15-351 / 15-650 / 02-613 Homework #6
Due: Friday, Dec. 4 by 11:59pm

You may discuss these problems with your current classmates, but you must write up your solutions independently, without using common notes or worksheets. You must indicate at the top of your homework who you worked with. Your write up should be clear and concise. You are trying to convince a skeptical reader that your answers are correct. Avoid pseudocode if possible. Your homework should be submitted via **GradeScope** as a typeset PDF. A LaTeX tutorial and template are available on the class website if you choose to use that system to typeset.

1. "**Thom's Problem.**" On graduate school visit days, professors need to meet with visiting students to explain to them how great CMU is. Visiting students have certain professors they want to meet with and professors have certain visiting students they want to impress. Over visit day, there are meetings at various fixed time slots, and some professors and students are only available at certain times. Your task is to find an assignment of students, professors, and times that allows as many meetings to take place as possible.

   More formally, we have:

   - A set of visiting students $S = \{s_1, \dots, s_m\}$.
   - A set of professors $P = \{p_1, \dots, t_n\}$.
   - A set of time slots $T = \{t_1, \dots, t_k\}$.
   - A collection of sets $A_{p_i} \subseteq T$, where A_{p_i} gives the time slots that professor p_i is available.
   - A collection of sets $A_{s_i} \subseteq T$, where A_{s_i} gives the time slots that student s_i is available.
   - A collection of sets $M_{p_i} \subseteq S$, where M_{p_i} gives the students with whom professor p_i is interested in meeting.
   - A collection of sets $M_{s_i} \subseteq P$, where M_{s_i} gives the professors that visiting student s_i wants to meet.

   Professor p_i will only meet with students on her list M_{p_i} and student s_i will only meet with professors on their list M_{s_i}. (We'll relax these requirements later.) A meeting can only happen at time t_i if both the student and professor are available. At any time, each student can meet with at most 1 professor and each professor can meet with at most 1 student.

   Example input:

Professor Meeting Preferences:

	Steve	Tom	Uri	Wei	Xin	Yang	Zhang
Prof. Abe	x			x		x	
Prof. Bob	x	x	x				
Prof. Carl	x			x		x	
Prof. Dave	x	x	x	x	x	x	x

Student Meeting Preferences:

	Prof. Abe	Prof. Bob	Prof. Carl	Prof. Dave
Steve			x	
Tom		x	x	
Uri	x	x		x
Wei			x	
Xin			x	
Yang			x	
Zhang			x	

Everyone's availability:

	2:00-2:30	2:30-3:00	3:00-3:30	3:30-4:00	4:00-4:30
Prof. Abe	x			x	
Prof. Bob	x	x	x		
Prof. Carl	x			x	
Prof. Dave		x		x	x
Steve	x	x		x	x
Tom		x	x	x	x
Uri	x		x	x	x
Wei	x	x		x	x
Xin	x		x	x	x
Yang	x	x	x	x	x
Zhang	x	x	x	x	x

(a) Write an integer linear program that finds maximum the number of (prof, student, time) meetings that can happen.

(b) Explain how to get the actual meeting schedule from your integer linear program.

(c) Suppose you are now also given, for each professor, a "bigshot score" b_{p_i} that is a positive number that says how important it is to satisfy that professor's demands. For every meeting that professor p_i takes with a student in her preference list M_{p_i}, you get b_{p_i} points. Describe how to modify your integer linear program to find a schedule that maximizes the number of bigshot points your schedule gets.

For the following NP-completeness proof problems, you can only assume that Vertex Cover, Set Cover, Independent Set, 3-SAT, Hamiltonian Cycle, and Hamiltonian Path, Traveling Salesman Problem, which we discussed in class, are NP-complete.

2. Let Integer Linear Programming be the decision problem asking whether a given maximization integer linear program has a solution of objective value $\geq$ a given k. Prove Integer Linear Programming is NP-complete.

3. A double-Hamiltonian traversal in an undirected graph G is a closed walk that goes through every vertex in G exactly *twice*.

   Prove that the problem of testing whether a graph has a double-Hamiltonian traversal is NP-complete.

4. Suppose $G = (V, E)$ is an undirected graph. A *strongly independent set* is a subset S of vertices such that for any two vertices $u, v \in S$ there is no path of length ≤ 2 between u and v. Consider the following Strongly Independent Set problem: Given an undirected graph $G = (V, E)$ and an integer k, does G have a strongly independent set of size k?

   Prove that the Strongly Independent Set problem is NP-complete.

5. The Subgraph-Isomorphism Problem takes two undirected graphs G_1 and G_2 and asks whether G_1 is a subgraph of G_2. In other words, we ask whether there is an injective function f to map the vertices of G_1 to the vertices of G_2 such that there is an edge $\{u, v\}$ in G_1 exactly when $\{f(u), f(v)\}$ is in G_2.

   Prove that Subgraph-Isomorphism Problem is NP-complete.

6. The Required Paths Subgraph problem takes as input an undirected graph G with nodes $v_1, \ldots, v_n$, an $n \times n$ symmetric matrix R of natural numbers, and an integer b, and asks "Is there a set S of b edges of G with the following property: between every pair of nodes v_i and v_j, $i \neq j$, there are at least R_{ij} disjoint paths (that is, paths sharing no other node except for the endpoints) using edges in S."

   Prove that this problem is NP-complete.