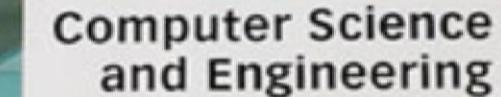


EECS 3101

Prof. Andy Mirzaian

A photograph of a red and white directional sign for York University's Computer Science and Engineering department. The sign is mounted on a metal pole and features the York University logo (a stylized 'U') at the top, followed by the text 'Computer Science and Engineering' and '120 Campus Walk'.

Computer Science
and Engineering

120 Campus Walk

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NP-Completeness

STUDY MATERIAL:

- [CLRS] chapter 34

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TOPICS

- Preliminaries
- Complexity Classes P, NP, co-NP, NP-complete
- Polynomial Assignment Project & Exam Completeness
- NP-Complete Problems
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Preliminaries

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Foundations & Limitations

Cantor's Continuum Hypothesis (CH):

There is no set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$.

True
or
False ?



1877	1903	1928	1937	1963
Georg Cantor posed CH	Russell wrote Frege about the Paradox	David Hilbert posed the Entscheidungsproblem (the decision problem on first-order logic)	Alan Turing published <i>On Computable Numbers with an Application to the Entscheidungsproblem</i>	Paul Cohen + Gödel proved CH is independent of axioms of set theory

Un-Computable Problems

No algorithm exists for certain computational problems, such as:

- **Diophantine Equations** [Diophantus of Alexandria, 3rd century AD]
Does a multi-variable polynomial with integer coefficients, e.g.,

$$x^3y^2z^2 + 7xy^4z^3 - 3x^2y^5z = 8,$$

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have an **integer** valued solution for its variables?

David Hilbert [1900]: Hilbert's 10th problem asked is this problem solvable.

Matiyasevich [1970] proved the unsolvability of general diophantine equations.

- **The Halting Problem** [Alan M. Turing, 1936]

– he was then a math student at Cambridge, England.

There were no **digital computers** or **programming languages!**

Arguably, these things came about *exactly because* of Turing's brilliant thoughts.

Other contributors: Emil Post [1920], Kurt Gödel [1930], Alonzo Church [1935].

- There are infinitely many uncomputable problems, e.g.,
Buffer Overflow, Post's Correspondence, ...

Computational Complexity Classes

- So, there are computationally unsolvable problems.
- Computable problems themselves have vastly different computational complexity.
- **Complexity Classes** classify problems by their computational complexity.

Computable
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$\text{EXP} = \text{exponential}$
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 $P = \text{polynomial}$

Uncomputable

Linear Programming

Given linearly constrained (in-)equalities & a linear objective function on many variables with integer coefficients, find **real** values for the variables that satisfy the constraints and optimize the objective function.

$$\begin{array}{ll}\text{minimize} & 3x + 5y - 2z \\ \text{subject to:} & 4x - y + 7z \leq 8 \\ & 2x + 4y = 3 \\ & 5y + 3z \leq 9 \\ & x \geq 0\end{array}$$

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- Computational Complexity of LP = polynomial [Leonid Khachyian 1979]
- LP is a versatile computational model for problem formulation.
 - Virtually all problems we have studied so far can be modeled as LP problems (with the exception of 0/1 Knapsack) and have polynomial time complexity, e.g., Max Flow, Min Cut, Shortest Paths, ...

Integer Linear Programming

Given linearly constrained (in-)equalities & a linear objective function on many variables with integer coefficients, find **integer** values for the variables that satisfy the constraints and optimize the objective function.

$$\begin{aligned} \text{minimize} \quad & 3x + 5y - 2z \\ \text{subject to:} \quad & 4x - y + 7z \leq 8 \end{aligned}$$

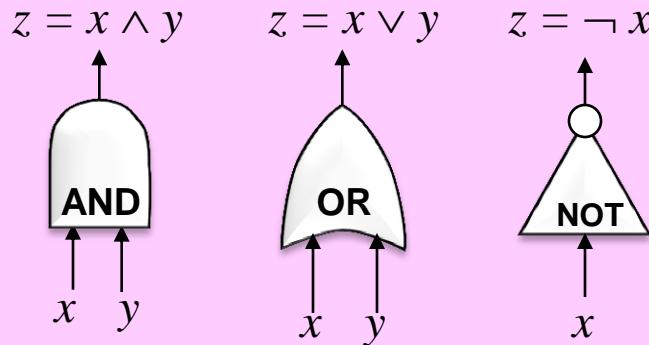
$$\begin{aligned} & 2x + 4y = 3 \\ & 5y + 3z \leq 9 \end{aligned}$$

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x, y, z are integers

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- Computational Complexity of ILP: polynomial or exponential ???
- ILP is at least as powerful and versatile as LP.
- **Search problems** can be cast as ILP, e.g.,
 - 0/1 Knapsack, Graph Coloring, Matching, Max Cut,
 - Traveling Salesman Problem, Boolean Formula Satisfiability,

Circuit SAT



Combinatorial Circuit
(a DAG of logical gates)

is satisfiable

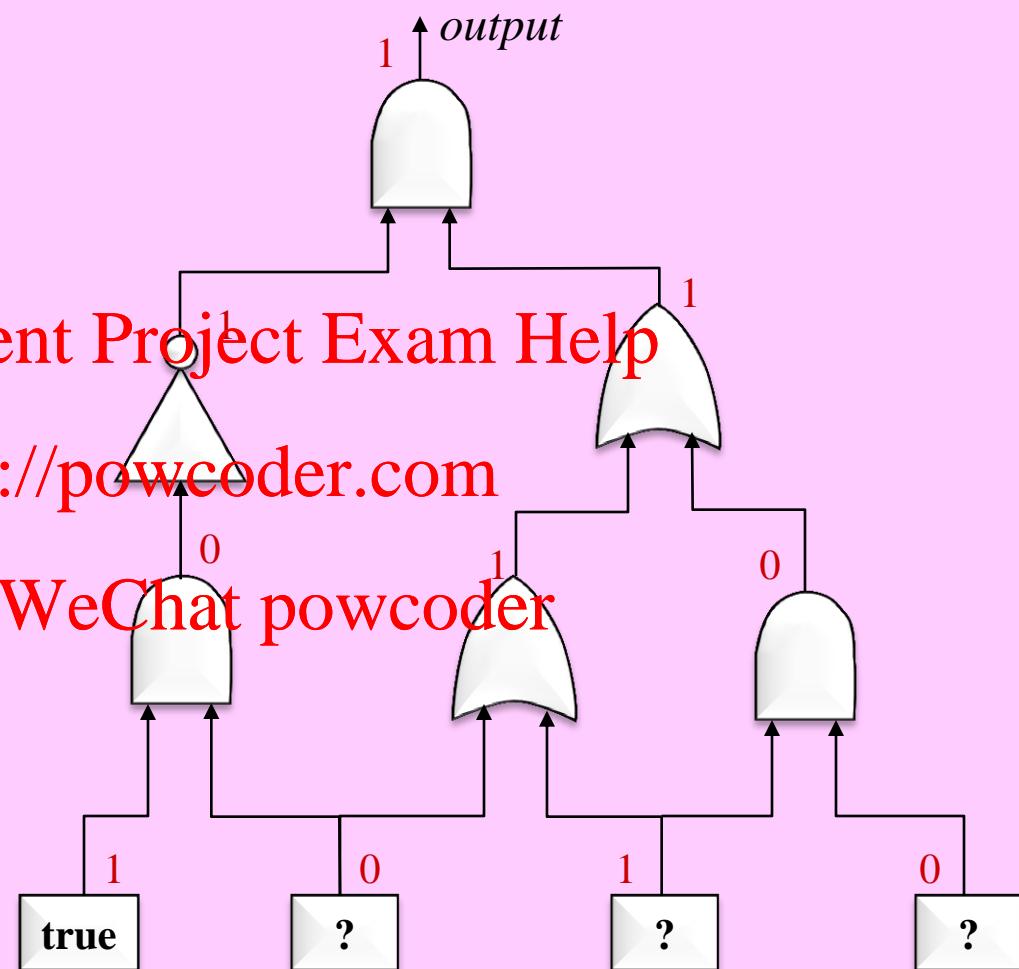
if and only if

there is a truth assignment
to its variable input gates
that makes the output true.

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For a given truth assignment,
evaluate gate outputs in topological order.

Boolean Formula SAT

$$\varphi = (x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (z \vee \bar{x}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (w \vee \bar{x} \vee z)$$

is an instance of **CNF-SAT** or **SAT** for short:

a Boolean formula in conjunctive normal form (CNF),

i.e., a conjunction (\wedge) of a number of **clauses** (the parentheses);

each clause is disjunction (\vee) of some **literals**

each literal is a Boolean **variable** or negation of a Boolean variable.

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- **k SAT**: instances are in k -CNF, i.e., each clause has $\leq k$ literals.
- **3SAT**: instances are in 3-CNF. (e.g., φ above)
- A Boolean formula is **satisfiable**, (Is φ satisfiable?)
if there is a truth assignment to its variables that makes the formula true.
- A Boolean formula is a **tautology**, (Is φ a tautology?)
if every truth assignment of its variables makes the formula true.
(Negation of a tautology is unsatisfiable.)

2SAT $\in \mathbf{P}$

Input: a boolean 2-CNF formula Φ .

Question: Is Φ satisfiable?

Step 1:

Idea: $(x \vee y) \equiv (\bar{x} \Rightarrow y) \equiv (\bar{y} \Rightarrow x)$

Construct the directed graph $G = (V, E)$:

$V = \{x, \bar{x} \mid x \text{ is a variable in } \Phi\}$.

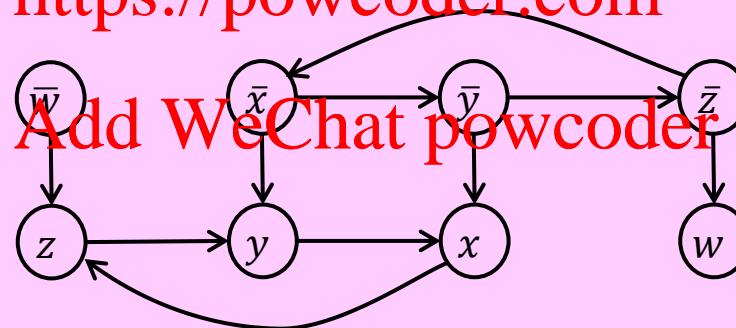
$E = \{ \bar{x} \rightarrow y, \bar{y} \rightarrow x \mid (x \vee y) \text{ is a clause in } \Phi \}$

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Example: $\Phi = (x \vee y) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (z \vee \bar{x}) \wedge (w \vee z)$

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$G:$



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Question: Can we assign T or F labels to vertices of G such that

- no opposite literal pair (i.e., x and \bar{x}) is labeled the same, and
- no edge has its incident vertices labeled as: $T \rightarrow F$?

2SAT $\in \mathbf{P}$

CLAIM: Φ is satisfiable \Leftrightarrow No SCC of G contains a variable and its negation.

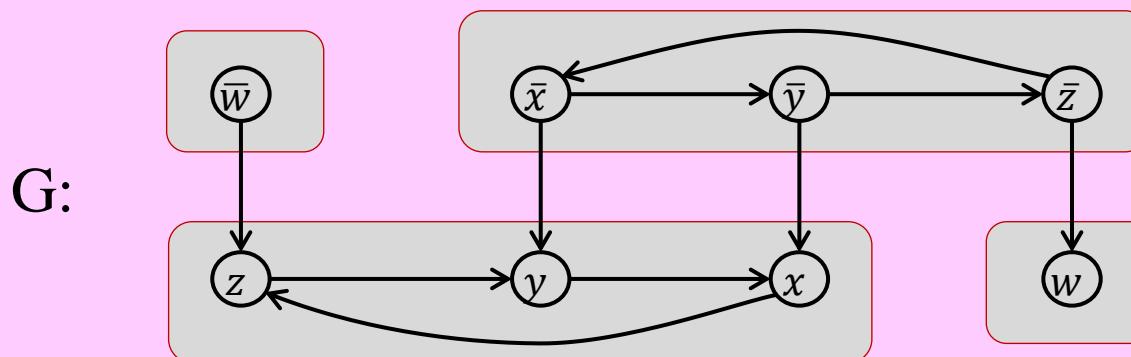
- **Proof sketch:** See Exercise 10.

Proof of $[\Rightarrow]$: In a ~~Assignment Project Exam Help~~ in the same cycle, hence in the same SCC, must have the same truth value (all true or all false).

Proof of $[\Leftarrow]$: In the SCC component DAG, if an SCC node has in-degree 0, its “negated” SCC node must have out-degree 0. Why?

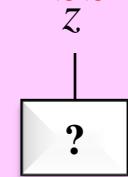
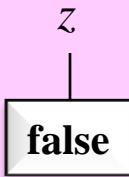
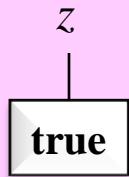
What truth values would you assign to such a pair of SCC nodes?

Adapt the SCC algorithm to get a full truth assignment in linear time.



Circuit SAT → SAT

Give the output of each gate a distinct name.
Convert each gate to an equivalent small CNF-SAT.
Then take their conjunction, including the circuit output.



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$$(z)$$

$$(\bar{z})$$

$$(\bar{z} \vee x)$$

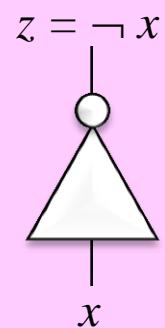
$$(\bar{z} \vee y)$$

$$(z \vee \bar{x} \vee \bar{y})$$

$$(z \vee \bar{x})$$

$$(z \vee \bar{y})$$

$$(\bar{z} \vee x \vee y)$$



Solution Construction vs Verification

Circuit-SAT, SAT, ILP, LP ... are (Combinatorial) Search Problems.

Search Problem. Instance I:

Construction: given I, output a valid solution S for instance I.

Verification: given (I, S), verify that S is a valid solution for instance I.

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Obvious: Construction is at least as hard as Verification.

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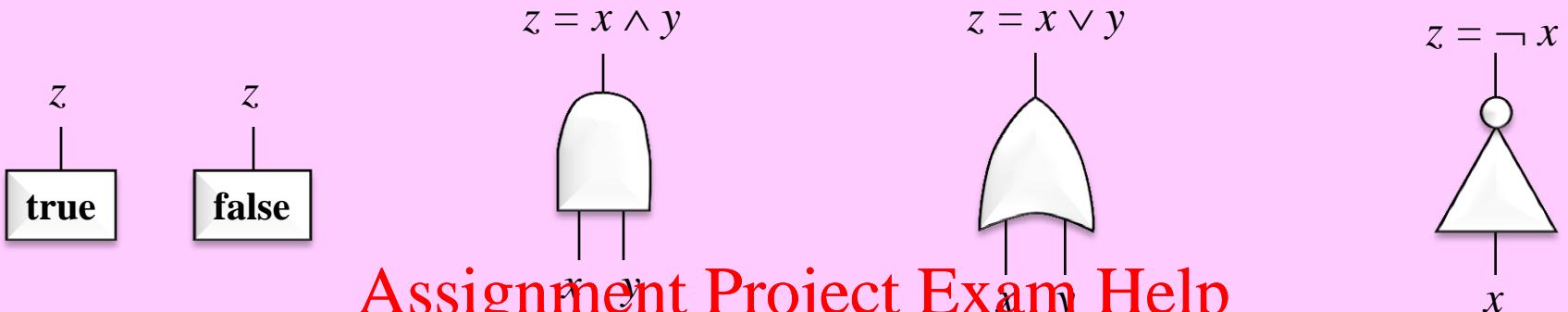
Question: Is Construction strictly harder than Verification?

Example: Consider establishing a Mathematical Fact.

Construction: Provide a Proof.

Verification: Given a “proof”, verify its validity.

Circuit SAT Verification \longrightarrow LP



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$$z = 1$$

$$z = 0$$

$$\frac{z \leq x}{z \leq y} \quad \frac{z \geq x}{z \geq y}$$

$$\frac{z \geq x + y - 1}{z \leq x + y}$$

$$z = 1 - x$$

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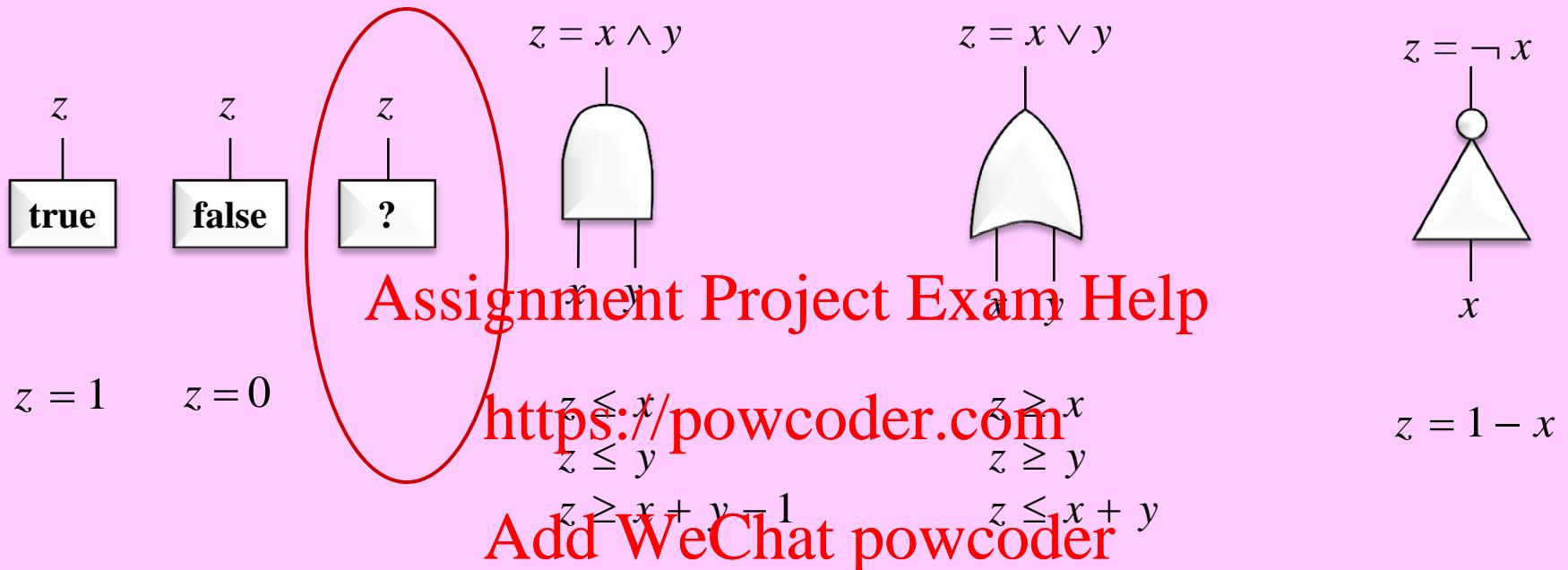
Plus, for all gate-output variables z , add the constraints:

$$0 \leq z \leq 1.$$

Objective: Maximize z_{OUT} (the circuit output).

Now we have an instance of LP

Circuit SAT Construction \longrightarrow ILP



Plus, for all gate-output variables z , add the constraints:

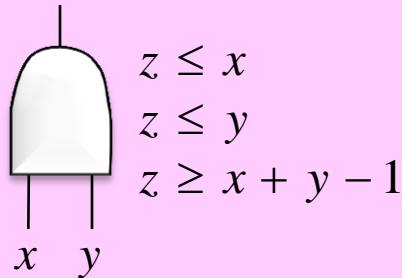
$0 \leq z \leq 1$, and z is an integer.

Objective: Maximize z_{OUT} (the circuit output).

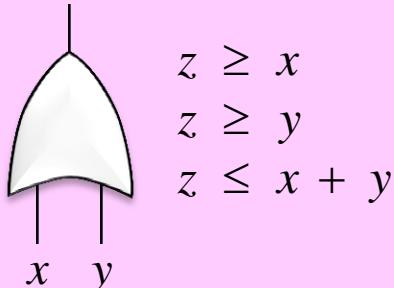
Now we have an instance of ILP.

Why ILP and not LP?

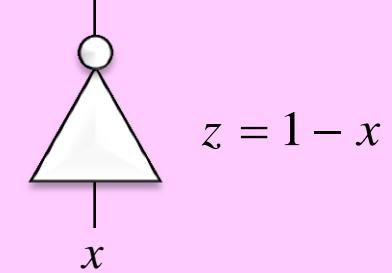
$$z = x \wedge y$$



$$z = x \vee y$$



$$z = \neg x$$



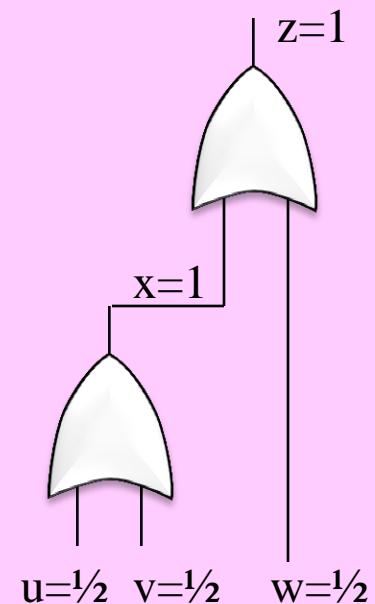
Plus the constraints $0 \leq z \leq 1$, for every variable z.

Integrality constraints are essential.
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Consider the circuit for an ~~And~~ satisfiable 3SAT instance.
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 Ignore the integrality constraints and assign the value $\frac{1}{2} = 1 - \frac{1}{2}$ to each variable and its negation.

Part of the circuit for an arbitrary clause $(u \vee v \vee w)$:

Conjunction of all these “1” clause outputs will be a “1” !



Is the circuit satisfiable or not ?!

Construction HARD Verification EASY ???

EASY

Linear Programming

Search Problem Verification

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3SAT Verification

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Integer Linear Programming

HARD

Search Problem Construction

3SAT Construction

Complexity Classes:

P, NP, co-NP, NP-complete

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Optimization vs Decision Problems

Optimization Problem: output is an optimum solution to the input instance

Shortest Path:

Instance: $\langle G, s, t \rangle$

Output: Shortest path in graph G from vertex s to t .

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Decision Problem: <https://powcoder.com>

the output is a “yes” or “no” answer.

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Decision version of Shortest Path:

Instance: $\langle G, s, t, K \rangle$

Question: Does graph G have a path from vertex s to t with length at most K ?

For a maximization problem, “at most” is replaced by “at least”.

Why Decision Problems ?

1. Optimization version of a problem is at least as hard as its decision version.

Example: To answer whether G has a path of length at most K from s to t, we can find the shortest path from s to t and compare its length with K.

2. So, if we can establish the fact that the decision version is hard, that would imply that the optimization version must also be hard.

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3. With uniform “yes/no” output type, it is more convenient to transform (or reduce) one decision problem to another, as we shall see.

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4. The purpose of these reductions is to establish lower-bound (as was done in the Sorting/Selection Slides), rather than obtaining an algorithmic upper-bound (as was done by reducing bipartite matching to max-flow).

5. By the way, the converse of (2) is usually true as well:

If the decision version is easy, so is the optimization version.

Example: with integer numeric input, do binary search on K in the decision version to find the shortest path.

The Complexity Class P

P = Deterministic **Polynomial** time:

the class of problems that admit a deterministic algorithm whose running time is $O(n^d)$ for some constant d, where n is the input size.

[Alan Cobham 1964] [Jack Edmonds 1965]

A problem is **tractable** (easy) if it belongs to P,
otherwise it is **intractable** (hard).

Justification (polynomial vs exponential):

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- P is preserved under many important computational models, e.g., RAM, Turing Machine, etc.
- Polynomials are the smallest class that contain linear functions & are closed under: composition, addition, multiplication.
- P scales nicely (multiplicative rather than additive)
- Dilemma: $O(n^{1000})$ vs $O(1.1^n)$! Explain.

The Complexity Class NP

NP = Non-deterministic Polynomial time:

the class of decision problems that admit a non-deterministic polynomial time algorithm.

Decision problem **A**ssignment Project Exam Help

Decision language of A: $L_A = \{ \langle I \rangle \mid I \text{ is a “yes” instance of problem } A \}$
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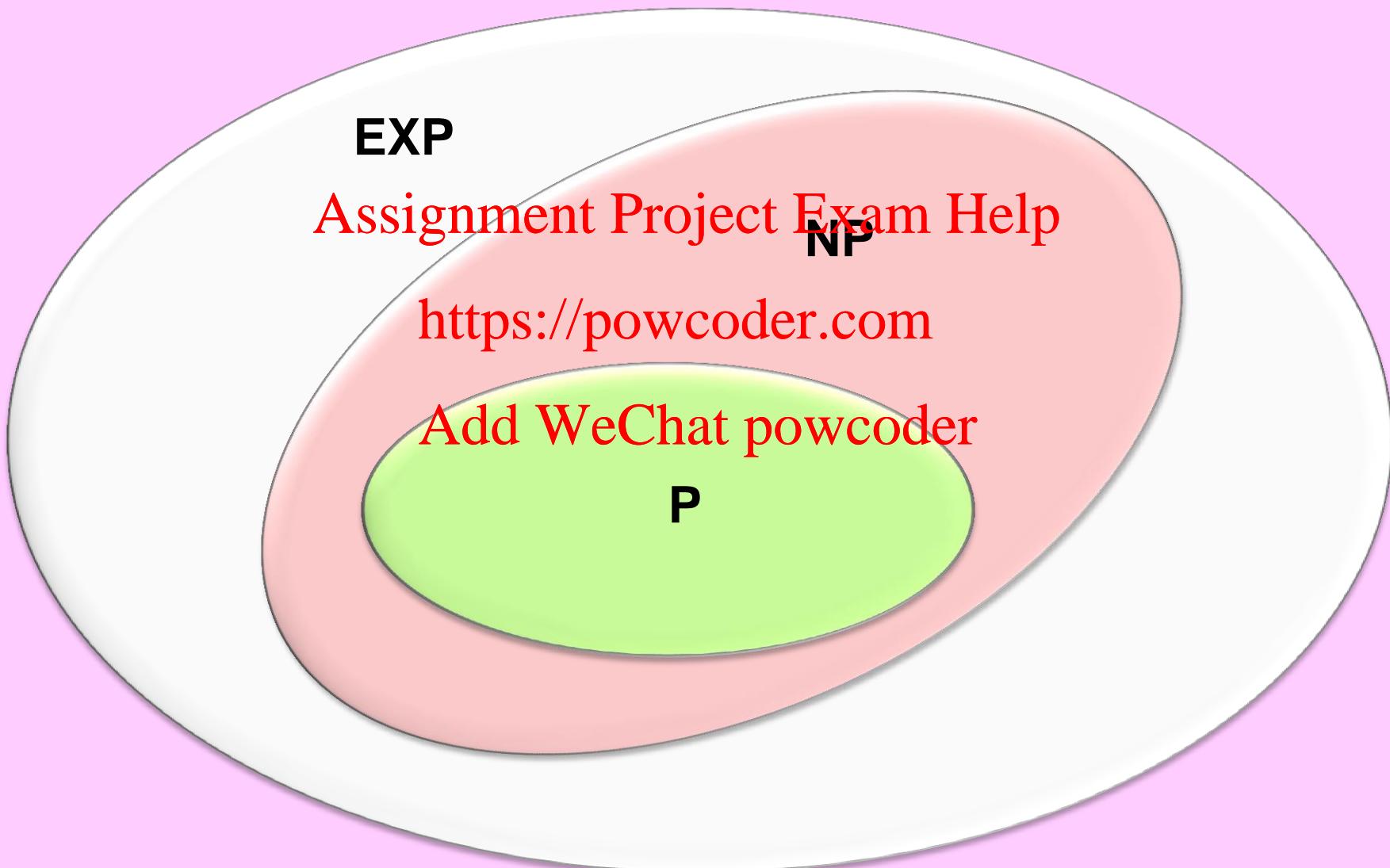
$A \in NP$

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- $\Leftrightarrow L_A = \{ \langle I \rangle \mid ALG(I) = \text{“yes”} \}$
for some non-deterministic polynomial time algorithm ALG
(ALG gives no termination guarantee on “no” instances!)
- $\Leftrightarrow L_A = \{ \langle I \rangle \mid \exists \text{ solution certificate } S, |S| \in \text{poly}(|I|), Verify(I, S) = \text{“yes”} \}$
for some deterministic algorithm $Verify$ with running time $\in \text{poly}(|I|)$.
(Note: the “guessed” solution certificate S cannot be too large.
It must satisfy: $|S| \in \text{poly}(|I|)$. See Exercise 9 at the end of these slides.)

P, NP, EXP

“P vs NP” \equiv “polynomial time Construction vs Verification”



The \$1,000,000 Question

- $P \subseteq NP$
- [Steven Cook, 1971]: Is $P = NP$ or $P \neq NP$?
This has turned out to be one of the most challenging open problems in all of mathematics and computer science!
- At the turn of the 20th century, The Clay Mathematics Institute posted 7 “Millenium Problems” with an award of \$1,000,000 for the solution of any one of them. [The “P vs NP” question is one of them.](https://powcoder.com)
- $P \neq NP \Leftrightarrow$ there exist problems in $NP - P$.
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The quest in search of such problems led to the question
“what are the hardest problems in NP?”
This gave rise to the class of **NP-complete** problems; the hardest in NP.
- [Steven Cook, 1971] discovered the first NP-complete problem: SAT
- [Richard Karp, 1972] published a long list of other NP-complete problems.
These were seemingly intractable (?) highly researched problems, with important applications in science technology and engineering.
- By now there are thousands of published NP-complete problems.

Search/OPT Problems: HARD vs EASY

Hard Problems (NP-complete) Easy Problems (in P)

Integer Linear Programming

3SAT

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Minimum Spanning Path

Longest Path

3D Matching

0-1 Knapsack

Independent Set

Hamiltonian Cycle

Max Cut

Linear Programming

2SAT

Minimum Spanning Tree

Shortest Path

Matching

Fractional Knapsack

Independent Set on trees

Eulerian Cycle

Min Cut

Some (unresolved) exceptions:

- Graph Isomorphism \leftarrow quasi-poly time [L. Babai, STOC2016]
- Integer Factoring \leftarrow poly-time quantum alg. [P. Shor, FOCS1994]

Complements of P and NP

$$\text{co-P} = \{A \mid (\text{complement of } A) = \bar{A} \in P\}$$

$$\text{co-NP} = \{A \mid (\text{complement of } A) = \bar{A} \in \text{NP}\}$$

P is closed under complementation [$P = \text{co-P}$]:

$$\forall A: A \in P \Leftrightarrow \bar{A} \in P$$



NP is not known to be closed under complementation.

$$P = \text{co-P}, \quad P \subseteq \text{NP}, \quad \text{co-P} \subseteq \text{co-NP}.$$

$$P \subseteq NP \cap co-NP$$



Open Questions:

$P = NP ?$

$P = NP \cap co-NP ?$

$NP = co-NP ?$

Polynomial Reducibility

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&

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NP-Completeness
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Hardest Problems in NP

- NP-complete = the class of hardest problems in NP.
- How do we show that a problem $A \in \text{NP}$ is NP-complete, i.e., A is at least as hard as any other problem in NP ?
- That is, if A is tractable, then **every** problem in NP is also tractable.
In other words: $A \in P \Rightarrow \text{NP} = P$.
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- **Polynomial Reducibility**, denoted “ \leq_P ” is the tool to use.
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Polynomial Reducibility

$A \leq_P B$: Problem A is polynomially reducible to Problem B,

if there is a polynomial-time computable transformation f from instances of A to instances of B that preserves membership:

1. $I \in A \Leftrightarrow f(I) \in B$, and
2. $f(I)$ can be computed in $\text{poly}(|I|)$ time.
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NOTE:

- the direction of reduction is from A to B (not the reverse).
- $I \in A \Rightarrow f(I) \in B$, &
- $I \notin A \Rightarrow f(I) \notin B$, &
- \exists a reduction algorithm that computes $f(I)$ in $\text{poly}(|I|)$ time, &
- length of instance $f(I)$ is also $\text{poly}(|I|)$.

Polynomial Reducibility

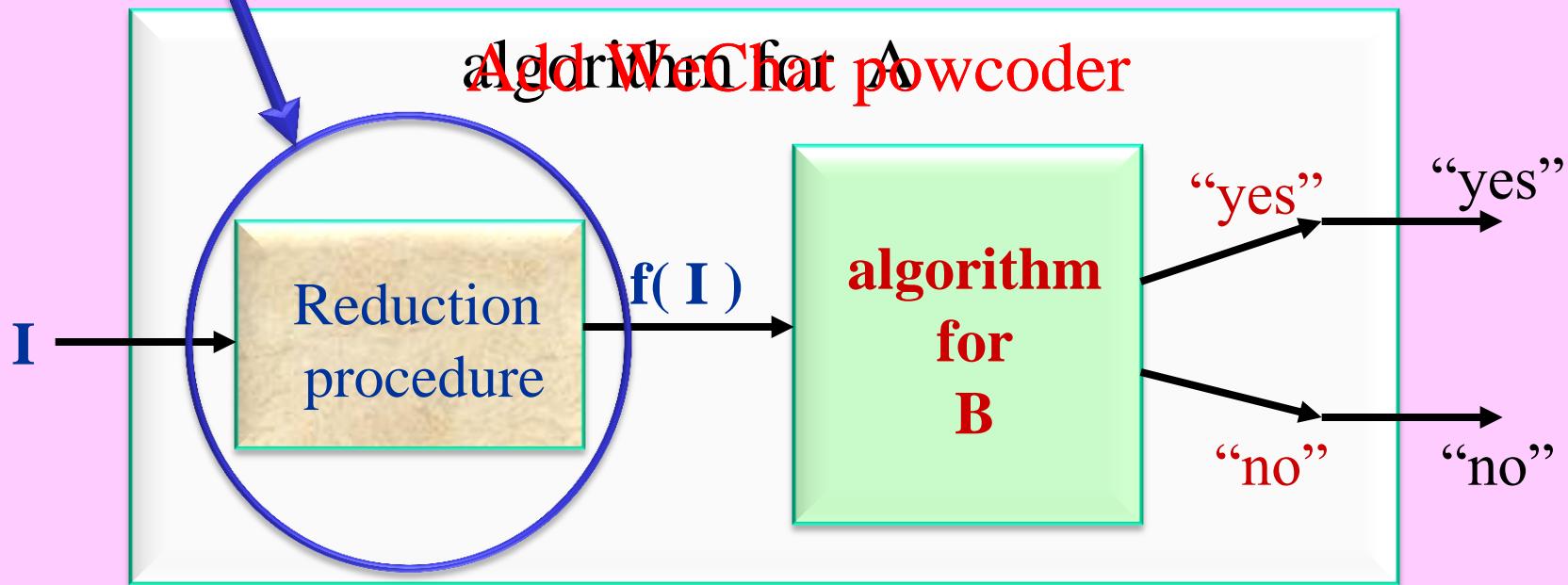
$A \leq_P B$

says:

A is not harder than B (to within a polynomial), &
B is at least as hard as A (the contra-positive)

- $B \in P \Rightarrow A \in P$
- $A \notin P \Rightarrow B \notin P$.

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NP-hard & NP-complete

B is NP-hard if every problem in NP polynomially reduces to B:

$$\forall A \in NP: A \leq_P B$$

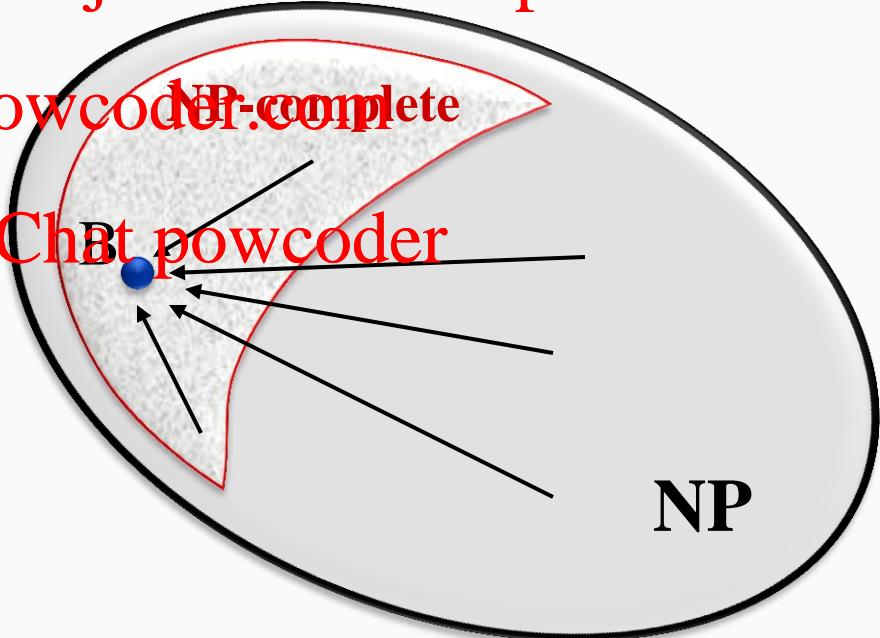
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B is NP-complete if

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1. $B \in NP$, and
2. B is NP-hard.



Circuit SAT is NP-complete

This is our first problem to be proved NP-complete.

We must show two things:

1. Circuit SAT \in NP,
2. Circuit SAT is NP-hard: $\forall A \in \text{NP}: A \leq_P \text{Circuit SAT}$.

Proof of 1. Circuit SAT \in NP: Assignment Project Exam Help

We have already seen this.

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An instance of the problem is a combinatorial circuit
of size n (# wires and gates)

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A certificate is a truth assignment to the input gates.

In time $\text{poly}(n)$ (in fact **in $O(n)$ time**) we can deterministically **verify**
whether the given certificate satisfies the circuit output gate:

evaluate gate outputs in topological order.

Circuit SAT is NP-complete

This is our first problem to be proved NP-complete.

We must show two things:

1. Circuit SAT \in NP,
2. Circuit SAT is NP-hard: $\forall A \in \text{NP}: A \leq_P \text{Circuit SAT}$.

Proof of 2 [sketch] Circuit SAT is NP-hard: **Assignment Project Exam Help**

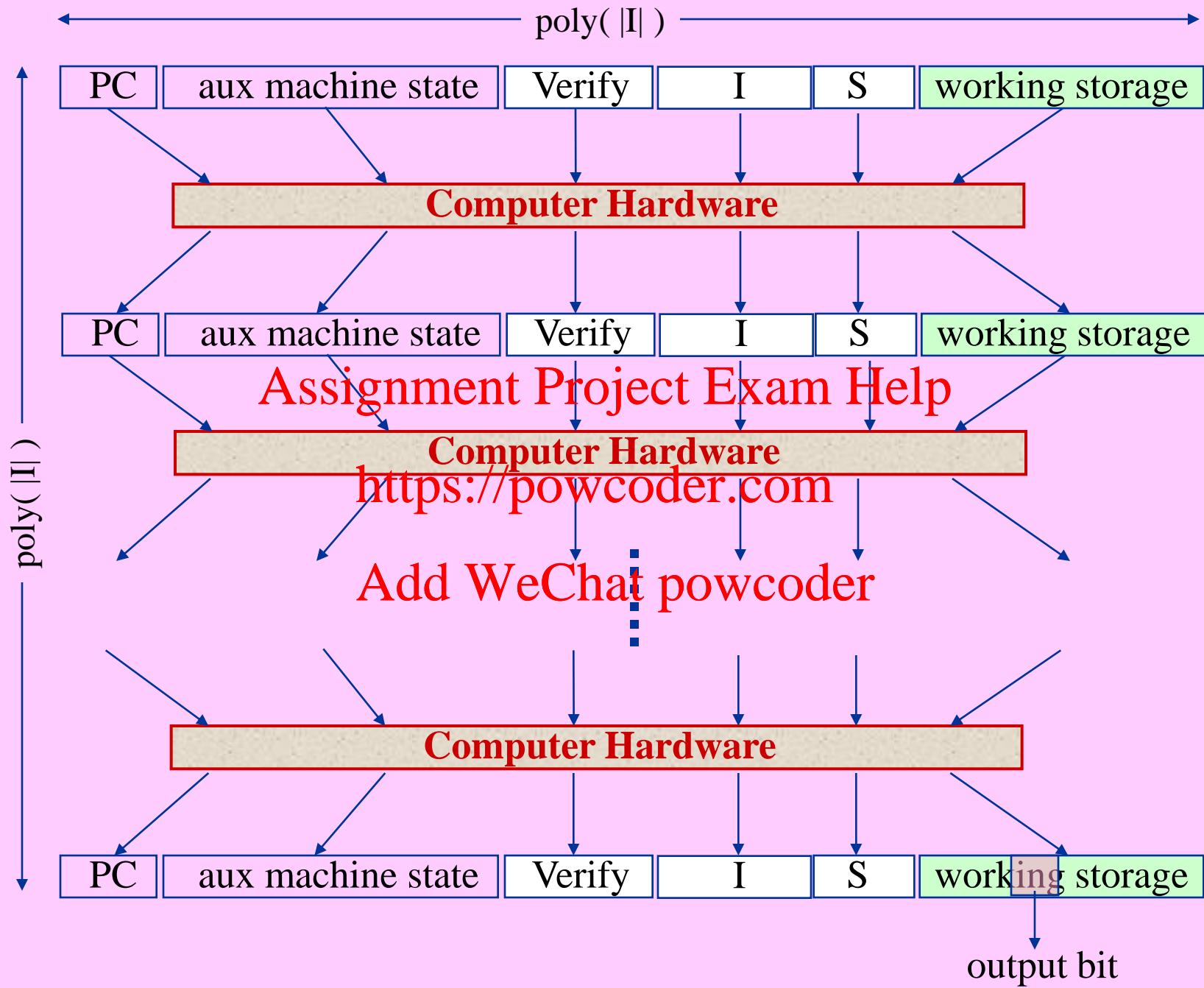
Consider an arbitrary problem $A \in \text{NP}$.

\exists a deterministic algorithm $\text{Verify}(I, S)$ that for any certificate S (encoded in binary) & $|S| = \text{poly}(|I|)$, runs in $\text{poly}(|I|)$ time and verifies whether S is a valid solution for instance I of problem A .

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Reduction Procedure: transform $\text{Verify}(I, S)$, for any given instance I , to a circuit C with the following properties:

- a) The reduction procedure constructs C in $\text{poly}(|I|)$ time,
- b) C has input bits I and S , where the I bits are given and fixed (0 or 1), and the S bits are unknown variables (? bits),
- c) $\text{Verify}(I, S) = \text{"yes"} \Leftrightarrow$ the truth assignment S to the variable input bits satisfies C . (P.T.O.)



Other NP-complete Problems

There is a shortcut on how to prove other problems are NP-complete.

FACT 1: \leq_P is transitive.

Proof: $A \leq_P B$ and $B \leq_P C \Rightarrow A \leq_P C$

(because polynomials are closed under composition)

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$A \leq_P B : x \in A \Leftrightarrow f(x) \in B$

for some $f : f(x)$ is computable in $O(n^d)$ time, $n = |x|$ & some constant d .

$B \leq_P C : x \in B \Leftrightarrow g(x) \in C$

for some $g : g(x)$ is computable in $O(n^c)$ time, $n = |x|$ & some constant c .

↓

$A \leq_P C : x \in A \Leftrightarrow h(x) \in C$

for $h(x) = g(f(x))$ computable in $O(n^{dc})$ time, $n = |x|$ & the constant dc .

Other NP-complete Problems

There is a shortcut on how to prove other problems are NP-complete.

FACT 1: \leq_P is transitive.

FACT 2: Suppose $A \leq_P B$. Then

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A is NP-hard $\Rightarrow B$ is NP-hard.

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Proof: $[\forall C \in NP: C \leq_P A]$ and $[A \leq_P B] \Rightarrow [\forall C \in NP: C \leq_P B]$

How to prove a new problem B is NP-hard:

Select a known NP-complete problem A and show $A \leq_P B$.

If you have choices, pick a problem A that “resembles” B as much as possible to facilitate an easier reduction.

NP-completeness & P vs NP ?

FACT 3: The following statements are equivalent:

- (1) $P = NP$,
- (2) All NP-complete problems are in P,
- (3) Some NP-complete problem is in P.

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Proof: (1) \Rightarrow (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (1): $B \text{ is NP-hard} \Rightarrow [\forall A \in NP: A \leq_P B]$

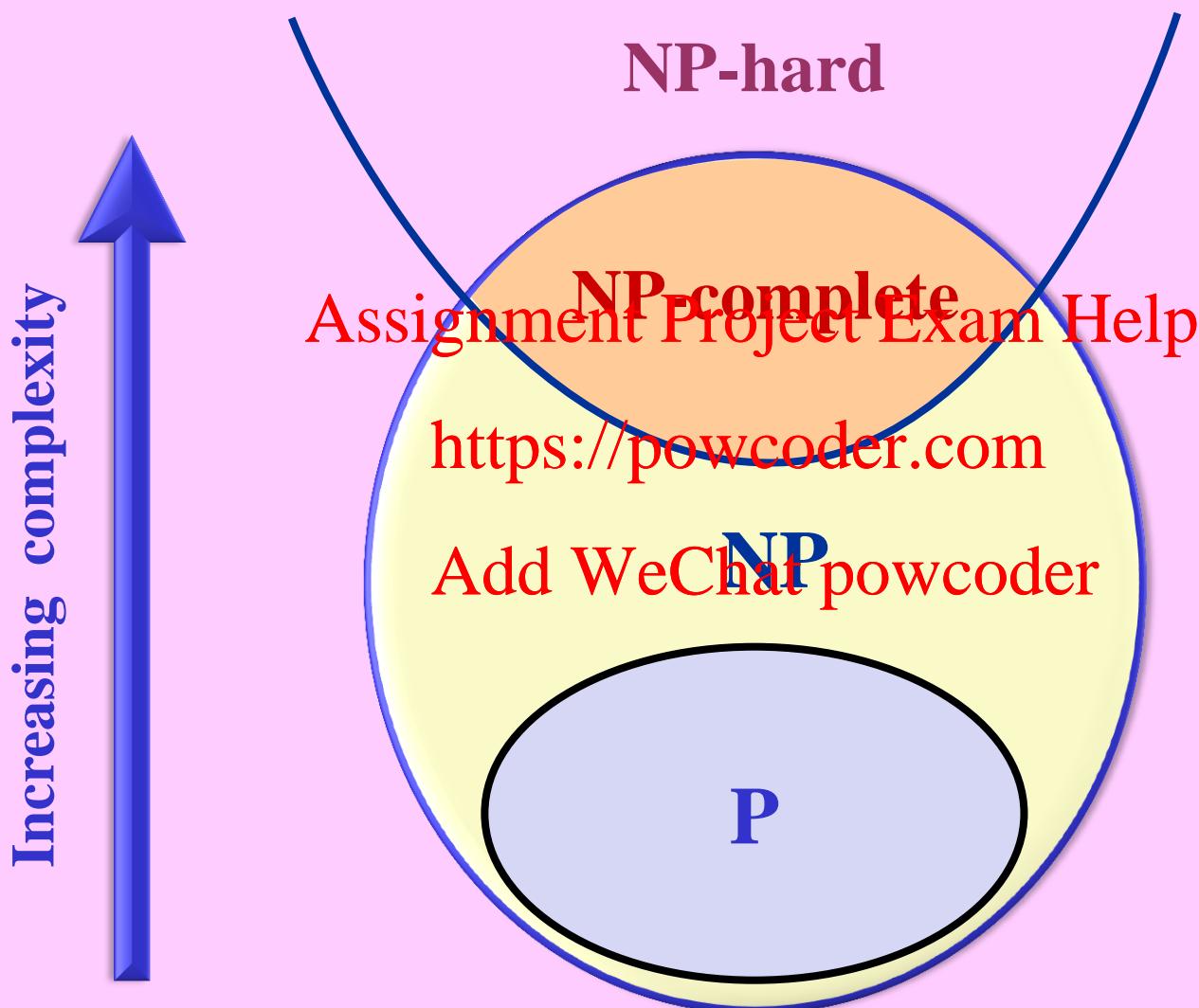
$A \leq_P B$ and $B \in P \Rightarrow A \in P$.

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If a single NP-complete problem is tractable, then all are!

This is a **strong but inconclusive** evidence that $P \neq NP$,
since despite decades of intense research by experts,
no polynomial-time algorithm has been found for any one
of a multitude of well known NP-complete problems.

If $P \neq NP$

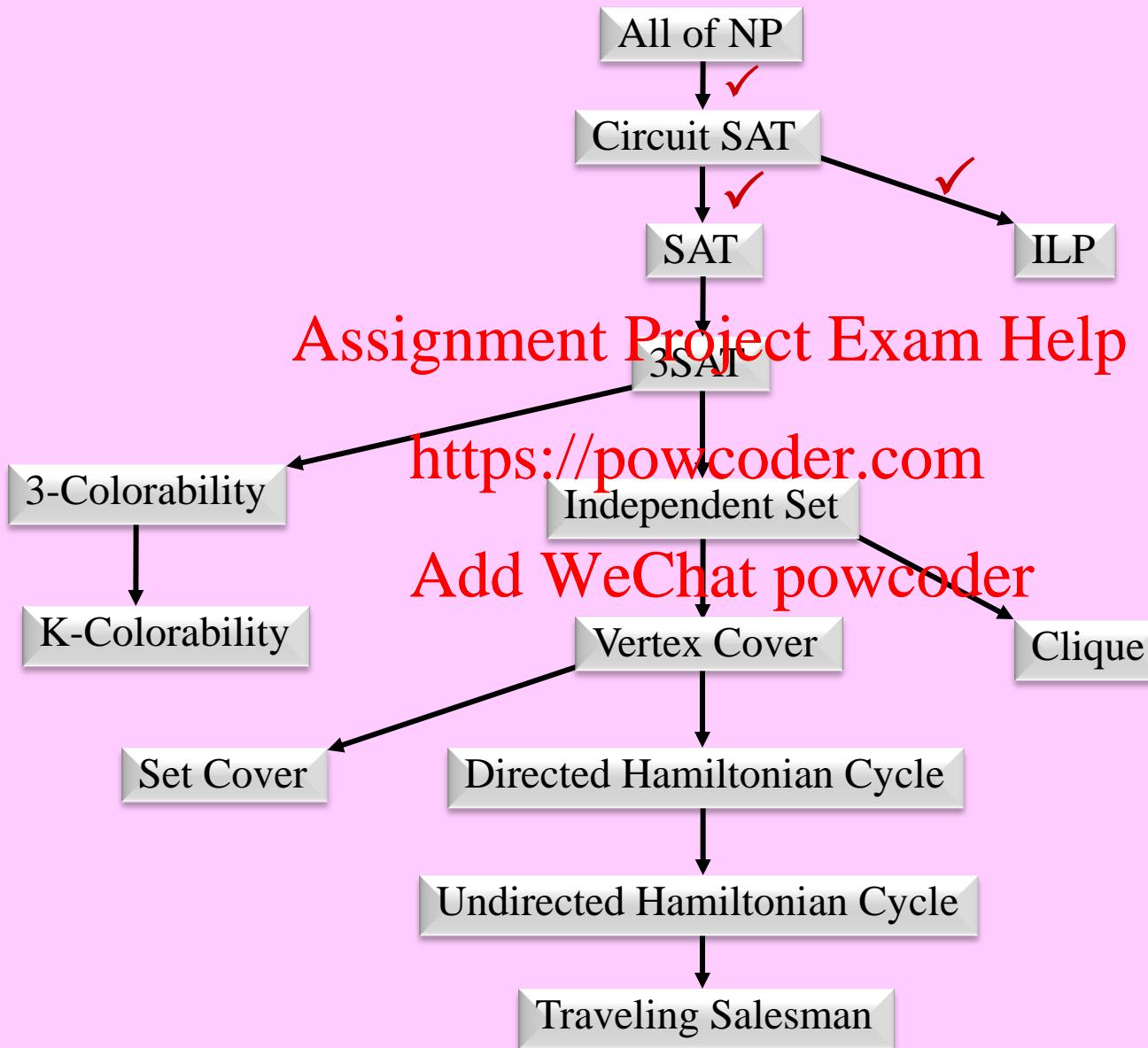


Additional Assignment Project Exam Help **NP-Complete** <https://powcoder.com>

Problems

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Some Reductions



These are all in NP

To prove a problem is NP-complete, we must show 2 things:

- (1) It is in NP (i.e., it has a polynomial-time verification algorithm)
- (2) It is NP-hard (the indicated reductions will show this)

(1) The listed problems are all in NP. (All except ILP are easy to show.)

Some examples:

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3SAT ∈ NP:

Verify that a given truth assignment satisfies the 3SAT instance:
<https://powcoder.com>
Done in linear time by evaluating each clause.

3-Colorability ∈ NP: Add WeChat powcoder

Given a coloring of vertices of a graph, verify that at most 3 colors are used, and verify that for each edge, its two incident vertices have different colors.

Undirected Hamiltonian Cycle ∈ NP:

Given a permutation certificate of vertices of a graph, verify that it forms a Hamiltonian cycle, i.e., a cycle in the graph that visits each vertex exactly once: Check that the given certificate is indeed a permutation, and that there is an edge in the graph between each successive pair of vertices in the permutation (cyclically).

SAT \leq_p 3SAT

In **polynomial time** transform an instance of SAT to an instance of 3SAT:
Convert each clause with $k > 3$ literals to $k-2$ clauses, each with 3 literals,
using some new Boolean variables:

$$\varphi = (a_1 \vee a_2 \vee \dots \vee a_k)$$



$$\psi = (a_1 \vee a_2 \vee x_1)(\bar{x}_1 \vee a_3 \vee x_2)(\bar{x}_2 \vee a_4 \vee x_3) \dots (\bar{x}_{k-4} \vee a_{k-2} \vee x_{k-3})(\bar{x}_{k-3} \vee a_{k-1} \vee a_k)$$

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CLAIM: φ is satisfied $\Leftrightarrow \exists$ truth assignment to x_i 's for which ψ is satisfied.

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Proof of [\Rightarrow]: If φ is satisfied, then some a_i must be true.

Set x_1, \dots, x_{i-2} to true and the rest false. That satisfies ψ .

Proof of [\Leftarrow]: If ψ is satisfied, then at least one a_i must be true.

Otherwise, the 1st clause of ψ forces x_1 to be true,

then the 2nd clause forces x_2 to be true, ...

eventually falsifying the last clause of ψ ; a contradiction.

3-Colorability

The 3-Colorability Problem:

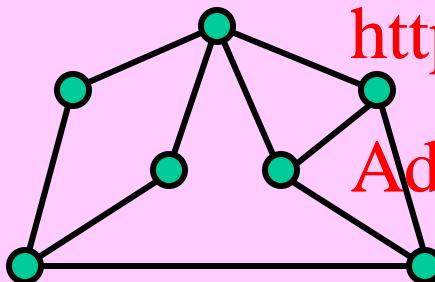
Given a graph G ,

is it possible to color each vertex of G by one of 3 colors (say red, blue, green) such that no edge of G has its both end vertices colored the same?

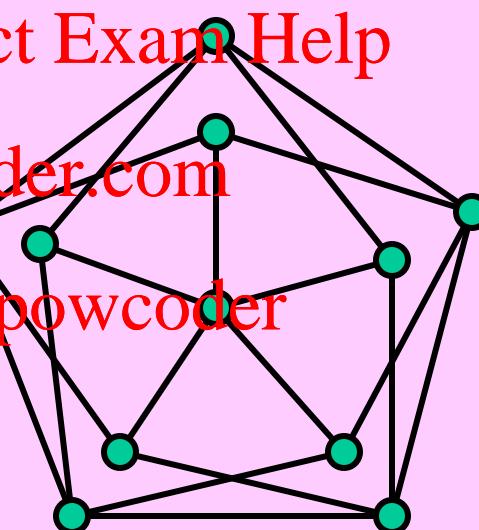
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3-colorable



Not 3-colorable

3SAT \leq_P **3-Colorability**

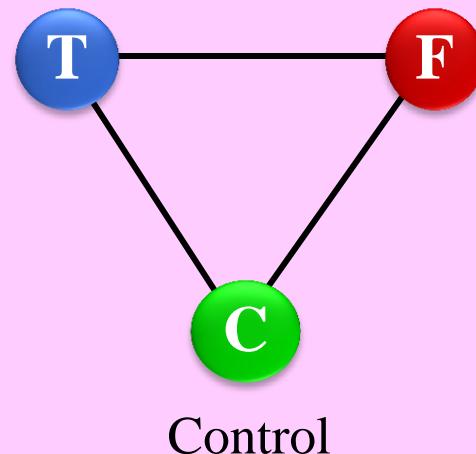
Reduction: Given any 3-CNF-SAT formula Φ , in time polynomial in $|\Phi|$, transforms Φ to a graph G such that

Φ is satisfiable $\Leftrightarrow G$ is 3-colorable.

G will be constructed from Φ
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by assembling suitable pieces together

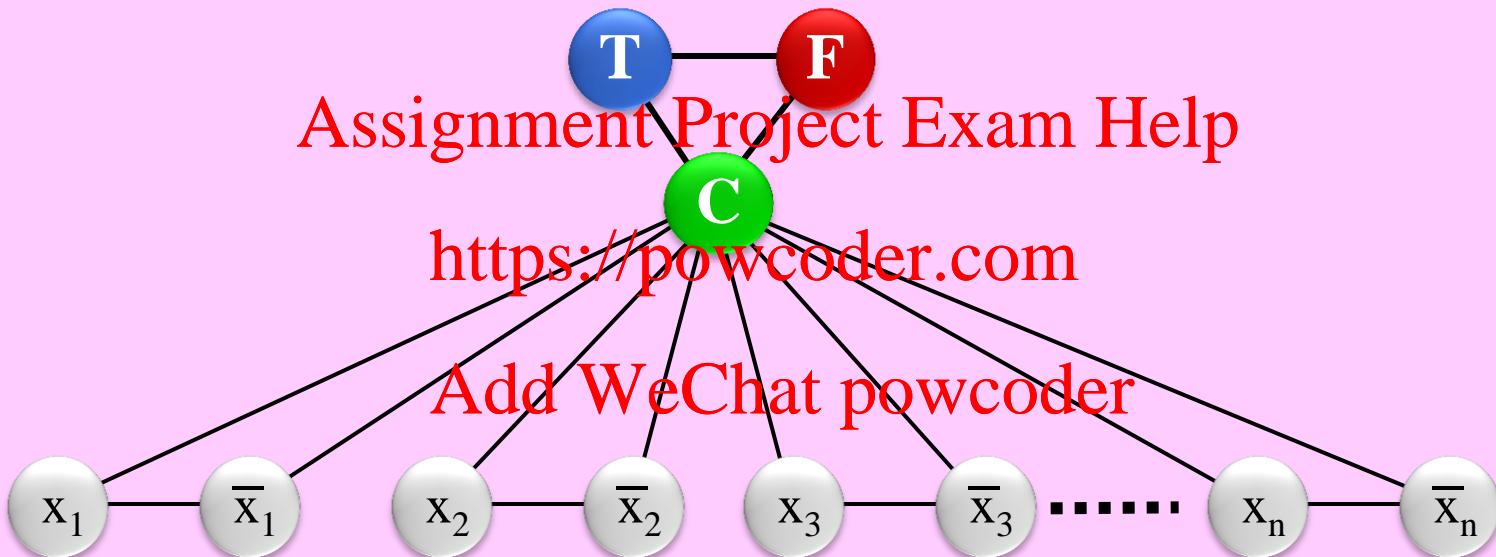
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STEP 1: The control assembly
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3SAT \leq_P 3-Colorability

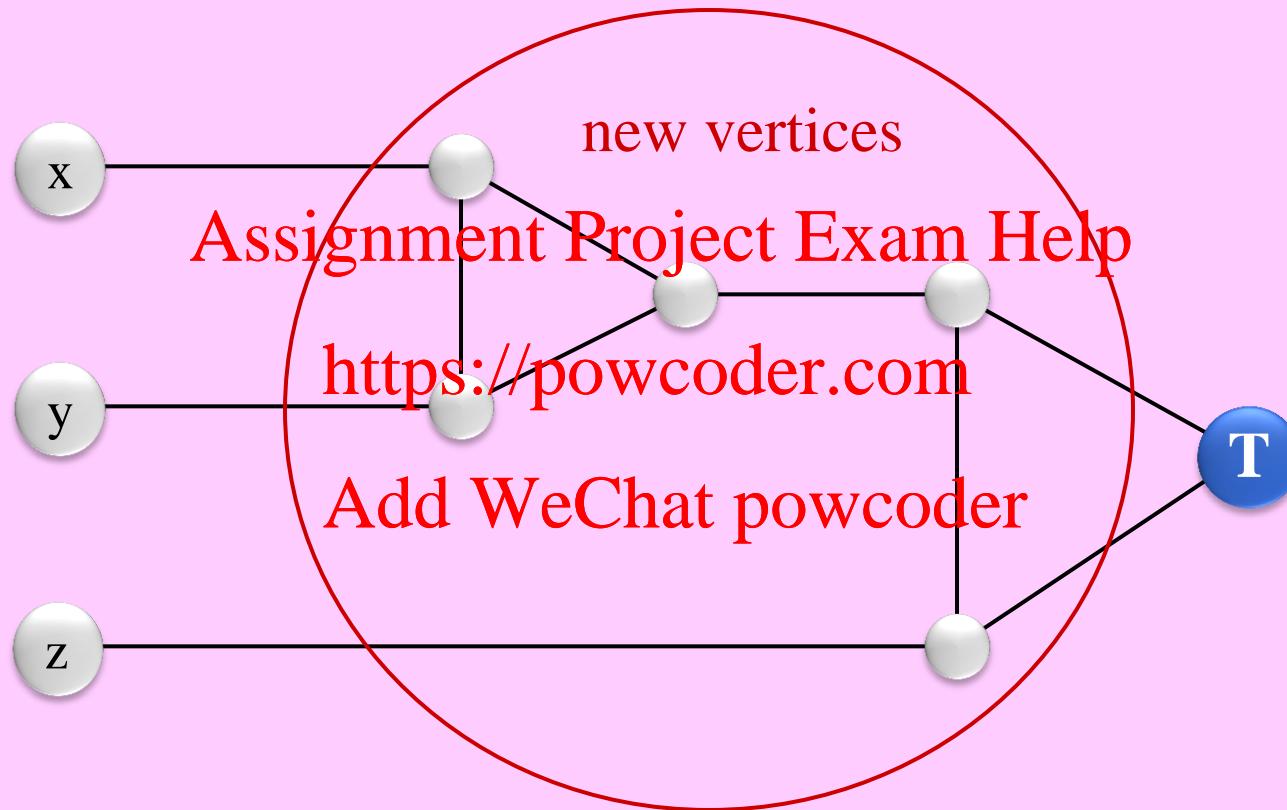
STEP 2: The assembly of Boolean literals



Each literal is forced a color T or F and its negation the opposite (F or T).
This forces a truth assignment.

3SAT \leq_P 3-Colorability

STEP 3: The assembly for each clause $(x \vee y \vee z)$

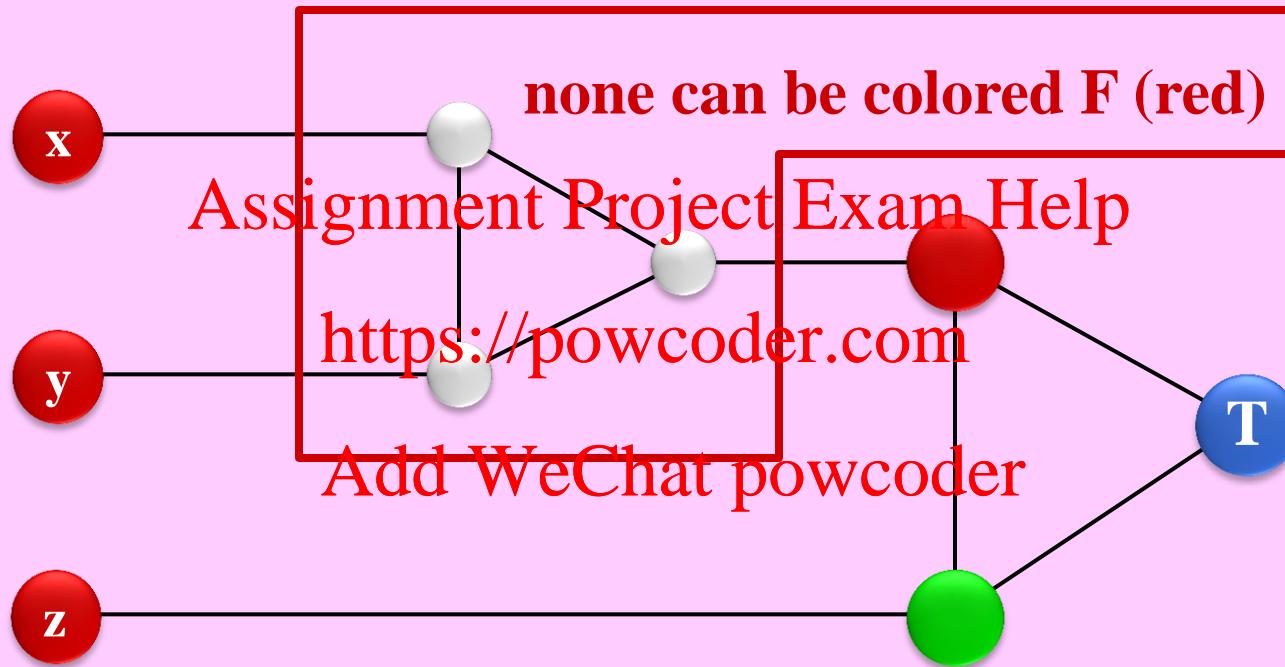


CLAIM: Suppose nodes x , y , z each get a color (T or F). Then, it is possible to complete a 3 coloring of the assembly \Leftrightarrow at least one of x , y , z got the color T.

3SAT \leq_P 3-Colorability

[\Rightarrow]: Suppose all x, y, z are colored F.

Then it's impossible to complete a 3 coloring of the assembly.



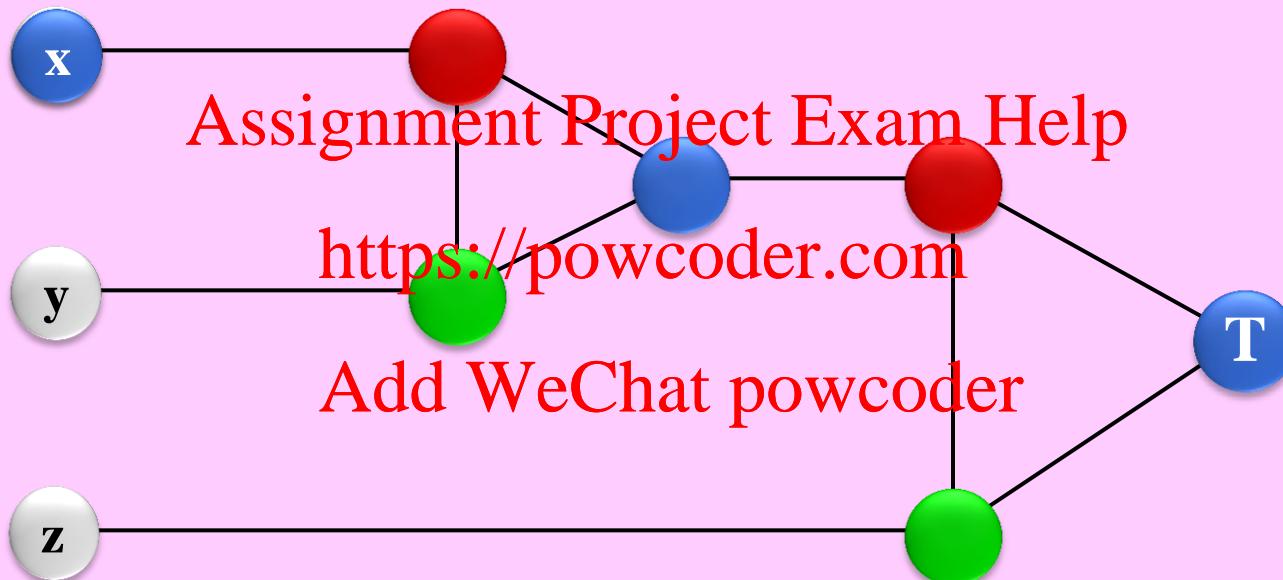
CLAIM: Suppose nodes x, y, z each get a color (T or F). Then, it is possible to complete a 3 coloring of the assembly
 \Leftrightarrow at least one of x, y, z got the color T.

3SAT \leq_P 3-Colorability

[\Leftarrow]: Suppose at least one of x, y, z is colored T.

Then it's possible to complete a 3 coloring of the assembly.

CASE 1: x is colored T: (the case that y is colored T is symmetric)



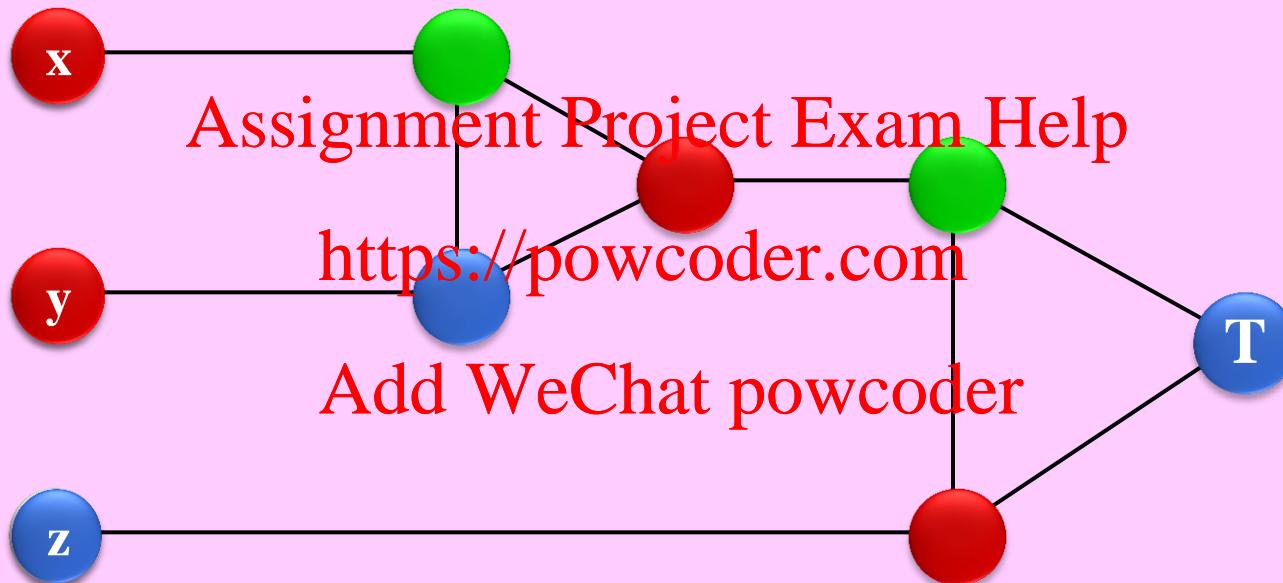
CLAIM: Suppose nodes x, y, z each get a color (T or F). Then, it is possible to complete a 3 coloring of the assembly
 \Leftrightarrow at least one of x, y, z got the color T.

3SAT \leq_P 3-Colorability

[\Leftarrow]: Suppose at least one of x , y , z is colored T.

Then it's possible to complete a 3 coloring of the assembly.

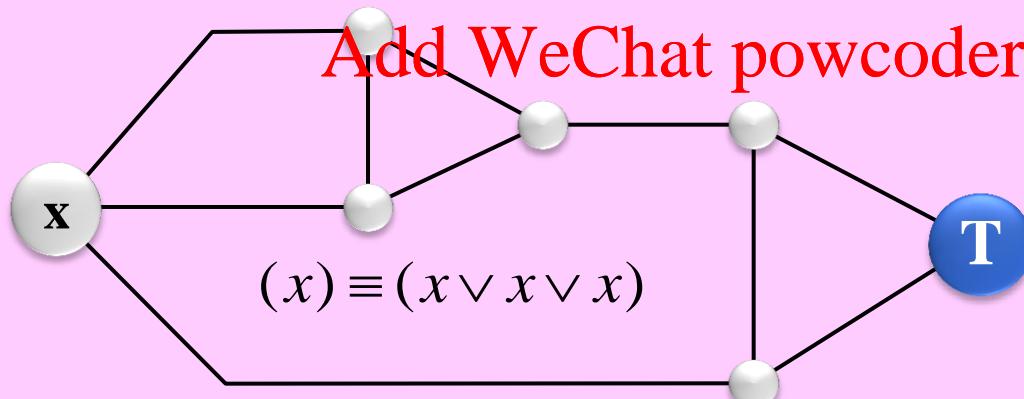
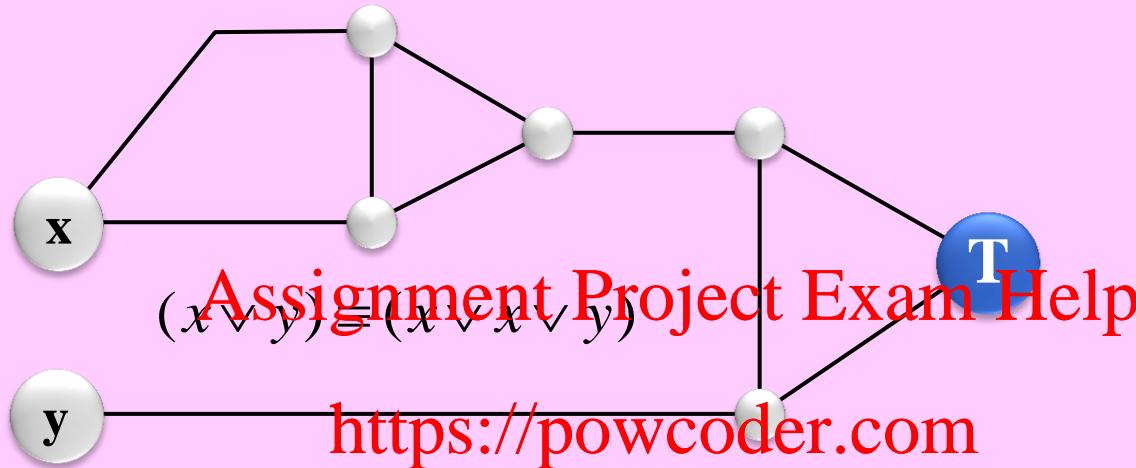
CASE 2: x and y are colored F and z is colored T:



CLAIM: Suppose nodes x , y , z each get a color (T or F). Then, it is possible to complete a 3 coloring of the assembly
 \Leftrightarrow at least one of x , y , z got the color T.

3SAT \leq_P 3-Colorability

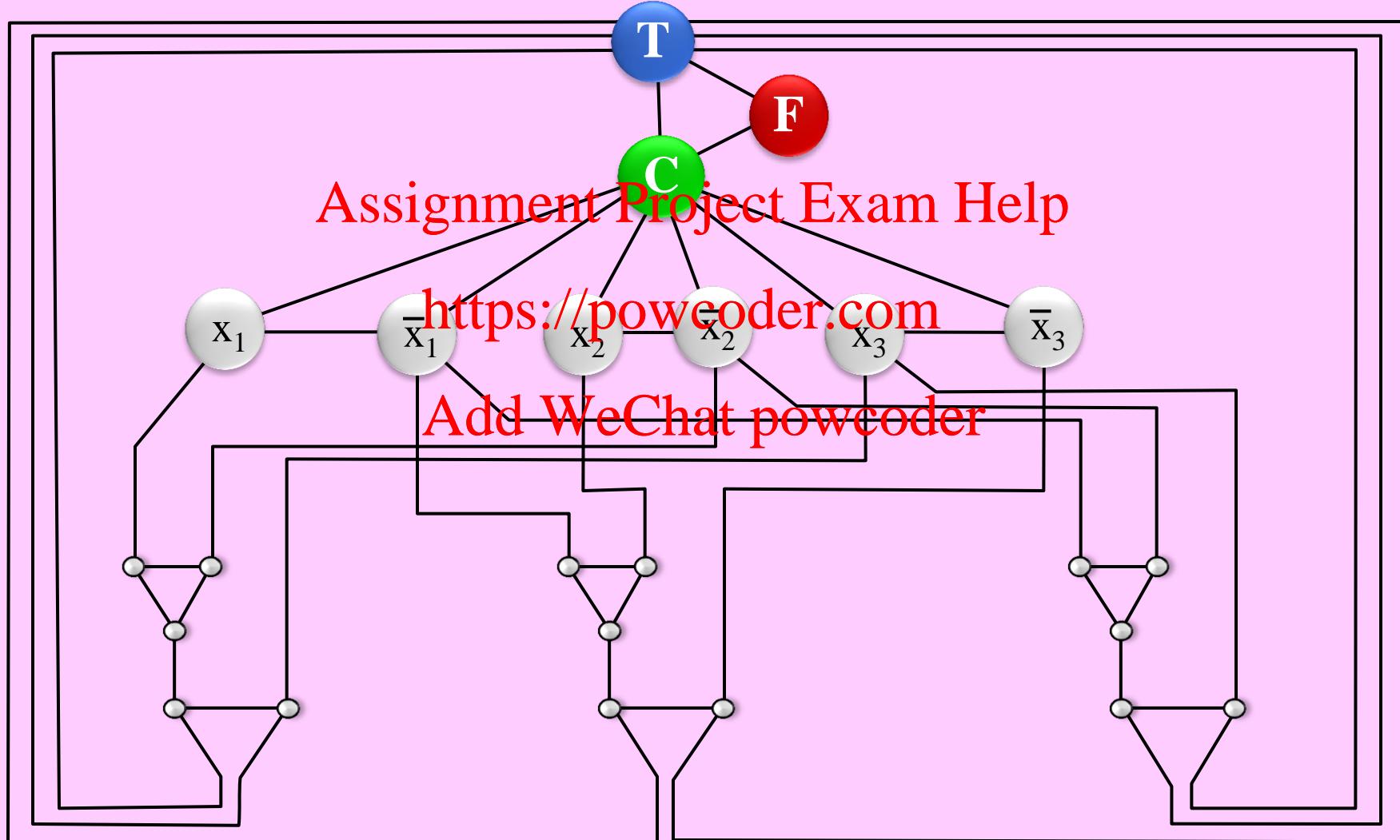
Clauses with less than 3 literals: “fuse” some literal “entry” points.



[These can be further simplified.]

3SAT \leq_P 3-Colorability

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$



3SAT \leq_P Independent Set

The Independent Set Problem:

An independent set in a graph is a set of vertices, no pair of which are adjacent. Given a graph G and integer K, does G have an independent set of size $\geq K$?

Reduction: Convert each clause with L literals to a clique of L vertices.

Add an edge between each literal and its negation. $K = \# \text{ clauses}$.

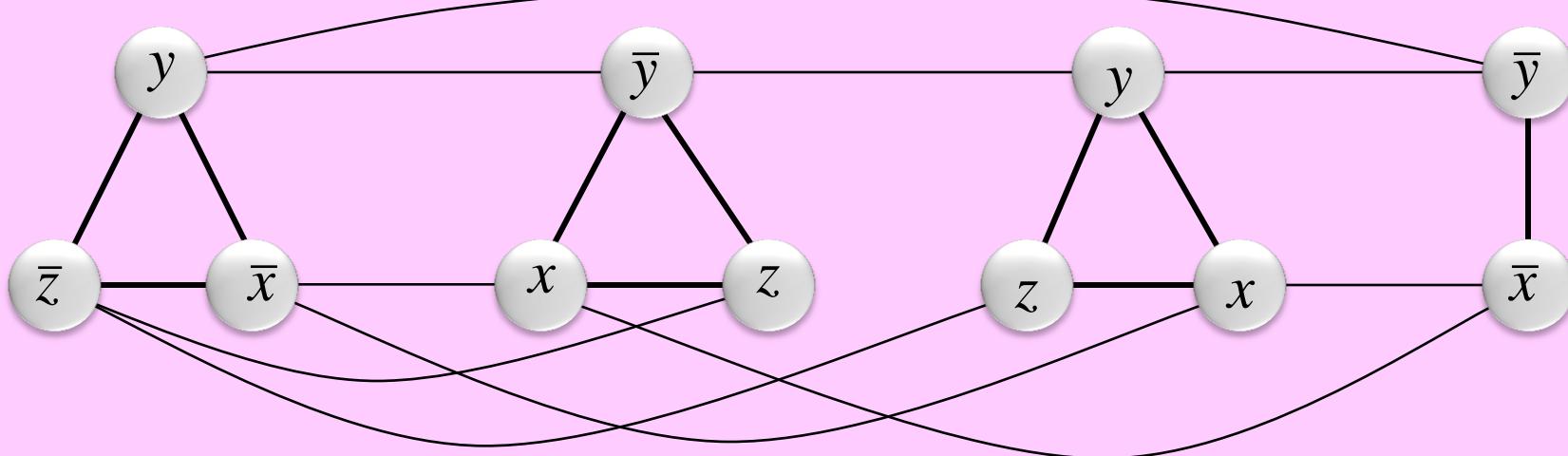
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$$(\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y})$$

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Independent Set \leq_P Clique

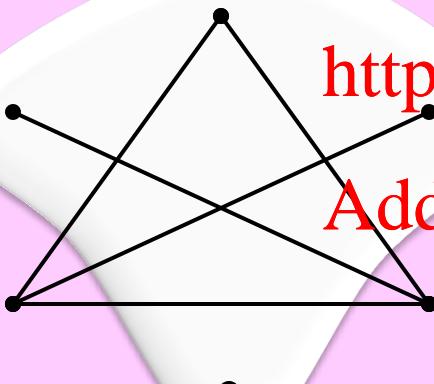
The Clique Problem:

A clique in a graph is a complete sub-graph,
i.e., a set of vertices, every pair of which are adjacent.

Given a graph G and integer K , does G have a clique of size $\geq K$?

Reduction: S is an independent set in $G \Leftrightarrow S$ is a clique in the complement \bar{G} .

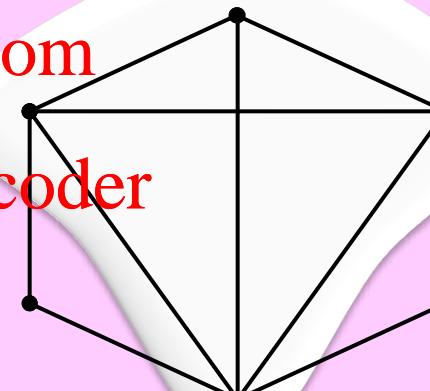
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An independent set
of size 4 in G .

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A 4-clique in \bar{G} .

Independent Set \leq_P Vertex Cover

The Vertex Cover Problem:

A vertex cover in a graph is a subset C of vertices that covers every edge, i.e., each edge of the graph has at least one end in C .

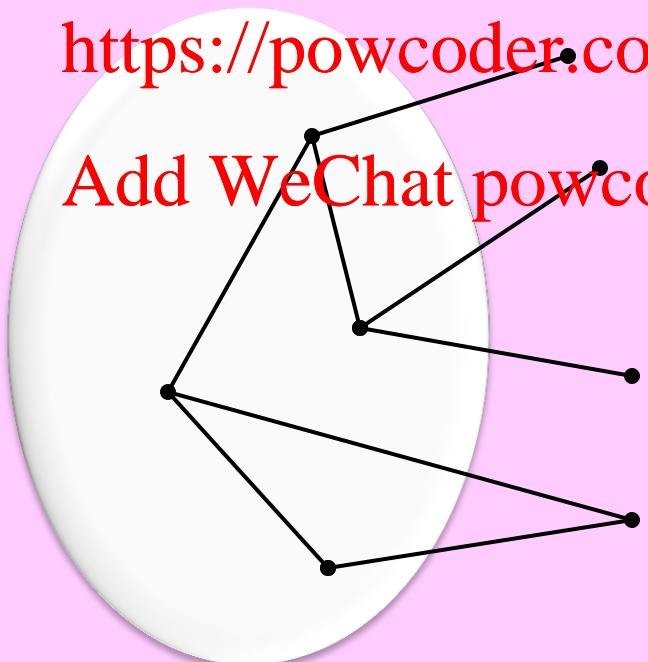
Given a graph $G = (V, E)$ and integer K , does G have a vertex cover of size $\leq K$?

Reduction: $C \subseteq V$ is a vertex cover in $G \Leftrightarrow V - C$ is an independent set in G .

G has an independent set of size $\geq K \Rightarrow G$ has a vertex cover of size $\leq |V| - K$.

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Vertex Cover \leq_P Set Cover

The Set Cover Problem:

Instance: A set $X = \{x_1, x_2, \dots, x_n\}$ and a collection $S = \{S_1, S_2, \dots, S_m\}$ of subsets of X such that S covers X , i.e., $S_1 \cup S_2 \cup \dots \cup S_m = X$.

A **cover set** is any sub-collection $C \subseteq S$ that also covers X .

Question: Given $\langle X, S, K \rangle$, is there a cover set C of size $\leq K$?

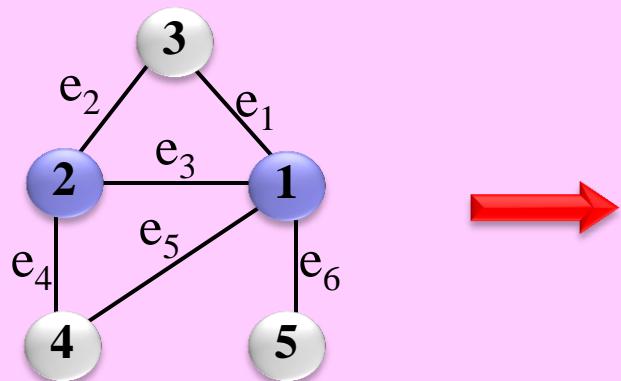
Reduction: $\langle G, K \rangle \rightarrow \langle X, S, K \rangle$ Assignment Project Exam Help

$$X = E(G), \quad S = \{S_v \mid v \in V(G)\}$$

$$S_v = \{e \in E(G) \mid e \text{ is incident to } v \text{ in } G\}, \text{ for } v \in V(G).$$

G has a vertex cover of size $\leq K \Leftrightarrow \langle X, S \rangle$ has a set cover of size $\leq K$.

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$$\begin{aligned} X &= \{e_1, e_2, e_3, e_4, e_5, e_6\} \\ S &= \{S_1, S_2, S_3, S_4, S_5\} \\ S_1 &= \{e_1, e_3, e_5, e_6\}, \quad S_4 = \{e_4, e_5\} \\ S_2 &= \{e_2, e_3, e_4\}, \quad S_5 = \{e_6\} \\ S_3 &= \{e_1, e_2\} \end{aligned}$$

$\{1, 2\}$ is a vertex cover in G

$\{S_1, S_2\}$ is a set cover of X

Hamiltonian Cycle

The Undirected (Directed) Hamiltonian Cycle Problem:

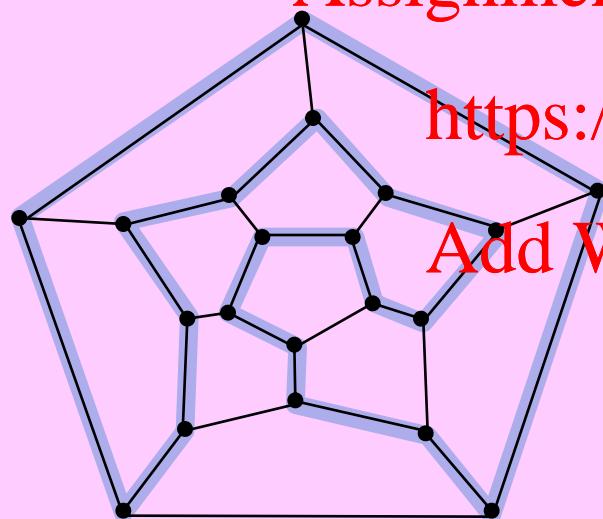
Instance: An undirected (directed) graph $G = (V, E)$.

Question: Is G Hamiltonian, i.e., does G have a Hamiltonian cycle; a cycle that goes through each vertex exactly once?

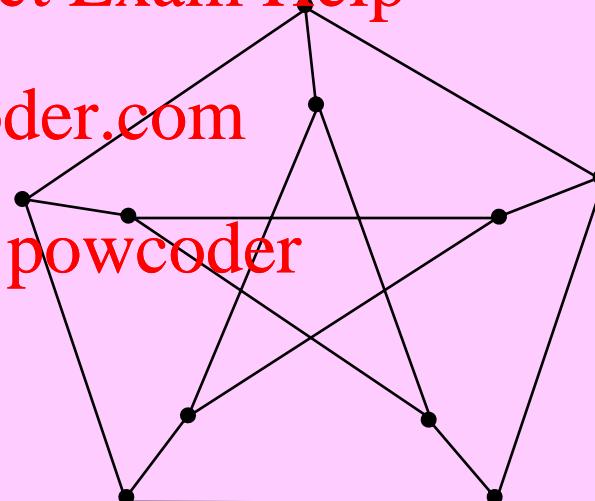
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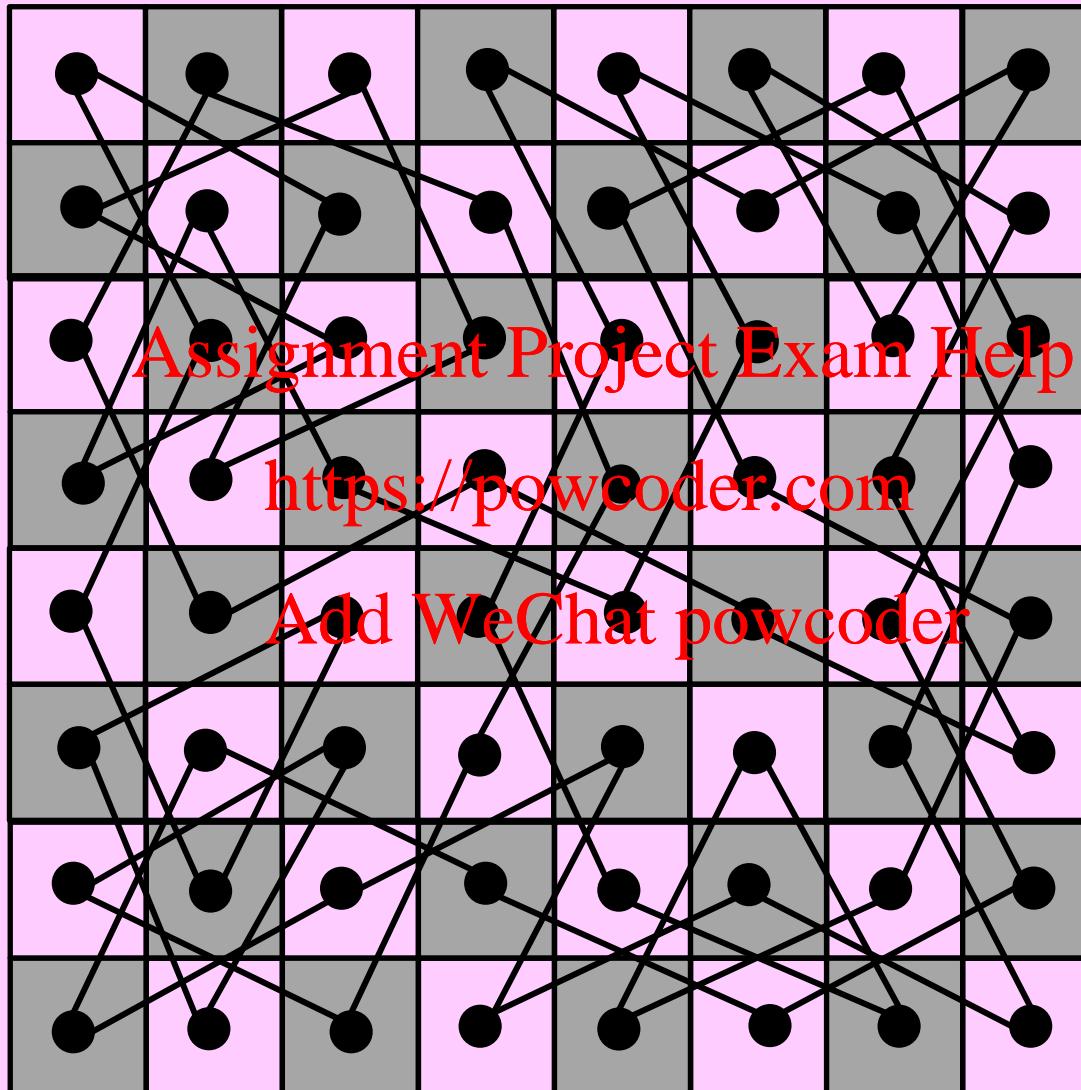


Hamiltonian
(skeleton of dodecahedron)



Non-Hamiltonian
(Peterson graph)

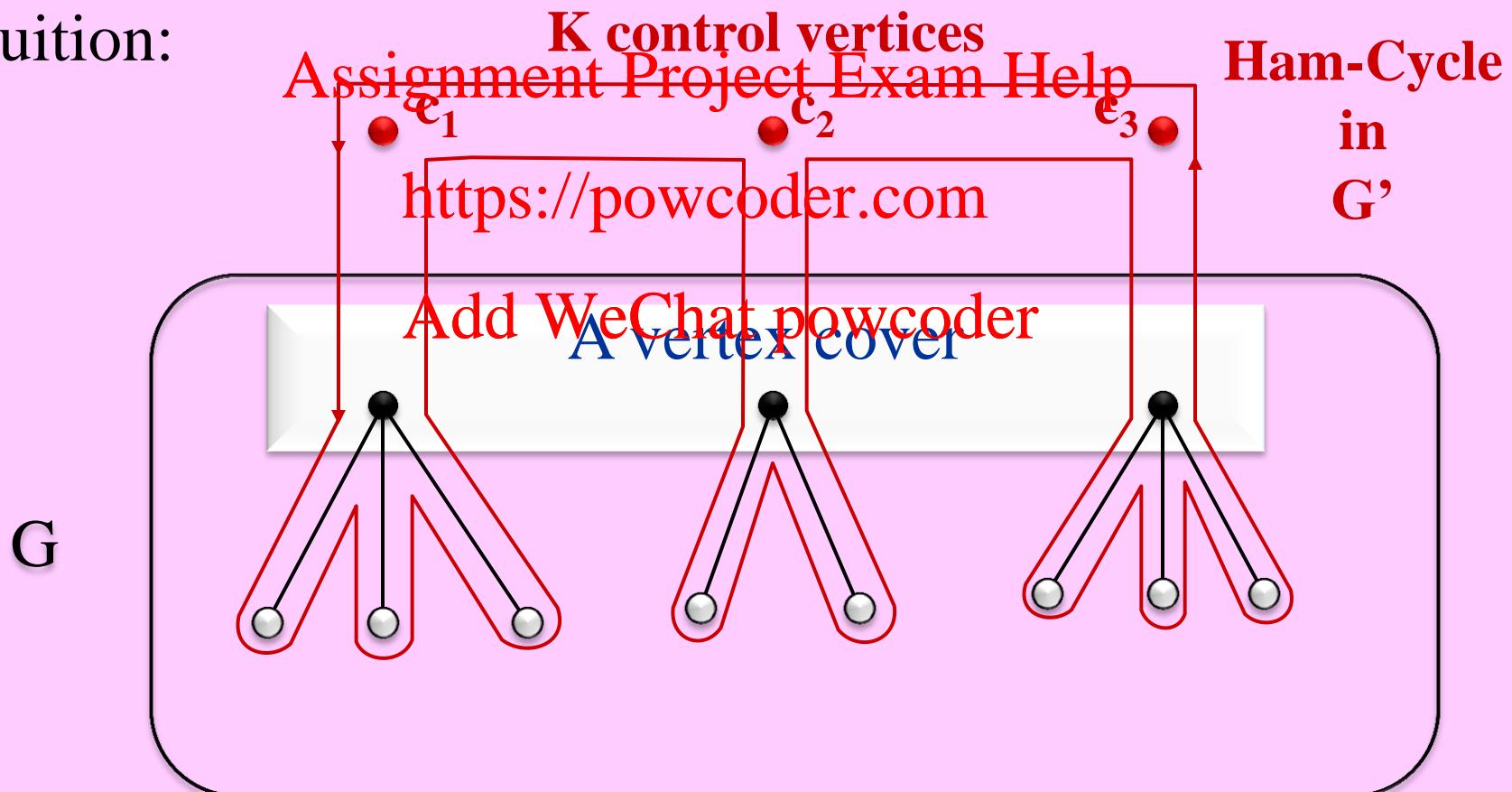
Knight's Tour on the chess board



Vertex Cover \leq_P Directed Ham-Cycle

$\langle G, K \rangle$: does G have a vertex cover of size $\leq K$?
→ $\langle G' \rangle$: is digraph G' Hamiltonian?

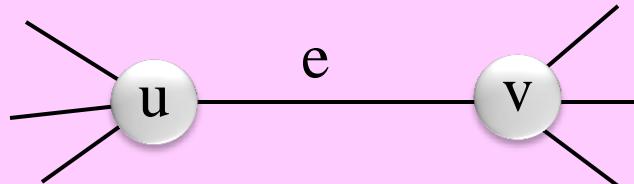
Intuition:



Vertex Cover \leq_P Directed Ham-Cycle

STEP 1: edge assembly A_e

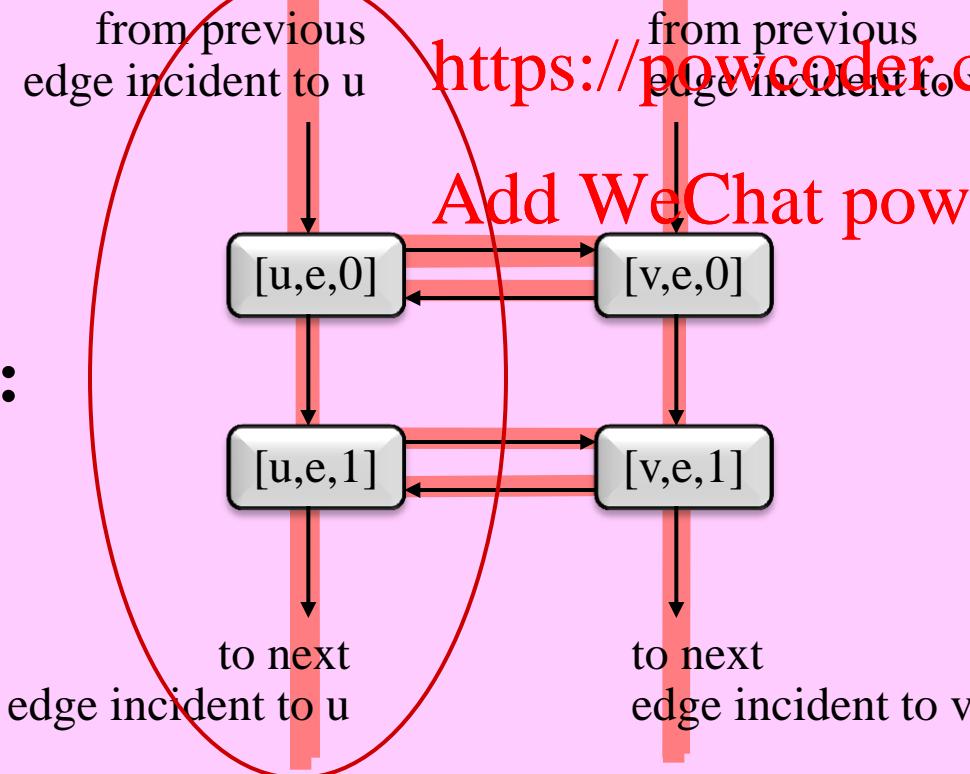
G:



forms a sequential list of edges incident to node u

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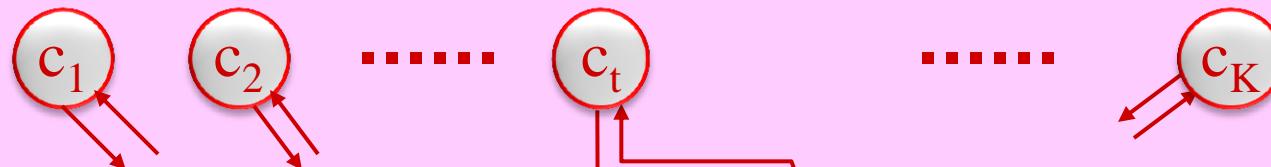
$A_e :$



Case 1:
Impossible.
[u,e,1] can't
be visited.
(3 other similar cases)

Vertex Cover \leq_p Directed Ham-Cycle

STEP 2: assembly for control nodes c_1, c_2, \dots, c_K



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edge-list
assembly
of node u
from step 1

[u, e₁, 0]

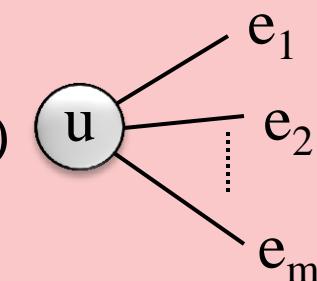
[u, e₁, 1]

[u, e_m, 0]

[u, e_m, 1]

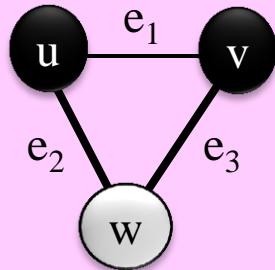
G:

$\forall u \in V(G)$

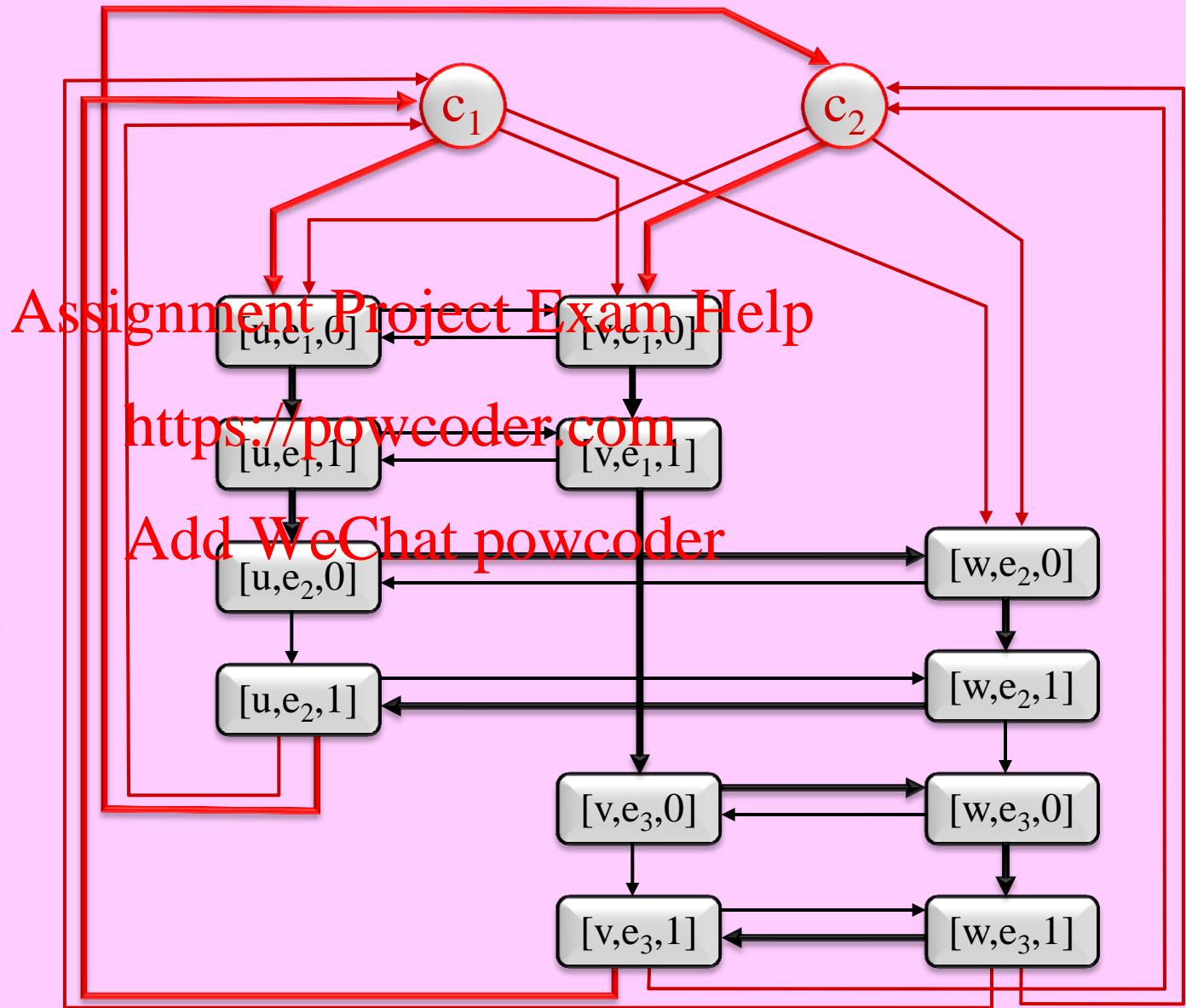


Vertex Cover \leq_P Directed Ham-Cycle

Vertex Cover
 $\{u, v\}$

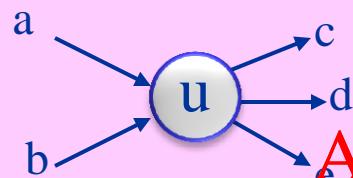


Thick edges
 show
 Ham-Cycle

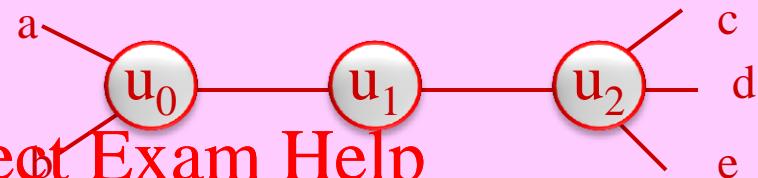


Directed Ham-Cycle \leq_P Undirected Ham-Cycle

Digraph G:



Undirected graph G':



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CLAIM: G has a directed Ham-cycle \Leftrightarrow G' has an undirected Ham-cycle.

Hamiltonian Cycle \leq_P TSP

The Traveling Salesman Problem:

Instance: An $n \times n$ weighted adjacency matrix $D = (d_{ij})$ of a **complete** weighted graph on n vertices with integer edge lengths d_{ij} , and an integer K .

Feasible Sol: A TSP tour is any Hamiltonian cycle in this complete graph.

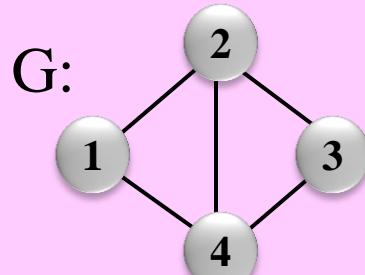
Question: Is there a TSP tour of length (or distance) $\leq K$?

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$$G = (V, E) \rightarrow n = |V|, d_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 2 & \text{if } (i, j) \notin E \end{cases}$$

distance matrix for $i \neq j$

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D =

0	1	2	1
1	0	1	1
2	1	0	1
1	1	1	0

CLAIM: G has a Hamiltonian cycle \Leftrightarrow D has a TSP tour of length $\leq n$.

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Exercises
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1. Merlin in King Arthur's Court:

In the court of King Arthur there dwelt 150 knights and 150 ladies-in-waiting. The king decided to marry them off, but the trouble was that some pairs hated each other so much that they would not even get married, let alone speak! King Arthur tried several times to pair them off but each time he ran into conflicts. So he summoned Merlin the Wizard and ordered him to find a pairing in which every pair was willing to marry. Now Merlin had supernatural powers and he saw immediately that none of the $150!$ possible pairings was feasible, and this he told the king. But Merlin was not only a great wizard, but a suspicious character as well, and King Arthur did not quite trust him. “*Find a pairing or I shall sentence you to be imprisoned in a cave forever!*” said Arthur.

Fortunately for Merlin, he could use his supernatural powers to browse forthcoming scientific literature, and he found several papers in the early 20th century that gave the reason why such a pairing could not exist. He went back to the King when all the knights and ladies were present, and asked a certain 56 ladies to stand on one side of the king and 95 knights on the other side, and asked: “*Is any one of you ladies, willing to marry any of these knights?*”, and when all said “*No!*”, Merlin said: “*O King, how can you command me to find a husband for each of these 56 ladies among the remaining 55 knights?*” So the king, whose courtly education did include the pigeonhole principle, saw that in this case Merlin had spoken the truth and he graciously dismissed him.

Some time elapsed and the king noticed that at the dinners served for the 150 knights at the famous round table, neighbors often quarreled and even fought. Arthur found this bad for the digestion and so once again he summoned Merlin and ordered him to find a way to seat the 150 knights around the table so that each of them should sit between two friends. Again, using his supernatural powers Merlin saw immediately that none of the $150!$ seatings would do, and this he reported to the king. Again, the king bade him find one or explain why it was impossible. “*Oh I wish there were some simple reason I could give to you! With some luck there could be a knight having only one friend, and so you too could see immediately that what you demand from me is impossible. But alas!, there is no such simple reason here, and I cannot explain to you mortals why no such seating exists, unless you are ready to spend the rest of your life listening to my arguments!*” The king was naturally unwilling to do that and so Merlin has lived imprisoned in a cave ever since.

[Explain the above paragraphs in terms of known facts about two graph problems.]

2. Optimization versus Decision:

TSP: given a distance matrix and a budget K , is there a TSP tour of length $\leq K$.

TSP-OPT: given a distance matrix, find the shortest TSP tour.

Show that if TSP can be solved in polynomial time, then so can TSP-OPT.

3. Search versus Decision:

Suppose you have an algorithm that runs in polynomial time and answers whether a given graph has a Hamiltonian cycle. Show that you can use such an algorithm as a subroutine to develop a polynomial time algorithm that returns the actual Hamiltonian cycle if there exists one

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4. Hamiltonian Path Problem:

Given a graph $G = (V, E)$, does it contain a simple spanning path, i.e., a path that visits every node exactly once? Such a path consists of $|V| - 1$ edges; it can start at any node and can end at any other node of G .

Show that the Hamiltonian Path Problem is NP-complete.

[Hint: reduce from the Hamiltonian Cycle Problem.]

5. Hamiltonian Path in a DAG:

Describe a polynomial-time algorithm that finds whether a given DAG contains a (directed) Hamiltonian path.

6. K-Spanning Tree Problem:

Does a given undirected graph $G = (V, E)$ have a spanning tree in which each node has degree $\leq K$?

For any fixed $K \geq 2$, show that the K-Spanning Tree Problem is NP-complete.

[Hint: start with $K=2$ and consider the relationship with the Hamiltonian Path Problem.]

7. Proving NP-completeness by generalization:

For each of the problems below, prove that it is NP-complete by showing that it is a **generalization** of some NP-complete problem we have seen in these Slides.

- (a) **Subgraph Isomorphism:** Given two undirected graphs G and H, is G a subgraph of H? (that is, by deleting some vertices and edges of H we obtain a graph that is , up to renaming of vertices, identical to G).
- (b) **Longest Path:** Given a graph G and an integer K, does G have a simple path of length at least K^2 ?
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- (c) **Dense Subgraph:** Given a graph G and two integers α and β , does G have a set of α vertices with at least β edges between them?
- (d) **Sparse Subgraph:** Given a graph G and two integers α and β , does G have a set of α vertices with at most β edges between them?
- (e) **Reliable Network:** We are given two $n \times n$ matrices, a *distance* matrix $D = (d_{ij})$ and a *connectivity requirement* matrix $R = (r_{ij})$, as well as a budget K . We must find a graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$ such that
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(1) the total cost of all edges in G (sum of d_{ij} , over all edges $(i,j) \in E$) is K or less, and
(2) between any two distinct vertices i and j there are at least r_{ij} vertex-disjoint paths.
[Hint: Suppose that all d_{ij} 's are 1 or 2, $K = n$, and r_{ij} 's are 2. Which well known NP-complete problem is this?]

8. In P or NP-complete?

Determine which of the following problems are NP-complete and which are solvable in polynomial time. In each problem you are given an undirected graph $G = (V, E)$ and:

- (a) A set of nodes $L \subseteq V$, and you must find a spanning tree such that its set of leaves includes the set L .
- (b) A set of nodes $L \subseteq V$, and you must find a spanning tree such that its set of leaves is precisely the set L .
- (c) A set of nodes $L \subseteq V$, and you must find a spanning tree such that its set of leaves is included in the set L .
- (d) An integer K , and you must find a spanning tree with K or fewer leaves.
- (e) An integer K , and you must find a spanning tree with K or more leaves.
- (f) An integer K , and you must find a spanning tree with exactly K leaves.

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[Hint: All the NP-completeness proofs are by generalization, except for one.]

9. Integer Linear Programming:

We have already shown that ILP is NP-hard by a reduction from 3SAT. To complete the proof that ILP is NP-complete, show that ILP is in NP. [Hint: Be careful! Given an instance of ILP feasibility, you must show that if it has a feasible solution, then it must have one that is only polynomially long in the length of the input instance, and hence, it can be written down in polynomial time.]

- 10. 2SAT:** Complete the details of the proof that $2\text{SAT} \in \text{P}$. In fact, show that 2SAT can be solved in linear time.
- 11. Tautologies:** Determine whether a given Boolean formula is a tautology.
Show that this problem is co-NP-complete (i.e, is in co-NP and is NP-hard).
- 12. Max2SAT:** Given a 2SAT formula Φ and an integer K , is there a truth assignment that satisfies at least K clauses of Φ ?

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Consider the following 10 clauses:

$(x), (y), (z), (w), (\neg x \vee \neg y), (\neg y \vee \neg z), (\neg z \vee \neg x), (x \vee \neg w), (y \vee \neg w), (z \vee \neg w)$.

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- (a) Show that if $(x \vee y \vee z)$ is true, then at least 7 of these 10 clauses can be satisfied.
(b) Show that if $(x \vee y \vee z)$ is false, then at most 6 of these 10 clauses can be satisfied.
(c) Use (a) and (b) and a reduction from 3SAT to show that Max2SAT is NP-complete.

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13. Boolean-SAT, CNF-SAT & DNF-SAT:

A boolean formula is in CNF (conjunctive normal form) if it is the conjunction (\wedge) of a number of clauses, where each clause is the disjunction (\vee) of some literals.

A boolean formula is in DNF (disjunctive normal form) if it is the disjunction (\vee) of a number of clauses, where each clause is the conjunction (\wedge) of some literals.

- (a) Show a polynomial-time reduction from Boolean-SAT to CNF-SAT that takes any boolean-formula Φ and converts it to a CNF-formula Ψ (with possible additional variables) such that Φ is satisfiable if and only if Ψ is satisfiable. [Hint: start with the parse tree of Φ , and convert each node to an equivalent collection of CNF clauses.]
- (b) We have shown that CNF-SAT is NP-complete. Show that DNF-SAT $\in P$.
- (c) Show a reduction from CNF-SAT to DNF-SAT. [Hint: Use the Truth Table Method.]
- (d) Do parts (b) and (c) imply that DNF-SAT is NP-hard and in P, and hence P=NP?

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14. Restricted 3SAT-1:

Show that 3SAT remains NP-complete even when we restrict to formulas in which each literal appears at most twice.

[Hint: replace all m appearances of a variable a by $a_1, a_2, a_3, \dots, a_m$; then add the following clauses: $(\bar{a}_1 \vee a_2) \wedge (\bar{a}_2 \vee a_3) \wedge \dots \wedge (\bar{a}_{m-1} \vee a_m) \wedge (\bar{a}_m \vee a_1)$.]

15. Restricted 3SAT-2:

Consider a special case of 3SAT in which all clauses have exactly 3 literals, and each variable and its negation together appear exactly 3 times. Show that this problem can be solved in polynomial time. [Hint: Create a bipartite graph with clauses on the left, variables on the right, and edges whenever a variable appears in a clause. Then use Hall's Matching Theorem (see exercises at the end of our Graph Slides).]

16. 3D Matching: given a set of n girls n boys and n pets, and a set of m girl-boy-pet triples, is there a subset of n triples in which every girl and boy and pet appear exactly once?

Show that 3D Matching is NP-complete. [Hint: reduce from 3SAT.]

17. The Hitting Set Problem:

We are given a set $X = \{x_1, x_2, \dots, x_n\}$, a collection $S = \{S_1, S_2, \dots, S_m\}$ of subsets of X , and a budget K .

A **hitting set** is any subset $H \subseteq X$ that intersects every S_t , i.e., $H \cap S_t \neq \emptyset$ for all $t = 1..m$.

Question: Given (X, S, K) , is there a hitting set H of size $\leq K$?

Show that the Hitting Set Problem is NP-complete.

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18. Clique + Independent Set: Prove that the following problem is NP-complete: Given an undirected graph G and an integer K , return a clique of size K *as well as* an independent set of size K , provided both exist.

19. The Maximum Common Subgraph Problem:

We are given two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and a budget K . Determine whether there is a graph H with K nodes that is a subgraph of both G_1 and G_2 (up to renaming of nodes). Show that this problem is NP-complete.

20. In task scheduling, it is common to use a graph representation with a node for each task and a directed edge from task i to task j if i is a precondition to j. This directed graph depicts the precedence constraints in the scheduling problem. Clearly, a schedule is possible if and only if the graph is acyclic; if it isn't, we'd like to identify the smallest number of constraints that must be dropped so as to make it acyclic. Given a digraph G, a subset $E' \subseteq E(G)$ is called a **feedback arc set** if removal of edges E' renders G acyclic.

Feedback Arc Set (FAS): Given a digraph G and a budget K, find a feedback arc set of $\leq K$ edges, if one exists.

(a) Show that FAS is in NP

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FAS can be shown to be NP-complete by a reduction from Vertex Cover. Given an instance (G, K) of Vertex Cover, where G is an undirected graph and we want a vertex cover of size $\leq K$, we construct an instance (G', K) of FAS as follows. If $V(G) = \{v_1, v_2, \dots, v_n\}$, then $V(G') = \{w_i, w'_i, \dots, w_n\} \cup \{w'_1, w'_2, \dots, w'_n\}$ and $E(G')$ consists of the following $n + 2|E(G)|$ directed edges:

- (i) (w_i, w'_i) for all $i = 1..n$,
- (ii) (w'_i, w_j) and (w'_j, w_i) for every $(v_i, v_j) \in E(G)$.

(b) Show that if G contains a vertex cover of size K, then G' contains a feedback arc set of size K.

(c) Show that if G' contains a feedback arc set of size K, then G contains a vertex cover of size (at most) K. [Hint: Given a feedback arc set of size K in G' , you may need to first modify it slightly to obtain another one which is of a more convenient form, but is of same size or smaller. Then argue that G must contain a vertex cover of the same size as the modified feedback arc set.]

21. [CLRS, Problem 34-4, page 1104] Scheduling with Profits and Deadlines:

Suppose you have one machine and a set of n jobs J_1, J_2, \dots, J_n . Each job J_k has a processing time t_k , a profit p_k , and a deadline d_k . The machine can process only one job at a time, and job J_k must run uninterruptedly for t_k consecutive time units. If you complete job J_k by its deadline d_k , you receive a profit p_k , but if you complete it after its deadline, you receive no profit. As an optimization problem, you are given the processing times, profits, and deadlines for a set of n jobs, and you wish to find a schedule that completes all the jobs and returns the greatest amount of profit.

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(a) State this problem as a decision problem.

(b) Show that the decision problem is NP-complete.
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(c) Give a polynomial-time algorithm for the decision problem, assuming that all processing times are integers in the range $[1..n]$. [Hint: use dynamic programming.]

(d) Give a polynomial-time algorithm for the optimization problem, assuming that all processing times are integers in the range $[1..n]$.

22. Node-Disjoint Paths Problem: Given an undirected graph in which some vertices have been specially marked: a certain number of “sources” s_1, s_2, \dots, s_k and an equal number of “destinations” t_1, t_2, \dots, t_k . The goal is to find k node-disjoint paths (that is, paths that have no nodes in common) where the i^{th} path goes from s_i to t_i .

Show that this problem is NP-complete.

[Hints: Here is a sequence of progressively stronger hints.

- (i) Reduce from 3SAT.
- (ii) For a 3SAT formula with m clauses and n variables, use $k = m + n$ sources (and destinations). Introduce one source/destination pair (s_x, t_x) for each variable x , and one source/destination pair (s_c, t_c) for each clause c .
- (iii) For each 3SAT clause, introduce two other vertices, one for each literal occurring in that clause and one for its complement.
- (iv) Note that if the path from s_c to t_c goes through an intermediate vertex representing, say, an occurrence of variable x , then no other path can go through that vertex. What vertex would you like the other path to be forced to go through instead?]

23. The Set Partitioning Problem: given a set S of n integers, can S be partitioned into two subsets A and $B = S - A$ such that the sum of integers in A is equal to the sum of integers in B ? Show that this problem is NP-complete.

24. The Knapsack Problem: Given a set of n pairs of positive integers $S = \{(v_i, w_i) \mid i = 1..n\}$, a target value V and a knapsack capacity W (also both positive integers), is there a subset C of S such that $\sum\{v_i \mid (v_i, w_i) \in C\} \geq V$ and $\sum\{w_i \mid (v_i, w_i) \in C\} \leq W$? Show that this problem is NP-complete.

25. DNA Sequencing Assignment Project Exam Help One experiment procedure for identifying a new DNA sequence repeatedly probes it to determine which k -mers (contiguous substrings of length k) it contains. Based on these, the full sequence must then be reconstructed.

Let's now formulate this as a combinatorial problem. For any string x (the DNA sequence), let $\Gamma(x)$ denote the multiset of all of its k -mers. In particular, $\Gamma(x)$ contains exactly $|x| - k + 1$ elements.

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The reconstruction problem is now easy to state: given a multiset of k -length strings, find a string x such that $\Gamma(x)$ is exactly this multiset.

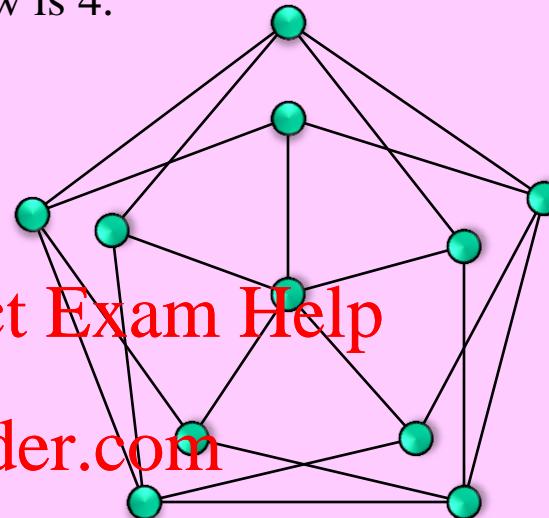
(a) Show that the reconstruction problem *reduces to* the Hamiltonian Path Problem.

[Hint: Construct a digraph with one node for each k -mer, and with an edge from a to b if the last $k - 1$ characters of a match the first $k - 1$ characters of b .]

(b) But in fact, there is much better news. Show that the same problem also *reduces to* the Eulerean Path Problem. [Hint: This time, use one directed edge for each k -mer.]

26. Chromatic Number of a graph:

The chromatic number of a graph is the minimum number of colors needed to color the vertices of the graph so that no pair of adjacent vertices are colored the same.
Show the chromatic number of the graph below is 4.



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27. Triangle-free graphs with high chromatic number:

Show that there are triangle-free graphs (i.e., with no clique subgraph of size 3) that have arbitrarily large chromatic number.

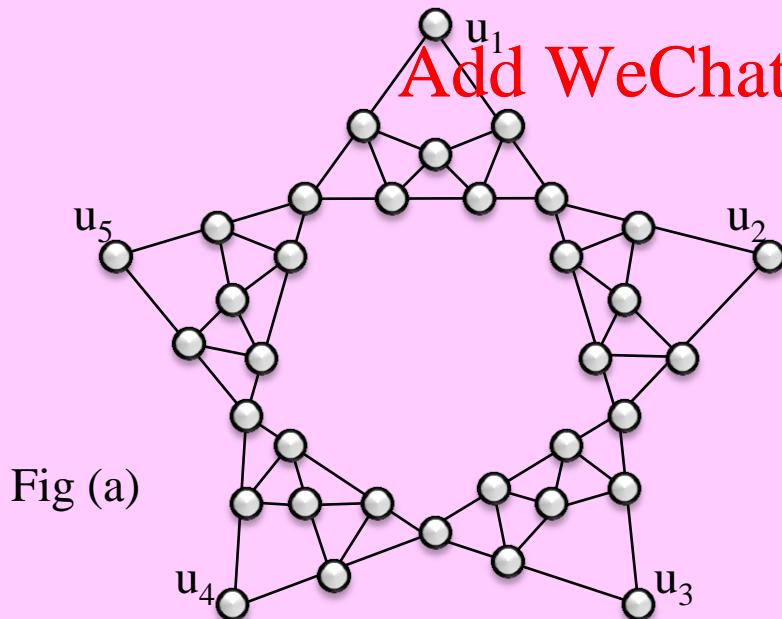
[Hint: Look at the graph above. We started with the pentagon, then doubled up the vertex set, then added the central vertex. Using that pattern, recursively build up larger triangle-free graphs with increasing chromatic numbers.]

28. Degree-Restricted 3-Colorability: Show that the 3-colorability problem remains NP-complete even if we restrict it to graphs of maximum vertex degree 4.

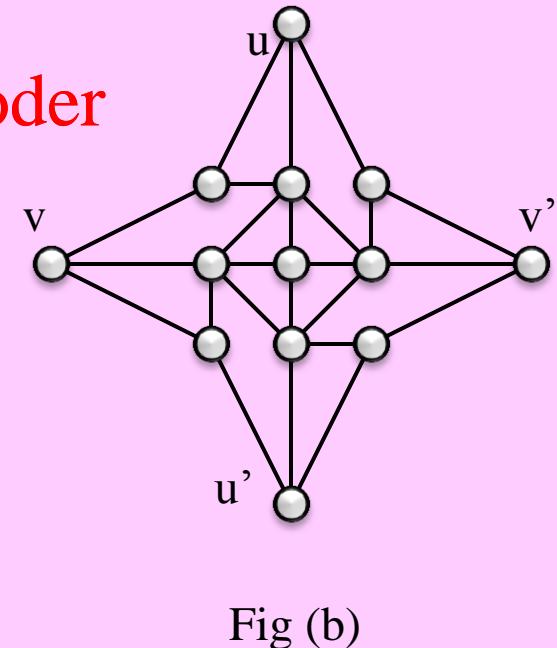
[Hint: Show that the “5-star” graph shown in Fig (a) below is 3 colorable, and in any valid 3 coloring of that graph all 5 vertices u_1, \dots, u_5 have the same color. Analogously define a “d-star” for any integer $d > 4$.]

29. Planar 3-Colorability: Show that the 3-colorability problem remains NP-complete even if we restrict it to planar graphs of maximum vertex degree 4.

[Hint: Show that the graph in Fig (b) below is 3 colorable. Furthermore, any coloring of the vertices $\{u, u', v, v'\}$ can be extended to a 3 coloring of that graph if and only if vertices u and u' have the same color and vertices v and v' have the same color. Combine this with the previous exercise.]



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30. 3-Colorability of circle intersections: Consider n circles in the plane such that no 2 of them are tangent and no 3 of them pass through the same point. Furthermore, assume each circle intersects at least one other circle.

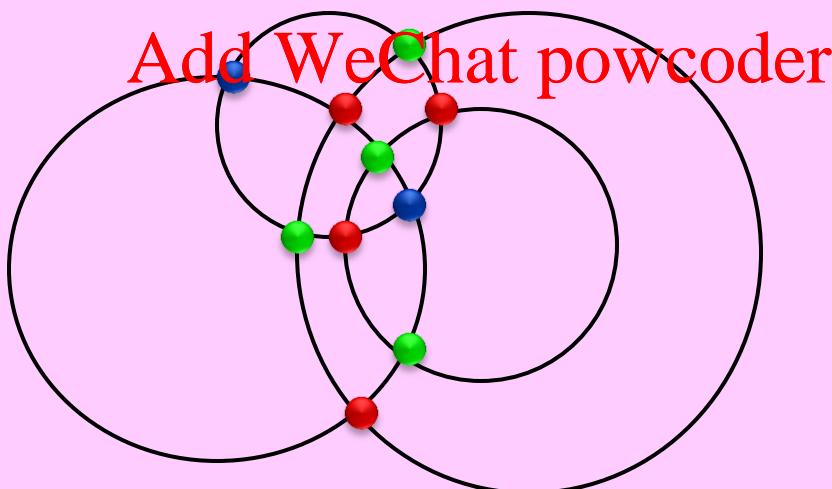
Now consider the planar (multi-) graph whose vertices are the circle intersection points, and whose edges are the arcs of the circles whose both end points are vertices and do not contain any other circle intersection point in their relative interior.
(The figure below shows an example.)

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Is it true that all such graphs are 3 colorable?

If yes prove it, if not give a counter-example.

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