Midterm

Name: ____________________

Andrew ID: ____________________

Problem	Score	Max
1		20
2		20
3		30
4		30
Total		100

Notes:

- Precision and thoroughness are both appreciated. Reaching the correct answer without the appropriate justification or incorrect reasoning will be penalized, heavily. Overly long solutions, while discouraged, wont be penalized.
- We have proof-read the problem set multiple times to ensure there are no bugs or missing information. Some details are however left out intentionally and are for you to figure out. So, if you really think some piece of information is missing and need to make an assumption to solve the problem — please go ahead; there is no need to run it by the instructors. Do not forget to mention the assumption made for solving the problem.

1. **(20 pts)** Suppose that a 2D image $f(\mathbf{x})$ has 2D Fourier transform $F(\mathbf{u})$.

   Let A be a 2×2 invertible matrix and, via the SVD, let $A = USV^T$ where U and V are rotation matrices and $S = \text{diag}[s_1, s_2]$ is a diagonal matrix.

   Show that the 2D Fourier transform of $f(A\mathbf{x})$ can be written as $c_0 F(B\mathbf{u})$, where c_0 is a scalar and B is a 2×2 matrix. Derive an expression for c_0 and B in terms of the singular values and singular vectors of A.

   Solution. Recall that if R is a rotation matrix, then

$$f(R\mathbf{x}) \xleftrightarrow{FT} F(R\mathbf{u}).$$

   Hence,

$$f_1(\mathbf{x}) = f(U\mathbf{x}) \xleftrightarrow{FT} F_1(\mathbf{u}) = f(U\mathbf{u})$$

   Next,

$$\begin{aligned} f_2(\mathbf{x}) = f(US\mathbf{x}) = f_1(S\mathbf{x}) = f_1(s_1 x, s_2 y) \xleftrightarrow{FT} F_2(\mathbf{u}) &= \frac{1}{s_1 s_2} F_1(u/s_1, v/s_2) \\ &= \frac{1}{s_1 s_2} F_1(S^{-1}\mathbf{u}) \\ &= \frac{1}{s_1 s_2} F(US^{-1}\mathbf{u}) \end{aligned}$$

   Note that both s_1 and s_2 are non-negative (singular values are by definition) and non-zero (since A is invertible).

   Finally,

$$f_3(\mathbf{x}) = f(USV^T\mathbf{x}) = f_2(V^T\mathbf{x}) \xleftrightarrow{FT} F_3(\mathbf{u}) = F_2(V^T\mathbf{u})$$

   Putting it all together we get,

$$f(A\mathbf{x}) = f(USV^T\mathbf{x}) \xleftrightarrow{FT} \frac{1}{s_1 s_2} F(US^{-1}V^T\mathbf{u})$$

   Hence, $c_0 = 1/(s_1 s_2) = 1/|\det(A)|$ and $B = US^{-1}V^T = (A^{-1})^T$.

   Alternate solution. We can directly compute the FT of $g(\mathbf{x}) = f(A\mathbf{x})$ as follows:

$$\begin{aligned} G(\mathbf{u}) = \int g(\mathbf{x}) e^{-j2\pi \mathbf{u}^\top \mathbf{x}} d\mathbf{x} &= \int f(A\mathbf{x}) e^{-j2\pi \mathbf{u}^\top \mathbf{x}} d\mathbf{x} \\ &= \int f(\mathbf{z}) e^{-j2\pi \mathbf{u}^\top A^{-1}\mathbf{z}} |\det(A^{-1})| d\mathbf{z} \quad \text{(substituting } A\mathbf{x} = \mathbf{z}) \\ &= \frac{1}{|\det(A)|} F(A^{-\top}\mathbf{u}) \end{aligned}$$

   Hence, $c_0 = 1/(s_1 s_2) = 1/|\det(A)|$ and $B = US^{-1}V^T = (A^{-1})^T$.

2. **(20 pts)** Suppose that the signal $\mathbf{x} = \{x[0], \ldots, x[n], \ldots, x[N-1]\}$ has DCT-II coefficients $\mathbf{d} = \{d[0], \ldots, d[k], \ldots, d[N-1]\}$.

   Let
$$\widetilde{\mathbf{x}} = \{x[N-1], x[N-2], ..., x[1], x[0]\},$$
   that is, $\widetilde{x}[n] = x[N-1-n]$.

   Compute the DCT-II coefficients of $\widetilde{\mathbf{x}}$ in terms of $\mathbf{d}$.

   Solution.

$$\mathbf{d}[k] = 2\sum_{m=0}^{N-1} \mathbf{x}[m] \cos\left(\frac{\pi}{N}k\left(m + \frac{1}{2}\right)\right)$$
$$\tilde{\mathbf{d}}[k] = 2\sum_{l=0}^{N-1} \tilde{\mathbf{x}}[l] \cos\left(\frac{\pi}{N}k\left(N + \frac{1}{2}\right)\right)$$

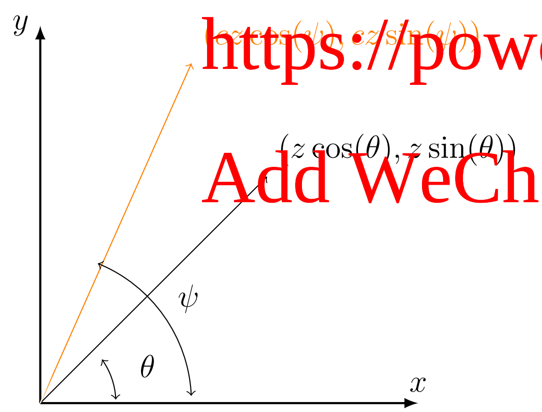
Note $\tilde{\mathbf{x}}[l] = \mathbf{x}[N-1-l]$. Let $m = N-1-l$, then $\tilde{\mathbf{x}}[l] = \mathbf{x}[m]$, and $l = N-1-m$.

$$\begin{aligned}
\tilde{\mathbf{d}}[k] &= 2\sum_{m=0}^{N-1} \mathbf{x}[m] \cos\left(\frac{\pi}{N}k\left((N-1-m) + \frac{1}{2}\right)\right) \\
&= 2\sum_{m=0}^{N-1} \mathbf{x}[m] \cos\left(\pi k - \frac{\pi}{N}k\left(m + \frac{1}{2}\right)\right) \\
&= 2\sum_{m=0}^{N-1} \mathbf{x}[m] \left(\sin(\pi k)\sin\left(\frac{\pi}{N}k\left(m + \frac{1}{2}\right)\right) + \cos(\pi k)\cos\left(\frac{\pi}{N}k\left(m + \frac{1}{2}\right)\right)\right) \\
&= (-1)^k\, 2\sum_{m=0}^{N-1} \mathbf{x}[m] \cos\left(\frac{\pi}{N}k\left(m + \frac{1}{2}\right)\right) \\
&= (-1)^k\, \mathbf{d}[k]
\end{aligned}$$

3. **(30 pts)** Suppose that image $f(x, y)$ has radon transform $r(\alpha, \theta)$.

- Find the radon transform of $f(\frac{x}{a}, \frac{y}{a})$, $\widetilde{r}_1(\alpha, \theta)$. Express $\widetilde{r}_1(\alpha, \theta)$ in terms of r and a.
- Find the radon transform of $f(\frac{x}{a}, \frac{y}{b})$, $\widetilde{r}_2(\alpha, \theta)$. Express $\widetilde{r}_2(\alpha, \theta)$ in terms of r, a, and b.

Solution: Observe that scaling of space leads to scaling in frequency. By Fourier slice theorem, 1D FT of line integrals are slices of the 2D FT of the image. When we compute line integrals of scaled images, then we are simply measuring along slices of the scaled 2D FT. Since, radial lines remain radial and linear under scaling, it is clear that we should be able to relate the radon transform directly.



$$f(x, y) \xrightarrow{\text{FT}} F(u, v) \tag{1}$$

$$f\left(\frac{x}{a}, \frac{y}{b}\right) \xrightarrow{FT} |a||b|F(au, bv) \tag{2}$$

$$g_\theta(\alpha) \xrightarrow{FT} G_\theta(z) = F(\cos(\theta)z, \sin(\theta)z) \tag{3}$$

$$\tilde{g}_\theta(\alpha) \xrightarrow{FT} \tilde{G}_\theta(z) = |a||b|F(a\cos(\theta)z, b\sin(\theta)z) \tag{4}$$

$$\tag{5}$$

Let $\cos(\psi) = \frac{a}{c}\cos(\theta)$ and $\sin(\psi) = \frac{b}{c}\sin(\theta)$ for some scalar c. To find ψ using a, b, θ, we have $\tan(\psi) = \frac{b\sin(\theta)}{a\cos(\theta)} = \frac{b}{a}\tan(\theta)$.

From $\cos^2(\psi)+\sin^2(\psi) = 1$, we know that $c = \sqrt{a^2\cos^2(\theta) + b^2\sin^2(\theta)}$. ψ is well defined because by triangular inequality $c \geq a\cos(\theta), c \geq b\sin(\theta)$, $|\cos(\psi)| \leq 1, |\sin(\psi)| \leq 1$

$$
\begin{aligned}
F(a\cos(\theta)z, b\sin(\theta)z) &= F\left(\frac{a}{c}\cos(\theta)(cz), \frac{b}{c}\sin(\theta)(cz)\right) &(6)\\
&= F(\cos(\psi)(cz), \sin(\psi)(cz)) &(7)
\end{aligned}
$$

From a geometric perspective, we are stretching (or shrinking) the unit vector $[z\cos(\theta), z\sin(\theta)]$ by a on the x-direction, and by b on the y-direction. The new vector $[az\cos(\theta), bz\sin(\theta)]$ now has length $c = \sqrt{a^2\cos^2(\theta) + b^2\sin^2(\theta)}$. The new angle ψ has $\tan(\psi) = \frac{b\sin(\theta)}{a\cos(\theta)}$. Using Eq.(5) and (7) we have:

$$
\begin{aligned}
\tilde{g}_\theta(\alpha) &\xrightarrow{FT} \tilde{G}_\theta(z) = |a||b|F(\cos(\psi)(cz), \sin(\psi)(cz)) &(8)\\
\tilde{g}_\theta(\alpha) &\xrightarrow{FT} |a||b|G_\psi(cz) &(9)\\
\tilde{g}_\theta(\alpha) &= \frac{|a||b|}{|c|} g_\psi\left(\frac{\alpha}{c}\right) &(10)
\end{aligned}
$$

For the special case $a = b$, we have $\tan(\psi) = \tan(\theta)$ therefore $\psi = \theta$ and $c = \sqrt{a^2\cos^2(\theta) + a^2\sin^2(\theta)} = |a|$

$$
\begin{aligned}
\tilde{g}_\theta(\alpha) &= \frac{|a||b|}{|c|} g_\psi\left(\frac{\alpha}{c}\right)\\
&= |a| g_\theta\left(\frac{\alpha}{a}\right)
\end{aligned}
$$

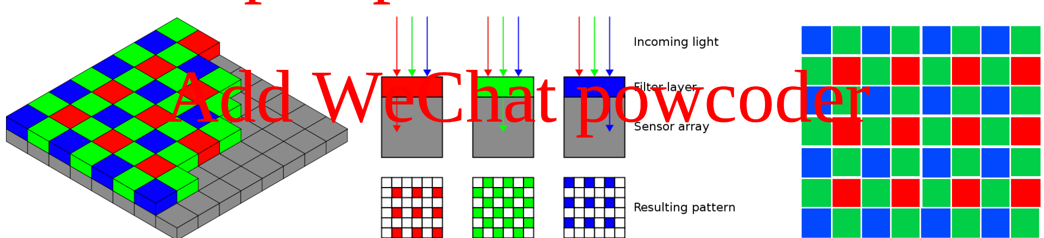
4. **(30 pts)** Derive the adjoint operators for the linear operators that appear in the following image processing problems.

   - In the super-resolution problem, we encounter the downsampling operator A_S that transforms an $N \times N$ high-resolution image to a $(N/D) \times (N/D)$ low-resolution image, where N, D and N/D are all integers. The operator $A_S : \mathbb{R}^{N \times N} \mapsto \mathbb{R}^{\frac{N}{D} \times \frac{N}{D}}$ is defined as follows.

   $$h = A_S(f) \implies h[x, y] = \sum_{i=1}^{D} \sum_{j=1}^{D} f[(x-1)D + i, (y-1)D + j]$$

   Derive the adjoint operator A_S^*.

   - Cameras sense RGB images with a single sensor using the following design. Each pixel on the sensor has a color filter that allows it to sense either red or green or blue channel. In essence, in each pixel, we select one of the color channels and sense the image in the color channel. Shown below is the so-called Bayer Pattern which refers to the specific arrangement of color filters on a sensor. In each 2×2 patch of pixels, we select two green pixels and one pixel each for red and blue, as shown here.



(a) The Bayer arrangement of color filters on the pixel array of an image sensor

(b) Profile/cross-section of sensor

(c) Color code for pixels on the sensor

Figure 1: Images and text courtesy of Wikipedia.

Suppose that you are given a three color image $I \in \mathbb{R}^{N \times N \times 3}$ where the third-dimension refers to the color channel in the order of red, green and blue. The mosiacking operator $A_M : \mathbb{R}^{N \times N \times 3} \mapsto \mathbb{R}^{N \times N}$ is defined as follows:

$$h = A_M(I) \implies h[x, y] = \begin{cases} I[x, y, 3] & x, y \text{ are both odd} \\ I[x, y, 1] & x, y \text{ are both even} \\ I[x, y, 2] & \text{otherwise} \end{cases}$$

Derive the adjoint operator A_M^*.

Solution. Recall that, given a linear operator A, its adjoint A^* is defined as follows:

$$\langle Af, g\rangle = \langle f, A^* g\rangle, \forall f, g.$$

Super resolution. Given $f \in \mathbb{R}^{N\times N}$ and $g \in \mathbb{R}^{N/D\times N/D}$, we can write:

$$\begin{aligned}
\langle Af, g\rangle &= \sum_{x,y} (Af)[x,y] g[x,y] \\
&= \sum_{x,y=1}^{N/D} \sum_{i,j=1}^{D} f[(x-1)D+i, (y-1)D+j]\ g[x,y] \\
&= \sum_{m,n=1}^{N} f[m,n] g\left[1 + \left\lfloor \frac{m-1}{D} \right\rfloor, 1 + \left\lfloor \frac{n-1}{D} \right\rfloor\right] \\
&= \langle f, A_S^* g\rangle
\end{aligned}$$

where

$$(A_S^* g)[m,n] = g\left[1 + \left\lfloor \frac{m-1}{D} \right\rfloor, 1 + \left\lfloor \frac{n-1}{D} \right\rfloor\right], \qquad 1 \le m, n \le N.$$

Basically, $A_S g$ is an image where each entry of g is "upsampled" into $D \times D$ block.

Mosaicking. Lets define the color selectionf function $I[x,y]$ as follows.

$$I[x,y] = \begin{cases} 3 & x, y \text{ are both odd} \\ 1 & x, y \text{ are both even} \\ 2 & \text{otherwise} \end{cases}$$

Hence,

$$(A_M f)[x,y] = f[x,y,I[x,y]].$$

Now, given $f \in \mathbb{R}^{N\times N\times 3}$ and $g \in \mathbb{R}^{N\times N}$, we can write:

$$\begin{aligned}
\langle A_M f, g\rangle &= \sum_{x,y} (Af)[x,y]\ g[x,y] \\
&= \sum_{x,y} f[x,y,I[x,y]]\ g[x,y] \\
&= \sum_{x,y,c} f[x,y,c]\ g[x,y]\ \mathrm{Ind}(I[x,y] = c) \quad (\mathrm{Ind}(\cdot) \text{ is the indicator function}) \\
&= \langle f, A_M^* g\rangle
\end{aligned}$$

where

$$(A_S^* g)[x,y,c] = g[x,y]\ \mathrm{Ind}(I[x,y] == c)$$

Hence, $A_S g$ is a three-color image with 2 zeros at each pixel, at the colors that were not measured.