15-351 / 15-650 / 02-613 (Fall 2020): Some Solutions Sketches for the Practice Exam

1

1. False. When the graph is a spanning tree, every edge is used.
2. False. We need a reduction from 3-SAT to X.
3. The minimum capacity of a cut = the maximum possible flow.
4. False. See Quiz 5 Q8.
5. The output of X is always at most $3 \cdot OPT$ where OPT is the optimal value of the minimization problem.
6. $|V| - k$.

2

(a) We follow the DFS tree, except that we go over and then back i[illegible] every time we find a back-edge.

(b) We visit nodes in any topological order. When we visit a node u, its parents and other ancestors must have been visited and their distances have been found. So we set

$$dist(u) = \min_{v \in par(u)} dist(v) + d(u,v)$$

3

Let $OPT(k,i)$ be the minimum maximum weights if we hold the first i books using k shelves. The final answer is $OPT(K,n)$. Base: $OPT(k > 0, 0) = 0$ and $OPT(0, i \geq 1) = +\infty$. Recurrence:

$$OPT(k,i) = \min_{1 \leq j \leq i} \left\{ \max \left\{ OPT(k-1, j-1), \sum_{p=j}^{i} w_p \right\} \right\}$$

Running time: there are $O(nk)$ subproblems, each taking $O(n)$ time, so the overall running time is $O(n^2 k)$.

4

Let $OPT(j)$ be the maximum score we can get from the first j rows and columns (position on the diagonal). Base case: $OPT(0) = 0$, and recurrence:

$$OPT(j) = \max_{i: 1 \leq i < j} \{-\alpha + f(B[i \ldots j]) + OPT(i)\}$$

The final answer is $OPT(n)$.

Runtime is $O(n^2)$ if a assume every call to f takes constant time.

5

We show INDEPENDENT SET $\leq_P$ HYPER-COMMUNITY. For a INDEPENDENT SET instance with graph $G = (V,E)$ and parameter k, we create a HYPER-COMMUNITY instance $(G' = (V,E'),k)$ by inversing the edge set, that is, an edge is in E' if and only if it is not in E. Since the graph in HYPER-COMMUNITY is directed, we replace every undirected edge in E' with two directed edges with opposite directions. In this way, $U \subseteq V$ is an independent set if and only if U is a hyper community in G'.

6

We show HAMILTONIAN CYCLE $\leq_P$ REQUIRED PATHS SUBGRAPH.

Reduction: Assume the Hamiltonian Cycle instance is $G = (V,E)$. We set $R_{ij} = 2$ for any two different vertices and set $R_{ii} = 0$ for any vertex. And we set $b = |V|$.

Correctness: If there is a Hamiltonian Cycle, this cycle contains exactly b edges and the set of these edges is a feasible solution to the REQUIRED PATHS SUBGRAPH instance, because every pair of vertices are in a cycle. Conversely, assume a set of b or fewer edges satisfies the constraints of the REQUIRED PATHS SUBGRAPH instance, This means every pair of vertices are connected and are in a cycle, indicating the set has exactly b edges. (Otherwise, the graph is a tree without cycles or is not a connected component.) This further implies there is exactly one cycle which contains all vertices because there is exactly one cycle. Hence, the b edges form a Hamiltonian cycle.

7

The k persons in P for different persons in D are disjoint. In the graph, we have a super-source s, a super-sink t, one node d_i for the ith person in D, one node p_i for the ith person in P, and two auxiliary sinks t^+, t^- controlling the number of smokers. About edges:

1. s is connected to every d_i with capacity k
2. d_i is connected to p_j iff the two persons share at least m mutations, and the capacity is one if connected
3. p_j is connected to t^+ if he's a smoker, and to t^- otherwise, with capacity 1 in either case
4. t^+ and t^- are connected to t with capacities $k|D|/3$ and $+\infty$, respectively

We select p_j if and only if a flow passes p_j.

8

We show two possible reductions, one from SET COVER and another from VERTEX COVER.

Possible solution #1: We show SET COVER $\leq_P$ DOMINATING SET.

Reduction: For SET COVER instance $(U, \{S_i\}_i, k)$, we create a graph $G = (V,E)$. We create a node s_i for each set S_i and a node u_i for each element in U. We connect nodes s_i and u_j by an edge with capacity if the jth element in U is in S_i. In addition, we create an extra s node that is connected to any other s nodes but not to any u nodes. We also connect every pair of s nodes. And the DOMINATING SET instance is (G,k).

Correctness: We first show, if the SET COVER instance has answer yes, then the answer to the DOMINATING SET is also yes. To show this, we construct a subset of nodes by select node s_i iff set S_i is selected.

Conversely, we must have selected some s nodes in order to cover the extra s node. If we selected a u node, we replace it with any s nodes to which it connects. After a series of transformations, all nodes are covered and the total number of nodes doesn't increase. So the corresponding set of subsets is a valid set cover.

Possible solution #2 (sketch): We reduce Vertex Cover (VC) to DS to show that it is NP-complete. Given an instance of VC, with graph $G = (V,E)$ and parameter k, construct a new graph G'. First add all vertices and edges of G to G'. Then for each edge $e(u,v) \in E$, add a node w to V' and add edge (u,w) and (v,w) to E'.