Coursework

Assignment weighting
15%

Assignment released
Monday, Week 8

Assignment due
Tuesday, Week 11, 23:59

Penalties for late submission
Late submissions will incur a 5% reduction in the mark for every working day (or part thereof) that the assignment is late and a mark of zero for submission more than 5 working days late.

How to submit

- Use the "Coursework (written solution)" link on Blackboard to submit your written solution.
- Use the "Coursework (MATLAB)" link on Blackboard to submit your MATLAB files in a single ZIP archive.

Feedback

You will receive feedback via Blackboard. The intention is to provide feedback two weeks after the submission deadline, but this may be changed due to extenuating circumstances forms submitted by other students.

Unfair means
The assignment should be completed individually. You should not discuss the assignment with other students and should not work together in completing the assignment. The assignment must be wholly your own work. References must be provided to any other work that is used as part of this assignment. Any suspicions of the use of unfair means will be investigated and may lead to penalties. See https://www.sheffield.ac.uk/ssid/unfair-means/index for guidance.

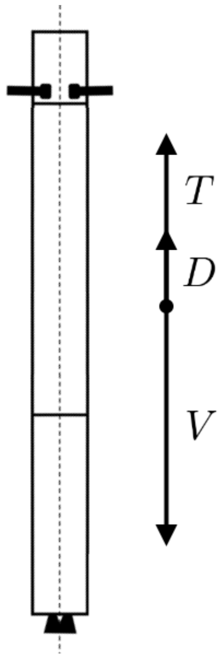
Extenuating circumstances
If you have medical or personal circumstances that cause you to be unable to submit this assignment on time or that may have affected your performance, please, complete an extenuating circumstances form and submit it to acse-support@sheffield.ac.uk along with the evidence of the circumstances. See https://www.sheffield.ac.uk/ssid/forms/circs for more information.

MATLAB help

- MATLAB Onramp
- Solving Ordinary Differential Equations with MATLAB
- Solving Boundary Value Problems with MATLAB
- Tutorial on solving BVPs with BVP4C

Assignment briefing

Problem 1. Consider the optimal landing of a rocket booster considered in Lecture 1. In Problem Set 1, we saw that, if the rocket is oriented vertically, the model becomes



$$
\begin{aligned}
\dot{r} &= -V \\
\dot{V} &= -\frac{1}{m}(T + D(r,V)) + g(r) \\
\dot{m} &= -\frac{c}{g_0}T
\end{aligned}
$$

where

- r is the radial distance from the Earth's centre;
- V is the speed;
- m is the rocket mass;
- $T \in [0, T_M]$ is the thrust magnitude (control);
- $D(r,V)$ is the drag force (a function of r and V);
- $g(r)$ is the gravitational acceleration at r;
- g_0 is the gravitational acceleration at $R_\oplus$ (Earth radius);
- c is a constant that depends on the engine type.

Assume that the booster's speed is 1100 km/h when its altitude (the distance to the surface of Earth) is 5 km. The mass of an empty booster is $25{,}600$ kg and it has $2{,}500$ kg of fuel.

a. Show that the optimal control for the minimum-time problem is "bang-bang". Write down the costate equations and boundary conditions. **[10 marks]**

b. Show that the optimal control for the minimum-fuel problem is "bang-bang". Write down the costate equations and boundary conditions. **[10 marks]**

c. The last two equations describing the booster dynamics imply

$$
\dot{V} = \frac{g_0}{c}\frac{\dot{m}}{m} - \frac{D(r,V)}{m} + g(r) = \frac{g_0}{c}\frac{d}{dt}\left[\ln m(t)\right] - \frac{D(r,V)}{m} + g(r).
$$

Assuming that $D(r,V) < mg(r)$ (the gravity is stronger than the drag), show that the consumed fuel is a monotone increasing function of the final time. (*Hint:* You need to integrate both sides from 0 to t_f and use $V(t_f) = 0$.) What does this observation allow us to conclude about the relation between the solutions of the minimum-time and minimum-fuel problems?

[15 marks]

d. The calculations above suggest that the optimal minimum-time control has the form

$$
T_*(t) = \begin{cases} 0, & t \in [0, t_s], \\ T_M, & t \in [t_s, t_f]. \end{cases}
$$

Write a MATLAB program that solves the state equations for given t_s and t_f. Use this program to find the optimal values of t_s and t_f. Plot the optimal state.

When solving the state equations, assume that the drag force is given by

$$D(r, V) = \frac{1}{2} C_D \rho(r) S V^2,$$

where

- $C_D = 0.3$ is the drag coefficient;
- $\rho(r) \equiv 1.225$ kg/m^3 is the air density (for simplicity we take it constant);
- $S = 10.75$ m^2 is the reference area.

The gravitational acceleration is given by

$$g(r) = g_0 \frac{R_\oplus^2}{r^2},$$

where

- $g_0 = 9.8$ m/s^2 is the acceleration on Earth;
- $R_\oplus = 6{,}371$ km is the Earth radius.

For the selected type of thruster, $T_M = 1{,}375.6$ kN and $c = \frac{1}{443}$ s^{-1}.

[15 marks]

Problem 2. Consider the plant

$$\dot{x}_1 = x_1 + x_2, \qquad x_1(0) = 2,$$
$$\dot{x}_2 = x_1 + u, \qquad x_2(0) = 2,$$

and the cost

$$J(u) = \frac{\alpha}{2}[x_1(2) - 4]^2 + \frac{\beta}{2}[x_2(2) - 6]^2 + \frac{1}{2}\int_0^2 \left[x_1^2(t) + x_2^2(t) + u^2(t)\right] dt.$$

a. Let $\alpha = 0$ and $x_1(2) = 4$. Express the optimal control in terms of the optimal costate. Write down the corresponding two-point boundary value problem. By solving this problem in MATLAB (use `bvp4c` or `bvp5c`), find the optimal states and costates for $\beta = 0, 0.1, 1, 10, 100$. Plot your results in four subplots each showing one state/costate component for all β. For example, `subplot(2,2,1)` must show x_1 for different β.

   On the same axes but using dashed lines, plot the optimal states and costates for the case when $\alpha = 0 = \beta$, $x_1(2) = 4$, and $x_2(2) = 6$. Add a legend, a title, and labels to each subplot. Explain what you observe and why. **[25 marks]**

b. Let $\alpha = 1 = \beta$ (free endpoint). Write down the boundary value problem to find the optimal control that guarantees

   $$(x_1(2) - 1)^2 + (x_2(2) - 1)^2 = 1.$$

   Using MATLAB, find and plot the optimal states, costates, and control. What are the values of $x_1(2)$ and $x_2(2)$? Do they satisfy the terminal condition? **[25 marks]**