ANLY-601 Pattern Recognition

Homework 1
Due Tuesday, January 29, 2018

Use only your course notes — no internet or texts.

1. **Moments of Gaussian Densities (10 points)**

   Consider the one-dimensional Gaussian pdf

   $$p(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\ \exp - \left(\frac{(x-m)^2}{2\,\sigma^2} \right) \quad .$$

   Use the fact that

   $$\int_{-\infty}^{\infty} \exp - (\alpha\, u^2)\ \ du \ = \ \sqrt{\frac{\pi}{\alpha}}$$

   and the identity

   $$\int u^2 \exp - (\alpha\, u^2)\ \ du \ = \ - \frac{d}{d\alpha} \int \exp - (\alpha\, u^2)\ \ du$$

   to show that the *even* central moments of the Gaussian density are

   $$E\,[(x-m)^n] \ = \ (1 \cdot 3 \cdot 5 \cdot \ldots \cdot n-1)\ \sigma^n \ \text{ for } n \text{ even} \quad .$$

   Use symmetry arguments (hint: antisymmetric integrand over symmetric bounds) to show that the *odd* central moments are all zero

   $$E\,[(x-m)^n] \ = \ 0 \quad \text{for } n \text{ odd}$$

2. **Conditional and Unconditional Variance (10 points)**

   In class we showed the relationship between conditional means and unconditional means. Specifically for random variables $x \in R^N$ and $y \in R^M$, the conditional mean of x is

   $$E[x|y] \ = \ \int x\, p(x|y)\ d^N x$$

   and the *unconditional mean* is

   $$\begin{aligned} E[x] \ &= \ \int x\, p(x)\ d^N x \ = \ \int x \left(\int p(x|y)\, p(y)\ d^M y \right)\ d^N x \\ &= \ \int \left(\int x\, p(x|y)\, d^N x \right)\, p(y)\, d^M y \ = \ E_y[\, E_x[x|y]\,] \quad . \end{aligned}$$

   The relationship between the conditional variance and the unconditional variance is a bit more interesting. For simplicity, take $x \in R$ and $y \in R$ (scalar random variables). The conditional variance is

   $$\mathrm{var}(x|y) \ = \ \int (x - E[x|y])^2\, p(x|y)\, dx \quad . \tag{1}$$

(Note that like the mean, the conditional variance is a function of x_2.) Show that the unconditional variance is related to the condition variance by

$$\mathrm{var}(x) \;=\; \int (x - E[x])^2\, p(x)\, dx \;=\; E_y[\,\mathrm{var}_x(x|y)\,] \;+\; \mathrm{var}_y(\,E[x|y]\,). \tag{2}$$

Your derivation must show explicitly what $\mathrm{var}_y(\,E[x|y]\,)$ means in terms of integral averages over quantities.

(Hint: Rewrite

$$\begin{aligned}
(x - E[x])^2 \;&=\; (x - E[x] + E[x|y] - E[x|y])^2 \;=\; (x - E[x|y] + E[x|y] - E[x])^2 \\
&=\; (x - E[x|y])^2 + (E[x|y] - E[x])^2 \\
&\quad\; + 2\,(x - E[x|y])(E[x|y] - E[x]) \quad .
\end{aligned}$$

)

3. **A Maximum likelihood estimation (5 points)**

   This problem has an interesting practical origin, that I'll explain after you hand your solution back.

   I have a bag filled with m plastic balls, numbered consecutively $1, 2, \ldots, m$. I don't tell you what the value of m is; I want you to make a (statistically informed) guess.

   So I give you one piece of data. I reach into the bag and pull out one of the balls at random (i.e. with probability $1/m$) and hand it to you. It has the value "19" printed on it.

   Let's compute the *maximum likelihood* estimate of the total number of balls m. Mathematically, this is the value of m that maximizes $p(x = 19|m)$. Start by building a likelihood function — since there is one ball with each number $1, 2, 3, \ldots, 19, \ldots, m$, any number on a ball in the range $1 - m$ can be observed with equal probability

   $$p(1|m) = p(2|m) = \cdots = p(m|m) = 1/m \quad .$$

   Note also that it's not possible to observe a number on a ball greater than (the unknown) m

   $$p(n|m) = 0 \text{ for } n > m \quad .$$

   These two pieces of information fix the likelihood function $p(x|m)$. Given this information, what is the value of m that maximizes the likelihood of the data $p(19|m)$?