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DYNAMIC PROGRAMMING: OPTIMAL STATIC BINARY SEARCH TREES

This lecture note describes an application of the dynamic programming paradigm on computing the optimal static binary search tree of n keys to minimize the expected search time for a given search probability distribution.

Suppose we are given a collection of n keys $K_1 < K_2 < \cdots < K_n$ which are to be stored in a binary search tree. After the tree has been constructed, only search operations will be performed — i.e. there will be no insertions or deletions. We are also given a probability density function P where $P(i)$ is the probability of searching for key K_i. There are many different binary search trees where the n given keys can be stored. For a particular tree T with these keys, the average number of comparisons to find a key, for the given probability density is

$$\sum_{i=1}^{n} P(i) \cdot (\ depth_T(K_i) + 1\ ),$$

where $depth_T(K_i)$ denotes the depth of the node where K_i is stored in T. The problem we would like to solve is to find, among all the possible binary search trees that contain the n keys, one which *minimizes* this quantity. Such a tree is called an *optimal (static) binary search tree*. Note that there may be several optimal binary trees for the given density function. This is why we speak of an optim*al*, rather than the optim*um* binary search tree.

A simple way to accomplish this is to try out all possible binary trees with n nodes, computing the average number of comparisons to find a key in each tree considered, and selecting a tree with the minimum average. Unfortunately, this simple strategy is ridiculously inefficient because there are too many trees to try out. In particular, there are $\binom{2n}{n}/(n+1)$ different binary trees with n nodes (if interested in the derivation of this formula, see Knuth, vol. I, pp. 388-389). Thus, if there are 20 keys, we have to try out 131,282,408,400 different trees. Computing the average number of comparisons in each at the rather astonishing speed of 1μsec per tree, will still take 2188 hours or approximately 91 days and nights of computing to find an optimal binary search tree (for just 20 keys)! Fortunately, there is a much more efficient, if less straightforward, way to find an optimal binary search tree. Let T be a binary search tree that contains K_i , K_{i+1} , $\cdots$, K_j for some $1 \le i \le j \le n$. We shall see shortly why it is useful to consider trees that contain subsets of successive keys. We define the *cost* of T as,

$$c(T) = \sum_{l=i}^{j} P(l) \cdot (\ depth_T(K_l) + 1\ )$$

Hence, if T contains all n keys (i.e. $i = 1$ and $j = n$), the cost of T is precisely the expected number of comparisons to find a key for the given density function.† Thus, we can rephrase our problem as follows: Given a density function for the n keys, find a minimum cost tree with n nodes.

Before giving the algorithm to find an optimal binary search tree, we prove two key facts.

† This is not so if T is missing some of the keys, because in that case the probabilities of the keys that are in T do not sum to 1 — that is, P is not a proper density function relative to the set of keys in the tree.

Lemma 1: Let T be a binary search tree containing keys $K_i, K_{i+1}, \cdots, K_j$, T_L and T_R be the left and right subtrees of T respectively. Then

$$c(T) = c(T_L) + c(T_R) + \sum_{l=i}^{j} P(l)\,.$$

Proof: This is an easy consequence of the definition of cost of a tree. You should prove it on your own. □

Lemma 2: Let T be a binary search tree that has minimum cost among all trees containing keys $K_i, K_{i+1}, \cdots, K_j$ and let K_m be the key at the root of T (so $i \le m \le j$). Then T_L, the left subtree of T, is a binary search tree that has minimum cost among all trees containing keys $K_i, \cdots, K_{m-1}$; and T_R, the right subtree of T, is a binary search tree that has minimum cost among all trees containing keys $K_{m+1}, \cdots, K_j$.

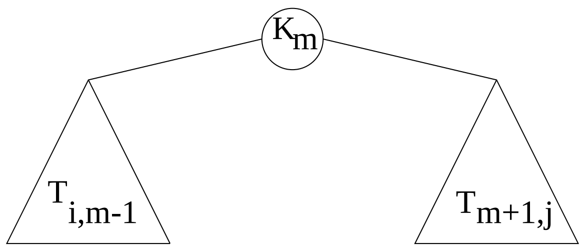
Proof: We prove the contrapositive. That is, if either the left or right subtree of T fails to satisfy the minimality property asserted in the lemma we show that T does not really have the minimum possible cost among all trees that contain $K_i, \cdots, K_j$.
Let T'_L and T'_R be minimum cost binary search trees that contain $K_i, \cdots, K_{m-1}$ and $K_{m+1}, \cdots, K_j$ respectively. Thus, $c(T'_L) \le c(T_L)$ and $c(T'_R) \le c(T_R)$ (*).
Further, let T' be the tree with key K_m in the root, and left and right subtrees T'_L and T'_R respectively. Evidently, T' is a binary search tree that contains keys $K_i, \cdots, K_j$. If T_L or T_R do not have the minimality property asserted by the lemma, it must be that either $c(T_L) > c(T'_L)$, or $c(T_R) > c(T'_R)$. This, together with (*) implies that $c(T_L) + c(T_R) > c(T'_L) + c(T'_R)$. From this and Lemma 1 we have,

$$c(T) = c(T_L) + c(T_R) + \sum_{l=i}^{j} P(l) \;>\; c(T'_L) + c(T'_R) + \sum_{l=i}^{j} P(l) = c(T').$$

Thus, $c(T) > c(T')$ and T is not a minimum cost binary search tree among all trees that contain keys $K_i, \cdots, K_j$. □

Computing an Optimal Binary Search Tree

Lemma 2 is the basis of an efficient algorithm to find an optimal binary search tree. Let T_{ij} denote an optimal binary search tree containing keys $K_i, K_{i+1}, \cdots, K_j$. Then T_{1n} is precisely the optimal binary search tree that we want to construct. Lemma 2 says that T_{ij} must be of the following form:



That is, its root has key K_m for some m, $i \le m \le j$, and its subtrees are $T_{i,m-1}$ and $T_{m+1,j}$, i.e. minimum cost subtrees containing keys $K_i, \cdots, K_{m-1}$ and $K_{m+1}, \cdots, K_j$ respectively. But $T_{i,m-1}$ and $T_{m+1,j}$ are "smaller" trees than T_{ij}. This suggests proceeding inductively, starting with small minimum cost trees (each containing just one key) and progressively building larger and larger minimum cost trees, until we have a minimum cost tree with n nodes which is what we are looking for.

More specifically, we start the induction with minimum cost trees each of which contains exactly one key, and proceed by constructing minimum cost trees with $2, 3, \ldots, n$ successive keys. Note that there are exactly $n - d + 1$ groups of d successive keys for each $d = 1, \cdots, n$. Thus, instead of considering *all* possible trees with n nodes we consider only n (minimum cost) trees with 1 node each, $n - 1$ (minimum cost) trees with 2 nodes each, $\cdots$, 1 minimum cost tree with n nodes, i.e. a total of $n(n+1)/2$ trees — much fewer than $\binom{2n}{n}/(n+1)$.

So now the question is how to compute these trees T_{ij} inductively. The basis of the induction, i.e. when $j - i = 0$ is trivial. In this case we have $j = i$ and the minimum cost binary search tree that stores K_i (in fact the only such tree!) is a single node containing K_i; its cost is $c(T_{ii}) = P(i)$.

For the inductive step, take $j - i > 0$ and assume that we have already computed all the T_{uv}'s and their costs, for $v - u < j - i$. Let T_{imj} be the tree with K_m in the root, and left and right subtrees $T_{i,m-1}$ and $T_{m+1,j}$ respectively. As we saw before, Lemma 2 implies that T_{ij} is the minimum cost tree among the T_{imj}'s. Thus we can find T_{ij} simply by trying out all the T_{imj}'s for $m = i, i+1, \cdots, j$. In fact Lemma 1 tells us how to compute $c(T_{imj})$ efficiently, so that "trying out" each possible m will not take too long: Since $(m-1) - i$ and $j - (m+1)$ are both $< j - i$, we have already (inductively) computed $T_{i,m-1}$ and $T_{m+1,j}$ as well as their costs, $c(T_{i,m-1})$ and $c(T_{m+1,j})$. Lemma 1 then tells us how to get $c(T_{imj})$ in terms of these. Note that when $m = i$ the left subtree of T_{imj} is $T_{i,i-1}$ and when $m = j$ the right subtree of T_{imj} is $T_{j+1,j}$. We define T_{ij} to be empty if $i > j$, and the cost of an empty tree to be 0.

Figure 1 shows this algorithm in pseudo-code. The algorithm takes as input an array *Prob*$[1..n]$, which specifies the probability density (i.e. *Prob*$[i] = P(i)$). It computes two two-dimensional arrays, *Root* and *Cost*, where *Root*$[i, j]$ is the root of T_{ij}, and *Cost*$[i, j] = c(T_{ij})$, for $1 \le i \le j \le n$.† To help compute *Root* and *Cost* the algorithm maintains a third array, *SumOfProb*, where *SumOfProb*$[i] = \sum_{l=1}^{i} P(l)$ for $1 \le i \le n$, and *SumOfProb*$[0] = 0$. Note that $\sum_{l=i}^{j} P(l) =$ *SumOfProb*$[j]$ $-$ *SumOfProb*$[i-1]$.

The algorithm of Figure 1 does not explicitly construct an optimal binary search tree but such a tree is implicit in the information in array *Root*. As an exercise you should write an algorithm which, given *Root* and an array *Key*$[1..n]$, where *Key*$[i] = K_i$, constructs an optimal binary search tree.

It is not hard to show that this algorithm has worst case time complexity $\Theta(n^3)$. A slight modification of this algorithm leads to a $\Theta(n^2)$ complexity (if interested, see D.E. Knuth, "Optimum binary search trees," *Acta Informatica*, vol. 1 (1971), pp. 14-25.)

Unsuccessful Searches

In this discussion we have only considered successful searches. However, if we take into account unsuccessful searches, maybe the constructed tree is no longer optimal. Fortunately, this

† For technical reasons that will become apparent when you look at the algorithm carefully we need to set *Cost*$[i, i-1] = 0$ for $1 \le i \le n+1$. Recall that $T_{i,i-1}$ is empty and thus has cost 0.

```
**algorithm** $OptimalBST\ (\ Prob[1..n]\ )$;
**begin**
(* initialization *)
**for** $i \leftarrow 1$ **to** $n+1$ **do** $Cost[i, i-1] \leftarrow 0$;
$SumOfProb[0] \leftarrow 0$;
**for** $i \leftarrow 1$ **to** $n$ **do begin**
   $SumOfProb[i] \leftarrow Prob[i] + SumOfProb[i-1]$;
   $Root[i, i] \leftarrow i$;
   $Cost[i, i] \leftarrow Prob[i]$
**end**;

**for** $d \leftarrow 1$ **to** $n-1$ **do**  (* compute info about trees with $d+1$ consecutive keys *)
  **for** $i \leftarrow 1$ **to** $n-d$ **do begin**  (* compute $Root[i, j]$ and $Cost[i, j]$ *)
     $j \leftarrow i + d$;
     $MinCost \leftarrow +\infty$;
     **for** $m \leftarrow i$ **to** $j$ **do begin**   (* find $m$ between $i$ and $j$ so that $c(T_{imj})$ is minimum *)
        $c \leftarrow Cost[i, m-1] + Cost[m+1, j] + SumOfProb[j] - SumOfProb[i-1]$;
        **if** $c < MinCost$ **then begin**
             $MinCost \leftarrow c$;
             $r \leftarrow m$
        **end**
     **end**;
     $Root[i, j] \leftarrow r$;
     $Cost[i, j] \leftarrow MinCost$
  **end**
**end**
```

Figure 1: Algorithm for optimal binary search tree.

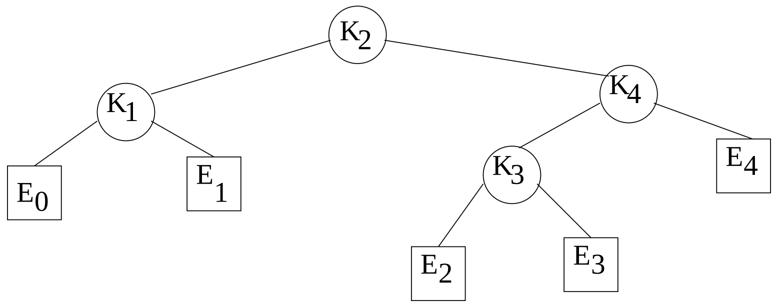
problem can be taken care of in a straightforward manner. To find an optimal binary search tree in the case where both successful and unsuccessful searches are taken into account, we must know the probability density for both successful and unsuccessful searches. So, in addition to $P(i)$ we are also given $Q(i)$ for $0 \le i \le n$, where

$Q(0)$ = prob. of searching for keys $< K_1$;

$Q(i)$ = prob. of searching for keys x, $K_i < x < K_{i+1}$, for $1 \le i < n$;

$Q(n)$ = prob. of searching for keys $> K_n$.

In each binary search tree containing keys $K_1, K_2, \cdots, K_n$ we add $n+1$ external nodes $E_0, E_1, \cdots, E_n$. This is illustrated below; the external nodes are drawn in boxes, as usual.



The average number of comparisons for a successful *or* unsuccessful search in such a tree T is

$$\sum_{i=1}^{n} P(i) \cdot (\, depth_T(K_i) + 1\,) + \sum_{i=0}^{n} Q(i) \cdot depth_T(E_i).$$

The left term is the contribution to the average number of comparisons by the successful searches and the right term is the contribution to the average number of comparisons by the unsuccessful searches. Now we want to find a tree that minimizes this quantity.

We can proceed exactly as before, except that the definition of the cost of a tree T with consecutive keys $K_i, K_{i+1}, \cdots, K_j$ is slightly modified to account for the unsuccessful searches. Namely, it becomes,

$$c'(T) = \sum_{l=i}^{j} P(l) \cdot (\, depth_T(K_l) + 1\,) + \sum_{l=i-1}^{j} Q(l) \cdot depth_T(E_l).$$

With this cost function Lemma 1 is slightly different:

Lemma 1': Let T be a binary search tree containing keys $K_i, K_{i+1}, \cdots, K_j$, T_L and T_R be the left and right subtrees of T respectively. Then

$$c'(T) = c'(T_L) + c'(T_R) + Q(i-1) + \sum_{l=i}^{j} (P(l) + Q(l)).$$

Everything else works out exactly as before. In particular, Lemma 2 is still valid (check this!). As an exercise show how to modify the algorithm in Figure 1 to account for these changes.

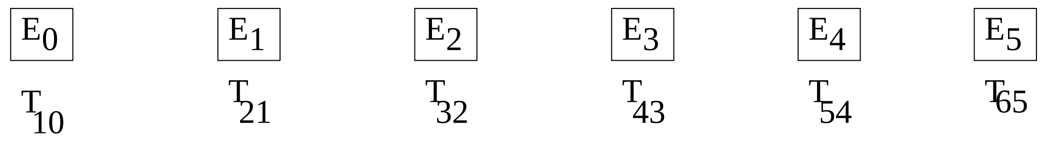
Example: Suppose we want to find an optimal binary search tree for the dictionary {begin, end, goto, repeat, until} for the following probabilities of searching for these keys ($P(i)$'s) and keys alphabetically ‘‘in between’’ (the $Q(i)$'s):

	K_1 = begin	K_2 = end	K_3 = goto	K_4 = repeat	K_5 = until
	$P(1) = .1$	$P(2) = .1$	$P(3) = .05$	$P(4) = .05$	$P(5) = .05$
$Q(0) = .05$	$Q(1) = .05$	$Q(2) = .2$	$Q(3) = .2$	$Q(4) = .1$	$Q(5) = .05$

The trees T_{ij} as computed by the algorithm on this example are shown in the table below with their costs. The computation proceeds row by row (*i.e.* to compute the trees and costs in row d, all the trees up to row $d-1$ must have been previously computed). The optimal binary search tree $T_{1,5}$ is shown on the last row.

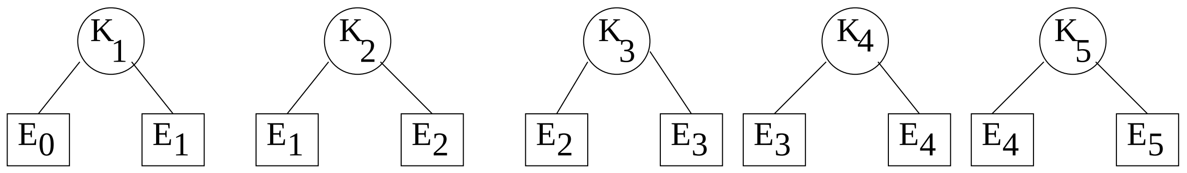
You are strongly encouraged to trace this example *carefully*. Even if you *think* you understand the algorithm from the previous abstract discussion, you may be surprised at how much better you'll understand it after working out an example.

d = -1 :



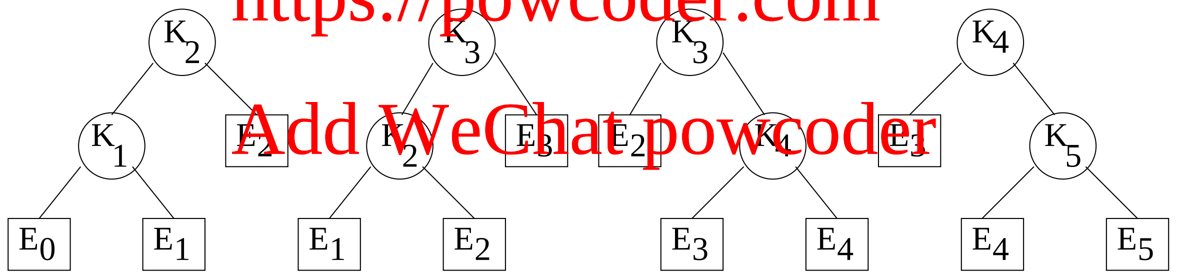
c(T_{10})=0 c(T_{21})=0 c(T_{32})=0 c(T_{43})=0 c(T_{54})=0 c(T_{65})=0

d = 0 :



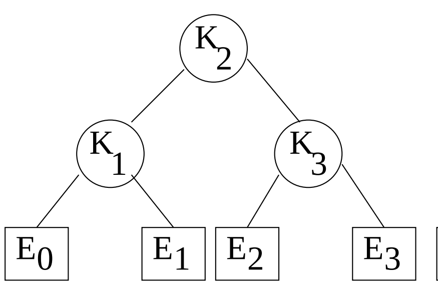
c(T_{11})=.2 c(T_{22})=.35 c(T_{33})=.45 c(T_{44})=.35 c(T_{55})=.2

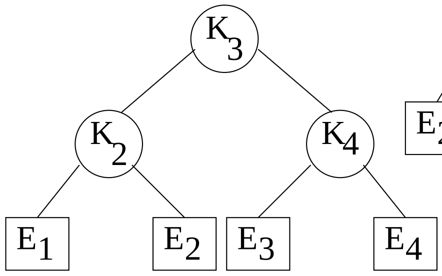
d = 1 :

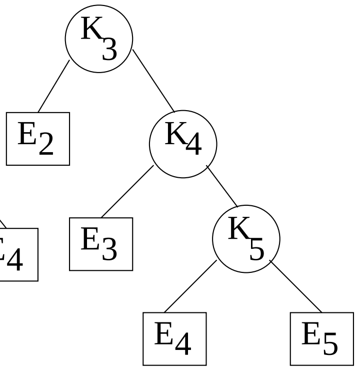


c(T_{12}) = .7 c(T_{23}) = .95 c(T_{34}) = .95 c(T_{45}) = .65

d = 2 :

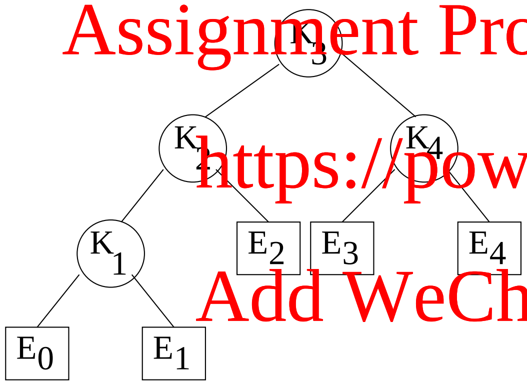


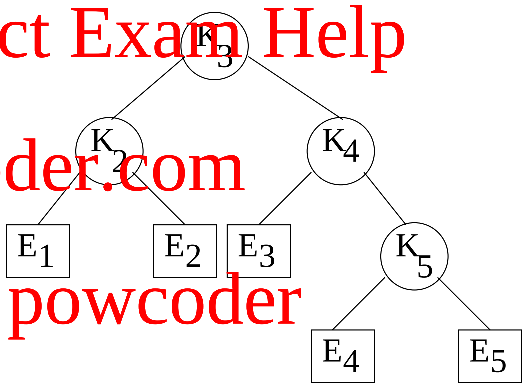




$c(T_{13}) = 1.4$ $c(T_{24}) = 1.45$ $c(T_{35}) = 1.35$

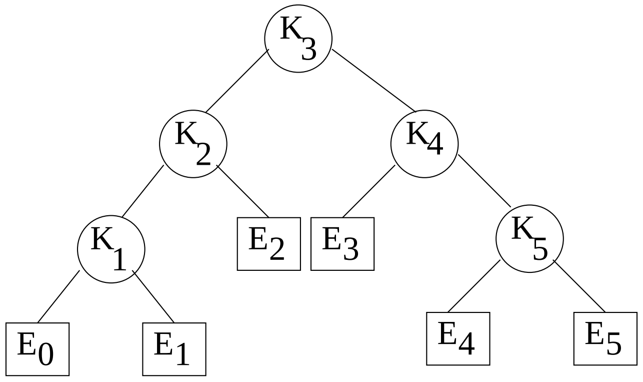
d = 3 :





$c(T_{14}) = 1.95$ $c(T_{25}) = 1.85$

d = 4 :



$c(T_{15}) = 2.35$