In the following weeks, we will cover materials in Lecture 28-34 in the textbook

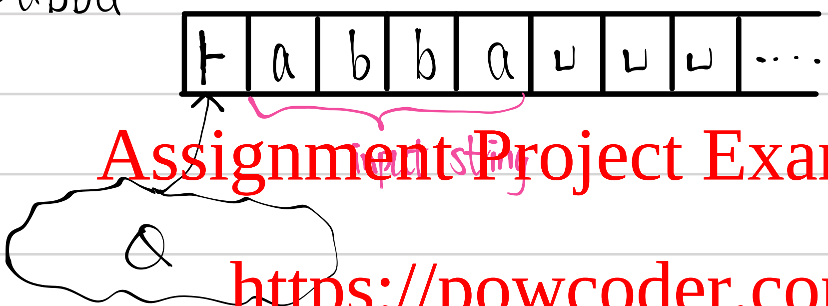
02/24 Wed Introduction to Turing Machines by Xiaoyun Fu

Turing Machines are the most powerful machines.

1. Informal Description of Turing Machines (TM)

- deterministic
- a finite set of states (just like DFAs): Q
- a one end infinite tape
- a read/write tape head that can move left (L) and right (R) over the tape

$w = abba$



- the input string is initially written on the tape in the left most cells right next to the special left endmarker (symbol) $\vdash$.
- the infinitely many cells to the right of the input all contain a special blank symbol $\sqcup$
- the machine starts in its start state $s \in Q$ with its tape head pointing to the leftmost cell (reading the $\vdash$)
- a transition function δ defines how the machine works.

  in each step, the machine reads the symbol on the tape cell under its head, depending on that symbol and the current state and δ, it writes a new symbol on that tape cell (replace the old symbol with the new symbol), and moves its head either left or right by one cell according to δ, and enters a new state.

- the machine accepts the input string if it enters a special accept state t and rejects the input string if it enters a special reject state r.

  We call r and t halting states, once the machine enters either of them, we call the machine halts on the input string.

  in some cases, the machine may run infinitely without ever accepting or rejecting. We call it loops on the input in this case.

2. DFA vs TM

DFA	TM
• can store information using states (finite memory)	• can store information using both states and tape (infinite memory)
• Can have access to the input string only once	• can have access to the input string any time by storing it on tape
• the machine always read each symbol in input string exactly once.	• the machine may read no, partial, or all symbols in input for 0, 1, or more times.
• computation always stops when the last symbol in the input is read.	• computation may go on forever without giving an accept/reject answer.

3. Formal description of Turing Machines

a deterministic one-tape Turing Machine is a 9-tuple.

$M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$

a finite set of states — the input alphabet — the tape alphabet

- $\Sigma \subseteq \Gamma$: each input symbol can appear on the tape.
- $\sqcup \in \Gamma - \Sigma$: input strings do not contain the blank symbol

  $\vdash \in \Gamma - \Sigma$: - - - - - - - - the left endmarker
- $s \in Q$: the start state
- $t \in Q$: the accept state
- $r \in Q$: the reject state $\boxed{t \neq r}$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$

  $\delta(p, a) = (q, b, d)$: $p \in Q, a \in \Gamma, q \in Q, b \in \Gamma, d \in \{L, R\}$

  when in state p with the head reading symbol a on tape, write b in place of a, move the head in direction d by one cell, and enter state q

  when a = b, it means leave the symbol untouched (replace a with a)

① We restrict TMs s.t. the $\vdash$ is never overwritten with another symbol and the tape head never moves off the tape to the left of $\vdash$.

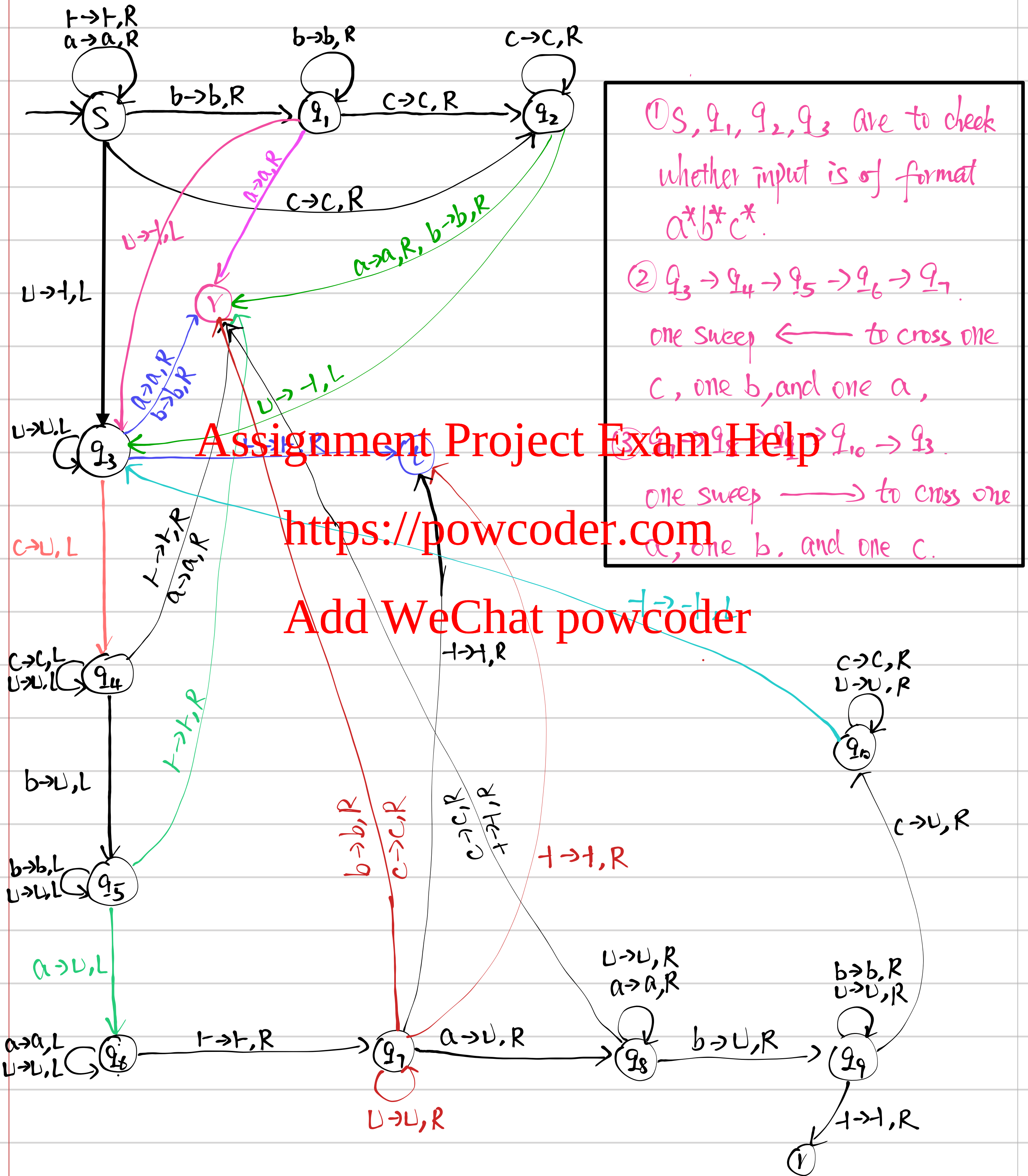
Always include $\delta(p, \vdash) = (q, \vdash, R)$ for all $p \in Q$. using some q you choose.

② We require that once the machine enters state t (or r), it never leaves it.

for all $b \in \Gamma$. $\delta(t, b) = (t, c, d)$ using some $c \in \Gamma$ and $d \in \{L, R\}$ you choose.

4. Example.

a TM M that accepts $L = \{a^n b^n c^n \mid n \geq 0\}$.



① s, q_1, q_2, q_3 are to check whether input is of format $a^*b^*c^*$.

② $q_3 \to q_4 \to q_5 \to q_6 \to q_7$.

one sweep ⟵ to cross one c, one b, and one a,

③ $q_7 \to q_8 \to q_9 \to q_{10} \to q_3$.

one sweep ⟶ to cross one a, one b, and one c.

5. Configurations and acceptance.

$y\sqcup^{\omega}$ denote the current tape content.

$y \in \Gamma^*$, $\sqcup^{\omega}$ all blank symbols on cells to the right of y.

a configuration is a global situation the machine is in.

an element of $Q \times \{y\sqcup^{\omega} \mid y \in \Gamma^*\} \times \mathbb{N}$

(p, z, n)

current tape head position counting from the left

current state

current tape content

start configuration on input $x \in \Sigma^*$ is

$(s, \vdash x\sqcup^{\omega}, 0)$

means pointing to $\vdash$ initially

transfer from one configuration to the next one by taking a step according to δ.

$$(p, z, n) \xrightarrow[M]{1} \begin{cases} (q, s_b^n(z), n-1) & \text{if } \delta(p, z_n) = (q, b, L) \\ (q, s_b^n(z), n+1) & \text{if } \delta(p, z_n) = (q, b, R) \end{cases}$$

$s_b^n(z)$ is the string obtained by substituting b for z_n at position n.

$$s_b^4(\overset{0}{\vdash}\overset{1}{b}\overset{2}{a}\overset{3}{a}\overset{4}{a}\overset{5}{c}\cdots) = \vdash baabc$$

- configuration $\alpha \xrightarrow[M]{0} \alpha$
- configuration $\alpha \xrightarrow[M]{n+1} \beta$ if $\alpha \xrightarrow[M]{n} \gamma \xrightarrow[M]{1} \beta$ for some γ
- $\alpha \xrightarrow[M]{*} \beta$ if $\alpha \xrightarrow[M]{n} \beta$ for some $n \geq 0$

M accepts input $x \in \Sigma^*$ if

$(s, \vdash x \sqcup^\omega, 0) \xrightarrow[M]{*} (t, y, n)$ for some $y \in \Gamma^*$ and some $n \in \mathbb{N}$

after taking 0 or more steps

M rejects $x \in \Sigma^*$ if

$(s, \vdash x \sqcup^\omega, 0) \xrightarrow[M]{*} (r, y, n)$ for some $y \in \Gamma^*$ and some $n \in \mathbb{N}$