EECS 3101

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Machine Model & Time Complexity

STUDY MATERIAL:

- [CLRS] chapters 1, 2, 3
- Lecture Note 2

Example

Time complexity shows dependence of algorithm's running time on input size.

Let's assume: Computer speed = 10^6 IPS,
Input: a data base of size $n = 10^6$

Time Complexity	Execution time
n	1 sec.
$n \log n$	20 sec.
n^2	12 days
2^n	40 quadrillion (10^{15}) years

Machine Model

- **Algorithm Analysis:**
  - ➢ should reveal intrinsic properties of the algorithm itself.
  - ➢ should not depend on any computing platform, programming language, compiler, computer speed, etc.
- **Random Access machine (RAM):**
  an idealized computer model
- **Elementary steps:**
  - ➢ arithmetic: $+ \quad - \quad \times \quad \div$
  - ➢ logic: and or not
  - ➢ comparison: $= \quad < \quad > \quad \neq \quad \leq \quad \geq$
  - ➢ assigning a value to a scalar variable: $\leftarrow$
  - ➢ input/output a scalar variable
  - ➢ following a pointer or array indexing
  - ➢ on rare occasions: $\sqrt{\ }$, sin, cos, …

Time Complexity

- **Time complexity** shows dependence of algorithm's running time on input size.
  - Worst-case
  - Average or expected-case
  - Amortized (studied in EECS 4101)

- **What is it good for?**

  - Tells us how efficient our design is before its costly implementation.

  - Reveals inefficiency bottlenecks in the algorithm.
  - Can use it to compare efficiency of different algorithms that solve the same problem.

  - Is a tool to figure out the true complexity of the problem itself! How fast is the "fastest" algorithm for the problem?
  - Helps us classify problems by their time complexity.

Input size & Cost measure

- **Input size:**
  - **Bit size:** # bits in the input.

    This is often cumbersome but sometimes necessary when running time depends on numeric data values, e.g., factoring an n-bit integer.

  - **Combinatorial size:** e.g., # items in an array, a list, a tree, # vertices and # edges in a graph, …

- **Cost measure:**
  - **Bit cost:** charge one unit of time to each bit operation
  - **Arithmetic cost:** charge one unit of time to each elementary RAM step

- **Complexity:**
  - **Bit complexity**
  - **Arithmetic complexity**

Input size & Cost measure

- **Input size:**
  - **Bit size:** # bits in the input.

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  - **Combinatorial size:** e.g., # items in an array, a list, a tree, # vertices and # edges in a graph, …

- **Cost measure:**
  - **Bit cost:** charge one
  - **Arithmetic cost:** cha M step

We choose these unless stated otherwise

- **Complexity:**
  - **Bit complexity**
  - **Arithmetic complexity**

Time Complexity Analysis

Worst-case time complexity is a function of input size

$$T: \mathcal{N} \rightarrow \Re^{+}$$

T(n) = **maximum** # steps by the algorithm on any input of size n.

```
Algorithm  InversionCount(A[1..n])
    count ← 0
    for i ← 1 .. n  do
        for j ← i+1 .. n  do
            if  A[i] > A[j]  then  count++
    return count
end
```


$T(n) = 3n^2 + 6n + 4$

$= \Theta(n^2)$.

Incidentally, this can be improved to $\Theta(n \log n)$ time by divide-&-conquer

Which line do you prefer? The 2nd is simpler and platform independent.

Caution!

Algorithm **SUM**(A[1..n])
    $S \leftarrow 0$
    for $i \leftarrow 1 .. n$ **do**
        $S \leftarrow S + A[i]$
    return S
end

Algorithm **PRIME**(P)   § integer $P > 1$
    for $i \leftarrow 2 .. P - 1$ **do**
        if $P \bmod i = 0$ **then return** NO
    return YES
end

Worst Case: $\Theta(n)$ time

LINEAR

Worst Case: $\Theta(P)$ time. Linear?

input bits $n \approx \log P$

$\Rightarrow P \approx 2^n$

$\Rightarrow T(n) = \Theta(P) = \Theta(2^n)$.

EXPONENTIAL!

PRIMALITY TESTING: used in cryptography …
It was a long standing open problem whether there is any deterministic algorithm that solves this problem in time polynomial in the input bit size.
This was eventually answered affirmatively by:

M. Agrawal, N. Kayal, N. Saxena, "PRIMES is in P," Annals of Mathematics 160, pp: 781-793, 2004.

Asymptotic Notations

$T(n) = \Theta(f(n))$

$$T(n) = \cancel{23}\, n^3 \; \cancel{+ \; 5\, n^2 \log n \; + \; 7\, n \log^2 n \; + \; 4 \log n \; + \; 6.}$$

drop constant multiplicative factor

drop lower order terms

$$T(n) = \Theta(n^3)$$

Why do we want to do this?

1. Asymptotically (at very large values of n) the leading term largely determines function behaviour.
2. With a new computer technology (say, 10 times faster) the leading coefficient will change (be divided by 10). So, that coefficient is technology dependent any way!
3. This simplification is still capable of distinguishing between important but distinct complexity classes,
   e.g., linear vs. quadratic, or polynomial vs exponential.

Asymptotic Notations: Θ O Ω o ω

Rough, intuitive meaning worth remembering:

Theta	$f(n) = \Theta(g(n))$	$f(n) \approx c\, g(n)$
Big Oh	$f(n) = O(g(n))$	$f(n) \leq c\, g(n)$
Big Omega	$f(n) = \Omega(g(n))$	$f(n) \geq c\, g(n)$
Little Oh	$f(n) = o(g(n))$	$f(n) \ll c\, g(n)$
Little Omega	$f(n) = \omega(g(n))$	$f(n) \gg c\, g(n)$

Asymptotics by ratio limit

$L = \lim_{n \to \infty} f(n)/g(n)$. If L exists, then:

Theta	$f(n) = \Theta(g(n))$	$0 < L < \infty$
Big Oh	$f(n) = O(g(n))$	$L < \infty$
Big Omega	$f(n) = \Omega(g(n))$	$0 < L$
Little Oh	$f(n) = o(g(n))$	$L = 0$
Little Omega	$f(n) = \omega(g(n))$	$L = \infty$

$\forall$ & $\exists$ quantifiers

Logic: "and" commutes with "and" , "or" commutes with "or" ,
but "and" & "or" do not commute.

Similarly, same type quantifiers commute, but different types do not:

$\exists x\ \exists y:\ P(x,y) = \exists y\ \exists x:\ P(x,y) = \exists x,y:\ P(x,y)$

$\forall x\ \forall y:\ P(x,y) = \forall y\ \forall x:\ P(x,y) = \forall x,y:\ P(x,y)$

$\forall x\ \exists y:\ P(x,y) \neq \exists y\ \forall x:\ P(x,y)$

Counter-example for the 3rd:

LHS: for every person x, there is a date y, such that x is born on date y.

RHS: there is a date y, such that for every person x, x is born on date y.

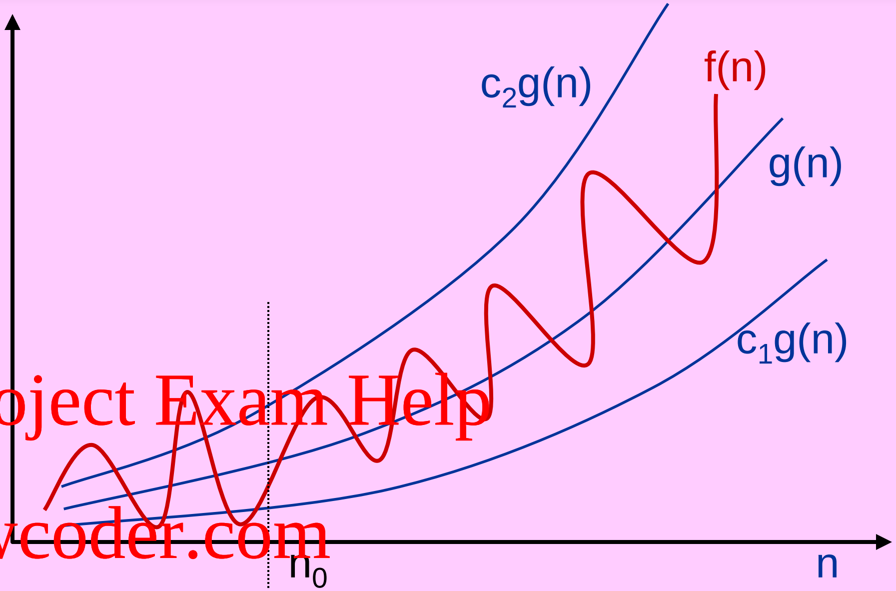
Each person has a birth date, but not every person is born on the same date!

Give natural examples for the following to show their differences:

1. $\forall x\ \exists y\ \forall z:\ P(x,y,z)$
2. $\forall x\ \forall z\ \exists y:\ P(x,y,z)$
3. $\exists y\ \forall x\ \forall z:\ P(x,y,z)$

Theta : Asymptotic Tight Bound

$f(n) = \Theta(g(n))$



$$\underbrace{\exists c_1, c_2, n_0}_{\in \Re^+} > 0 \;:\; \forall n \geq n_0, \quad c_1 g(n) \leq f(n) \leq c_2 g(n).$$

Theta : Example

$f(n) = \Theta(g(n))$ $\longleftrightarrow$ $\exists c_1, c_2, n_0 > 0 \; : \; \forall n \ge n_0, \quad c_1 g(n) \le f(n) \le c_2 g(n)$

$$\boxed{f(n) = 23n^3 - 10\, n^2 \log n + 7n + 6}$$

$f(n) = [23 - (10 \log n)/n + 7/n^2 + 6/n^3]n^3$ *factor out leading term*

$\forall n \ge 10 : \quad f(n) \le (23 + 0 + 7/100 + 6/1000)n^3$

$= (23 + 0 + 0.07 + 0.006)n^3$

$= 23.076\, n^3 < 24\, n^3$

$\forall n \ge 10 : \quad f(n) \ge (23 - \log 10 + 0 + 0)n^3 > (23 - \log 16)n^3 = 19n^3$

$\forall n \ge 10 : \quad 19\, n^3 \le f(n) \le 24\, n^3$

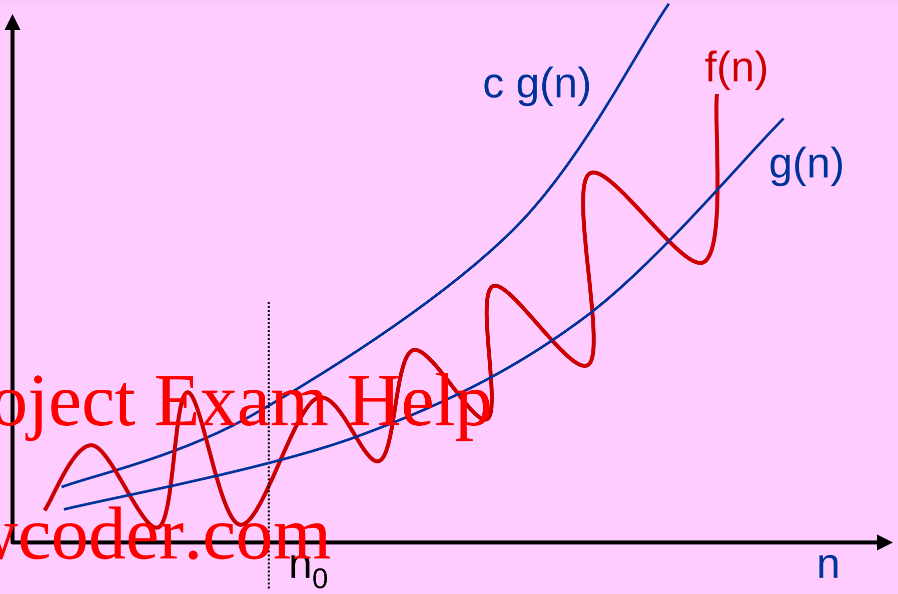
(Take $n_0 = 10$, $c_1 = 19$, $c_2 = 24$, $g(n) = n^3$)

$$\boxed{f(n) = \Theta(n^3)}$$

Now you see why we can
drop lower order terms &
the constant multiplicative factor.

Big Oh: Asymptotic Upper Bound

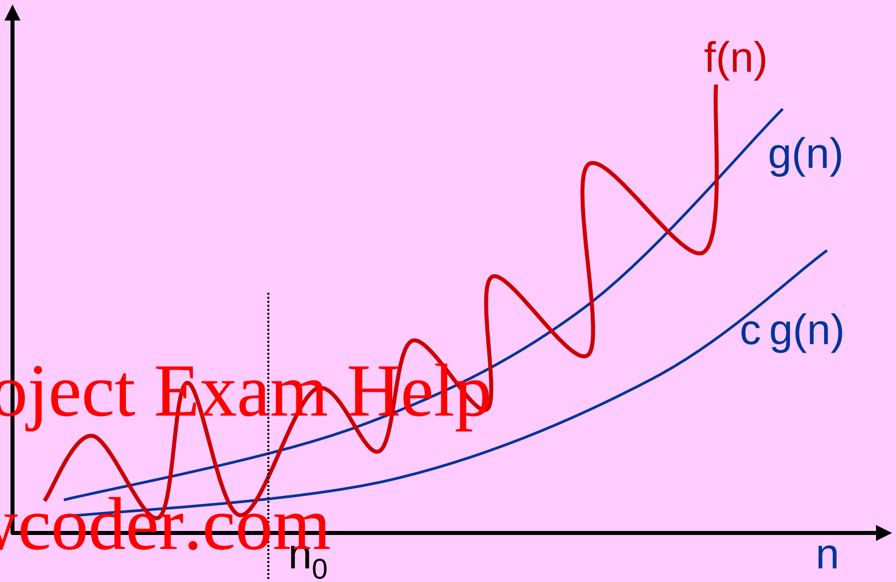
$$f(n) = O(g(n))$$



$$\underbrace{\exists c,\ n_0}_{\in \mathfrak{R}^+} > 0 \ :\ \forall n \geq n_0,\quad f(n) \leq c\, g(n).$$

Big Omega : Asymptotic Lower Bound

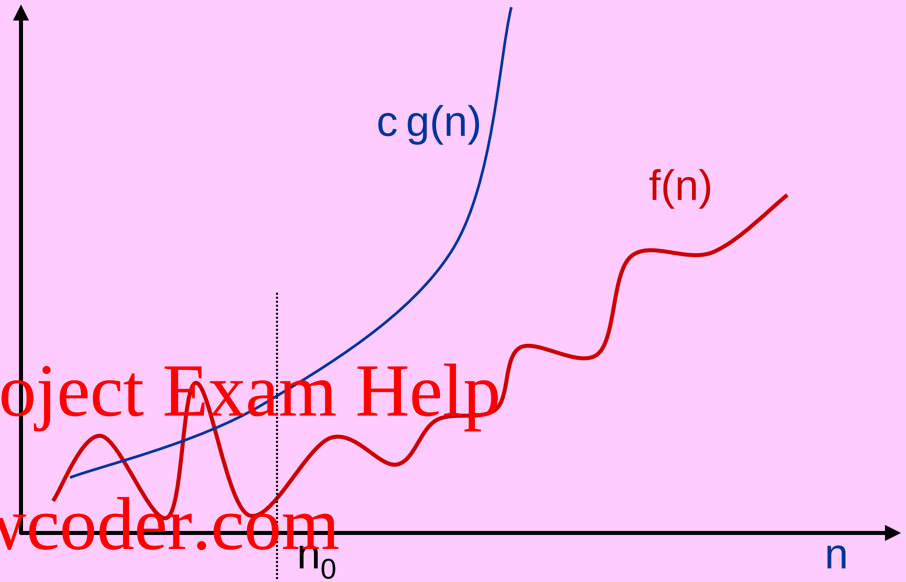
$$f(n) = \Omega(g(n))$$



$$\underbrace{\exists c, n_0}_{\in \Re^+} > 0 \ : \ \forall n \geq n_0 , \quad cg(n) \leq f(n).$$

Little oh : Non-tight Asymptotic Upper Bound

$f(n) = o(g(n))$

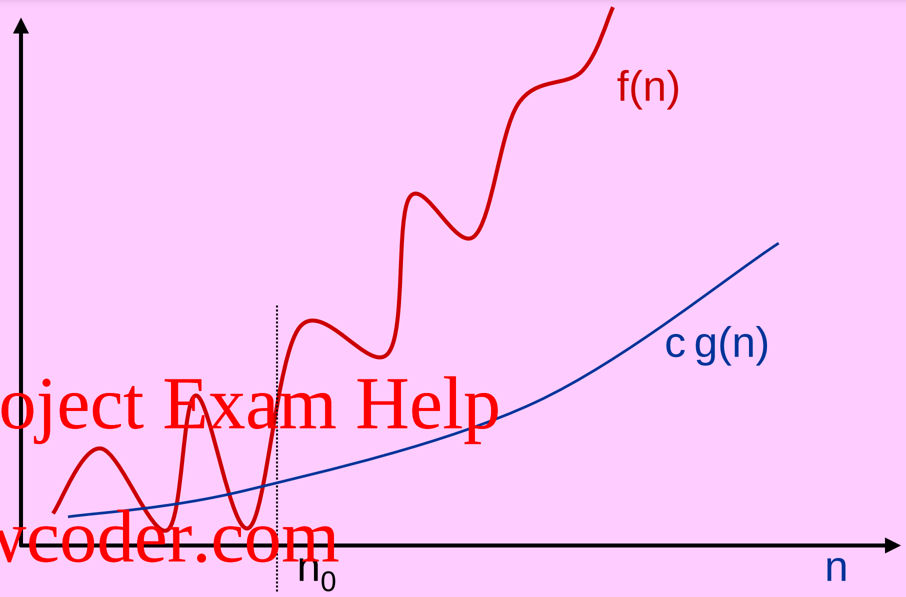


$\forall c > 0, \exists n_0 > 0 : \forall n \geq n_0, \ f(n) < c\, g(n).$

No matter how small $\in \Re^+$

Little omega : Non-tight Asymptotic Lower Bound

$f(n) = \omega(g(n))$



$\forall c > 0, \exists n_0 > 0 : \forall n \geq n_0, \quad f(n) > c\, g(n).$

No matter how large $\in \Re^+$

Definitions of Asymptotic Notations

$f(n) = \Theta(g(n))$	$\exists c_1, c_2 > 0, \exists n_0 > 0: \forall n \ge n_0,\ c_1 g(n) \le f(n) \le c_2 g(n)$
$f(n) = O(g(n))$	$\exists c > 0, \exists n_0 > 0: \forall n \ge n_0,\ f(n) \le c\, g(n)$
$f(n) = \Omega(g(n))$	$\exists c > 0, \exists n_0 > 0: \forall n \ge n_0,\ c\, g(n) \le f(n)$
$f(n) = o(g(n))$	$\forall c > 0, \exists n_0 > 0: \forall n \ge n_0,\ f(n) < c\, g(n)$
$f(n) = \omega(g(n))$	$\forall c > 0, \exists n_0 > 0: \forall n \ge n_0,\ c\, g(n) < f(n)$

Example Proof

$f(n) = o(g(n))$ $\quad \forall\, c > 0,\ \exists n_0 > 0:\ \forall n \geq n_0,\quad f(n) < c\, g(n)$

CLAIM: $n^2 \neq o(n)$.

Proof: Need to show: $\neg\,(\forall c > 0,\ \exists n_0 > 0:\ \forall n \geq n_0\ (n^2 < c\,n))$.

Move ¬ inside: $\Leftrightarrow\ (\exists\, c > 0,\ \forall n_0 > 0,\ \exists n \geq n_0\ (n^2 \geq c\,n))$.

$n \geq c$

Work inside-out:

$\exists n\ (n \geq n_0$ and $n \geq c)$

$\forall n_0 > 0,\ \exists n \geq \max\{n_0, c\}$

Now browse outside-in. Everything OK!

choose $c=1$, then $\forall\, n_0 > 0$, choose $n = \max\{n_0, c\}$

Any short cuts?

$f(n) = o(g(n))$ $\forall c > 0,\ \exists n_0 > 0:\ \forall n \geq n_0,\ f(n) < c\, g(n)$

If you are a starter on these notations, you are advised to exercise your predicate logic muscles for a while to get the hang of it!

However, in the next few slides, we will study some **FACTS & RULES** that help us manipulate and reason about asymptotic notations in most common situations with much ease.

A Classification of Functions

$\boldsymbol{Recall}$: $X < Y \Leftrightarrow \log X < \log Y$. $Also, \log(X^Y) = Y \log X$.

- **o(1)** e.g., $1/\omega(1)$, $(\log n)^{-3}$, n^{-2}, 2^{-3n}, 0.
- **O(1)** asymptotically upper-bounded by a constant
- **Θ(1)** e.g., $f(n) = 5 + 1/n$, $(2n+1)/(6n+\log n)$

- **Poly-Logarithmic:** $f(n) = \log^{\Theta(1)} n$ e.g., $\log^3 n$.
- **Polynomial:** $f(n) = n^{\Theta(1)}$ $\log f(n) = \Theta(1) \log n = \Theta(\log n)$ e.g., $f(n) = 3n + 1$, $n^{1.5} log^{-3} n$, n^{100} .
- **Super-Polynomial but Sub-Exponential:** $f(n) = n^{\omega(1)} \cap 2^{o(n)}$ $\log f(n) = \omega(\log n) \cap o(n)$ e.g., $f(n) = n^{log\, n}$, $2^{\sqrt{n}}$, $2^{\frac{n}{\log n}}$
- **Exponential or more:** $f(n) = 2^{\Omega(n)}$ $\log f(n) = \Omega(n)$ e.g., $f(n) = 23^{n \log n}$, 2^{n^2}, 2^{2^n} .

Order of Growth Rules

1. Below $X \ll Y$ means $X = o(Y)$:

$o(1) \ll \Theta(1) \ll$ Poly-Log $\ll$ Polynomial $\ll$ Exponential or more

$$o(1) \ll \Theta(1) \ll \log^{\Theta(1)} n \ll n^{\Theta(1)} \ll 2^{\Omega(n)}$$

Examples:

$$\log^{100} n = o(n^{0.01})$$

$$n^{100} = o((1.1)^{3n+1})$$

Asymptotic Relation Rules

2. **Skew Symmetric:** $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$
   $f(n) = o(g(n)) \Leftrightarrow g(n) = \omega(f(n))$

3. $f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n))$ & $f(n) = \Omega(g(n))$
   $\Leftrightarrow f(n) = O(g(n))$ & $g(n) = O(f(n))$

4. **Symmetric [only Θ]:** $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

5. **Reflexive:** $f(n) = \Theta(f(n))$, $f(n) = O(f(n))$, $f(n) = \Omega(f(n))$,
   $f(n) \neq o(f(n))$, $f(n) \neq \omega(f(n))$.

6. **Transitive:** $f(n) = O(g(n))$ & $g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
   works for Θ, Ω, o, ω too.

7. $f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$ & $f(n) \neq \Omega(g(n)) \Rightarrow f(n) \neq \Theta(g(n))$

8. $f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n))$ & $f(n) \neq O(g(n)) \Rightarrow f(n) \neq \Theta(g(n))$

9. $f(n) = g(n) + o(g(n)) \Rightarrow f(n) = \Theta(g(n))$ [can hide lower order terms]

Arithmetic Rules

10. Given: f(n) = O(F(n)), g(n) = O(G(n)), h(n) = Θ(H(n)), constant d > 0.

Sum: $f(n) + g(n) = O(F(n)) + O(G(n)) = O(\max\{ F(n), G(n) \})$.

Product: $f(n) \cdot g(n) = O(F(n)) \cdot O(G(n)) = O(F(n) \cdot G(n))$.
$f(n) \cdot h(n) = O(F(n)) \cdot \Theta(H(n)) = O(F(n) \cdot H(n))$.

Division: $$\frac{f(n)}{h(n)} = \frac{O(F(n))}{\Theta(H(n))} = O\left(\frac{F(n)}{H(n)}\right)$$

Power: $f(n)^d = O(F(n)^d)$.

In addition to "O", these rules also apply to Θ, Ω, o, ω.

Examples:

- $16\,n^2 + 9n\log n = O(n^2) + O(n\log n) = O(\max\{n^2, n\log n\}) = O(n^2)$.
- $n^2 = \Theta(n^2)$, $\log n = o(n^{0.01}) \Rightarrow n^2\log n = o(n^{2.01})$.
- $\frac{3n+7\log n}{4n\log n+9} = \frac{\Theta(n)}{\Theta(n\ \log n)} = \Theta\left(\frac{n}{\mathrm{n}\log n}\right) = \Theta\left(\frac{1}{\log n}\right)$.
- $3n + 1 = \Theta(n) \Rightarrow (3n+1)^{3.7} = \Theta(n^{3.7})$.

Proof of Rule of Sum

$f(n) = O(F(n))$ & $g(n) = O(G(n))$ $\Rightarrow$ $f(n)+g(n) = O(\max\{F(n),G(n)\})$

Proof: From the premise we have:

$$\exists n_1, c_1 > 0: \quad \forall n \geq n_1, \quad f(n) \leq c_1 F(n)$$
$$\exists n_2, c_2 > 0: \quad \forall n \geq n_2, \quad g(n) \leq c_2 G(n)$$

Now define $n_0 = \max\{n_1, n_2\} > 0$ & $c = c_1 + c_2 > 0$.

We get:

$$\begin{aligned} \forall n \geq n_0, \quad f(n)+g(n) &\leq c_1 F(n) + c_2 G(n) \\ &\leq c_1 \max\{F(n), G(n)\} + c_2 \max\{F(n), G(n)\} \\ &= (c_1 + c_2) \max\{F(n), G(n)\} \\ &= c \max\{F(n), G(n)\} \end{aligned}$$

We have shown:

$$\exists n_0, c > 0: \forall n \geq n_0, \quad f(n) + g(n) \leq c \max\{F(n), G(n)\}.$$

Therefore, $f(n) + g(n) = O(\max\{F(n), G(n)\})$.

Where did we go wrong!

"CLAIM" $\Theta(n) = \Theta(1)$.

"Proof": By repeated application of the max rule of sum: $\Theta(1) + \Theta(1) = \Theta(1)$.

$$
\begin{aligned}
\Theta(n) &= \overbrace{\Theta(1) + \Theta(1) + \Theta(1) + \cdots + \Theta(1) + \Theta(1) + \underbrace{\Theta(1) + \Theta(1)}}^{n\ \Theta(1)'s} \\
&= \Theta(1) + \Theta(1) + \Theta(1) + \cdots + \Theta(1) + \underbrace{\Theta(1) + \Theta(1)} \\
&= \Theta(1) + \Theta(1) + \Theta(1) + \cdots + \underbrace{\Theta(1) + \Theta(1)} \\
&= \cdots \\
&= \underbrace{\Theta(1) + \Theta(1)} \\
&= \Theta(1)
\end{aligned}
$$

Do you buy this? Where did we go wrong?

META RULE: Within an expression, you can apply the asymptotic simplification rules only a **constant** number of times!

WHY?

Algorithm Complexity
vs
Problem Complexity

Algorithm Time Complexity

T(n) = worst-case time complexity of **algorithm ALG**.

- **T(n) = O(f(n)):**
  You must prove that on **every** input of size n, for **<u>all sufficiently large</u>** n, ALG takes at most O(f(n)) time. That is, even the worst input of size n cannot force ALG to require more than O(f(n)) time.

- **T(n) = Ω(f(n)):**
  Demonstrate the existence of an input instance I_n of size n for **<u>infinitely many</u>** n, for which ALG requires at least Ω(f(n)) time.

Just one n or a finite # of sample n's won't do, because for all n beyond those samples, ALG could go on cruise speed!

- **T(n) = Θ(f(n)):**
  Do both!

Problem Time Complexity

T(n) = worst-case time complexity of **problem P**
is the time complexity of the “best” algorithm that solves P.

- **T(n) = O(f(n)):**
  Need to demonstrate (the existence of) an algorithm ALG that correctly solves problem P, and that worst-case time complexity of ALG is at most O(f(n)).

This is usually the much harder task

- **T(n) = Ω(f(n)):**
  Prove that for **every** algorithm ALG that correctly solves every instance of problem P, worst-case time complexity of ALG must be at least Ω(f(n)).

- **T(n) = Θ(f(n)):** Do both!

Example Problem: Sorting

Some sorting algorithms and their worst-case time complexities:

Quick-Sort:	$\Theta(n^2)$
Insertion-Sort:	$\Theta(n^2)$
Selection-Sort:	$\Theta(n^2)$
Merge-Sort:	$\Theta(n \log n)$
Heap-Sort:	$\Theta(n \log n)$

there are infinitely many sorting algorithms!

Shown in Slide 5: essentially every algorithm that solves the

SORTING Problem

requires at least

Ω(n log n)

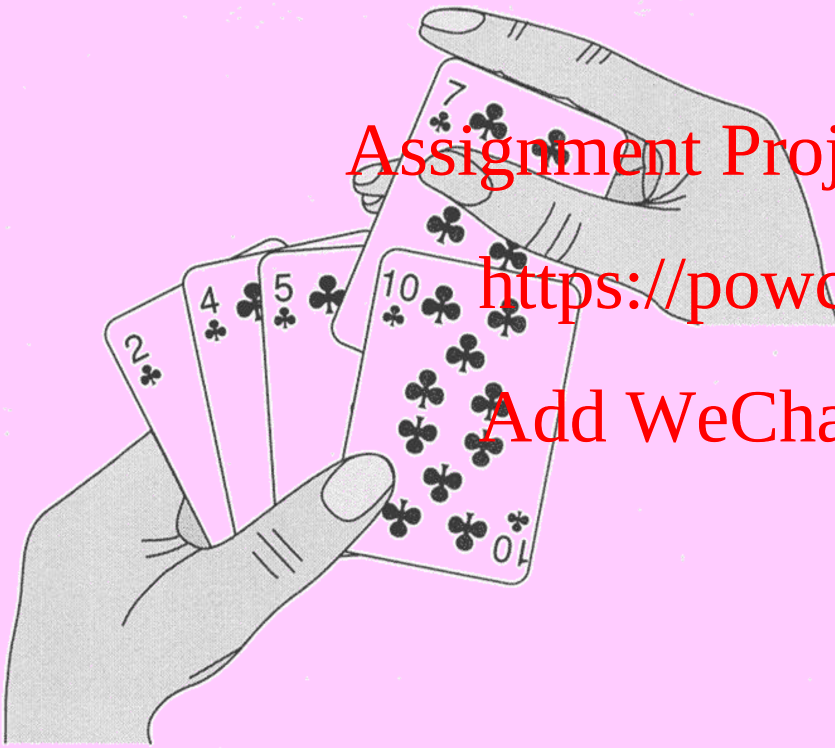
time in the worst-case.

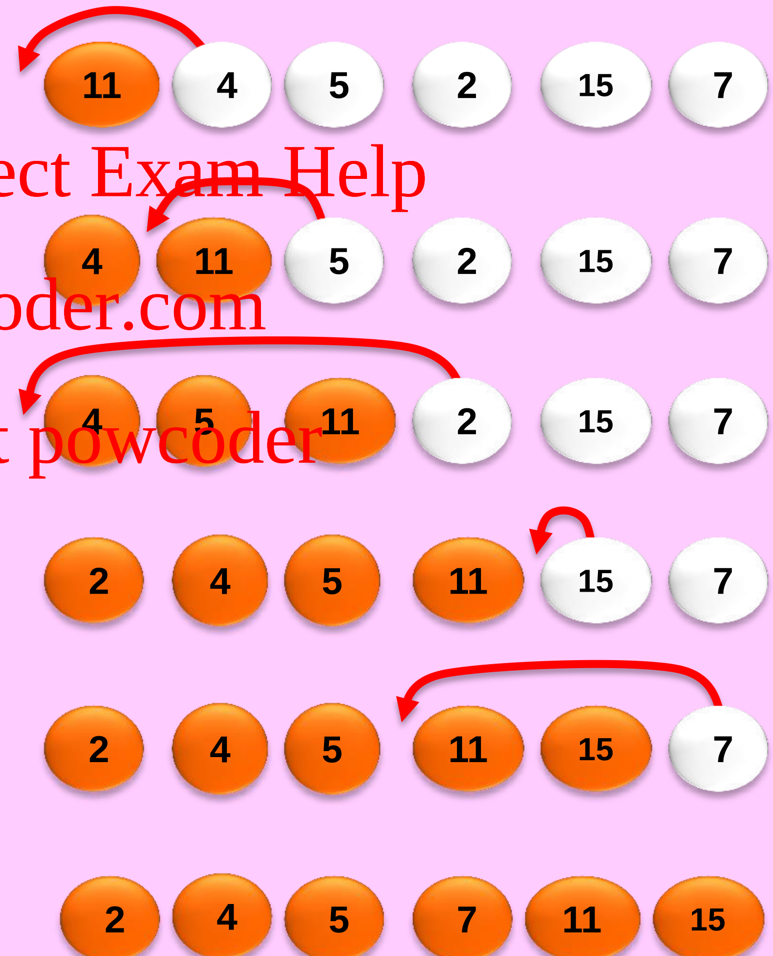
So, Merge-Sort and Heap-Sort are worst-case optimal, and

SORTING complexity is Θ(n log n).

Insertion Sort

an incremental algorithm





Insertion Sort: Time Complexity

```
Algorithm InsertionSort(A[1..n])
Pre-Condition: A[1..n] is an array of n numbers
Post-Condition: A[1..n] permuted in sorted order
    for i ← 2 .. n do
            LI: A[1..i –1] is sorted, A[i..n] is untouched.
        § insert A[i] into sorted prefix A[1..i–1] by right-cyclic-shift:
            key ← A[i]
            j ← i –1
            while j > 0 and A[j] > key do
                A[j+1] ← A[j]
                j ← j –1
            end-while
            A[j+1] ← key
    end-for
end
```


$$T(n)$$

$$= \Theta\left(\sum_{i=2}^{n} (1 + t_i + 1) \right)$$

$$= \Theta\left(n + \sum_{i=2}^{n} t_i \right)$$

$$= \Theta\left(n + \sum_{i=2}^{n} i \right)$$

$$= \Theta\left(n + n^2\right)$$

$$= \Theta\left(n^2\right).$$

Worst-case: $t_i = i$ iterations (reverse sorted).

$$\sum_{i=2}^{n} i = \frac{n(n+1)}{2} - 1 = \Theta\left(n^2\right).$$

Exercises

1. We listed a number of facts and showed the proofs of some of them. Prove the rest.
2. Give the most simplified answer to each of the following with a brief explanation.
   a) $f(n) = 4n^3 + 6n^2 \log n + 1200 = \Theta(?)$
   b) $f(n) = \dfrac{5n^6 + O(n^3 \log n)}{3n^2 \log n + O(n)} = \Theta(?)$
   c) $f(n) = 109n^2 + 10^{10^{10}} n \log n + 3 \cdot 2^n = \Theta(?)$
   d) $f(n) = n^{\frac{\log \log n}{\log n}} = \Theta(?)$
   e) Does $f(n) = o(n) + \Theta(n^2 \log n)$ imply $f(n) = \Omega(n \log \log n)$?
   f) Indicate whether or not each function below is $\omega(n \log n)$:

      $3n + 2, \quad 5n^2 / \log^3 n, \quad \log\big((n^2)!\big)$.
   g) Let $f(n) = 3n^6 / \log^{15} n + 7n\sqrt{n}$. Is $f(n) = n^{\Theta(1)}$?
3. Are there any differences between $\Theta(1)^n$ and $\Theta(2^n)$ and $2^{\Theta(n)}$? Explain.
4. Are any of the following implications always true? Prove or give a counter-example.
   a) $f(n) = \Theta(g(n)) \Rightarrow f(n) = cg(n) + o(g(n))$, for some real constant $c > 0$.
   b) $f(n) = \Theta(g(n)) \Rightarrow f(n) = cg(n) + O(g(n))$, for some real constant $c > 0$.
5. Show that $2^{O(\log \log n)} = \log^{O(1)} n$.
6. **Prove or disprove:** $n^5 = \Theta(n^{5+o(1)})$.

7. [CLRS: Problem 3-4, p. 62] Asymptotic notation properties

Let f(n) and g(n) be asymptotically positive functions.

Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

(b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

(c) $f(n) = O(g(n))$ implies $\log(f(n)) = O(\log(g(n)))$,

where $\log(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.

(d) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.

(e) $f(n) = O((f(n))^2)$.

(f) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.

(g) $f(n) = \Theta(f(n/2))$.

(h) $f(n) + o(f(n)) = \Theta(f(n))$.

8. We have: $2^{2^{\log\log n}} = 2^{\log n} = n,$ and

$$2^{2^{\log\log n - 1}} = \left(2^{2^{\log\log n}}\right)^{\frac{1}{2}} = \sqrt{n}\,.$$

So, is $2^{2^{\lfloor \log\log n \rfloor}}$ equal to $\Theta(n)$ or $\Theta(\sqrt{n}\,)$? Which one?

9. **Prove or disprove:** The following implications hold.

(a) $f(n) = \Theta(g(n)) \Rightarrow f(n)^{h(n)} = \Theta(\, g(n)^{h(n)}\,)$.

(b) $f(n) = \Theta(g(n))$ and $h(n) = \Theta(1) \Rightarrow f(n)^{h(n)} = \Theta(\, g(n)^{h(n)}\,)$.

[Note: Compare with "Power Rule".]

10. Is it true that for every pair of functions f(n) and g(n) we must have at least one of: $f(n) = O(g(n))$ or $g(n) = O(f(n))$?

[Hint: Consider $f(n) = \begin{cases} n & for\ odd\ n \\ n^2 & for\ even\ n \end{cases}$, $g(n) = \begin{cases} n^3 & for\ odd\ n \\ n & for\ even\ n \end{cases}$.]

11. Demonstrate two functions f: $\mathcal{N} \to \mathcal{N}$ and g: $\mathcal{N} \to \mathcal{N}$ with the following properties: Both f and g are strictly monotonically increasing, and $f \neq O(g)$, and $g \neq O(f)$.

12. Show that $\left(1 + \frac{1}{\Theta(n)}\right)^{\Theta(n)} = \Theta(1)$.

[*Hint*: $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = e \approx 2.7182$ *is Neper's constant.*]

13. Rank the following functions by asymptotic order of growth. Put them in equivalence classes based on Θ-equality. That is, f and g are in the same class iff $f = \Theta(g)$, and f is in an earlier class than g if $f = o(g)$. [*Hint: see also the previous exercise.*]

$$5, \quad \sqrt{\log n}, \quad \log n, \quad n^{\log n}, \quad \log^{n} n, \quad \log(n!), \quad (\lceil \log n \rceil)!$$

$$n^{3/\log n}, \quad n^{\log\log n}, \quad \sqrt{4^{n}}, \quad \sqrt{4}^{\sqrt{n}}, \quad 2^{2^{n+3}}, \quad 2^{2^{n-3}},$$

$$\left(\frac{n^4 + 4n + 1}{n^4 + 1}\right)^{6n^3+5}, \quad \left(\frac{n^2 + 4n + 1}{n^2 + 1}\right)^{n\log n}, \quad \left(\frac{2n+1}{2n}\right)^{2n^2} n^{-4}.$$

14. We say a function f: $\mathcal{N} \to \Re^+$ is **asymptotically additive** if $f(n) + f(m) = \Theta(f(n+m))$. Which of the following functions are asymptotically additive? Justify your answer.

   i) Logarithm: $\log n$, (Assume $n > 1$.)
   ii) Monomial: n^d, for any real constant $d \geq 0$ (this includes, e.g., $\sqrt{n}$ with d=0.5.)
   iii) Harmonic: $1/n$. (Assume $n > 0$.)
   iv) Exponential: a^n, for any real constant $a > 1$.

Prove or disprove: If f and g are asymptotically additive and are positive. then so is:

   i) $a \cdot f$, for any real constant $a > 0$,
   ii) $f + g$,
   iii) $f - g$, assuming $(f - g)(n)$ is always positive. Extra credit is given if $f - g$ is monotonically increasing.
   iv) $f \cdot g$.
   v) f^{g}.

15. The **Set Disjointness Problem** is defined as follows: We are given two sets A and B, each containing n arbitrary numbers. We want to determine whether their intersection is empty, i.e., whether they have any common element.

   a) Design the most efficient algorithm you can to solve this problem and analyze worst-case time complexity of your algorithm.
   b) What is the time complexity of the Set Disjointness Problem itself?
   [You are not yet equipped to answer part (b).
   The needed methods will be covered in Lecture Slide 5.]

END