_2]$. Then*

$$\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} f(z)dz = i(\theta_2 - \theta_1)\mathrm{Res}(f, z_0).$$

Proof. As f has a simple pole at z_0, we know that

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z), \text{ where } g \text{ is analytic in a disk about } z_0.$$

$$\Rightarrow \int_{C_\varepsilon} f(z)dz = a_{-1} \int_{C_\varepsilon} \frac{1}{z - z_0} dz + \int_{C_\varepsilon} g(z)dz.$$

1.
$$\int_{C_\varepsilon} \frac{1}{z - z_0} dz = \int_{\theta_1}^{\theta_2} \frac{1}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = i(\theta_2 - \theta_1).$$

2.
$$|g(z)| \leq M, \quad z \in C_\varepsilon \text{ if } \varepsilon \text{ is small.}$$
$$\Rightarrow \left| \int_{C_\varepsilon} g(z)dz \right| \leq M\varepsilon(\theta_2 - \theta_1) \to 0,\ \varepsilon \to 0^+.$$

□

Thus,

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} f(z)dz = i(\theta_2 - \theta_1)\text{Res}(f, z_o), \quad f(z) = \frac{e^{iz}}{z}$$
$$\Rightarrow \int_{\Gamma_{R,\varepsilon}} f(z)dz = 0\ \forall R, \varepsilon > 0$$

since f is analytic inside and on $\Gamma_{R,\varepsilon}$. Hence,

$$\Rightarrow I + (-i\pi)\text{Res}(f, 0) + 0 = 0$$
$$\Rightarrow I = i\pi\text{Res}(f, 0) = i\pi.$$

Remark 23. *It follows that*

$$2\int_0^\infty \frac{\sin(x)}{x}dx = PV\int_{-\infty}^\infty \frac{\sin(x)}{x}dx = \Im\left(\int_{-\infty}^\infty \frac{e^{ix}}{x}dx\right) = \pi$$
$$\Rightarrow \int_{-\infty}^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

18 Some More Examples of Integral Calculations

Integrals With Branch Points

Example 45. Compute $I = \int_0^\infty \frac{x^{-a}}{z+1}dx$, $0 < a < 1$.

Solution: We want to compute the following:

$$I = \lim_{R \to \infty} \lim_{\varepsilon \to 0^+} \int_\varepsilon^R \frac{x^{-a}}{x+1}dx$$

Let z^{-a} be the branch given as follows:

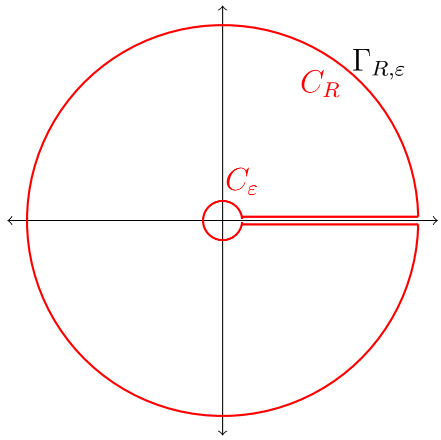
$$z^{-a} = e^{-a\log(z)} = e^{-a(\ln r + i\theta)} = r^{-a}e^{-ia\theta},\ 0 < \theta < 2\pi.$$

Then z^{-a} is analytic in $\mathbb{C} \setminus [0, \infty)$.

Now let $f(z) = \frac{z^{-a}}{z+1} = \frac{r^{-a}e^{-ia\theta}}{re^{i\theta}+1}$ and consider

$$\int\limits_{\Gamma_{R,\varepsilon}} f(z)dz, \ \Gamma_R, \varepsilon \text{ as in figure.}$$

We use $\theta = 0$ in the upper and $\theta = 2\pi$ on the lower side of the segment $[\varepsilon, R]$. So we have



$$f(z) = \frac{r^{-a}}{r+1} = \frac{x^{-a}}{x+1} \text{ on the 'upper segment', and similarly}$$
$$f(z) = \frac{r^{-a}e^{-ia2\pi}}{r+1} = \frac{x^{-a}e^{-ia2\pi}}{x+1} \text{ on the 'lower segment'.}$$

Bu the residue theorem, we have that

$$\int\limits_{\varepsilon}^{R} \frac{x^{-a}}{x+1}dx + \int\limits_{C_R} f(z)dz + \int\limits_{R}^{\varepsilon} \frac{x^{-a}e^{-ia2\pi}}{x+1}dx + \int\limits_{C_\varepsilon} f(z)dz = 2\pi i \operatorname{Res}(f,-1) \qquad (*)$$

Note that f has a simple pole at $z=-1$, so $\operatorname{Res}(f,-1) = \lim\limits_{z\to-1}(z+1)f(z) = z^{-a} = e^{-ia\pi}$.

So, from $(*)$,

$$(1-e^{-ia2\pi})\int\limits_{\varepsilon}^{R} \frac{x^{-a}}{x+1}dx + \int\limits_{C_R} f(z)dz + \int\limits_{C_\varepsilon} f(z)dz = 2\pi i e^{-ia\pi} \qquad (**)$$

But

$$\left|\int\limits_{C_R} f(z)dz\right| \le \frac{R^{-a}}{R-1}2\pi R \to 0, \quad R\to\infty$$

$$\left|\int\limits_{C_\varepsilon} f(z)dz\right| \le \frac{\varepsilon^{-a}}{1-\varepsilon}2\pi\varepsilon \to 0, \quad \varepsilon\to 0^+.$$

If we let $R\to+\infty$, $\varepsilon\to0^+$ in $(**)$, we see that

$$I = \frac{2\pi i e^{-ia\pi}}{1-e^{-ia2\pi}} = \frac{2\pi}{e^{ia\pi}-e^{-ia\pi}} = \frac{\pi}{\sin(a\pi)}.$$

Example 46. Compute $I = \int\limits_0^\infty \frac{dx}{(x+1)(x^2+2x+1)}$.

This takes a trick! (or real analysis!). Let $\log(z)$ be the branch of the logarithm as in the previous problem.

$$\log(z) = \ln|z| + i\theta,\ \theta \in [0, 2\pi)$$

Now, let

$$f(z) = \frac{\log(z)}{(z+1)(z^2+2z+2)} \text{ and consider } \int\limits_{\Gamma_{R,\varepsilon}} f(z)dz$$

where Γ_R, ε as in the previous problem. If we sum the contributions from the upper and lower side of the segment $[\varepsilon, R)$, the terms containing logarithms cancel.

$$\int\limits_\varepsilon^R \frac{\log(x)}{(x+1)(x^2+2x+2)}dx + \int\limits_R^\varepsilon \frac{\log(x)+2\pi i}{(x+1)(x^2+2x+2)}dx = -2\pi i \int\limits_\varepsilon^R \frac{1}{(x+1)(x^2+2x+2)}dx$$
$$= -2\pi i I, \text{ as } R \to \infty \text{ and } \varepsilon \to 0^+. \text{ It holds that}$$
$$\left| \int\limits_{C_R} \frac{\log(z)}{(z+1)(z^2+2z+2)}dz \right| \leq \frac{\ln R + 2\pi}{(R-1)(R^2-2R-2)} 2\pi R \to 0, \text{as } R \to \infty$$
$$\left| \int\limits_{C_\varepsilon} \frac{\log(z)}{(z+1)(z^2+2z+2)}dz \right| \leq \frac{\ln \varepsilon + 2\pi}{(1-\varepsilon)(2-2\varepsilon-\varepsilon^2)} 2\pi\varepsilon \to 0, \text{as } \varepsilon \to 0^+.$$

Note that f has poles at $z=-1$ and $z=-1\pm i$, and that $f(z) = \dfrac{\log(z)}{(z+1)(z-(-1-i))(z-(-1+i))}$. The residue theorem gives, after taking limits, that

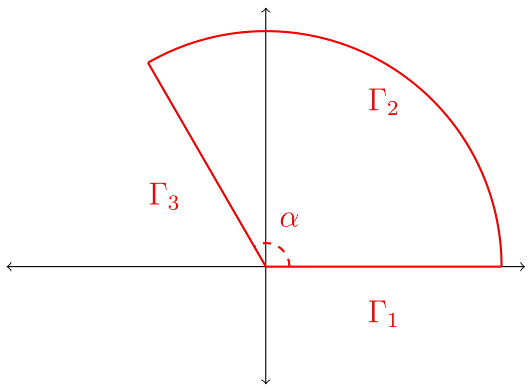
$$-2\pi i I = 2\pi i \sum \text{Res}(f, z_i) = 2\pi i(\text{Res}(f,-1) + \text{Res}(f,-1-i) + \text{Res}(f,-1+i))$$
$$\iff I = -\left(\frac{i\pi}{1} + \frac{\ln(\sqrt{2}) + \frac{i3\pi}{4}}{-2} + \frac{\ln(\sqrt{2}) + \frac{i5\pi}{4}}{-2} \right) = \frac{\ln(2)}{2}.$$

Remark 24. *The same idea works if you would like to compute*

$$\int\limits_0^\infty \frac{P(x)}{Q(x)}dx,\ \deg(Q) \geq \deg(P) + 2,\ Q \neq 0 \textit{ on } [0, \infty).$$

Example 47. Compute $I = \int\limits_0^\infty \frac{1}{x^3+1}dx$

Solution: Let $f(z) = \dfrac{1}{z^3+1}$ and consider



$$\int_{\Gamma_R} f(z)$$

with Γ_R as in figure. Choose $\alpha = 2\pi/3$, $R > 1$.

Note that

$$\int_{\Gamma_3} f(z)dz = \int_R^0 \frac{1}{r^3 e^{i3\alpha}+1}\alpha e^{i\alpha} dr = -e^{i\alpha}\int_0^R \frac{1}{r^3+1}dr$$

The residue theorem then gives

$$\int_{\Gamma_R} f(z)dz = 2\pi i \mathrm{Res}(f, e^{i\pi/3}) = 2\pi i \frac{1}{3z^2}\bigg|_{z=e^{i\pi/3}} = \frac{2\pi i}{3e^{i2\pi/3}}.$$

Standard estimate using ML-inequality and taking the limits gives the result:

$$\Rightarrow (1-e^{i2\pi/3})I = \frac{2\pi i}{3e^{i2\pi/3}} \iff I = \frac{2\pi i}{3(1-e^{i2\pi/3})e^{i2\pi/3}} = \frac{2\pi i}{2e^{i\pi/3}(e^{-i\pi/3}-e^{i\pi/3})e^{i2\pi/3}}$$
$$= \frac{2\pi i}{3\sin(\frac{\pi}{3})} = \frac{2\pi}{3\sqrt{3}}.$$

19 The Argument Principle

Theorem 59 (The Argument Principle)**.** *Let C be a simple positively oriented closed contour in $\mathbb{C}$. Suppose that f is analytic and non-zero on this contour. and meromorphic inside C. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f)$$

Where N_0 and N_p are the number of zeros and poles of f inside C respectively.

Proof. Let $G(z) = \dfrac{f'(z)}{f(z)}$. The function G is analytic on C, and has a zero of order m at $z_0 \in C$.

$$\Rightarrow f(z) = (z-z_0)^m g(z) \text{ where } \infty g \infty \text{ is analytic and nonzero at } z_0$$
$$\Rightarrow f'(z) = m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)$$
$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)} \Rightarrow G \text{ has a simple pole at } z_0 \text{ with residue } m.$$

If instead f has a pole of order k at z_p

$$\Rightarrow f(z) = \frac{g(z)}{(z - z_0)^k} \text{ where } g \text{ is analytic and nonzero at } z_p.$$

A similar computation shows that

$$G(z) = \frac{-k}{(z - z_p)} + \frac{g'(z)}{g(z)} \Rightarrow G \text{ has a simple pole at } z_p \text{ with residue } -k.$$

By the residue theorem,

$$\int\limits_C G(z)dz = \int\limits_C \frac{f'(z)}{f(z)}dz = 2\pi i(N_0(f) - N_p(f)).$$

$\square$

Remark 25. *Note that*