Midterm

Name: ______________________________

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Problem	Score	Max
1		20
2		20
3		30
4		30
Total		100

Notes:

- Precision and thoroughness are both appreciated. Reaching the correct answer without the appropriate justification or incorrect reasoning will be penalized, heavily. Overly long solutions, while discouraged, wont be penalized.
- We have proof-read the problem set multiple times to ensure there are no bugs or missing information. Some details are however left out intentionally and are for you to figure out. So, if you really think some piece of information is missing and need to make an assumption to solve the problem — please go ahead; there is no need to run it by the instructors. Do not forget to mention the assumption made for solving the problem.

1. **(5 pts)** Let the radon transform of an image $i_1(x, y)$ be $r_1(\alpha, \theta)$.

   Derive an expression for $r_f(\alpha, \theta)$, the radon transform of $f(x, y)$ defined as follows:

$$f(x, y) = \frac{d}{dx} i_1(x, y).$$

Solution. Note that $i_1(x, y) \xleftrightarrow{2DFT} I_1(u, v)$. Hence,

$$f(x, y) \xleftrightarrow{2DFT} F(u, v) = j2\pi u I_1(u, v).$$

Recall from the Fourier Slice theorem that

$$r(\alpha, \theta) = r_\theta(\alpha) \xleftrightarrow{1DFT} R_\theta(\omega) = I_1(\omega\cos\theta, \omega\sin\theta).$$

Let $r_f(\alpha, \theta) = g_\theta(\alpha) \xleftrightarrow{1DFT} G_\theta(\omega)$.
Hence,

$$\begin{aligned} G_\theta(\omega) &= j2\pi\omega\cos\theta I_1(\omega\cos\theta, \omega\sin\theta) \\ &= \cos\theta\,(j2\pi\omega \cdot R_\theta(\omega)) \\ &= \cos\theta \cdot \mathrm{FT}\left(\frac{d}{d\alpha} r_\theta(\alpha)\right) \end{aligned}$$

Hence, the radon transform of $f(x, y)$, denoted at $r_f(\alpha, \theta)$ is given as

$$r_f(\alpha, \theta) = \cos\theta \frac{d}{d\alpha} r_1(\alpha, \theta)$$

2. **(5 pts)** We derived in class that the solution to the regularized least squares problem:

$$\arg\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|^2 + \lambda\|\mathbf{x}\|^2$$

is

$$\widehat{\mathbf{x}} = (A^\top A + \lambda I)^{-1} A^T \mathbf{y}.$$

Let A be an $M \times N$ matrix. Suppose that we are given the singular value decomposition of A, i.e, we are given the left and right singular vectors $U \in \mathbb{R}^{M\times M}$ and $V \in \mathbb{R}^{N\times N}$, and the diagonal matrix of singular values $S \in \mathbb{R}^{M\times N}$ such that

$$A = USV^T.$$

Derive (and simply) an expression for $\widehat{\mathbf{x}}$ in terms of $\mathbf{y}, U, S, V$ and λ.

Solution.

$$\begin{aligned}
A^\top A &= VS^TU^\top USV^\top \\
&= V\Sigma V^\top \\
&\text{where } \Sigma \text{ is a } N\times N \text{ diagonal matrix with squares of the singular values of } A \\
A^\top A + \lambda I &= V\Sigma V^\top + \lambda I \\
&= V\Sigma V^\top + VV^\top \\
&= V(\Sigma + \lambda I)V^\top \\
(A^\top A + \lambda I)^{-1} &= (.V(\Sigma + I)V^\top)^{-1} \\
&= (V^\top)^{-1}(\Sigma + I)^{-1}(V)^{-1} \\
&= V(\Sigma + I)^{-1}V^\top \\
\widehat{\mathbf{x}} &= V(\Sigma + \lambda I)^{-1}V^\top VS^\top U^\top \mathbf{y} \\
&= V(\Sigma + \lambda I)^{-1}S^\top U^\top \mathbf{y}
\end{aligned}$$

Note that the matrix $(\Sigma + \lambda I)^{-1}S^\top$ is also a diagonal matrix whose (j,j)-th entry is

$$\frac{S_{jj}}{S_{jj}^2 + \lambda}.$$

3. **(5 pts)** Let $x[n]$ be a N_0-length DT signal. Let $d[k]$ be its DCT-II transform defined as

$$d[k] = 2\sum_{n=0}^{N_0-1} x[n]\cos\left(\frac{\pi k}{N_0}(n+1/2)\right).$$

Derive an expression for the energy of $x[n]$

$$\sum_{n=0}^{N_0-1} |x[n]|^2$$

in terms of its DCT-II coefficients $d[k]$.

Solution. From Parseval's theorem, we know that energy of a signal in TD is a scaled version of its energy of its DTFS coefficients.

In this case it means

$$2\sum_{n=0}^{N_0-1} |x[n]|^2 = \frac{1}{4N_0}\sum_{k=-N_0}^{3N_0-1} |X[k]|^2.$$

Recall that

$$d[N_0] = 0, \qquad d[k] = d[-k], \qquad d[k+2N_0] = -d[k].$$

Hence,

$$2\sum_{n=0}^{N_0-1} |x[n]|^2 = \frac{1}{4N_0}\left(2|d[0]|^2 + 4\sum_{k=1}^{N_0-1} |d[k]|^2\right)$$

Hence

$$\sum_{n=0}^{N_0-1} |x[n]|^2 = \frac{1}{4N_0}|d[0]|^2 + \frac{1}{2N_0}\sum_{k=1}^{N_0-1} |d[k]|^2.$$

4. **(5 pts)** Images are generally positive and this implies that their DC term is the most dominant Fourier coefficient. We will explore this in this problem.

   Let $x[n]$ be a real positive signal of length N. That is, $x[n] \geq 0$. Let $X[k]$ be the DTFS of the signal $x[n]$. Show that the

$$X[0] \geq |X[k]|.$$

Solution.

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \\
|X[k]| &= |\sum x[n] e^{-j2\pi kn/N}| \\
&\leq \sum |x[n] e^{-j2\pi kn/N}| \\
&= \sum |x[n]| \\
&= \sum x[n] \\
&= X[0]
\end{aligned}$$