Midterm

Name: ______________________________

Andrew ID: ______________________________

Problem	Score	Max
1		20
2		20
3		30
4		30
Total		100

Notes:

- Precision and thoroughness are both appreciated. Reaching the correct answer without the appropriate justification or incorrect reasoning will be penalized, heavily. Overly long solutions, while discouraged, wont be penalized.
- We have proof-read the problem set multiple times to ensure there are no bugs or missing information. Some details are however left out intentionally and are for you to figure out. So, if you really think some piece of information is missing and need to make an assumption to solve the problem — please go ahead; there is no need to run it by the instructors. Do not forget to mention the assumption made for solving the problem.

1. **(20 pts)** Suppose that a 2D image $f(\mathbf{x})$ has 2D Fourier transform $F(\mathbf{u})$.

   Let A be a 2×2 invertible matrix and, via the SVD, let $A = USV^T$ where U and V are rotation matrices and $S = \text{diag}[s_1, s_2]$ is a diagonal matrix.

   Show that the 2D Fourier transform of $f(A\mathbf{x})$ can be written as $c_0 F(B\mathbf{u})$, where c_0 is a scalar and B is a 2×2 matrix. Derive an expression for c_0 and B in terms of the singular values and singular vectors of A.

2. **(20 pts)** Suppose that the signal $\mathbf{x} = \{x[0], \dots, x[n], \dots, x[N-1]\}$ has DCT-II coefficients $\mathbf{d} = \{d[0], \dots, d[k], \dots, d[N-1]\}$.

   Let
   $$\widetilde{\mathbf{x}} = \{x[N-1], x[N-2], ..., x[1], x[0]\},$$
   that is, $\widetilde{x}[n] = x[N-1-n]$.

   Compute the DCT-II coefficients of $\widetilde{\mathbf{x}}$ in terms of $\mathbf{d}$.

3. **(30 pts)** Suppose that image $f(x, y)$ has radon transform $r(\alpha, \theta)$.

- Find the radon transform of $f(\frac{x}{a}, \frac{y}{a})$, $\widetilde{r}_1(\alpha, \theta)$. Express $\widetilde{r}_1(\alpha, \theta)$ in terms of r and a.
- Find the radon transform of $f(\frac{x}{a}, \frac{y}{b})$, $\widetilde{r}_2(\alpha, \theta)$. Express $\widetilde{r}_2(\alpha, \theta)$ in terms of r, a, and b.

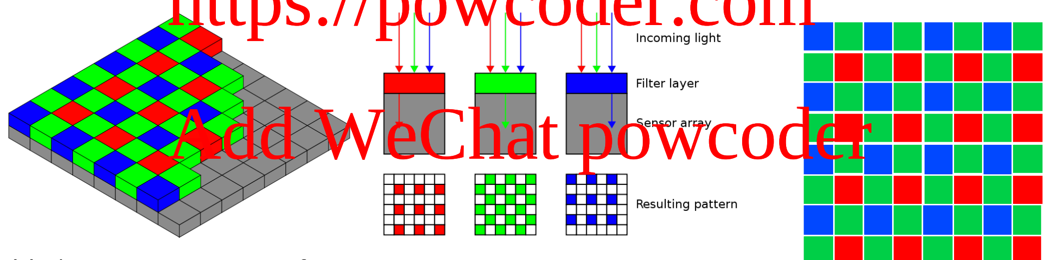
4. **(30 pts)** Derive the adjoint operators for the linear operators that appear in the following image processing problems.

- In the super-resolution problem, we encounter the downsampling operator A_S that transforms an $N \times N$ high-resolution image to a $(N/D) \times (N/D)$ low-resolution image, where N, D and N/D are all integers. The operator $A_S : \mathbb{R}^{N \times N} \mapsto \mathbb{R}^{\frac{N}{D} \times \frac{N}{D}}$ is defined as follows.

$$h = A_S(f) \implies h[x, y] = \sum_{i=1}^{D} \sum_{j=1}^{D} f[(x-1)D + i, (y-1)D + j]$$

Derive the adjoint operator A_S^*.

- Cameras sense RGB images with a single sensor using the following design. Each pixel on the sensor has a color filter that allows it to sense either red or green or blue channel. In essence, in each pixel, we select one of the color channels and sense the image in the color channel. Shown below is the so-called Bayer Pattern which refers to the specific arrangement of color filters on a sensor. In each 2×2 patch of pixels, we select two green pixels and one pixel each for red and blue, as shown here.



(a) The Bayer arrangement of color filters on the pixel array of an image sensor

(b) Profile/cross-section of sensor

(c) Color code for pixels on the sensor

Figure 1: Images and text courtesy of Wikipedia.

Suppose that you are given a three color image $I \in \mathbb{R}^{N \times N \times 3}$ where the third-dimension refers to the color channel in the order of red, green and blue. The mosiacking operator $A_M : \mathbb{R}^{N \times N \times 3} \mapsto \mathbb{R}^{N \times N}$ is defined as follows:

$$h = A_M(I) \implies h[x, y] = \begin{cases} I[x, y, 3] & x, y \text{ are both odd} \\ I[x, y, 1] & x, y \text{ are both even} \\ I[x, y, 2] & \text{otherwise} \end{cases}$$

Derive the adjoint operator A_M^*.