
Lecture Notes

Pouya Ashraf

In this document are lecture notes for the course complex analysis, which was given by Jörgen Östensson at Uppsala University in 2017. All figures are drawn with vector graphics directly in \LaTeX , so that you can zoom in if anything is unclear (fantastic, right?).

May 26, 2017

Du you appreciate that all relevant info for the course is more or less contained in this document so that you (possibly) don't have to buy the course literature? Do you want to increase the quality of my life marginally, as a thank you for the work I put into this? Send any amount of money with swish to 070-422 40 81

Contents

1	Introduction	1
	Polar form	2
	The Logarithm Function	3
2	Branches of the complex logarithm and complex powers	3
	Branches of the logarithm	3
	Complex Powers	4
	Trigonometric/Hyperbolic Functions	5
3	Topology of the Complex Plane	5
	Limits and Continuity	6
	The Complex Derivative, Analytic Functions	7
4	Cauchy-Riemann Equations, Harmonic Functions	8
	Inverse Mappings	10
	Harmonic Functions	11
5	Conformal Mappings and Stereographic Projections	12
	Conformal Mappings	12
	Stereographic Projections	13
	Möbius Mappings/Transformations	14
6	More on Möbius Transformations	16
	The Cross-Ratio	17
	Symmetry preserving property	18
7	Dirichlet's Problem	19
8	Complex Integration	21
	Contour Integrals	22
	ML-Inequality	23
9	Independence of Path, Cauchy's Integral Theorem	24
	Independence of Path	24
	Cauchy's Integral Theorem	26
10	Cauchy's Integral Formula and Applications	28
	Consequences of Cauchy's Generalized Integral Formula	30
11	The Fundamental Theorem of Algebra and Applications to Harmonic Functions	30
	The Mean-Value Property and Maximum Modulus Principle	31
12	More on Analytic Functions and Notions of Convergence	34
	Poisson's Integral Formula	34
	Sequences and Series of Functions, Uniform Convergence	35

1 Introduction

Definition 1. A complex number is a number on the form $x + iy$, where $x, y \in \mathbb{R}$. Two complex numbers $x_1 + iy_1$, $x_2 + iy_2$ are said to be equal iff. $x_1 = x_2$ and $y_1 = y_2$. The number x is called the real part, and the number y is called the imaginary part.

We write $x = \Re(x + iy)$, $y = \Im(x + iy)$. The set of complex numbers is denoted \mathbb{C} .

We define addition and multiplication as follows:

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2) \\ (x_1 + iy_1) \cdot (x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\end{aligned}$$

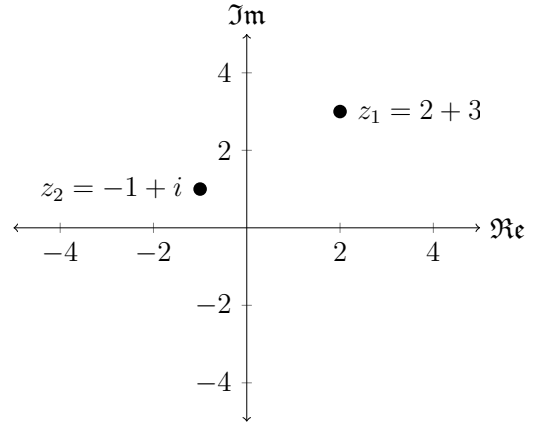
complex numbers are often denoted by z or w .

Definition 2. The complex conjugate of $z = x + iy$ is denoted \bar{z} and is defined by $\bar{z} = x - iy$. It holds that

$$\begin{aligned}\overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 \cdot z_2} &= \bar{z}_1 \cdot \bar{z}_2.\end{aligned}$$

Note also that $\Re(z) = \frac{z + \bar{z}}{2}$, $\Im(z) = \frac{z - \bar{z}}{2i}$.

It is natural to represent a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Thus geometric representation is called the complex/Argand plane.



Definition 3. The absolute value of a complex number $z = x + iy$ is denoted $|z|$, and is defined by $|z| = \sqrt{x^2 + y^2}$. It holds that

$$\begin{aligned}|z|^2 &= z \cdot \bar{z} \\ |z_1 \cdot z_2| &= |z_1| \cdot |z_2|\end{aligned}$$

Note also that every $z \in \mathbb{C}$, $z \neq 0$ has a multiplicative inverse $\frac{1}{z}$ given by $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Theorem 1 (The Triangle Inequality). *For $z_1, z_2 \in \mathbb{C}$ it holds that*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Corollary 1. *For $z_1, z_2 \in \mathbb{C}$, it holds that*

$$||z_1| - |z_2|| \leq |z_1| - |z_2|$$

Proof.

$$\begin{aligned}|z_1| &= |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2| \\ \Rightarrow |z_1| - |z_2| &\leq |z_1 - z_2|\end{aligned}$$

Now let $z_1 \longleftrightarrow z_2$

□

Polar form

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, so $\exists \theta$ s.t.

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta).$$

Therefore $z = x + iy$ can be written as follows:

$$z = |z|(\cos(\theta) + i \sin(\theta))$$

Note that $r = |z|$ is uniquely determined by z , but θ is **not**. θ is only unique up to integer multiples of 2π , i.e. if a particular θ suffices, then so does $\theta + 2\pi n$, $n \in \mathbb{Z}$. We let all these numbers be denoted by $\arg(z)$

It is practical to have a notation for one of these values of $\arg(z)$. The so called **principal value** of $\arg(z)$, denoted $\text{Arg}(z)$ is specified as the value of $\arg(z)$ which belongs to the interval $(-\pi, \pi]$.

Example 1.

$$\begin{aligned} \arg(1 + i) &= \left\{ \frac{\pi}{4} + 2\pi n \mid n \in \mathbb{Z} \right\} \\ \text{Arg}(1 + i) &= \frac{\pi}{4} \end{aligned}$$

Remark 1. One calls $\text{Arg}(z)$ a **branch** of $\arg(z)$. Note that $\text{Arg}(z)$ is 'discontinuous' along the negative real axis, which is called the branch cut of this function

Suppose $z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1))$, $z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 [(\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) \\ &\quad + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

Correctly interpreted, then, $|z_1 z_2| = |z_1| |z_2|$, and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Definition 4 (The exponential function). For $z = x + iy$, let $e^z := e^x(\cos(y) + i \sin(y))$

Remark 2. Note that e^z agrees with the 'usual' exponential function if $z \in \mathbb{R}$, i.e. the above definition extends the 'usual' exponential function to all of \mathbb{C} .

Note in particular that $e^{iy} = \cos(y) + i \sin(y)$, $y \in \mathbb{R}$ is called Euler's formula. In polar form, z can be written as $z = r(\cos(\theta) + i \sin(\theta)) = r e^{i\theta}$. Moreover, if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i\theta_1} e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

From this it follows that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$. Note also that $(e^{i\theta})^n = e^{i\theta} \cdot \dots \cdot e^{i\theta} = e^{in\theta}$, i.e. $(\cos(\theta) + i \sin(\theta))^n = (\cos(n\theta) + i \sin(n\theta))$ [de Moivre's formula].

The Logarithm Function

In real analysis, one defines the logarithm $\ln(x)$ as the inverse of the exponential function e^x , but the problem here is that e^z is not an injective function (and has no inverse).

Given $z \in \mathbb{C} \setminus \{0\}$ one chooses to define $\log(z)$ as the set of all $w \in \mathbb{C}$ whose image is z under the exponential function, i.e, $w = \log(z) \iff e^w = z$ (So $\log(z)$ is a multivalued function).

Write $z = re^{i\theta}$, $w = u + iv$. Then

$$\begin{aligned} e^w = z &\iff re^{i\theta} = e^r e^{iv} \\ &\iff u = \log(r) = \log(|z|) \end{aligned}$$

and

$$v = \theta + 2\pi k, \quad k \in \mathbb{Z} = \arg(z)$$

The explicit definition is:

Definition 5. For $z \neq 0$ we define $\log(z)$ as

$$\begin{aligned} \log(z) &= \ln(|z|) + i \arg(z) \\ &= \ln(|z|) + i \text{Arg}(z + 2\pi k), \quad k \in \mathbb{Z} \end{aligned}$$

Example 2. Compute $\log(1 + i)$

$$\begin{aligned} \log 1 + i &= \ln(|1 + i|) + i \arg(1 + i) \\ &= \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi k\right), \quad k \in \mathbb{Z} \\ &= \frac{1}{2} \ln 2 + i\left(\frac{\pi}{4} + 2\pi k\right), \quad k \in \mathbb{Z} \end{aligned}$$

2 Branches of the complex logarithm and complex powers

Branches of the logarithm

If we replace $\arg z$ by $\text{Arg}(z)$ in our definition of the logarithm from the previous section, we obtain a single-valued function; a so called 'branch' of $\log(z)$

$$\text{Log}(z) := \ln(|z|) + i \text{Arg}(z), \quad z \in \mathbb{C} \setminus \{0\}.$$

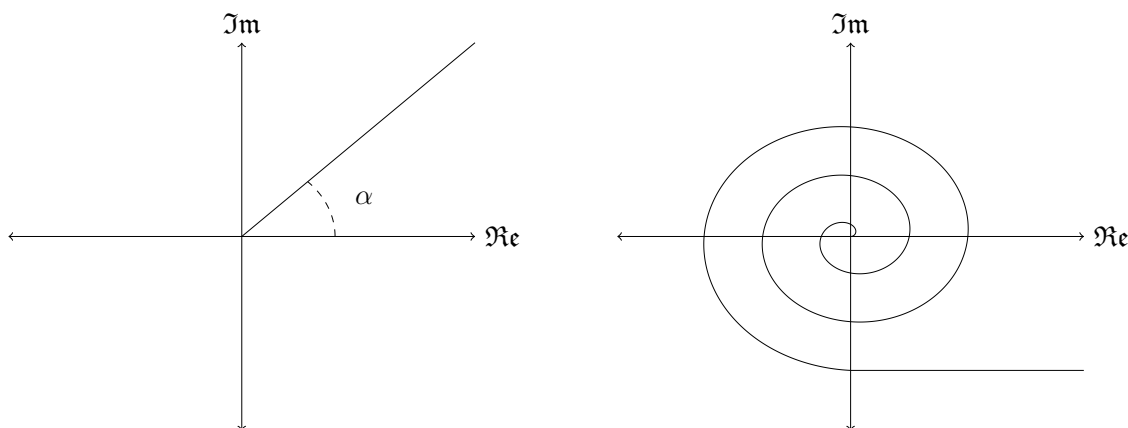
This particular logarithm is called the principal logarithm. Note that $\text{Log}(z)$ extends the 'usual' logarithm (defined on $(0, \infty)$) to $\mathbb{C} \setminus \{0\}$. Note also that $\text{Log}(z)$ is 'discontinuous' along the negative real line; its so called 'branch cut'.

We shall see that $\text{Log}(z)$ is differentiable in $\mathbb{C} \setminus (-\infty, 0]$.

There are other branches of the logarithm $\log(z)$, e.g. a branch cut at the angle α (left figure), resulting in the logarithm being defined as

$$\log(z) := \ln(|z|) + i \arg_{\alpha}(z), \arg_{\alpha}(z) \in (\alpha, \alpha + 2\pi]$$

We could also have a spiraling branch cut, like in the right right figure. This is not very useful, but that's not the point. The point is that we can define our branch cut however we want.



To understand/visualize a function $f : \mathbb{C} \rightarrow \mathbb{C}$ one can check how it maps various regions. One usually writes $f(z) = f(x + iy) = u + iv$ and then draw two copies of the complex plane \mathbb{C} ; a z -plane for the domain, and a w -plane for the range

Complex Powers

Given a complex number $z \in \mathbb{C}$, consider the equation $w^n = z$ (*). The set of all solutions w of (*) is denoted by $z^{\frac{1}{n}}$, and is called the n^{th} root of z . If $z = 0$, then $w = 0$. Suppose $z \neq 0$ and write $w = |w|e^{i\alpha}$, $z = |z|e^{i\theta}$ with help from de Moivre's formula. We then have that $|w|^n e^{in\alpha} = |z|e^{i\theta}$, and clearly

$$\begin{cases} |w|^n = |z| \\ n\alpha = \theta + 2k\pi, k \in \mathbb{Z} \end{cases} \iff \begin{cases} |w| = \sqrt[n]{|z|} \\ \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}. \end{cases}$$

Every $k \in \mathbb{Z}$ gives a solution of (*), but only $k \in \{0, 1, \dots, n-1\}$ solutions are unique; namely:

$$z^{\frac{1}{n}} = \sqrt[n]{|z|} e^{i(\frac{\theta}{n} + k\frac{2\pi}{n})}, k \in \{0, 1, \dots, n-1\}.$$

Suppose again that $z \neq 0$. For $n \in \mathbb{Z}$, it holds that $z^n = e^{n \log(z)}$ for every value of $\log(z)$. It is also true that $z^{\frac{1}{n}} = e^{\frac{1}{n} \log(z)}$, i.e. the right-hand-side

$$e^{n(\ln|z| + i \arg(z))} = e^{n \ln|z|} e^{ni \arg(z)} = |z|^n (e^{i \arg(z)})^n.$$

Definition 6. For $\alpha \in \mathbb{C}$, we let $z^\alpha = e^{\alpha \log(z)}$, $z \in \mathbb{C} \setminus \{0\}$.

This makes z^α (in general) a multi-valued function.

Example 3. Compute i^{-2i} .

Solution:

$$i^{-2i} = e^{-2i \log(i)} = e^{-2i \cdot i(\frac{\pi}{2} + k2\pi)} = e^{(4k+1)\pi}, \quad k \in \mathbb{Z}.$$

By choosing a branch of $\log(z)$ (i.e. of $\arg(z)$) in the definition of z^α , one obtains branches of z^α , for example, the principal branch of z^α is defined as follows: $z^\alpha = e^{\alpha \log(z)}$.

Example 4. The principal branch of $z^{\frac{1}{3}}$ is given as follows:

$$e^{\frac{1}{3} \text{Log}(z)} = e^{\frac{1}{3}(\ln|z| + i \arg(z))} = |z|^{\frac{1}{3}} e^{i \frac{\text{Arg}(z)}{3}}.$$

Trigonometric/Hyperbolic Functions

We have

$$\begin{cases} e^{iy} &= \cos(y) + i \sin(y) \\ e^{-iy} &= \cos(y) - i \sin(y) \end{cases} \quad \forall y \in \mathbb{R}$$

$$\Rightarrow \cos(y) = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin(y) = \frac{e^{iy} - e^{-iy}}{2i}, \quad \forall y \in \mathbb{R}.$$

Definition 7. For $z \in \mathbb{C}$ we define $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$. We also define \tan , \cot analogously to the real cases. We define the hyperbolic functions as $\cosh(z) = \frac{e^z + e^{-z}}{2}$ and $\sinh(z) = \frac{e^z - e^{-z}}{2i}$.

Example 5. Solve $\sin(z) = 2$.

Solution:

$$\begin{aligned} \sin(z) = 2 &\iff \frac{e^{iz} - e^{-iz}}{2i} = 2 \iff e^{iz} - e^{-iz} = 4i \\ &\iff (e^{iz})^2 - 4e^{iz} - 1 = 0 \iff e^{iz} = 2i \pm \sqrt{-4 + 1} \\ &\iff e^{iz} = 2i \pm i\sqrt{3} \\ iz &= \log(i(2 \pm \sqrt{3})) = \ln|2 \pm \sqrt{3}| + i \arg\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z} \\ &\iff z = \frac{\pi}{2} + 2k\pi - i \ln(2 \pm \sqrt{3}). \end{aligned}$$

3 Topology of the Complex Plane

The set $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is called the open disk with center z_0 and radius r . A subset \mathcal{M} of \mathbb{C} is called open if for every $z_0 \in \mathcal{M}$ there exists an $r > 0$ s.th. $D_r(z_0) \subseteq \mathcal{M}$.

Example 6. The disk $D_r(z_0)$ is open (hence the name 'open disk').

A subset \mathcal{L} of \mathbb{C} is called closed if its complement \mathcal{L}^c is open. $\mathcal{L}^c = \mathbb{C} \setminus \mathcal{L}$.

Example 7. $\{z \in \mathbb{C} : |z - z_0| \leq r\}$.

A point z_0 is called an interior point of \mathcal{M} if there is an $r > 0$ s.th. $D_r(z_0) \subseteq \mathcal{M}$. A point $z_0 \in \mathbb{C}$ is called a boundary point of \mathcal{M} if $\forall r > 0$ it holds that $D_r(z_0) \cap \mathcal{M} \neq \emptyset$ and $D_r(z_0) \cap \mathcal{M}^c \neq \emptyset$. The set of all interior points is denoted $\text{Int}(\mathcal{M})$ or \mathcal{M}° . The set of all boundary points is denoted $\partial\mathcal{M}$.

It holds that

- \mathcal{M} is closed $\iff \partial\mathcal{M} \subseteq \mathcal{M}$.
- \mathcal{M} is open $\iff \partial\mathcal{M} \subseteq \mathcal{M}^c$.

An open set \mathcal{M} is called path connected if every pair of points can be connected by a polygonal path contained in \mathcal{M} .

Remark 3. one can assume that the polygonal paths have segments parallel to the coordinate axes.

Definition 8. An open connected set is called a domain.

Theorem 2. Suppose that $u(x, y)$ is a real-valued function in a domain $D \in \mathbb{R}^2$. Suppose also that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

in all of D . Then u is constant in D .

A domain D is called simply connected if every closed curve in D can be, within D , continuously deformed to a point (this essentially means that D has no 'holes').

Limits and Continuity

Definition 9. A sequence $\{z_n\}_{n=1}^\infty$ of complex numbers is said to have the limit z_0 or to converge to z_0 and we write $\lim_{n \rightarrow \infty} z_n = z_0$ if for every $\varepsilon > 0$, $\exists n \geq N$ s.th. $|z - z_0| < \varepsilon \forall n \in \mathbb{N}$.

Remark 4.

$$z_n \rightarrow z_0 \iff \begin{cases} \Re(z_n) \rightarrow \Re(z_0) \\ \Im(z_n) \rightarrow \Im(z_0). \end{cases}$$

Definition 10. Let f be a function defined in some punctured neighbourhood of z_0 . We say that f has the limit w_0 as $z \rightarrow z_0$ and we write

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

if for every $\varepsilon > 0 \exists \delta > 0$ s.th. $0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

Theorem 3. For $z = x + iy$, let $u(x, y) = \Re(f(z))$, $v(x, y) = \Im(f(z))$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then,

$$\lim_{z \rightarrow z_0} f(z) = w_0 \iff \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0. \end{cases}$$

Proof. exercise. □

Definition 11. Let f be a function defined in a neighbourhood of z_0 . Then f is said to be continuous at z_0 if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ and a function is said to be continuous on the (open) set \mathcal{M} if it is continuous at every point of \mathcal{M} .

Theorem 4. If $\lim_{z \rightarrow z_0} f(z) = A$ and $\lim_{z \rightarrow z_0} g(z) = B$ then

- $\lim_{z \rightarrow z_0} f(z) + g(z) = A + B$
- $\lim_{z \rightarrow z_0} f(z)g(z) = AB$ $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$

Corollary 2. If f and g are continuous at z_0 , then so are $f + g$, fg , and $\frac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$.

Example 8. It's easy to show that $f(z) = \text{const.}$ and $g(z) = z$ are continuous in the entire complex plane \mathbb{C} . It follows that polynomials are continuous in \mathbb{C} . Rational functions are continuous everywhere except where the denominator is zero.

Example 9. $f(z) = e^z = e^x(\cos(y) + i \sin(y))$ is a continuous function in \mathbb{C} .

The Complex Derivative, Analytic Functions

Definition 12. Let f be a continuous function defined in a neighbourhood of a point z_0 . We say that f is differentiable at z_0 if the following limit exists

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

The limit is called the derivative of f at z_0 and is denoted $f'(z_0)$ or $\frac{\partial f}{\partial z}(z_0)$.

Remark 5. Δz is a complex number, so it can approach 0 'in many different ways'. In order for the derivative to exist, the result must be independent of how Δz gets to 0.

Example 10. $f(z) = \bar{z}$ is nowhere differentiable.

Proof.

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

Now let $\Delta z \rightarrow 0$ in two different ways. First let $\Delta z = \Delta x$; the limit will approach 0. Second, let $\Delta z = \Delta y$; the limit will approach -1 \square

Example 11. Let $n \in \mathbb{N}_{>0}$. The $\frac{d}{dz} z^n = n z^{n-1}$ since by the binomial theorem,

$$\frac{(z + \Delta z)^n - z^n}{\Delta z} = \frac{n z^{n-1} \Delta z + \binom{n}{k} z^{n-2} \Delta z^2 + \dots}{\Delta z} \rightarrow n z^{n-1}.$$

Theorem 5. If f and g are differentiable at z , then

- $(f + g)'(z) = f'(z) + g'(z)$
- $(cf)'(z) = cf'(z)$
- $(f \cdot g)'(z) = (f'g)(z) + (fg')(z)$
- $\left(\frac{f}{g}\right)'(z) = \frac{f'g - fg'}{g^2}(z).$

Also, the Chain-rule holds, i.e. if g is differentiable at z and f is differentiable at $g(z)$, then $(f \circ g)'(z) = f'(g(z))g'(z)$.

Definition 13. A complex-valued function f is said to be analytic in an open set G if it is differentiable at every point of G . We say that f is analytic at the point z_0 if it is differentiable in a neighbourhood of z_0 . If f is analytic in all of \mathbb{C} , we call f an 'entire' function.

Remark 6. The book also assumes $f'(z)$ to be continuous in G in order for f to be analytic in G . It turns out that the continuity of f is automatic.

4 Cauchy-Riemann Equations, Harmonic Functions

Suppose that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{f(x_0 + \Delta x + i(y_0 + \Delta y)) - f(x_0 + iy_0)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{(u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z} \end{aligned}$$

1: Consider $\Delta z = \Delta x$, i.e. $\Delta y = 0$

$$\begin{aligned}\Rightarrow f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0).\end{aligned}$$

2: Now let $\Delta z = \Delta y$, i.e. $\Delta x = 0$

$$\begin{aligned}\Rightarrow f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -i u_y(x_0, y_0) + v_y(x_0, y_0).\end{aligned}$$

It must therefore hold that $u_x + v_y = -i u_y + v_x$ which is equivalent to

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad (\text{Cauchy-Riemann Equations})$$

Theorem 6. *A necessary condition for $f = u + iv$ to be differentiable at $z_0 = x_0 + iy_0$ is that the Cauchy-Riemann Equations be satisfied by f at z_0 .*

Remark 7. *We also saw the following: If f is differentiable at z_0 , then the derivative is given by $f'(z_0) = u_x(x_0, y_0) + i v(x_0, y_0)$.*

The following provides a sufficient condition for differentiability:

Theorem 7. *Suppose that f is defined in an open set G containing $z_0 = x_0 + iy_0$. Suppose also that u_x, u_y, v_x, v_y exist in ' G ', are continuous at z_0 , and satisfy the Cauchy-Riemann Equations. Then f is differentiable at z_0 .*

Proof. In view of the continuity of the first partial derivatives at (x_0, y_0) , it follows that:

$$\begin{aligned}u(x_0, y_0) + u(x_0 + \Delta x, y_0 + \Delta y) &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2} \rho_1(\Delta x, \Delta y)\end{aligned}$$

and

$$\begin{aligned}v(x_0, y_0) + v(x_0 + \Delta x, y_0 + \Delta y) &= v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2} \rho_2(\Delta x, \Delta y)\end{aligned}$$

[See Calculus; C^1 implies differentiability]

$$\begin{aligned}\Rightarrow f(z_0 + \Delta z) - f(z_0) &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y \\ &\quad + i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y) \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \\ &= u_x(x_0, y_0)\Delta x + i v_x(x_0, y_0)\Delta x \\ &\quad + u_y(x_0, y_0)\Delta y + i v_y(x_0, y_0)\Delta y \\ &\quad + |\Delta z|(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)).\end{aligned}$$

$\rho_1, \rho_2 \rightarrow 0$ as $\Delta z \rightarrow 0$, which implies that

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, and is equal to

$$= u_x(x_0 + y_0) + iv_x(x_0, y_0).$$

□

Example 12. Show that e^z is entire and find its derivative.

Solution: $e^z = e^x(\cos(y) + i \sin(y))$

$$\begin{cases} u(x, y) = e^x \cos(y) \\ v(x, y) = e^x \sin(y) \end{cases} \Rightarrow \begin{cases} u_x = e^x \cos(y) = v_y \\ u_y = -e^x \sin(y) = -v_x \end{cases}$$

thus, $u, v \in C^1$ and satisfy the Cauchy-Riemann Equations, which implies that e^z is entire. Moreover, $\frac{d}{dz}e^z = u_x + iv_x = u + iv = e^z$.

Remark 8. By the uniqueness of so called 'analytic continuation', the above definition of e^z is the only one which makes e^z entire. (See uniqueness principle in book, p.156).

Example 13. Since e^{iz} and e^{-iz} are both entire, so are of course $\cos(z)$ and $\sin(z)$. Moreover,

$$\frac{d}{dz} \sin(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos(z). \quad (\text{Chain rule})$$

Inverse Mappings

Suppose f is analytic (with $f'(z)$ continuous) in a domain D . Consider a mapping

$$(x, y) \rightarrow \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

as a mapping of $D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Its Jacobian matrix is given by

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

It's determinant is given by

$$\det(J_f) = u_x v_y - u_y v_x = (u_x)^2 + (v_x)^2 = |f'(z)|^2$$

The inverse function theorem leads to the following:

Theorem 8 (Inverse Function Theorem). *Suppose that $f(z)$ is analytic on a domain D (with $f'(z)$ continuous), and also that $f'(z_0) \neq 0$. Then there is a neighbourhood U of z_0 and a neighbourhood V of $f(z_0)$ s.th. $f : U \rightarrow V$ is a bijection and the inverse mapping $f^{-1} : V \rightarrow U$ is analytic with derivative*

$$\frac{d}{dw}f^{-1}(w) = \frac{1}{f'(z)}, \quad w = f(z)$$

Example 14. $f(z) = e^z$ maps the region $\{z : \Im(z) \in (-\pi, \pi)\}$ to $\mathbb{C} \setminus (-\infty, 0]$. $f'(z) \neq 0$ and $f^{-1}(w) = \text{Log}(w)$ is defined in $\mathbb{C} \setminus (-\infty, 0]$. Then $\text{Log}(w)$ must be analytic in $\mathbb{C} \setminus (-\infty, 0]$. Moreover

$$\frac{d}{dw}\text{Log}(w) = \frac{1}{f'(z)} = \frac{1}{e^z} = \frac{1}{w}.$$

Clearly, $\frac{d}{dw}\text{Log}(w) = \frac{1}{w}$ for *any* branch of $\log(w)$ away from the cut.

Example 15.

$$\begin{aligned} f(z) &= z^\alpha = e^{\alpha \log(z)}; \alpha \in \mathbb{C} \\ \Rightarrow f'(z) &= e^{\alpha \log(z)} \frac{d}{dz}(\alpha \log(z)) \\ &= e^{\alpha \log(z)} \frac{\alpha}{z} = z^\alpha \frac{\alpha}{z} = \alpha z^{\alpha-1} \end{aligned}$$

Harmonic Functions

Definition 14. A real-valued function $\varphi(x, y)$ is said to be harmonic in a domain D if $\varphi \in C^2(D)$ and φ satisfies the so called Laplace-equation:

$$\Delta\varphi = \nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0 \quad \text{in } D$$

Theorem 9. *Suppose $f = u + iv$ is analytic in a domain D . Then u and v are harmonic in D*

Proof. We later prove that $u, v \in C^\infty$. (see theorem on page 114 in book).

$$\begin{cases} u_x = v_y \Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x \Rightarrow u_{yy} = -v_{xy} \end{cases}$$

since $v_{yx} = v_{xy}$, the result follows. Similarly, $v_{xx} + v_{yy} = 0$. □

Definition 15. If u is harmonic in a domain D and v is a harmonic function in D s.th. $f = u + iv$ is analytic in D , then we say that v is a harmonic conjugate of u in D .

Example 16. Construct an analytic function whose real-part is $u(x, y) = y^3 - 3x^2y$.

Solution: Note that u is harmonic:

$$\Delta u = u_{xx} + u_{yy} = -6y + 6y = 0.$$

If $f = u + iv$ is analytic, then by the Cauchy-Riemann Equations, we get that

$$v_y = -u_x = -6xy \quad (1)$$

$$v_x = -u_y = -3y^2 + 3x^2 \quad (2)$$

Now, integrate (1) w.r.t y , which gives us $v(x, y) = -3xy^2 + \Phi(x)$ and insert into (2)

$$-3y^2 + \Phi'(x) = -3y^2 + 3x^2$$

$$\Phi'(x) = 3x^2 \Rightarrow \Phi(x) = x^3 + c$$

Thus $v = -3xy^2 + x^3 + c$, so $f = u + iv = y^3 - 3x^2y + i(-3xy^2 + x^3 + c)$.

Note: $f(x + i0) = i(x^3 + c)$. Now put $g(z) = i(z^3 + c)$. g is an entire function, and agrees with f on \mathbb{R} . By the uniqueness principle, $f \equiv g$, so $f(z) = i(z^3 + c)$, $c \in \mathbb{R}$

5 Conformal Mappings and Stereographic Projections

Conformal Mappings

Let D be a domain in \mathbb{C} , $z_0 \in D$. Suppose $f : D \rightarrow \mathbb{C}$ is analytic with $f'(z_0) \neq 0$. Let $\gamma(t) = x(t) + iy(t)$ be a C^1 -curve in D that passes through $z_0 = z(0)$ with $\gamma'(0) \neq 0$. Then $(f \circ \gamma)(t) = f(\gamma(t))$ is a C^1 -curve through $(f \circ \gamma)(0) = f(z_0)$. Moreover,

$$\begin{aligned} (f \circ \gamma)'(0) &= \frac{d}{dt}(f(\gamma(t)))|_{t=0} = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \frac{\gamma(t) - \gamma(0)}{t} \\ &= f'(z_0)\gamma'(0). \end{aligned}$$

i.e. $(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$ is a tangent vector to $(f \circ \gamma)$ at $f(z_0)$. Note that $\arg(f \circ \gamma)'(0) = \arg(f'(z_0)) + \arg(\gamma'(0))$.

Remark 9. If γ_1, γ_2 are two C^1 -curves which intersect at z_0 , then the angle from $(f \circ \gamma_1)'(0)$ to $(f \circ \gamma_2)'(0)$ are the same as the angles from $\gamma_1'(0)$ to $\gamma_2'(0)$.

Definition 16. A C^1 -mapping $f : D \rightarrow \mathbb{C}$ is said to be conformal at z_0 if it satisfies Remark 9 (angles are preserved). If f maps the domain D Bijectively onto V and if f is conformal at every point of D , we call $f : D \rightarrow V$ a conformal mapping.

We have proven: if f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0 .

Example 17. 1. $f(z) = e^z$ is a conformal mapping at every point $z \in \mathbb{C}$, since $f'(z) = e^z \neq 0$

2. $f(z) = z^2$ is conformal at every point $z \in \mathbb{C} \setminus \{0\}$, since $f'(z) = 2z \neq 0$ iff. $z \neq 0$.

Stereographic Projections

Consider the unit sphere S in \mathbb{R}^3 (the Riemann-sphere).

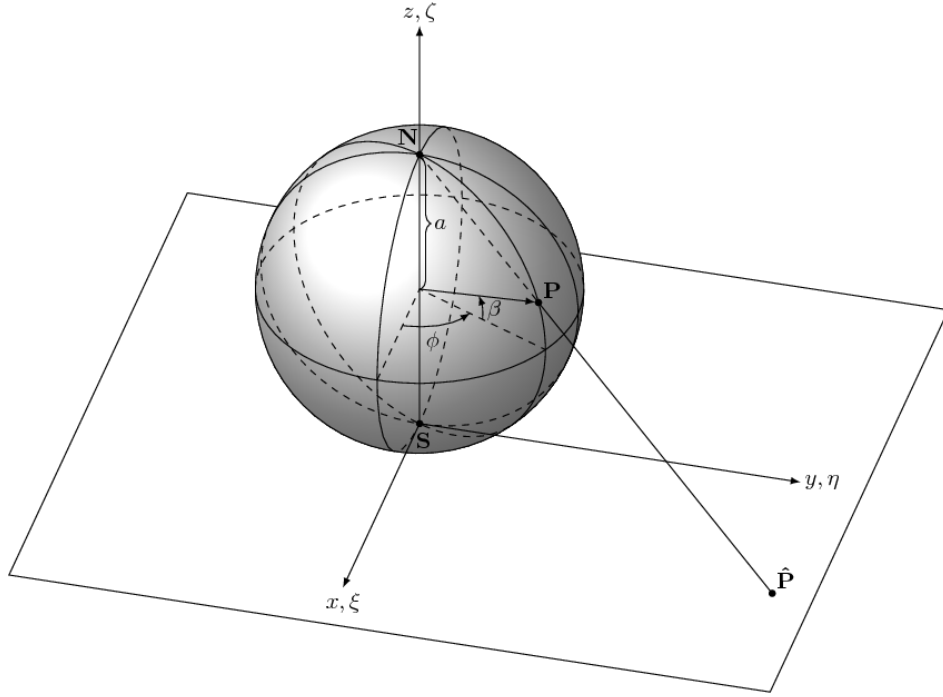


Figure 2: Stereographic projection

Given any point $P = (x, y, z) \in S$ other than N , we draw the line through N and P . We define the stereographic projection of P to be the point $\hat{P} = (\xi, \eta, 0)$ where the line intersects the plane at $\zeta = 0$.

Clearly, $(\xi, \eta, 0) = (0, 0, 1) + t[(x, y, z) - (0, 0, 1)]$, where t is given by the fact that $1 + t(z - 1) = 0$. Solve for t :

$$t = \frac{1}{1 - z}, \text{ i.e. } \hat{P} = \xi + i\eta = \frac{x + iy}{1 - z}. \quad (2)$$

Conversely, given $\hat{P} = \xi + i\eta \in \mathbb{C} \sim (\xi, \eta, 0)$, then the line through N and \hat{P} is given by $(x, y, z) = (0, 0, 1) + t[(\xi, \eta, 0) - (0, 0, 1)]$, $t \in \mathbb{R}$. It intersects S when

$$\begin{aligned} \xi^2 + \eta^2 + \zeta^2 = 1 &\iff (t\xi)^2 + (t\eta)^2 + (1 - t)^2 = 1 \\ &\iff t^2(\xi^2 + \eta^2 + \zeta^2) = 2t \\ &\iff t = 0 \quad \vee \quad t = \frac{2}{\xi^2 + \eta^2 + 1} = \frac{2}{|\zeta|^2 + 1} \end{aligned}$$

This corresponds to

$$P = \left(\frac{2\xi}{|\zeta|^2 + 1}, \frac{2\eta}{|\zeta|^2 + 1}, \frac{|\zeta|^2 + 1}{|\zeta|^2 + 1} \right).$$

Thus, the stereographic projection $s : S \setminus \{0\} \rightarrow \mathbb{C}$ defines a bijection. Letting $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane and define $s(N) = \infty$, then s becomes a bijective map $s : S \rightarrow \hat{\mathbb{C}}$.

Theorem 10. *Under stereographic projections, circles on S correspond to circles and lines on \mathbb{C} .*

Remark 10. *We refer to circles as lines in \mathbb{C} as 'Circles' in $\hat{\mathbb{C}}$, where lines are considered as circles through ∞ .*

Proof. The general equation for a circle or line in the plane $\hat{P} = \xi + i\eta$ is as follows:

$$A(\xi^2 + \eta^2) + B\xi + C\eta + D = 0$$

Using (2), we get the following:

$$\begin{aligned} & A \left(\left(\frac{x}{1-z} \right)^2 + \left(\frac{y}{1-z} \right)^2 \right) + B \left(\frac{x}{1-z} \right) + C \left(\frac{y}{1-z} \right) + D = 0 \\ \iff & A(x^2 + y^2) + Bx(1-z) + Cy(1-z) + D(1-z)^2 = 0 \\ \iff & A(1-z)^2 + Bx(1-z) + Cy(1-z) + D(1-z)^2 = 0 \\ \iff & A(1-z) + Bx + Cy + D(1-z) = 0 \\ \iff & Bx + Cy + (A-D)z + D + A = 0. \end{aligned}$$

This is the equation for a plane in \mathbb{R}^3 , which intersects the Riemann-Sphere in a circle. \square

Möbius Mappings/Transformations

Definition 17. A Möbius transformation is a mapping on the form

$$T(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}$$

where $ad - bc \neq 0$ (so that T is not constant).

If $c = 0$ we let $T(\infty) = \infty$. Then $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is bijective. If $c \neq 0$, then $T : \mathbb{C} \setminus \{-\frac{d}{c}\} \rightarrow \mathbb{C} \setminus \{\frac{a}{c}\}$ is a bijection. Letting $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$ we extend T to a bijective mapping $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$.

1. Note that

$$\begin{aligned} T'(z) &= \frac{d}{dz} \left(\frac{az + b}{cz + d} \right) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} \\ &= \frac{ad - bc}{(cz + d)^2} \end{aligned}$$

thus $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a conformal mapping.

2. If

$$\begin{aligned}
T(z) &= \frac{ax+b}{cz+d}, \quad S(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \\
\Rightarrow (S \circ T)(z) &= S(T(z)) = \frac{\alpha T(z) + \beta}{\gamma T(z) + \delta} = \frac{\alpha \left(\frac{az+b}{cz+d} \right) + \beta}{\gamma \left(\frac{az+b}{cz+d} \right) + \delta} \\
&= \frac{(\alpha a + \beta c)z + (ab + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)}
\end{aligned}$$

so the composition of Möbius transformations is are Möbius transformations (so they form a group under the operation of composition). Note the following:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

Meaning that the composing transformations is 'the same' as multiplying matrices containing the coefficients of the transformations (by 'the same', we mean that there is an isomorphy between these two).

The inverse, $T^{-1} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, is given by solving the following:

$$z = T^{-1}(w) = \begin{cases} -\frac{dw+b}{cw+a}, & w \neq \frac{a}{c} \\ \infty, & w = \frac{a}{c} \\ -\frac{d}{c}, & w = \infty \end{cases}$$

Lemma 1. *If a Möbius Transformation T has more than 2 fixed points in $\hat{\mathbb{C}}$ (i.e. z s.th. $T(z) = z$) then it is the identity mapping.*

Proof. If $c = 0$ then

$$\begin{aligned}
T(z) &= \frac{az+b}{d} \\
T(z) = z &\iff \frac{az+b}{d} = z \iff (a-d)z + b = 0
\end{aligned}$$

so T has at most one fixed point in \mathbb{C} unless $a = d$ and $b = 0$, which is equivalent to $T(z)$ being the identity mapping. This implies that T has at most two fixed points in $\hat{\mathbb{C}}$ unless T is identically 0.

If $c \neq 0$ then

$$T(z) = z \iff \frac{az+b}{cz+d} = z \iff az^2 + (d-a)z - b = 0.$$

So T has at most 2 fixed points in \mathbb{C} unless $c = 0$ and $a = d$, $b = 0$. But $c \neq 0$, which is a contradiction and the result follows. \square

Theorem 11. If S, T are Möbius Transformation s.th. $S(z_i) = T(z_i)$ for three different z_i , then $S \equiv T$.

Proof. $T^{-1} \circ S^{-1}$ is a Möbius Transformation s.th. $(T^{-1} \circ S)(z_i) = z_i$, $i = 1, 2, 3$. This implies that $(T^{-1} \circ S)(z) = z$, $\forall z$ by lemma. So $S \equiv T$. \square

6 More on Möbius Transformations

Recall that a Möbius transformation is a mapping on the form

$$T(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

Some particular cases:

1. $T(z) = z + b$; a translation
2. $T(z) = az = |a|e^{i \arg(z)}$; a rotation & magnification
3. $T(z) = \frac{1}{z}$; an inversion

Note that if $c \neq 0$,

$$T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2} \frac{1}{z + \frac{d}{c}}$$

so every Möbius transformation is a composition of Möbius transformation of the type 1, 2, or 3.

Theorem 12. Every Möbius transformation maps 'circles' onto 'circles'

Remark 11. Recall that a 'circle' in $\hat{\mathbb{C}}$ is a circle or a line in \mathbb{C} . A line in \mathbb{C} is a 'circle' through ∞ in $\hat{\mathbb{C}}$.

Proof. It is easy to see that mappings on the form of 1 or 2 map circles onto circles and lines onto lines. It is enough to prove that inversion

$$T(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = u + iv$$

maps 'circles' onto 'circles'. [Note that $u^2 + v^2 = \frac{1}{x^2 + y^2}$].

A 'circle' in $\hat{\mathbb{C}}$ has equation

$$\begin{aligned} A(x^2 + y^2) + Cx + Dy + E = 0 &\iff A + \frac{Cx}{x^2 + y^2} + \frac{Dy}{x^2 + y^2} + \frac{E}{x^2 + y^2} = 0 \\ &\iff A + Cu + Dv + E(u^2 + v^2); \end{aligned}$$

Which is precisely the equation of a 'circle' in $\hat{\mathbb{C}}$. \square

Example 18. Determine the image of the disk $|z - 2| = 2$ under the transformation $T(z) = \frac{z}{2z - 8}$.

Solution: First determine the image of $c : |z - 2| = 2$.

$T(4) = \infty \Rightarrow T(z)$ must be a line

$$T(0) = 0, \quad T(2 + 2i) = \frac{2 + 2i}{2(2 + 2i) - 8} = \frac{2 + 2i}{-4 + 2i} = -\frac{1}{2} \frac{1 + i}{1 - i} = -\frac{i}{2}.$$

Since $T(2) = -\frac{1}{2}$, clearly $\{z : |z - 2| < 2\}$ maps to the left half-plane.

Given a 'circle' C_z in the z -plane, and a 'circle' C_w in the w -plane, can one find a Möbius transformation T s.th. $T(C_z) = C_w$? Yes!

The Cross-Ratio

Definition 18. Let $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be distinct points. Put

$$(z, z_1, z_2, z_3) := \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} \in \hat{\mathbb{C}}.$$

If some of the z_i is infinity, then the right-hand side should be interpreted as

$$(z, z_1, z_2, z_3) = \begin{cases} \frac{z_2 - z_3}{z - z_3} & z_1 = \infty \\ \frac{z - z_1}{z - z_3} & z_2 = \infty \\ \frac{z - z_1}{z_2 - z_1} & z_3 = \infty \end{cases}$$

(z, z_1, z_2, z_3) is called the cross-ratio of the points. Note that $S(z) := (z, z_1, z_2, z_3)$ is a Möbius transformation s.th. $S(z_1) = 0$, $S(z_2) = 1$, $S(z_3) = \infty$.

By an earlier proposition, it is the unique Möbius transformation that maps z_1, z_2, z_3 to $0, 1, \infty$.

Theorem 13. Given a triple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ of distinct points, and another triple $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ of distinct points, there is a unique Möbius transformation T s.th. $T(z_i) = w_i$, $i = 1, 2, 3$.

Proof. By our earlier proposition, there is at most one such T . We now prove that there is a unique such T by constructing a Möbius transformation T s.th. $T(z_i) = w_i$. Put $S(z) = (z, z_1, z_2, z_3)$, $U(w) = (w, w_1, w_2, w_3)$.

$$\begin{aligned} S(z_1) &\longrightarrow 0 \longleftarrow U(w_1) \\ S(z_2) &\longrightarrow 1 \longleftarrow U(w_2) \\ S(z_3) &\longrightarrow \infty \longleftarrow U(w_3) \end{aligned}$$

So the transformation $T(z) := (U^{-1} \circ S)(z) = U^{-1}(S(z))$. Furthermore

$$\begin{aligned} w = T(z) &\iff U^{-1}(S(z)) = w \iff U(w) = S(z) \\ &\iff (z, z_1, z_2, z_3) = (w, w_1, w_2, w_3). \end{aligned}$$

□

Let z_1, z_2, z_3 be three distinct points on a 'circle' C_z in $\hat{\mathbb{C}}$. Note that C_z is *oriented* by the order of z_1, z_2, z_3 . that is, C_z is acquiring an orientation by proceeding through z_1, z_2, z_3 in succession.

Since Möbius transformations are conformal, it maps the region to the left of C_z oriented by z_1, z_2, z_3 onto the region to the left of $C_w = T(C_z)$, if oriented by w_1, w_2, w_3 .

Example 19. Find a Möbius transformation that maps the region $|z| > 1$ onto $\Re(w) < 0$

Solution: Let $z_1 = 1, z_2 = -i, z_3 = -1$, and $w_1 = 0, w_2 = 1, w_3 = \infty$.

$$\begin{aligned} \frac{w - w_1}{w - \infty} \frac{i - \infty}{i - 0} &= \frac{z - 1}{z + 1} \frac{-i + 1}{-i - 1} \\ \iff w &= (-i) \frac{z - 1}{z + 1} \frac{1 - i}{1 + i} = \frac{1 - z}{1 + z}. \end{aligned}$$

Symmetry preserving property

Two points z_1, z_2 are said to be symmetric w.r.t. a line L if L is the perpendicular bisector of the line segment from z_1 to z_2 . This means that every circle or line through the points z_1 and z_2 intersect L orthogonally. This motivates the following definition:

Definition 19. Two points are said to be symmetric w.r.t. a circle if every circle or line through z_1, z_2 intersects the circle orthogonally. In particular, the center of the circle and ∞ are symmetric w.r.t the circle.

Theorem 14 (Symmetry Principle). *Let C_z be a circle or line in the z -plane and $w = T(z)$ be a Möbius transformation. Then the two points z_1, z_2 are symmetric w.r.t C_z iff.*

$$w_1 = T(z_1) \text{ and } w_2 = T(z_2) \text{ are symmetric w.r.t. } C_w = T(C_z).$$

Proof. Follows since Möbius transformations preserve the class of circles and lines as well as orthogonality.

□

Example 20. Let C be a proper circle in \mathbb{C} with center a and radius R . Given a point $\alpha \in \hat{\mathbb{C}}$ there is a unique point α^* s.th. α and α^* are symmetric w.r.t C . Indeed it holds that

$$(\alpha^* - \alpha) \overline{(\alpha - \alpha^*)} = R^2$$

or

$$\alpha^* = \alpha + \frac{R^2}{(\alpha - a)} = a + \frac{R^2}{|\alpha - a|^2}(\alpha - a)$$

Note: $\arg(\alpha^* - \alpha) = \arg(\alpha - a)$. $|\alpha^* - a||\alpha - a| = R^2$.

7 Dirichlet's Problem

Many applications involve solving Dirichlet's problem: Find a function $\varphi(x, y)$ defined on $D \cup \partial D$, $\varphi \in C^2$ in D s.th.

1. $\Delta\varphi = \varphi_{xx} + \varphi_{yy} = 0$ in D
2. $\varphi =$ some given function on ∂D .

This problem is easily solved in some special cases.

1. If the region we are interested in is the area between two points on the x -axis, and the temperature at the endpoints are given as A and B , then

$$\begin{cases} \Delta\varphi = 0 \text{ in } D \\ \varphi(a, y) = A \\ \varphi(x, b) = B \end{cases}$$

Let $\varphi(x, y) = \alpha x + \beta$, choose

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \iff \begin{cases} \alpha = \frac{B - A}{b - a} \\ \beta = \frac{Ab - Ba}{b - a} \end{cases}$$

$$\text{So } \varphi(x, y) = \frac{A - B}{b - a}x + \frac{Ab - Ba}{b - a}.$$

2. If the region we are interested in is say, the right half-plane, and the temperature at some point on the positive imaginary axis is given as A , and the temperature at some point on the negative imaginary axis is given as B , then

$\varphi(x, y) = \alpha \text{Arg}(z) + \beta$. choose α, β s.th.

$$\begin{cases} \alpha \cdot \frac{\pi}{2} + \beta = A \\ \alpha \cdot -\frac{\pi}{2} + \beta = B \end{cases} \iff \begin{cases} \alpha = \frac{A - B}{\pi} \\ \beta = \frac{A + B}{2}. \end{cases}$$

$$\varphi(x, y) = \frac{A - B}{\pi} \text{Arg}(z) + \frac{A + B}{2}.$$

3. If the region we are interested in lies between the positive real axis and a ray centered at the origin with argument α relative to the positive real axis

$$\varphi(x, y) = \frac{1}{\alpha} \text{Arg}(z).$$

4. If the region is the upper half-plane, and given a set of n ordered points x_i on the real axis, the temperature between the points x_i, x_{i+1} is given by a_i . Then the temperature distribution is given by

$$\varphi(x, y) = a_n + \frac{1}{\pi} \sum_{i=1}^n (a_{i-1} - a_i) \text{Arg}(z - x_i)$$

5. If the region is an annulus with inner radius r_1 and outer radius r_2 , and the temperature at these are A and B respectively, then the temperature distribution is given by

$$\varphi(x, y) = \alpha \ln |z| + \beta, \quad \alpha = \frac{B - A}{\ln(r_2) - \ln(r_1)}, \quad \beta = \frac{A \ln(r_2) - B \ln(r_1)}{\ln(r_2) - \ln(r_1)}.$$

How about more complicated Dirichlet problems? The idea is to map the more complicated problem to an easier problem using a conformal mapping, e.g. Möbius transformations.

Theorem 15. Suppose $f : D \rightarrow D'$ is analytic, $f = u + iv$, where D, D' are domains. If $\psi(u, v)$ is harmonic in D' , then $\phi(x, y) := \phi(u(x, y), v(x, y))$ (*) is harmonic.

Proof. Take $z_0 \in D$, then $w_0 = f(z_0) \in D'$, and since D' is open, there exists a disk $V \ni w_0$ s.th. $V \subseteq D'$. Since f is continuous we know that there is a disk $U \in D$ s.th. $f(U) \subseteq V$. Since ψ is harmonic in D' which is simply connected, there is an analytic functions in V s.th. $\psi = \Re(g)$. But then $g \circ f$ is analytic in U and moreover $\Re(g \circ f)(z) = \psi(u(x, y), v(x, y)) = \phi(x, y)$. Hence ϕ is harmonic in U . Since z_0 was chosen arbitrarily, this implies that ϕ is harmonic in D . \square

Suppose now that the function $f_D \rightarrow D'$ maps D bijectively on D' and that ∂D maps bijectively onto $\partial D'$. Suppose also that the boundary conditions for ψ in D' correspond to the boundary conditions for ϕ in D , i.e. (*) holds for $(x, y) \in \partial D$.

If we can solve the Dirichlet problem for $\psi \in D'$, we can also solve it for ϕ .

Example 21. Find a function ϕ which is harmonic inside the unit disk $|z| < 1$ s.th. $\phi(x, y) = 1$ on the upper half-circle and -1 on the lower half-circle.

Solution: The Möbius transformation taking

$$\begin{cases} -1 & \rightarrow 0 \\ 0 & \rightarrow 1 \\ 1 & \rightarrow \infty \end{cases} \quad \text{is given by} \quad \frac{w - 0}{w - \infty} \frac{1 - \infty}{1 - 0} = \frac{z + 1}{z - 1} \frac{0 - 1}{0 + 1}$$

$$w = -\frac{z + 1}{z - 1} = \frac{1 + z}{1 - z}.$$

$$\begin{aligned}\Rightarrow \psi(u, v) &= \frac{2}{\pi} \text{Arg}(w) \\ \Rightarrow \phi(x, y) &= \frac{2}{\pi} \text{Arg} \left(\frac{1+z}{1-z} \right) = \frac{2}{\pi} \text{Arg} \left(\frac{1+x+iy}{1-x-iy} \right) = \text{Arg} \left(\frac{2y}{1-x^2-y^2} \right)\end{aligned}$$

Theorem 16 (The Riemann Mapping Theorem). *Every simply connected domain $D \neq \mathbb{C}$ can be conformally mapped onto the open unit disk.*

Theorem 17 (Poisson Integral Formula for the disk). *Let f be a continuous function defined on the unit circle $|z| = 1$. Then*

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(e^{it}) dt$$

Proof. will follow from the Cauchy Integral Formula. □

8 Complex Integration

We shall now study so-called contour integral (or line integrals) of complex-valued functions. This theory of integration will teach more about the properties of analytic functions. There are infinitely many parameterizations for a given curve, e.g. change the 'direction'. Obviously, there are two natural directions of a smooth curve. A smooth curve with a specified directions is called a directed smooth curve.

Example 22. Give parameterizations of the following directed smooth curves

- (a) The line from z_1 to z_2 .
- (b) The circle if radius r and z_0
- (a) $z(t) = z_1 + t(z_2 - z_1)$, $0 \leq t \leq 1$
- (b) $z(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$

Definition 20. A contour γ is either a single point or a finite sequence $(\gamma_1, \gamma_2, \dots, \gamma_n)$ of oriented smooth curves s.th. the terminal point of the k^{th} curve is the starting point of the $k+1^{\text{th}}$ curve. The contour γ is said to be a closed contour if the terminal and initial points coincide. If these are the only self-intersections, we call γ a simple closed curve.

If γ is a smooth curve, the length of γ is given by

$$l(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \left| \frac{dz}{dt} \right| dt$$

Contour Integrals

We shall now see how to define the contour integral of a complex-valued function over a contour γ

$$\int_{\gamma} f(z)dz$$

where γ is a smooth curve.

Definition 21 (Properties of Contour Integrals).

- $\int_{\gamma} (f + g)(z)dz = \int_{\gamma} f(z)dz + \int_{\gamma} g(z)dz$
- $\int_{\gamma} cf(z)dz = c \int_{\gamma} f(z)dz$
- $\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$

Theorem 18. *If f is continuous along γ , then f is integrable.*

How do we compute it? First consider a segment of \mathbb{R}

$$\int_a^b f(t)dt, \quad f = u + iv$$

and u, v continuous on $[a, b]$. Let $F(t)$ be an anti-derivative of $f(t)$, i.e. $F(t) = U(t) + iV(t)$, where $U'(t) = u(t)$ and $V'(t) = v(t)$, so

$$\int_a^b f(t)dt = \int_a^b u(t) + iv(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt = [U(t)]_a^b + i[V(t)]_a^b = F(b) - F(a).$$

Theorem 19. *If f is continuous on $[a, b]$ and $F'(t) = f(t)$ for $t \in [a, b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

The integral along any directed smooth curve can be reduced to an integral on the above form by using a parameterization of the directed smooth curve $z(t, a \leq t \leq b)$.

Let $z_0 = z(t_0)$, $z_1 = z(t_1), \dots, z_n = z(t_n)$, where $a = t_0 < t_1 < t_2 < \dots < t_n = b$. then

$$\begin{aligned} \sum_{k=1}^n f(z_k) \Delta z_k &= \sum_{k=1}^n f(z(t_k))(z(t_k) - z(t_{k-1})) && \text{Riemann sum for } f(z(t))z'(t) \\ &= \sum_{k=1}^n f(z(t_k))z'(t_k)\Delta t_k \end{aligned}$$

This suggests the following theorem:

Theorem 20. *Let f be a continuous function on a directed smooth curve having admissible parameterization $z(t)$, $a \leq t \leq b$. Then*

$$\int_{\gamma} f(z) dx = \int_a^b f(z(t))z'(t) dt.$$

Example 23. compute

$$\int_{C_r} (z - z_0)^n dz, \quad n \in \mathbb{Z}$$

over $C_r = \{z : |z - z_0| = r\}$, oriented counter-clockwise.

Solution: Let $z(t) = re^{it}$, $0 \leq t \leq 2\pi$. then $z'(t) = ire^{it}$, so

$$\begin{aligned} \int_{C_r} (z - z_0)^n dz &= \int_0^{2\pi} r^n e^{it} ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ &= \begin{cases} ir^{n+1} \left[\frac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi}, & n \neq -1 \\ i2\pi, & n = -1 \end{cases} \end{aligned}$$

Definition 22. Suppose $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$ and let f be continuous on γ , then we let

$$\int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{\gamma_k} f(z) dz \quad \text{and if } \gamma = z_0 \text{ the integral vanishes identically.}$$

ML-Inequality

Suppose $|f(z)| \leq M \forall z \in \gamma$. Then

$$\left| \sum_{k=1}^n f(z_k) \Delta z_k \right| \leq \sum_{k=1}^n |f(z_k) \Delta z_k| \leq M \sum_{k=1}^n |\Delta z_k| \leq Ml(\gamma).$$

Letting $\mu(P_n) \rightarrow 0$ leaves us with

$$\left| \int_{\gamma} f(z) dz \right| \leq ML, \quad L = l(\gamma)$$

Theorem 21. Suppose f is continuous on γ and that $|f(z)| \leq M \forall z \in \gamma$. Then

$$\left| \int_{\gamma} f(z) dz \right| \leq ML, \quad L = l(\gamma)$$

9 Independence of Path, Cauchy's Integral Theorem

Independence of Path

Theorem 22. Suppose the f is continuous in a domain D , and that f has an anti-derivative F in D , i.e. $F'(z) = f(z) \forall z \in D$. Let γ be a contour in D with initial point z_i and terminal points z_t . Then

$$\int_{\gamma} f(z) dz = F(z_t) - F(z_i)$$

Proof.

$$\int_{\gamma} f(z) dz = \sum_k \int_{\gamma_k} f(z) dz = \sum_k \int_{\tau_{k-1}}^{\tau_k} f(z(t)) z'(t) dt$$

where $z(t)$, $\tau_{k-1} \leq t \leq \tau_k$ is a parameterization of γ_k . Now, $\frac{d}{dt} F(z(t)) = F'(z(t)) z'(t) = f(z(t)) z'(t)$. So by Theorem 18 from the previous chapter,

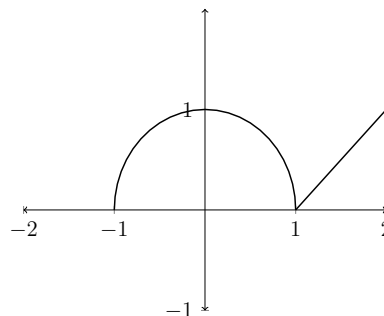
$$\int_{\tau_{k-1}}^{\tau_k} f(z(t)) z'(t) dt = [F(z(t))]_{\tau_{k-1}}^{\tau_k} = F(z(\tau_k)) - F(z(\tau_{k-1})).$$

Summing over k yields our result. □

Example 24. Compute

$$\int_{\gamma} \cos(z) dz$$

with γ as in figure.



Solution:

$$\int_{\gamma} \cos(z) dz = [\sin(z)]_{-1}^{2+i} = \sin(2+i) - \sin(-1).$$

Corollary 3. *If f is continuous in a domain D and has an anti-derivative in D , then*

$$\int_{\gamma} f(z) dz = 0$$

for any closed contour $\gamma \in D$.

Example 25. Let $f(z) = z^n$, $n \neq -1$, and let γ be any closed contour not passing through the origin. Then

$$\int_{\gamma} z^n dz = 0$$

since z^n has an anti-derivative $\frac{z^{n+1}}{n+1}$ in $\mathbb{C} \setminus \{0\}$.

Theorem 23. *Let f be continuous in a domain D . Then the following are equivalent:*

- (i) *f has an anti-derivative in D .*
- (ii) $\int_{\gamma} f(z) dz = 0$ *for every closed contour $\gamma \in D$.*
- (iii) *Contour integrals are independent of path in D . (i.e. if γ_1 and γ_2 are two contours with the same initial and terminal points, then the integrals give the same value).*

Proof.

(i) \Rightarrow (ii) Shown above.

(i) \Rightarrow (iii) Shown above.

(ii) \Rightarrow (iii) Given γ_1 and γ_2 , let $\gamma := \gamma_1 + (-\gamma_2) \Rightarrow$ the integral is identically zero.

(iii) \Rightarrow (i) Fix $z_0 \in D$. This implies that for any $z \in D$ there exists a polygonal path γ_z from z_0 to z . Define $F(z) = \int_{\gamma_z} f(\zeta) d\zeta$. $F(z)$ is well-defined, i.e. independent of choice of γ_z , by (iii).

$$\begin{aligned} \Rightarrow F(z + \Delta z) - F(z) &= \int_L f(\zeta) d\zeta = \int_L f(z) d\zeta + \int_L (f(\zeta) - f(z)) d\zeta \\ &= f(z) \Delta z + \int_L (f(\zeta) - f(z)) d\zeta \end{aligned}$$

i.e.

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z} \int_L (f(\zeta) - f(z)) d\zeta$$

But by the ML-Inequality

$$\left| \frac{1}{\Delta z} \int_L (f(\zeta) - f(z)) d\zeta \right| \leq \frac{1}{\Delta z} \max_{\zeta \in L} |f(\zeta) - f(z)| |\Delta z| \rightarrow 0$$

as $\Delta z \rightarrow 0$ by the continuity of f . This implies that $F'(z) = f(z)$.

□

Cauchy's Integral Theorem

Let γ be a simple closed contour in \mathbb{C} parameterized by $z(t)$, $a \leq t \leq b$.

$$\begin{aligned} \Rightarrow \int_{\gamma} f(z) dz &= \int_a^b f(z(t)) z'(t) dt = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))] \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) dt \\ &= \int_a^b \left[u(x(t), y(t)) \left(\frac{dx}{dt} \right) - v(x(t), y(t)) \left(\frac{dy}{dt} \right) \right] dt \\ &\quad + i \int_a^b \left[v(x(t), y(t)) \left(\frac{dx}{dt} \right) - u(x(t), y(t)) \left(\frac{dy}{dt} \right) \right] dt \end{aligned}$$

i.e.

$$\int_{\gamma} f(z) dz = \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy.$$

Theorem 24 (Green's Theorem). *Let $\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$ be a C^1 vector field defined on a simply connected domain D , and let γ be a positively oriented simple closed contour in D . Then*

$$\int_{\gamma} (F_1 dx + F_2 dy) = \iint_{\Omega} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Where $\gamma = \partial\Omega$.

Let us use this expression for the integral of $f(z)$ over γ :

$$\int_{\gamma} f(z) dz = \iint_{\Omega} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy.$$

If we assume $u, v \in C^1$. If we moreover assume that f is analytic in D , the entire integral vanishes identically in view of the Cauchy-Riemann Equations.

Theorem 25 (Cauchy's Integral Theorem). *Suppose that f is analytic in a simply connected domain D and let γ be any closed contour in D . Then*

$$\int_{\gamma} f(z) dz = 0$$

Remark 12. *The theorem generalizes the discussion thus far in two ways:*

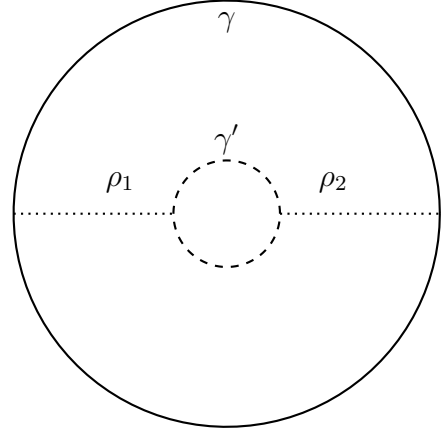
- (i) γ can be any closed contour, i.e. need not be simple.
- (ii) The assumption that $u, v \in C^1$ has been dropped.

The fact that the second assumption can be dropped was proven by Edward Goursat. The Theorem is therefore often called the Cauchy-Goursat Theorem.

Remark 13. *The Theorem implies that if f is analytic inside and on a simple closed contour, then the integral of f along said contour is identically zero.*

Theorem 26. *Suppose f is analytic in a simply connected domain. Then f has an anti-derivative, contour integrals are path-independent, and integrals over closed contours vanish identically.*

Example 26. Consider the contours γ (black) and γ' (dashed) in the figure. Suppose that f is analytic in a domain containing γ and γ' , and the region between these two contours. (In the equations below, γ_1, γ'_1 denote the parts of the contours 'above' ρ_1, ρ_2 and γ_2, γ'_2 denote the parts of the contours 'below' ρ_1, ρ_2). By Cauchy's Integral Theorem



$$\int_{\gamma_1} + \int_{\rho_1} + \int_{-\gamma'_1} + \int_{\rho_2} = 0 \qquad \int_{\gamma_2} + \int_{\rho_2} + \int_{-\gamma'_2} + \int_{\rho_1} = 0$$

$$\Rightarrow \int_{\gamma_1} + \int_{\gamma_2} = \int_{-\gamma'_1} + \int_{-\gamma'_2}$$

$$\int_{\gamma} = \int_{\gamma'}.$$

So we can 'deform' the contour γ into γ' without affecting the integral. This illustrates the 'Deformation Theorem'.

10 Cauchy's Integral Formula and Applications

Theorem 27 (Cauchy's Integral Formula). *Suppose f is analytic in a simply connected domain D . Let γ be a simple closed positively oriented contour in D . Moreover, let $z_0 \in \gamma$. then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

Proof. $\frac{f(z)}{z - z_0}$ is analytic in $D \setminus \{z_0\}$. As in the last example from the previous chapter, we see that

$$\begin{aligned} \int_{\gamma} \frac{f(z)}{z - z_0} dz &= \int_{C_r} \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z_0)}{z - z_0} dz + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \\ &= 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz. \end{aligned}$$

Note: Clearly, the second term above is independent of r . So it is enough to show that

$$\lim_{r \rightarrow 0} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

Let $M_r := \max_{z \in C_r} |f(z) - f(z_0)|$. f is continuous, which implies that $M_r \rightarrow 0$ as $r \rightarrow 0^+$. So, by the ML-Inequality:

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{M_r}{r} 2\pi r \rightarrow 0.$$

□

Remark 14. $f(z_0)$ is determined by $f(z)$, $z \in \gamma$. so, Cauchy's formula says that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \quad z \text{ interior to } \gamma.$$

By differentiability under the integral sign, it seems plausible that:

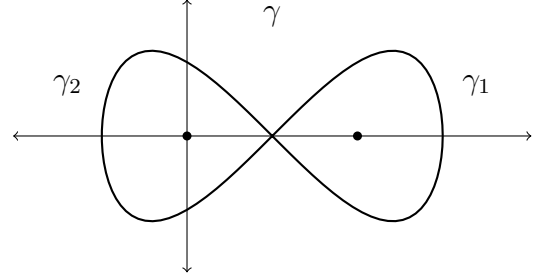
Theorem 28 (Cauchy's Generalized Integral Formula). *If f is analytic inside and on a simple closed positively oriented contour γ , and z is a point interior to γ , then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Proof. The proof is by induction by using the definition of the derivative.. □

Example 27. Compute

$$\int_{\gamma} \frac{2z+1}{z(z-1)^2} dz$$



where γ is as in the figure.

Note that $\gamma = \gamma_1 + \gamma_2$, and γ_1, γ_2 are simple closed contours.

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma_1} \frac{2z+1}{z(z-1)^2} dz + \int_{\gamma_2} \frac{2z+1}{z(z-1)^2} dz \\ &= 2\pi i \frac{d}{dz} \left(\frac{2z+1}{z} \right) \Big|_{z=1} - 2\pi i \left(\frac{2z+1}{(z-1)^2} \right) \Big|_{z=0} \\ &= -4\pi i. \end{aligned}$$

Theorem 29. If f is analytic in a domain D , then all derivatives f, f', f'', \dots exist in D and are analytic in D .

Proof. Apply Cauchy's Generalized Integral Formula in a positively oriented contour (circle) around an arbitrary point $z_0 \in D$. □

So, in particular, if f is analytic, then f' is analytic as well.

Suppose f is analytic. Then $f'(z) = u_x + iv_x = v_y - iu_y$ by Cauchy-Riemann equations. Since f' is analytic, hence continuous, it follows that $u, v \in C^1$. Since f'' exists and is continuous; and moreover $f''(z) = u_{zz} + iv_{xx} = v_{xy} - iu_{xy}$. It follows that all second order derivatives are continuous, etc.

Theorem 30. If $f = u + iv$ is analytic, then $u, v \in C^\infty(D)$.

Remark 15. This completes the proof that u, v are harmonic functions. Suppose f is continuous in D and that

$$\int_{\gamma} f(z) dz = 0$$

for all closed contours $\gamma \in D$. Independence of path theorem implies that f has an anti-derivative in D , i.e. $F' = f$ in D . Since F is analytic, this implies that $F' = f$ is analytic.

Theorem 31 (Morera). *If f is continuous in a domain d , and*

$$\int_{\gamma} f(z) dz = 0$$

for all closed contours $\gamma \in D$, then f is analytic in D

Consequences of Cauchy's Generalized Integral Formula

Theorem 32 (Cauchy Estimate). *Let f be analytic inside and on a circle C_R of radius R centered at z_0 . Suppose $|f(z)| \leq M \forall z \in C_R$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \quad \forall n \in \mathbb{N}$$

Proof. Give C_r a positive orientation. Apply Cauchy's Generalized Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad \zeta \in C_r$$

$$\left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| \leq \frac{M}{R^{n+1}}$$

so, by the ML-Inequality

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi i} \frac{M}{r^{n+1}} 2\pi R = \frac{n!M}{R^n}.$$

□

Suppose f is an entire function, $|f(z)| \leq M \forall z \in \mathbb{C}$. Then a Cauchy-estimate implies

$$|f'(z_0)| \leq \frac{M}{R} \forall R \Rightarrow |f'(z_0)| = 0$$

i.e. $f'(z_0) = 0$. Since z_0 was arbitrary $f'(z) = 0 \forall z \in \mathbb{C} \Rightarrow f$ constant in \mathbb{C} .

Theorem 33 (Liouville's Theorem). *The only bounded entire functions are the constant functions.*

11 The Fundamental Theorem of Algebra and Applications to Harmonic Functions

Theorem 34 (Fundamental Theorem of Algebra). *Every non-constant polynomial with complex coefficients has at least one (1) zero.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_n \neq 0$. Suppose $p(z)$ has no zeros. Put $f(z) = (p(z))^{-1}$. Then f is entire. We next show that $f(z)$ is bounded.

1.

$$\begin{aligned} p(z) &= z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right), \quad p(z) \rightarrow a_n \text{ as } z \rightarrow \infty \\ &\Rightarrow \exists \varphi : |z| \geq \varphi \Rightarrow \left| \frac{p(z)}{z^n} \right| \geq \frac{|a_n|}{z^n} \\ &\Rightarrow |f(z)| = \frac{1}{|p(z)|} \leq \frac{2}{|z|^n |a_n|} \leq \frac{2}{\varphi^n |a_n|}, \quad |z| \geq \varphi \end{aligned}$$

2. For $|z| \leq \varphi$ the function $|f(z)|$ is a continuous functions on a compact set, which means it must have an upper bound said set ($|z| \leq \varphi$). This implies that $\frac{1}{p(z)}$ is a bounded entire function, hence constant by Liouville's Theorem. This in turn implies $p(z)$ is constant, meaning $n = 0$.

In other words: The only polynomials without zeros are the constant polynomials. \square

The Mean-Value Property and Maximum Modulus Principle

Suppose f is analytic inside and on a circle C_R of radius R centered at z_0 . By Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} dz.$$

Parameterize $C_R : z(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$.

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it} dt = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + Re^{it}) dt. \quad (*)$$

(*) is called the 'mean value property'.

Lemma 2. Suppose f is analytic in a disk centered at z_0 , sat $D(z_0)$. Suppose also that

$$\max_{z \in D(z_0)} |f(z)| = |f(z_0)|$$

Then

$$|f(z)| = |f(z_0)| \quad \forall z \in D(z_0)$$

Proof. Suppose $|f(z)|$ is not constant. Then $\exists z_1 \in D(z_0)$ s.th. $|f(z_1)| < |f(z_0)|$. Let C_R be the circle with center in z_0 passing through z_1 . By assumption, $|f(z)| \leq |f(z_0)| \quad \forall z \in$

C_R . Since f is continuous, there exists a segment of C_R containing z_1 in which $|f(z)| < |f(z_0)|$. Say that $|f(z)| < |f(z_0)| - 2\pi\varepsilon$ on a segment of length δ .

$$\begin{aligned} \Rightarrow |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) dt \right| && \text{By the mean-value-property} \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})| dt \\ &< \frac{1}{2\pi} (|f(z_0)|(2\pi - \delta) + (|f(z_0)| - 2\pi\varepsilon)\delta) \\ &= |f(z_0)| - \varepsilon\delta. \end{aligned}$$

Which is a contradiction. □

Theorem 35 (The Maximum Modulus Principle). *If f is analytic in a domain D and $|f(z)|$ attains its maximum value at a point $z_0 \in D$, then f is constant in D .*

Proof. We show that $|f|$ is constant in D . The result then follows from the Cauchy-Riemann Equations. Suppose that $|f(z)|$ is not constant. This implies

$$\exists z_1 \in D : |f(z_1)| < |f(z_0)|.$$

Let γ be a polygonal path from z_0 to z . We consider now the values of $|f(z)|$ for z on γ , starting at z_0 . Then $\exists w \in \gamma$ s.th.

- (i) $|f(z)| = |f(z_0)|$, $\forall z$ preceding $z \in \gamma$.
- (ii) \exists points $z \in \gamma$ arbitrarily close to $w \in \gamma$ s.th. $|f(z)| < |f(z_0)|$

follows from the supremum property of \mathbb{R} . Parameterize γ s.th. $z = z(t)$, $t \in [0, 1]$, $z(0) = z_0$, $z(1) = z_1$. Let M be the set

$$M = \{z \in [0, 1] : |f(z(t))| < |f(z)|\}.$$

We know that $M \neq \emptyset$ since $1 \in M$, and M is bounded below by 0. Let $\alpha = \inf(M)$ (greatest lower bound). Put $w = f(z(\alpha))$.

- (i) α is a lower bound implies that for all $t \in M$ it holds that $\alpha \leq t$

$$\begin{aligned} \Rightarrow t < \alpha &\Rightarrow t \notin M, \text{ i.e. } |f(z(t))| = |f(z)| \\ \Rightarrow (i) &\text{ holds true.} \end{aligned}$$

- (ii) α is the greatest lower bound

$$\Rightarrow \forall \beta > \alpha \exists t \in M : t < \beta,$$

i.e. there are points t with $|f(z(0))| < |f(z_0)|$. So (ii) holds.

Since f is continuous, (i) implies $|f(w)| = |f(z_0)|$. There exists $D(w)$ contained in D . By lemma, $|f(z)|$ is constant in $D(w)$, but this contradicts (ii). Thus the assumption the $|f|$ is not constant is false. □

Now, suppose f is analytic in a bounded domain D , and that f is continuous up to ∂D . This implies that $|f|$ attains a maximum value in \bar{D} .

Theorem 36. *A function which is analytic in a bounded domain D and continuous up to ∂D attains its maximum and minimum modulus on the boundary*

Recall that if ϕ is a harmonic function in a simply connected domain D , then there exists a function f such that $\Re(f) = \phi$. Now, we prove the above theorem:

Proof. If there is such an f , say $f = \phi + i\psi$, then $f' = \phi_x + i\psi_y = \phi_x - i\phi_y$. So, put $g(z) = \phi_x - i\phi_y$. Then

$$\begin{cases} (\phi_x)_x = (-\phi_y)_y; & \phi \text{ harmonic} \\ (\phi_x)_y = -(-\phi_y)_x; & \phi \in C^2 \end{cases} \Rightarrow \text{Cauchy-Riemann Equations satisfied}$$

Also, g 's real and imaginary parts are C^1 since $\phi \in C^2$. This implies the following:

- $g(z)$ is analytic in a simply connected domain.
- g has an anti-derivative $G \in D$, $G = u + iv$, since $G' = g$.
- $u_x - iu_y = \phi_x - i\phi_y$
- $(\phi - u)_x = (\phi - u)_y = 0 \Rightarrow (\phi - u)$ is some constant $c \in \mathbb{R}$

Now put $f(z) = g(z) + c$ and the result follows. □

Theorem 37 (Maximum Principle for Harmonic Functions). *If ϕ is harmonic in a simply connected domain D and attains its maximum or minimum at some point inside the domain, then ϕ is constant.*

Proof. Let ϕ be harmonic in a simply connected domain D and let $f = \phi + i\psi$ be analytic in D . Then

$$|e^f| = |e^{\phi+i\psi}| = |e^\phi|.$$

The exponential function is monotonically increasing, so if ϕ has maximum in D , then so will $|e^f|$. This implies that e^f is constant by the Maximum Modulus Principle. □

Theorem 38. *If ϕ is harmonic in a bounded simply connected domain D , and continuous up to ∂D , then ϕ attains its maximum and minimum on ∂D .*

Proof. Since ϕ attains its minimum precisely when $-\phi$ attains its maximum, this implies the corresponding minimum principle. □

12 More on Analytic Functions and Notions of Convergence

We begin by making a remark on Theorems 37 and 38 from the last chapter. In the formulation of the two theorems, the assumption of simple connectedness is not necessary (the first theorem gives an analog of a Lemma from the same chapter, then argue similarly as in the proof of the Maximum Modulus Principle).

Recall now the Dirichlet problem; the problem of finding a function $\phi(x, y)$ which is harmonic in a domain D , and continuous up to ∂D , with given values on ∂D . An interesting question to pose is: 'Is there a solution to Dirichlet's Problem, and if so: is it unique?'. The above implies uniqueness for bounded domains.

Theorem 39. *Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be harmonic in a bounded domain D , and continuous up to ∂D , s.th. $\phi_1 = \phi_2$ on ∂D . Then $\phi_1 \equiv \phi_2$ in D .*

Proof. Let $\phi = \phi_1 - \phi_2$. By Theorem 38, ϕ is harmonic and attains both maximum and minimum on ∂D . But $\phi = 0$ on ∂D , so $\phi_1 = \phi_2$. \square

An explicit solution can be found if e.g. D is a disk.

Poisson's Integral Formula

Suppose $f = \phi + i\psi$ is analytic inside and on the circle C_R centered at the origin and with radius R . then, by Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{\zeta - z} dz.$$

Let z^* , z be symmetric w.r.t. C_R , i.e.

$$(z^* - 0)\overline{(z - 0)} = R^2 \iff z^* = \frac{R^2}{\bar{z}}.$$

Then $z^* > R$, so by Cauchy's theorem

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_R} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z^*} \right) dz = \frac{1}{2\pi i} \oint_{C_R} \frac{R^2 - |z|^2}{(\zeta - z)(R^2 - \bar{z}\zeta)} f(\zeta) d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - \bar{z}Re^{it})} f(Re^{it}) iRe^{it} dt \\ &= \frac{R^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2} dt \end{aligned}$$

Taking the real part and identifying (x, y) with re^{it} yields

$$\phi(re^{it}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(Re^{it})}{R^2 - 2R \cos(z - \theta) + r^2} dt$$

Sequences and Series of Functions, Uniform Convergence

In Section 3 we defined what is meant by convergence of a sequence of complex numbers. Recall that the sequence $\{a_n\}_{n=0}^\infty$ has a limit a if for $\varepsilon > 0 \exists N : n > N \Rightarrow |a_n - a| < \varepsilon$.

Definition 23. A series is a formal expression of the form

$$c_0 + c_1 + c_2 + \dots + c_j + \dots \iff \sum_{j=0}^{\infty} c_j \quad c_j \in \mathbb{C}$$

The n^{th} partial sum of the series, s_n , is the sum

$$\sum_{j=0}^n c_j$$

of the first $n + 1$ terms. If $\{s_n\}_{n=0}^\infty$ has a limit s , then the series is said to converge to s . In this case, we say that

$$s = \sum_{j=0}^{\infty} c_j.$$

If the series does not converge, it is said to diverge.

Example 28. The following holds:

$$\sum_{j=0}^{\infty} c^j = \frac{1}{1 - c}, \quad |c| < 1.$$

Proof.

$$\begin{aligned} (1 - c)(1 + c + \dots + c^{n-1} + c^n) &= 1 + c + \dots + c^{n-1} + c^n - (c + c^2 + \dots + c^n + c^{n+1}) \\ &= 1 - c^{n+1} \end{aligned}$$

i.e.

$$\frac{1}{1 - c} - (1 + c + \dots + c^{n-1} + c^n) = \frac{c^{n+1}}{1 - c}.$$

The result follows, since $\frac{c^{n+1}}{1 - c} \rightarrow 0$ as $n \rightarrow \infty$ if $|c| < 1$ □

Theorem 40 (Comparison Theorem). *Suppose that $|c_j| \leq M_j$, $\forall j \in \mathbb{N}$. Then*

$$\sum_{j=0}^{\infty} M_j \text{ convergent} \Rightarrow \sum_{j=0}^{\infty} c_j \text{ convergent}.$$

The proof of the above theorem is rather straight forward if one is familiar with the Cauchy Convergence Criterion.

Definition 24 (Cauchy Convergence Criterion).

$$\begin{aligned} \{a_n\}_{n=0}^{\infty} \text{ converges} &\iff \{a_n\}_{n=0}^{\infty} \text{ is Cauchy.} \\ \varepsilon > 0 \Rightarrow \exists N \in \mathbb{N} : n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon \end{aligned}$$

Definition 25 (Absolute Convergence). The series

$$\sum_{j=0}^{\infty} c_j$$

is said to be absolutely convergent if

$$\sum_{j=0}^{\infty} |c_j|$$

is convergent. An absolutely convergent series is convergent by a trivial use of the comparison test.

Theorem 41 (Ratio Test). *Suppose that the terms of the series*

$$\sum_{j=0}^{\infty} c_j$$

have the following property: $\frac{|c_{j+1}|}{|c_j|} \rightarrow L$ as $j \rightarrow \infty$. Then the given series converge if $L < 1$ and diverges if $L > 1$.

Example 29. Show that $\sum_{j=0}^{\infty} \frac{4^j}{j!}$ converges.

Solution: Put $c_j = \frac{4^j}{j!}$. Then

$$\frac{|c_{j+1}|}{|c_j|} = \frac{4^{j+1}}{(j+1)!} \frac{j!}{4^j} = \frac{4}{j+1} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Convergence follows from the ratio test.

Definition 26 (Pointwise Convergence). Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions defined on the set $E \subseteq \mathbb{C}$. Then we say that the sequence converges pointwise to f on E if for each $z \in E$ the sequence of numbers $\{f_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$, i.e. for every $z \in E$ and for every $\varepsilon > 0$:

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon.$$

Definition 27 (Uniform convergence). Let $\{f_n\}_{n=0}^{\infty}$ be defined on $E \subseteq \mathbb{C}$. We say that the sequence converges uniformly to f on E if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon \forall z \in E$$

Remark 16. The above definition is equivalent to $\sup_{z \in E} |f_n(z) - f(z)| \rightarrow 0$ as $n \rightarrow \infty$.

Why care about uniform convergence? Because it preserves properties of functions under the limit operation.

Theorem 42. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of continuous functions on a set $E \subseteq \mathbb{C}$, and suppose it converges uniformly to f on E . Then f is continuous on E .

Proof. Take $z_0 \in E$ and let $\varepsilon > 0$ be given. We want to show that

$$z \in E, |z - z_0| < \delta \Rightarrow |f_n(z_0) - f(z)| < \varepsilon.$$

First choose N so large that $|f_n(z) - f(z)| < \varepsilon/3 \forall z \in E$. This is possible since the sequence is uniformly convergent. Since f_N is continuous at z_0 , there exists δ s.t. $|f_N(z_0) - f_N(z)| < \varepsilon/3$ if $|z - z_0| < \delta$. For such z it then holds that

$$\begin{aligned} |f(z_0) - f(z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

13 Power Series and Taylor Series