Name - Sagar Powar MFML - Assigment-1 (1)

BITS ID - 2025AA05421

Q.1

a) Given system
$$4x+y+z=6$$
$$x+3y+z=5$$
$$x+y+2z=4$$

form the Augmented matrix

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 6 \\ 1 & 3 & 1 & 5 \\ 1 & 1 & 2 & 4 \end{array}\right]$$

(1) swap Rows (1) & (2)
$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 5 \\ 4 & 1 & 1 & 6 \\ 1 & 1 & 2 & 4 \end{array}\right]$$

(2) $\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 4R_1}$
$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 5 \\ 0 & -11 & -3 & -14 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

(3) $\xrightarrow{R_2 \leftarrow R_2/-11}$
$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 5 \\ 0 & 1 & 3/11 & 14/11 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

(4) $\xrightarrow{R_3 \leftarrow R_3 + 2R_2}$
$$\begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & 1 & 3/11 & 14/11 \\ 0 & 0 & 17/11 & 17/11 \end{bmatrix}$$

(5) using Backward substitution,

from $R_3 \rightarrow \frac{17}{11}z = \frac{17}{11} \Rightarrow \boxed{z=1}$

from $R_2 \rightarrow y + \frac{3}{11} = \frac{14}{11} \Rightarrow \boxed{y=1}$ Solution

from $R_1 \rightarrow x + 3(1) + 1 = 5 \Rightarrow \boxed{x=1}$

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Q.1

b) <u>Eigen-decomposition method:</u> $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$

(1) find the eigenvalues (λ)

$$\det(A - \lambda I) = 0$$

i.e.

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

using first-Row expansion,

$$(4-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 1 \end{vmatrix} = 0 \quad \text{— (I)}$$

find minors,

(1) $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 1 = \underline{\lambda^2 - 5\lambda + 5}$

(2) $\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} = 1(2-\lambda) - 1 = \underline{1-\lambda}$

(3) $\begin{vmatrix} 1 & 3-\lambda \\ 1 & 1 \end{vmatrix} = 1 - (3-\lambda) = \underline{\lambda - 2}$

substitute back in (I)

$$(4-\lambda)(\lambda^2 - 5\lambda + 5) - (1-\lambda) + (\lambda - 2) = 0$$

$$4\lambda^2 - \lambda^3 - 20\lambda + 5\lambda^2 + \underline{20} - 5\lambda \underline{-1} + \lambda + \lambda \underline{-2} = 0$$

$$-\lambda^3 + 9\lambda^2 - 23\lambda + 17 = 0$$

$$\lambda^3 - 9\lambda^2 + 23\lambda - 17 = 0$$

(2) <u>Find roots</u>

Test $\lambda = 1$; $1 - 9 + 23 - 17 = -2 \neq 0$

Test $\lambda = 2$; $8 - 36 + 46 - 17 = 1 \neq 0$

Test $\lambda = 3$ $27 - 81 + 69 - 17 = -2 \neq 0$

hence irrational eigen values $\cancel{\lambda_1}$, $\cancel{\lambda_2}$, $\cancel{\lambda_3}$

Name-sagar powar ID-2025AAOS421 (3)

~~Find P^{-1}~~

~~$\det(P) = \begin{vmatrix} 1 & 1 & 4 \\ -1 & -1 & 3 \\ 0 & 0 & 5 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 5(-1+1) = 0$~~

~~columns ① 4② are linearly dependent as duplicates,~~

Find out values numerically,

$\lambda_1 \approx 1.324$, $\lambda_2 \approx 2.460$, $\lambda_3 \approx 5.216$.

③ Find eigen-vectors

for each λ, solve $(A-\lambda I)V = 0$

a) $\lambda = 1.324$

$$A - \lambda I = \begin{bmatrix} 4-1.324 & 1 & 1 \\ 1 & 3-1.324 & 1 \\ 1 & 1 & 2-1.324 \end{bmatrix} = \begin{bmatrix} 2.676 & 1 & 1 \\ 1 & 1.676 & 1 \\ 1 & 1 & 0.676 \end{bmatrix}$$

solve,

$$(A-\lambda I)V_1 = 0 \quad \begin{bmatrix} 2.676 & 1 & 1 \\ 1 & 1.676 & 1 \\ 1 & 1 & 0.676 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solving this

$$V_1 \approx \begin{bmatrix} 0.742 \\ -0.618 \\ -0.257 \end{bmatrix}$$

(b) $\lambda = 2.46$

solve $(A-\lambda I)V_2 = 0$

$$\Rightarrow \begin{bmatrix} 4-2.46 & 1 & 1 \\ 1 & 3-2.46 & 1 \\ 1 & 1 & 2-2.46 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.54 & 1 & 1 \\ 1 & 0.54 & 1 \\ 1 & 1 & -0.46 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this,

$$V_2 \approx \begin{bmatrix} -0.263 \\ 0.529 \\ -0.866 \end{bmatrix}$$

(i) $\lambda \approx 5.216$,

$$(A-\lambda I)V_3 = 0$$

$$\Rightarrow \begin{bmatrix} 4-5.216 & 1 & 1 \\ 1 & 3-5.216 & 1 \\ 1 & 1 & 2-5.216 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1.216 & 1 & 1 \\ 1 & -2.216 & 1 \\ 1 & 1 & -3.216 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solving this,

$$V_3 \approx \begin{bmatrix} 0.616 \\ 0.582 \\ 0.530 \end{bmatrix}$$

Now, solve using Eigen-Decomposition method

$Ax = b$ given, $A = PDP^{-1}$

$x = A^{-1}b$ $A^{-1} = PD^{-1}P^{-1}$

$\boxed{x = PD^{-1}P^{-1}b}$

since A is symmetric, eigen vectors are orthonormal.

$\boxed{P^{-1} = P^T}$

so, $\boxed{x = PD^{-1}P^Tb}$

① Compute $D^{-1} = \begin{bmatrix} \frac{1}{1.324} & 0 & 0 \\ 0 & \frac{1}{2.460} & 0 \\ 0 & 0 & \frac{1}{5.216} \end{bmatrix} \approx \begin{bmatrix} 0.755 & 0 & 0 \\ 0 & 0.407 & 0 \\ 0 & 0 & 0.192 \end{bmatrix}$

② Compute $c = P^Tb = \begin{bmatrix} 0.742 & -0.618 & -0.257 \\ -0.283 & 0.529 & -0.806 \\ 0.616 & 0.582 & 0.530 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix}$

$$c \approx \begin{bmatrix} 0.276 \\ -1.656 \\ 8.724 \end{bmatrix}$$

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(3) Compute $d = D^{-1}c$

$$d = \begin{bmatrix} 0.755 & 0 & 0 \\ 0 & 0.407 & 0 \\ 0 & 0 & 0.192 \end{bmatrix} \begin{bmatrix} 0.276 \\ -1.656 \\ 8.724 \end{bmatrix} = \begin{bmatrix} 0.208 \\ -0.674 \\ 1.676 \end{bmatrix}$$

(4) Compute $x = Pd$

$$x = \begin{bmatrix} 0.742 & -0.263 & 0.616 \\ -0.618 & 0.529 & 0.582 \\ -0.257 & -0.806 & 0.530 \end{bmatrix} \begin{bmatrix} 0.208 \\ -0.674 \\ 1.676 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.00 \\ 1.00 \end{bmatrix}$$

Final solution

$$\boxed{x = 1, \; y = 1, \; z = 1}$$

Q.1 (c) <u>Cholesky Decomposition</u> :-

(1) Check for symmetry & positive Definiteness

- $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ since $A^T = A$, the matrix is symmetric

a) Minor (first) $4 > 0$

b) Second minor $\det \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 12 - 1 = 11 > 0$

c) Third minor $\det(A) = 17 > 0$

matrix is <u>symmetric</u> & <u>positive definite</u>.

so <u>cholesky form</u> can be applied.

(2) $\boxed{A = LL^T}$

where $\underline{L}$ is lower triangular $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$$LL^T = \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{11} l_{21} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{11} l_{31} & l_{21} l_{31} + l_{22} l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

(2) Now, match enteries with $\underline{A}$

solve for 'L' element by element.

(a) $a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}} = \sqrt{4} \Rightarrow \boxed{l_{11} = 2}$

(b) $l_{11} l_{21} = a_{21} = 1 \Rightarrow 2 l_{21} = 1 \Rightarrow \boxed{l_{21} = 1/2}$

(c) $a_{31} = l_{11} l_{31} \Rightarrow l_{31} = \frac{a_{31}}{l_{11}} = \frac{1}{2} \Rightarrow \boxed{l_{31} = \frac{1}{2}}$

(d) $a_{22} = l_{21}^2 + l_{22}^2 = 3 \Rightarrow \frac{1}{4} + l_{22}^2 = 3 \Rightarrow \boxed{l_{22} = \frac{\sqrt{11}}{2}}$

(e) $a_{32} = 1 = l_{21} l_{31} + l_{22} l_{32} \Rightarrow \frac{1}{2} \cdot \frac{1}{2} + \frac{\sqrt{11}}{2} \cdot l_{32} = 1$

$\hookrightarrow \cancel{l_{32} = \sqrt{}}\ \frac{\sqrt{11}}{2} l_{32} = \frac{3}{4} \Rightarrow \boxed{l_{32} = \frac{3}{2\sqrt{11}}}$

(f) $a_{33} = 2 = l_{31}^2 + l_{32}^2 + l_{33}^2$

$= \frac{1}{4} + \frac{9}{44} + l_{33}^2 = 2 \Rightarrow \boxed{l_{33} = \sqrt{\frac{17}{11}}}$

so,
$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1/2 & \sqrt{11}/2 & 0 \\ 1/2 & \frac{3}{2\sqrt{11}} & \sqrt{17/11} \end{bmatrix}$$

(3) Solve system $\boxed{Ax=b}$, $A = LL^T$

so $\boxed{L(L^Tx)=b}$

$\boxed{\text{Step-1}}$ solve $\underline{Ly=b}$ & $\underline{y=L^Tx}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1/2 & \sqrt{11}/2 & 0 \\ 1/2 & \frac{3}{2\sqrt{11}} & \sqrt{\frac{17}{11}} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

Row 1: $2y_1 = 6 \Rightarrow \boxed{y_1 = 3}$

Row 2: $\frac{1}{2}y_1 + \frac{\sqrt{11}}{2}y_2 = 5$

$\therefore \frac{1\times 3}{2} + \frac{\sqrt{11}}{2} y_2 = 5 \Rightarrow \frac{\sqrt{11}}{2}y_2 = \frac{7}{2} \Rightarrow \boxed{y_2 = \frac{7}{\sqrt{11}}}$

Row 3: $\frac{1}{2}y_1 + \frac{3}{2\sqrt{11}}y_2 + \sqrt{\frac{17}{11}}y_3 = 4$

$\frac{3}{2} + \frac{3}{2\sqrt{11}} \times \frac{7}{\sqrt{11}} + \sqrt{\frac{17}{11}}y_3 = 4$

$\frac{3}{2} + \frac{21}{22} + \sqrt{\frac{17}{11}}y_3 = 4 \Rightarrow \boxed{y_3 = \frac{\sqrt{17}}{\sqrt{11}}}$

$\boxed{\text{Step 2}}$ Backward substitution: $(L^Tx = y)$

$$L^T = \begin{bmatrix} 2 & 1/2 & 1/2 \\ 0 & \frac{\sqrt{11}}{2} & \frac{3}{2\sqrt{11}} \\ 0 & 0 & \sqrt{\frac{17}{11}} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7/\sqrt{11} \\ \sqrt{17}/\sqrt{11} \end{bmatrix}$$

Row 3: $\sqrt{\frac{17}{11}}z = \sqrt{\frac{17}{11}} \Rightarrow \boxed{z=1}$

Row 2: $\frac{\sqrt{11}}{2}y + \frac{3}{2\sqrt{11}} = \frac{7}{\sqrt{11}} \Rightarrow \frac{\sqrt{11}}{2}y = \frac{7}{\sqrt{11}} - \frac{3}{2\sqrt{11}} \Rightarrow \boxed{y=1}$

Row 1: $2x + \frac{1}{2}(1) + \frac{1}{2}(1) = 3 \Rightarrow 2x = 2 \Rightarrow \boxed{x=1}$

$\boxed{\text{Solution} \Rightarrow x=1, y=1, z=1}$

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Q.1

d) Comparison on computational cost

Method	Complexity	Cost	Comments
(1) Gaussian Elimination	simple	$O(n^3)$	Most efficient for single solution.
(2) Eigen Decomposition	complex	$O(n^3)$	Highest general cost. finding eigen values is complex & iterative.
(3) Cholesky Decomposition	Moderate	$O(n^3/3)$	efficient for symmetric positive definite matrix.

Conclusion:

(1) for this problem, matrix (3x3), Gaussian elimination is the most practical & simplest method.

(2) ED - is complex & unnecessary to solve the linear system equations.

(3) Cholesky Decomposition is most efficient & numerically stable method here.

Q.2

a) Given,

$$M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Inner product must satisfy 3 properties.

(1) Symmetry

$$\langle x, y \rangle = \langle y, x \rangle$$

This requires matrix M to be symmetric. i.e. $M^T = M$

$$M^T = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = M$$

symmetry holds.

(2) Linearity check

for all x_1, x_2, y & scalar c

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

$$\langle cx, y \rangle = c \langle x, y \rangle$$

This holds because matrix multiplication is linear

(3) Positive-Definiteness

$$\langle x, x \rangle = x^T M x > 0 \quad \text{for } x \neq 0$$

Matrix M is block diagonal with positive definite blocks.

a.) Top-left block $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow \det = 6-1 = 5 > 0$

b) Bottom-right Block $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \det = 8-1 = 7 > 0$

M is positive definite.

Conclusion - $\boxed{\langle x, y \rangle = x^T M y \text{ is a valid inner product on } R^4}$

Q.2

b) Compute $\langle u,v\rangle, \langle v,w\rangle, \langle v,w\rangle$

We alway compute $\langle a,b\rangle = a^T M b$

(1) compute $\langle u,v\rangle = u^T(Mv)$

a) Find $Mv = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix}$

Now, Dot with u

b) $\langle u,v\rangle = u^T(Mv) = (1,2,0,1)\cdot(-1,3,5,3)$

$= -1+6+0+3 = \boxed{8}$

(2) Compute $\langle u,w\rangle$.

a) find $Mw = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \\ 2 \end{bmatrix}$

b) $\langle u,w\rangle = u^T(Mw) = (1,2,0\,1)\cdot(2\ -1\ 8\ 2)$

$= 2-2+0+2 = \boxed{2}$

(3) compute $\langle v,w\rangle$

a) find $Mw = (2\ -1\ 8\ 2)$

b) find $\langle v,w\rangle = v^T(Mw) = (0\ 1\ 1\ 1)\cdot(2\ -1\ 8\ 2)$

$= 0-1+8+2 = \boxed{9}$

final answers,

$$\boxed{\langle u,v\rangle = 8,\ \langle u,w\rangle = 2,\ \langle v,w\rangle = 9}$$

Q.2 c

Compute norms $\|u\|, \|v\|, \|w\|$

$$\|x\| = \sqrt{\langle x, x\rangle}$$

① <u>Norm of u</u>

$$Mu = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 2 \end{bmatrix}$$

$$\langle u, u\rangle = (1, 2, 0, 1) \cdot (0, 5, 1, 2)$$
$$= 0 + 10 + 0 + 2 = 12$$

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{12} = \boxed{2\sqrt{3}}$$

② <u>Norm of V</u>

$$Mv = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

$$\langle v, v\rangle = (0 \cdot 1\ 1\ 1) \cdot (-1\ 3\ 5\ 3) \cdot$$
$$= 0 + 3 + 5 + 3 = 11$$

$$\boxed{\|v\| = \sqrt{11}}$$

③ <u>Norm of W.</u>

$$\langle w, w\rangle = (1\ 0\ 2\ 0) \cdot (2\ -1\ 8\ 2)$$
$$= 2 + 0 + 16 + 0 = 18$$

$$\boxed{\|w\| = \sqrt{18} = 3\sqrt{2}}$$

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Q.2

d) show that u, v, w are linearly independant.

Solve the equation,

$$au + bv + cw = 0$$

$$a\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\2\\0\end{bmatrix} = 0$$

This gives,

$$a + c = 0 \quad -(1)$$
$$2a + b = 0 \quad -(2)$$
$$b + 2c = 0 \quad -(3)$$
$$a + b = 0 \quad -(4)$$

from (4), $b = -a$, (2) becomes $2a - a = 0 \Rightarrow \boxed{a = 0}$

from (1), $a + c = 0 \Rightarrow \boxed{c = 0}$

from (3), $b + 2c = 0 \Rightarrow \boxed{b = 0}$

that is, $\boxed{a = b = c = 0}$

<u>conclusion</u>

$\boxed{u, v, w \text{ are linearly independant}}$

Q2

e) obtain orthonormal set using Gram-schmidt process

$\langle u,u\rangle = 12$, $\langle u,v\rangle = 8$, $\langle u,w\rangle = 2$

step ① First orthonormal vector

$$\boxed{e_1 = \frac{u}{\|u\|} = \frac{1}{\sqrt{12}}\begin{pmatrix}1\\2\\0\\1\end{pmatrix}}$$

step ② Orthogonalize v with respect to u

ⓐ $v_2 = v - \frac{\langle v,u\rangle}{\langle u,u\rangle} u$ —— (I)

$$\text{proj.}(v) = \frac{\langle v,u\rangle}{\langle u,u\rangle} u = \frac{8}{12} u = \frac{2}{3} u = \frac{2}{3}\begin{pmatrix}1\\2\\0\\1\end{pmatrix} = \begin{pmatrix}2/3\\4/3\\0\\2/3\end{pmatrix}$$

ⓑ $$v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \begin{bmatrix}2/3\\4/3\\0\\2/3\end{bmatrix} = \begin{bmatrix}-2/3\\-1/3\\1\\1/3\end{bmatrix}$$

ⓒ Compute norm of v_2

$$Mv_2 = \begin{bmatrix}2 & -1 & 0 & 0\\-1 & 3 & 0 & 0\\0 & 0 & 4 & 1\\0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}-2/3\\-1/3\\1\\1/3\end{bmatrix} = \begin{bmatrix}-1\\-1/3\\13/3\\5/3\end{bmatrix}$$

$$\langle v_2,v_2\rangle = v_2^T(Mv_2) = \left(\frac{-2}{3}\ \frac{-1}{3}\ 1\ \frac{1}{3}\right)\cdot\left(-1\ -1/3\ 13/3\ 5/3\right)$$

$$= \frac{2}{3} + \frac{1}{9} + \frac{13}{3} + \frac{5}{9}$$

$$= \frac{51}{9} = \frac{17}{3}$$

ⓓ Normalize v_2

$$\|v_2\| = \sqrt{\langle v_2,v_2\rangle} = \sqrt{17/3}$$

$$\boxed{e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{17/3}}\begin{bmatrix}-2/3\\-1/3\\1\\1/3\end{bmatrix}}$$

e. **Step 3: Orthogonalize w**

(3.1) Find $\langle w, u\rangle$

$$Mu = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 2 \\ -1 \\ 8 \\ 2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 2 \end{bmatrix}$$

$$\langle w, u\rangle = w^T(Mu) = (1\ 0\ 2\ 0)\cdot(0\ 5\ 1\ 2) = 2$$

(3.2) First projection

$$\frac{\langle w, u\rangle}{\langle u, u\rangle}u = \frac{2}{12}u = \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 0 \\ 1/6 \end{bmatrix} \quad \text{--- (I)}$$

(3.3) Find $\langle w, v_2\rangle$

$$\langle w, v_2\rangle = w^T(Mv_2) = (1\ 0\ 2\ 0)\cdot\left(-1\ \ -\tfrac{1}{3}\ \ \tfrac{13}{3}\ \ \tfrac{5}{3}\right)$$

$$= -1 + 0 + \frac{26}{3} + 0 = \frac{23}{3}$$

(3.4) Second projection

$$\frac{\langle w, v_2\rangle}{\langle v_2, v_2\rangle} = \frac{23/3}{17/3} = \frac{23}{17} \quad \text{--- (II)}$$

(3.5) Find w_3

$$w_3 = w - \text{proj}_u(w) - \text{proj}_{v_2}(w)$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/6 \\ 1/3 \\ 0 \\ 1/6 \end{bmatrix} - \frac{23}{17}\begin{bmatrix} -2/3 \\ -1/3 \\ 1 \\ 1/3 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 59/34 \\ 2/17 \\ 11/17 \\ -21/34 \end{bmatrix}$$

Q.2 e

Step 3.6

Compute $\|w_3\| = \sqrt{\langle w_3, w_3\rangle}$

$\langle w_3, w_3\rangle = w_3^T (M w_3)$

① Find

$$M w_3 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 59/34 \\ 2/17 \\ 11/17 \\ -21/34 \end{bmatrix} = \cancel{\begin{bmatrix} -7/34 \\ 20/34 \\ 13/17 \\ 10/17 \end{bmatrix}} \begin{bmatrix} 57/17 \\ -47/34 \\ 67/34 \\ -10/17 \end{bmatrix}$$

$$\langle w_3, w_3\rangle = w_3^T (M w_3)$$

$$= \begin{bmatrix} \frac{59}{34} & \frac{2}{17} & \frac{11}{17} & \frac{-21}{34} \end{bmatrix} \cdot \begin{bmatrix} \frac{57}{17} & \frac{-47}{34} & \frac{67}{34} & \frac{-10}{17} \end{bmatrix}$$

$$= \frac{2108}{289}$$

Step 3.7

Norm of $w_3 = \|w_3\| = \sqrt{\frac{2108}{289}} = \frac{\sqrt{2108}}{17}$

Step 3.8

Normalized vector e_3

$$e_3 = \frac{w_3}{\|w_3\|} = \frac{17}{\sqrt{2108}} \begin{bmatrix} 59/34 \\ 2/17 \\ 11/17 \\ -21/34 \end{bmatrix}$$

Final Answer:

$$\boxed{e_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \qquad \boxed{e_2 = \frac{1}{\sqrt{17/3}} \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \\ 1/3 \end{bmatrix}}$$

$$\boxed{e_3 = \frac{17}{\sqrt{2108}} \begin{bmatrix} 59/34 \\ 2/17 \\ 11/17 \\ -21/34 \end{bmatrix}}$$