

You may use your class notes, the text, or any calculus books — please use no other references (including internet or other statistics texts). If you use Mathematica to derive results, you must include the notebook with your solution so I can see what you did.

1. Estimating the parameter of a binomial distribution

I give you a coin and ask you to estimate the probability of a toss resulting in heads. You decide to construct the estimate by tossing the coin N times and counting the number of heads n .

- (a) The probability of recording n heads from N tosses is given by the *binomial distribution*

$$p(n|N, \alpha) = \binom{N}{n} \alpha^n (1 - \alpha)^{(N-n)} = \frac{N!}{n! (N - n)!} \alpha^n (1 - \alpha)^{(N-n)} \quad (1)$$

where α is the probability of heads coming up on any particular toss. It is the quantity we want to estimate.

Derive the maximum likelihood estimate of α , i.e. the value of α that maximizes the log probability (the log of Eqn. (1)) of throwing the coin N times and observing n heads. (Notice that the data for this experiment is the *single number* n .)

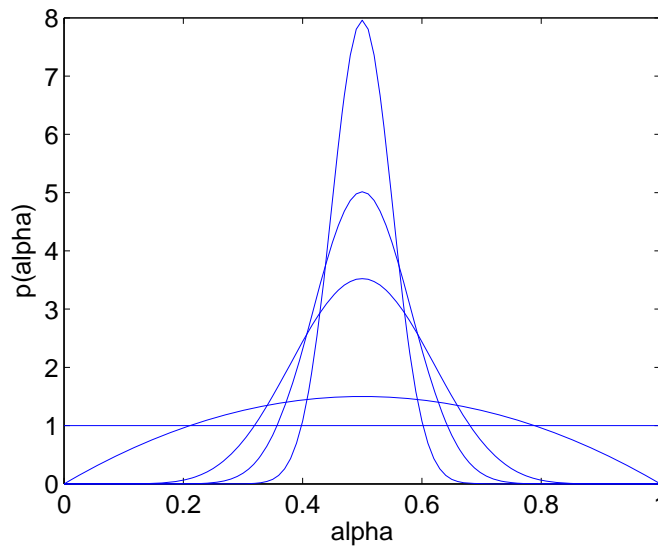
- (b) Suppose now you decide to construct a MAP estimate of α . Show that the beta distribution

$$p(\alpha) = \frac{1}{B(a, b)} \alpha^{a-1} (1 - \alpha)^{b-1} \quad \text{with } 1 \leq a, b \leq \infty \quad (2)$$

is a conjugate prior¹ for the likelihood function in (1) **and derive the MAP estimate of α** . (The normalization factor $B(a, b)$ is the Euler beta function – but you don't need knowledge of this function to complete the problem.)

- (c) Since there's no particular reason to believe that the coin I hand you is grossly unfair, you opt to pick a and b so that the prior $p(\alpha)$ is maximum at $1/2$ and symmetric about that value. To achieve this, you set $a = b$. For this choice, it's clear that with $a = b = 1$ the prior is flat. For larger values of $a = b$, the prior distribution gets progressively more peaked up about $\alpha = 1/2$; the variance of the *prior* is $\text{var}(\alpha) = 1/(4(2a + 1))$. The prior distribution is plotted below for several choices of $a = b$.

¹Recall from your notes that for a *conjugate prior* $p(\alpha)$, the posterior density $p(\alpha|D) \propto p(D|\alpha) p(\alpha)$ is of the same algebraic form (here a beta distribution) as the prior density $p(\alpha)$.



Use the formula for the MAP estimate of α that you derived in part (b), together with the above information about the beta distribution to discuss how the value of a (with $b = a$) is the prior distribution effects the MAP estimate of α relative to the maximum likelihood estimate of α . That is, discuss how the change in the shape of the prior distribution with increasing a is reflected in the MAP estimate.

2. Fitting Constrained Gaussian Models

If you're asked to fit a Gaussian distribution to a set of m , n -dimensional data points $D = (x^{(1)}, x^{(2)}, \dots, x^{(m)})$, you know that the maximum likelihood estimate of the mean and covariance for the model are

$$\hat{\mu} = \frac{1}{m} \sum_{i=1}^m x^{(i)}$$

and

$$\hat{\Sigma} = \frac{1}{m} \sum_{i=1}^m (x^{(i)} - \hat{\mu})(x^{(i)} - \hat{\mu})^T$$

respectively.

Suppose now that you are told that the *model covariance* matrix is constrained to be a (positive) *constant times the identity matrix*

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 & 0 & \dots \\ 0 & \sigma^2 & 0 & \dots \\ 0 & 0 & \sigma^2 & \dots \\ \vdots & & & \ddots \end{pmatrix} . \quad (3)$$

- Write down the log-likelihood L of the set of n -dimensional data vectors D under this model.
- Derive the maximum likelihood estimate of Σ .

3. Error, Bias, Variance and Paintball

You're making final selections for members for your paintball team in preparation for the national championship. You have your final two contestants fire six shots into a target with the results below:



(A)

(B)

Assignment Project Exam Help

Which contestant do you choose for your team? Suppose the guns have adjustable sights, does that change your decision? Discuss in statistical terms.

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