Q1. Consider the following system of linear equations:

$4x$ +y+z=6

x +3y+z=5

x +y+2z=4.

Solve this system using three different methods:

(a) Form the augmented matrix corresponding to the system, reduce it to echelon form, and determine the solution using backward substitution. [0.5M]

Solution:

The augmented matrix = $\begin{bmatrix} 4 & 1 & 1 & 6 \\ 1 & 3 & 1 & 5 \\ 1 & 1 & 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 & 5 \\ 4 & 1 & 1 & 6 \\ 1 & 1 & 2 & 4 \end{bmatrix} R1 \leftrightarrow R2 \quad \approx \begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & -11 & -3 & -14 \\ 0 & -2 & 1 & -1 \end{bmatrix} \approx \begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & -22 & -6 & -28 \\ 0 & 22 & -11 & 11 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & -22 & -6 & -28 \\ 0 & 0 & -17 & -17 \end{bmatrix}$$

z=1, −11y−3(1)=−14⇒y=1, x+3(1)+1=5⇒x=1.

(x,y,z)=(1,1,1) is the solution. [either 0.5 or 0.]

(b) Compute the eigen-decomposition of the coefficient matrix, and then use this decomposition to solve the given system of equations. [1.5M]

First to find the eigen values of the coefficient matrix $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

$$\det(A - \lambda \mathrm{I}) = \begin{bmatrix} 4-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = (4-\lambda)[(3-\lambda)(2-\lambda)-1]-(1-\lambda)+(\lambda-2)$$

$$= (4-\lambda)(\lambda^2-5\lambda+5)+2\lambda-3$$

$$= -\lambda^3+9\lambda^2-24\lambda+17$$

Let $P(\lambda) = -\lambda^3+9\lambda^2-24\lambda+17$

By trial and error method:

We find $P(1) = -1^3+9(1)^2-24(1)+17 = 1 > 0$

$P(2) = -2^3+9(2)^2-24(2)+17 = -3 < 0$

Hence one root lies between 1 and 2

Now $P(1.5) = -(1{,}5)^3+9(1.5)^2-24(1.5)+17 = -2.125 < 0$

Hence the root lies between 1 and 1.5.

$P(1.125) = -0.0332 < 0$

Hence the root lies between 1 and 1.125

$P(1.0625) = 0.4612 > 0$

Hence the root lies between 1.0625 and 1.125.

$P(1.09375) = 0.2092 > 0$

The root lies in between 1.09375 and 1.125.

The midpoint of 1.09372 and 1.125 is 1.109375

We will stop this procedure with this accuracy and assume that 1.1094 is a root.

If students have stopped with 1.0625 or found out the root by any other method please award full mark. [0.5M]

Some students might have further iterate and get the root more accurate as.1.12. Now I will assume that one eigen value is 1.11 and find the other two eigen values by factorizing the polynomial

$$\lambda^3 - 9\lambda^2 + 24\lambda - 17 = (\lambda - 1.11)(\lambda^2 - 7.89\lambda + 15.244)$$

Solving this quadratic equation we get the eigen values are 1.11, 4.51, 3.38.

Hence the diagonal matrix D = $\begin{bmatrix} 4.51 & 0 & 0 \\ 0 & 3.38 & 0 \\ 0 & 0 & 1.11 \end{bmatrix}$

$(A - 4the.51I)X = 0$ implies $X = \begin{bmatrix} 1 \\ 1.15 \\ 0.77 \end{bmatrix}$

$(A - 3.38I)X = 0$ implies $X = \begin{bmatrix} 1 \\ 0.48 \\ -1.29 \end{bmatrix}$

$(A - 1.11I)X = 0$ implies $X = \begin{bmatrix} 1 \\ -1.63 \\ 1 \end{bmatrix}$

Hence the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 1.15 & 0.48 & -1.63 \\ 0.77 & -1.29 & 1 \end{bmatrix}$

Note: Students might have written columns of D and P in some other order. Please see whether eigen vector is matching with corresponding eigen value column.

$$D^{-1} = \begin{bmatrix} \frac{1}{4.51} & 0 & 0 \\ 0 & \frac{1}{3.38} & 0 \\ 0 & 0 & \frac{1}{1.11} \end{bmatrix} = \begin{bmatrix} 0.222 & 0 & 0 \\ 0 & 0.296 & 0 \\ 0 & 0 & 0.901 \end{bmatrix}$$

To find P^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1.5 & 0.48 & -1.63 & 0 & 1 & 0 \\ 0.77 & -1.29 & 1 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1.02 & -3.13 & -1.5 & 1 & 0 \\ 0 & -2.06 & 0.23 & -0.77 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & -2.07 & -0.47 & 0.98 & 0 \\ 0 & 1 & 3.07 & 1.47 & -0.98 & 0 \\ 0 & 0 & 6.56 & 2.26 & -2.02 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.24 & 0.34 & 0.32 \\ 0 & 1 & 0 & 0.41 & -0.03 & -0.47 \\ 0 & 0 & 1 & 0.34 & -0.31 & 0.15 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 0.24 & 0.34 & 0.32 \\ 0.41 & -0.03 & -0.47 \\ 0.34 & -0.31 & 0.15 \end{bmatrix}$$

Since $A = PDP^{-1}$, $X = A^{-1}b = PD^{-1}P^{-1}b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$(approximately).

(c) Compute the Cholesky decomposition of the coefficient matrix, and use this factorization to solve the system. [1.5M]

Let

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$l_{11} = \sqrt{4} = 2$$
$$l_{21} = \tfrac{1}{2}, \quad l_{31} = \tfrac{1}{2}$$

$$l_{22} = \sqrt{3 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$$

$$l_{32} = \frac{1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{l_{22}} = \frac{3}{2\sqrt{11}}$$

$$l_{33} = \sqrt{2 - \left(\frac{1}{2}\right)^2 - \left(\frac{3}{2\sqrt{11}}\right)^2} = \sqrt{\frac{14}{11}}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{11}}{2} & 0 \\ \frac{1}{2} & \frac{3}{2\sqrt{11}} & \sqrt{\frac{14}{11}} \end{bmatrix}$$

Solve $L\mathbf{y} = \mathbf{b}$ (forward substitution)

$$\begin{aligned} 2y_1 &= 6 & &\Rightarrow y_1 = 3 \\ \frac{1}{2}(3) + \frac{\sqrt{11}}{2}y_2 &= 5 & &\Rightarrow y_2 = \frac{7}{\sqrt{11}} \\ \frac{1}{2}(3) + \frac{3}{2\sqrt{11}}\left(\frac{7}{\sqrt{11}}\right) + \sqrt{\frac{14}{11}}y_3 &= 4 & &\Rightarrow y_3 = \sqrt{\frac{11}{14}} \end{aligned}$$

Solve $L^T\mathbf{x} = \mathbf{y}$ (back substitution)

$$\begin{aligned} \sqrt{\frac{14}{11}}z &= \sqrt{\frac{11}{14}} & &\Rightarrow z = \frac{11}{14} \\ \frac{\sqrt{11}}{2}y + \frac{3}{2\sqrt{11}}z &= \frac{7}{\sqrt{11}} & &\Rightarrow y = \frac{13}{14} \\ 2x + \frac{1}{2}y + \frac{1}{2}z &= 3 & &\Rightarrow x = 1 \end{aligned}$$

The solution is $X = \begin{bmatrix} 1 \\ \frac{13}{14} \\ \frac{11}{14} \end{bmatrix}$.

(d) Compare the three methods above and comment on for this problem which method is more efficient in terms of computational cost. [0.5M]

For this problem solving the system of equations using Cholesky decomposition is the best.

Solving the system of equations using the eigen decomposition is the worst.

Solution of Q2.

(a) In order to verify < > is an inner product it is enough to prove that M is a symmetric positive definite matrix

Now $M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Clearly $M^T = M$. Hence M is symmetric.

Now we shall prove that M is positive definite.

Now $\det[2] = 2 > 0$.

$\det\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 6 - 1 = 5 > 0$

$\det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = (2 * 12) + \big(1 * (-4)\big) = 24 - 4 = 20 > 0$

$$\det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = det\begin{bmatrix} -1 & 3 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

$$= det\begin{bmatrix} -1 & 3 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix} = 40 > 0.$$

Hence M is positive definite. Hence < > is an inner product.

If students directly verify that < > is an inner product, please award full mark. [1M]

(b) [Completely correct 1 Mark. Partially correct ½ mark.]

$$u = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, w = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

$$< u, v > = [1 \quad 2 \quad 0 \quad 1] \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} >$$

$$= [1 \quad 2 \quad 0 \quad 1] \begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = -1 + 6 + 0 + 3 = 8.$$

$$< u, w > = < \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} > = [1 \quad 2 \quad 0 \quad 1] \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= [1 \quad 2 \quad 0 \quad 1] \begin{bmatrix} 2 \\ -1 \\ 8 \\ 2 \end{bmatrix} = 2 - 2 + 0 + 2 = 2.$$

Solution of Q2c. [Completely correct 1M. Partially correct ½ M]

$$\left\| \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \right\| = \sqrt{<\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\0\\1\end{bmatrix}>} = \sqrt{\begin{bmatrix}1 & 2 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 0\\-1 & 3 & 0 & 0\\0 & 0 & 4 & 1\\0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\2\\0\\1\end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix}1 & 2 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\5\\1\\2\end{bmatrix}} = \sqrt{12} = 2\sqrt{3}.$$

$$\left\| \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\| = \sqrt{<\begin{bmatrix}0\\1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix}>} = \sqrt{\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 0\\-1 & 3 & 0 & 0\\0 & 0 & 4 & 1\\0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}0\\1\\1\\1\end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\3\\5\\3\end{bmatrix}} = \sqrt{11}.$$

$$\left\| \begin{bmatrix}1\\0\\2\\0\end{bmatrix} \right\| = \sqrt{<\begin{bmatrix}1\\0\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\2\\0\end{bmatrix}>} = \sqrt{\begin{bmatrix}1 & 0 & 2 & 0\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 0\\-1 & 3 & 0 & 0\\0 & 0 & 4 & 1\\0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\0\\2\\0\end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix}1 & 0 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\-1\\8\\2\end{bmatrix}} = \sqrt{18}.$$

(d) To prove $\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$ *are linearly independent*. [1 or 0M]

Let a$\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 2\\1 & 1 & 0\end{bmatrix} \approx \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 4\\0 & 0 & 1\end{bmatrix} \approx \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 4\\0 & 0 & 1\end{bmatrix}$$

Since every column has pivot element, a = b = c = 0.

(e) Applying Gram-Schmidt method to orthonormalize the vectors $\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$.

$$v_1 = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}, u_1 = \frac{1}{2\sqrt{3}}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\\\frac{1}{2\sqrt{3}}\end{bmatrix}.$$

$$v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{1}{2\sqrt{3}} < \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}\frac{1}{2\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\\\frac{1}{2\sqrt{3}}\end{bmatrix} > \begin{bmatrix}\frac{1}{2\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\\\frac{1}{2\sqrt{3}}\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{8}{2\sqrt{3}}\begin{bmatrix}\frac{1}{2\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\\\frac{1}{2\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{-2}{3}\\\frac{-1}{3}\\1\\\frac{1}{3}\end{bmatrix}$$

$$< v_2, v_2 > = \frac{17}{3} \quad Hence \ \left||v_2|\right| = \sqrt{\frac{17}{3}}$$

$$u_2 = \sqrt{\frac{3}{17}}\begin{bmatrix}\frac{-2}{3}\\\frac{-1}{3}\\1\\\frac{1}{3}\end{bmatrix}$$ [1M]

$$v_3 = \begin{bmatrix}1\\0\\2\\0\end{bmatrix} - < \begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{6}}\end{bmatrix} > \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{6}}\end{bmatrix} - < \begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{6}}\\0\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix} > \begin{bmatrix}-\frac{1}{\sqrt{6}}\\0\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}\frac{59}{34}\\\frac{2}{17}\\\frac{11}{17}\\-\frac{21}{34}\end{bmatrix}$$

$$\langle v_3, v_3\rangle = \frac{2108}{289}, \| v_3 \| = \frac{\sqrt{2108}}{17}.$$

$$u_3 = \frac{17}{\sqrt{2108}}\begin{bmatrix}\frac{59}{34}\\\frac{2}{17}\\\frac{11}{17}\\-\frac{21}{34}\end{bmatrix}$$ [1M]

The orthonormal vectors are u_1, u_2, u_3.

$$w_1 = w - \mathrm{proj}_{\,u}\, w - \mathrm{proj}_{\,v_1}\, w = \begin{pmatrix} \frac{59}{34} \\ \frac{2}{17} \\ \frac{11}{17} \\ -\frac{21}{34} \end{pmatrix}.$$
