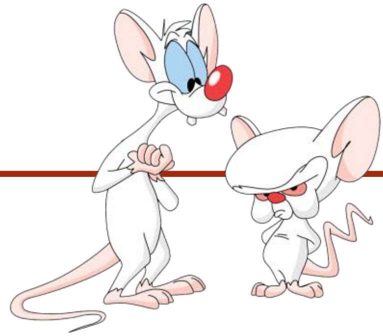
COMP 250

INTRODUCTION TO COMPUTER SCIENCE

Week 9-3 : Recursion 2 (Binary Search)

Giulia Alberini, Fall 2020

WHAT ARE WE GOING TO DO IN THIS VIDEO?



- More recursive algorithms
  - Decimal to Binary conversion
  - Power function
- Binary Search

RECALL: DECIMAL TO BINARY (ITERATIVE)

ALGORITHM

Constructing Base 2 Expansions

procedure $BinaryExpansion(n)$

$k := 0$

While $n > 0$

$\quad a_k := n\%2$

$\quad$ n$:= n/2$

$\quad k := k + 1$

return $(a_{k-1}, \ldots, a_1, a_0)$

Recall that a decimal number n requires approximately $\log_2 n$ bits for its binary representation.

DECIMAL TO BINARY (RECURSIVE)

ALGORITHM

Constructing Base 2 Expansions

procedure *BinaryExpansion*(n)

If $n > 0$:

    BinaryExpansion$(n/2)$

    print$(n\%2)$

Also in this case, there are $\log_2 n$ recursive calls

POWER (x^n) – DEFINITION

- Definition of power

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ times}}$$

- Inductive definition:
  - Base clause:

$$x^0 = 1$$

  - Inductive clause:

$$x^n = x \cdot x^{n-1}$$

POWER (x^n) – ITERATIVE 1

Let x a positive integer and let n be a positive number.


```
power(x, n) {
   int result =1;
   for(int i=1; i<=n; i++) {
      result = result *x;
   }
   return result;
}
```

POWER (x^n) – RECURSIVE


```
power(x, n) {
    if(n==0) {
        return 1;
    } else {
        return x*power(x,n-1);
    }
}
```

POWER() – CAN WE DO BETTER?

More interesting approach using recursion:

$$x^{18} = x^9 * x^9$$

$$x^9 = x^4 * x^4 * x$$

$$x^4 = x^2 * x^2$$

$$x^2 = x * x$$

POWER (x^n) – RECURSIVE 2

```
power( x, n) {
   if (n == 0)
      return 1;
   else if (n == 1)
      return x;
   else{
      tmp = power(x, n/2);
      if (n%2==0)
         return tmp*tmp;     // one multiplication
      else
         return tmp*tmp*x;   // two multiplications
   }
}
```

A SIMILAR IDEA CAN BE IMPLEMENTED ITERATIVELY

IDEA: Let's use the binary expansion of n, say $n = (a_{k-1}, \dots, a_1, a_0)_2$.

Note that:

$$x^n = x^{a_{k-1}2^{k-1} + \cdots + a_1 2 + a_0} = x^{a_{k-1}2^{k-1}} \dots x^{a_1 2} \cdot x^{a_0}$$

This shows how to compute x^n: we only need to compute the values of $x, x^2, (x^2)^2 = x^4, \dots, x^{2^k}$. Once we have these terms we multiply the terms x^{2^j}, where $a_j = 1$.

POWER (x^n) – ITERATIVE 2

Let $n = (a_{k-1}, \ldots, a_1, a_0)_2$

```
power(x, n) {
    result = 1;
    pow = x;
    if(n%2 == 1)
        result = x;
    n = n/2;
    while(n != 0) {                  // log2(n) − 1 iterations
        pow = pow * pow;             // 1 multiplication
        if(n%2 == 1)
            result = result * pow;   // 1 multiplication
        n = n/2;
    }
    return result;
}
```

POWER (x^n) – ITERATIVE 2

Let $n = (a_{k-1}, \ldots, a_1, a_0)_2$

```
power(x, n) {
   result = 1;
   pow = x;
   if(n%2 == 1)
      result = x;
   n = n/2;
   while(n != 0) {          // log2(n) - 1 iterations
      pow = pow * pow;      // 1 multiplication
      if(n%2 == 1)
         result = result * pow; // 1 multiplication
      n = n/2;
   }
   return result;
}
```

POWER (x^n) – ITERATIVE 2

Let $n = (a_{k-1}, \ldots, a_1, a_0)_2$

```
power(x, n) {
   result = 1;
   pow = x;
   if(n%2 == 1)
      result = x;
   n = n/2;
   while(n != 0) {              // log2(n) − 1 iterations
      pow = pow * pow;          // 1 multiplication
      if(n%2 == 1)
         result = result * pow; // 1 multiplication
      n = n/2;
   }
   return result;
}
```

EXAMPLE: x^{243}

$$n = (243)_{10} = (11110011)_2$$

Q: How many multiplications do we need?

EXAMPLE: x^{243}

$$n = (243)_{10} = (11110011)_2$$

Q: How many multiplications do we need?

A: Recursive method: 5*2 + 2*1 = 12.

Iterative method: 7 + 5 = 12

The highest order bit in the recursive method is the base case, and doesn't require a multiplication.
The lowest order bit in the iterative method does not require multiplication.

EXAMPLE: x^{243}

$$n = (243)_{10} = (11110011)_2$$

Q: How many multiplications do we need?

A: $O(\log n)$

OBSERVATIONS

The second approach we looked at uses fewer multiplications than the first one, and thus the second approach *seems faster.*

Q: Is this indeed the case?

A: No. Why not ?

OBSERVATIONS

Hint: Let x be a positive integer with M digits.

- x^2 has about ? digits.

- x^3 has about ? digits.

- :
- x^n has about ? digits.

OBSERVATIONS

Hint: Let x be a positive integer with M digits.

- x^2 has about 2M digits.

- x^3 has about 3M digits.
- :

- x^n has about $n * \mathrm{M}$ digits.

We cannot assume that multiplication takes 'constant' time.

Taking large powers gives very large numbers and multiplications becomes more expensive.

BINARY SEARCH

SEARCHING A LIST

- Goal: find a given element in a list.

- Solution: go through all the elements in the list and check whether the element is there (*linear search*).

- Could we do this any faster if the list was sorted to begin with?

Think of how you search for a term in an index. Do you start at the beginning and then scan through to the end? (No.)

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BINARY SEARCH

- Inputs:
  - A sorted list.
  - The element we are looking for (the *key*)
- IDEA: First compare the key with the element in the middle of the list
  - If the key is less than the middle element, we only need to search the first half of the list, so we continue searching on this smaller list.
  - If the key is greater than the middle element, we only need to search the second half of the list, so we continue searching on this smaller list.
  - If the key equals the middle element, we have a match – return its index.

http://www.oxfordmathcenter.com/drupal7/node/31

EXAMPLE

- Search for 25

-4	1	5	6	14	23	31	35	52	70

EXAMPLE

- Search for 25
- Look at the middle element and compare

-4	1	5	6	14	23	31	35	52	70

EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half

-4	1	5	6	14	23	31	35	52	70

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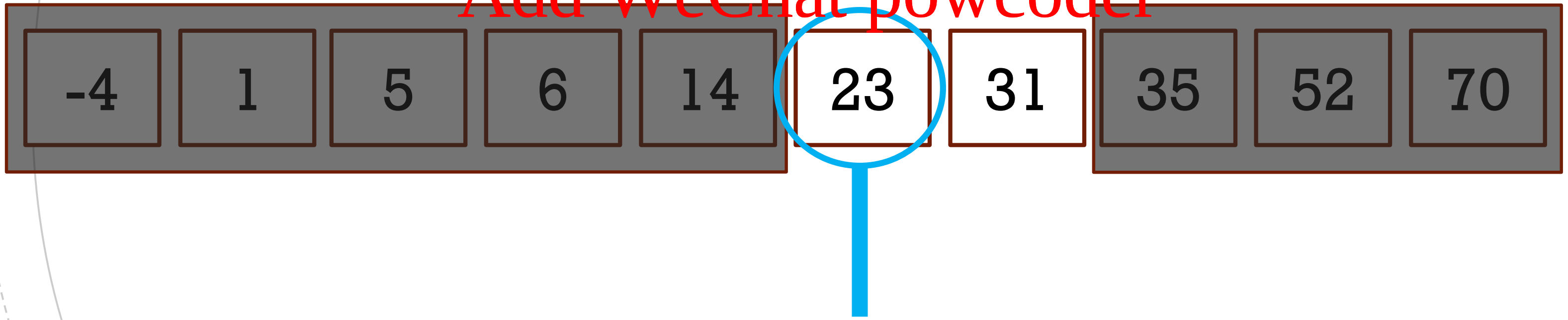

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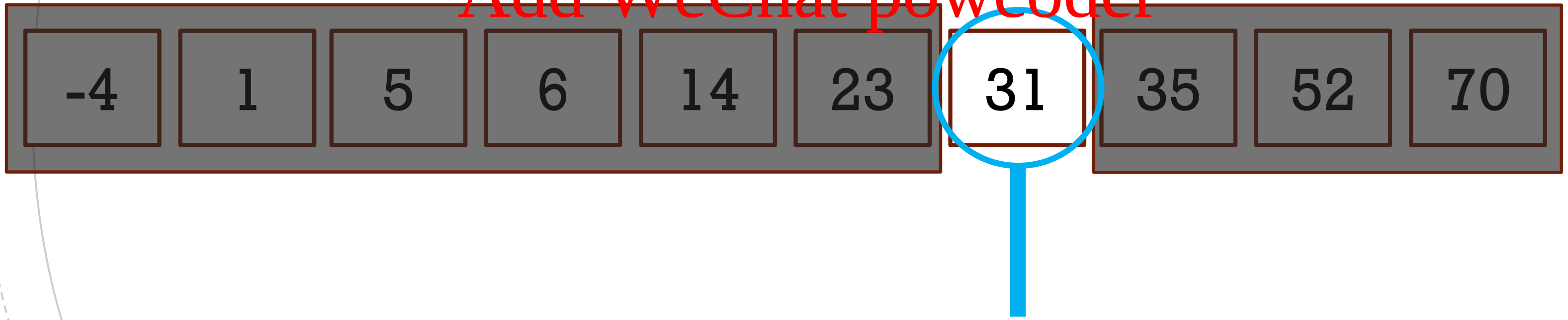

EXAMPLE

- Search for 25
- Look at the middle element and compare
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-4	1	5	6	14	23	31	35	52	70

EXAMPLE

- Search for 25
- There are no more elements in the list → the element is not there! Return -1.

-4	1	5	6	14	23	31	35	52	70

IMPLEMENT BINARY SEARCH

- Idea: keep track of the left and right indices denoting the section of the list that needs to be searched.

- What is the index of the element that we compare to the key as a function of the left and right indices?

BACK TO EXAMPLE

- Search for 25 (initialize left and right)

-4	1	5	6	14	23	31	35	52	70

left = 0

right = 9

right = size -1

BACK TO EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)

mid = (left+right)/2

mid = 4

-4	1	5	6	14	23	31	35	52	70

left = 0

right = 9

BACK TO EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update left)

mid = 4

-4	1	5	6	14	23	31	35	52	70

left = 5

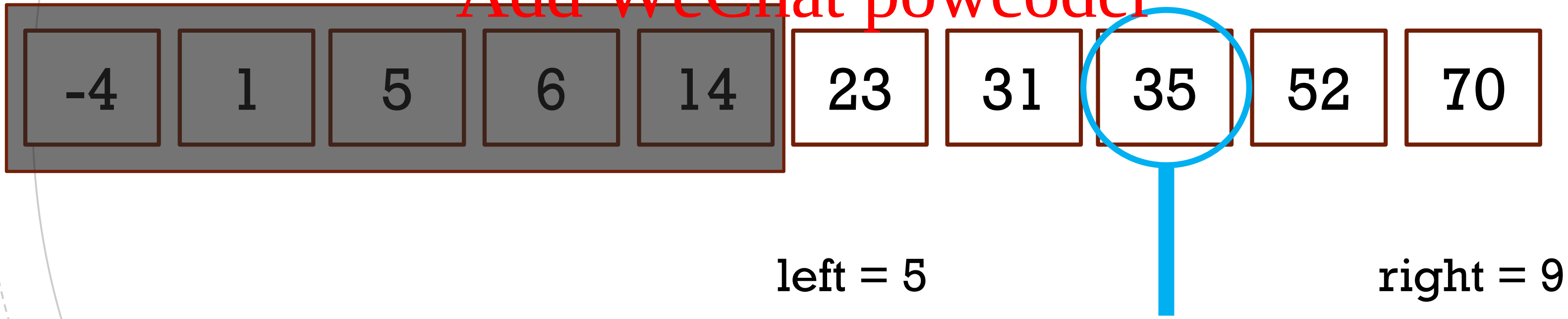
right = 9

left = mid +1

EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)

mid = (left+right)/2
= 7

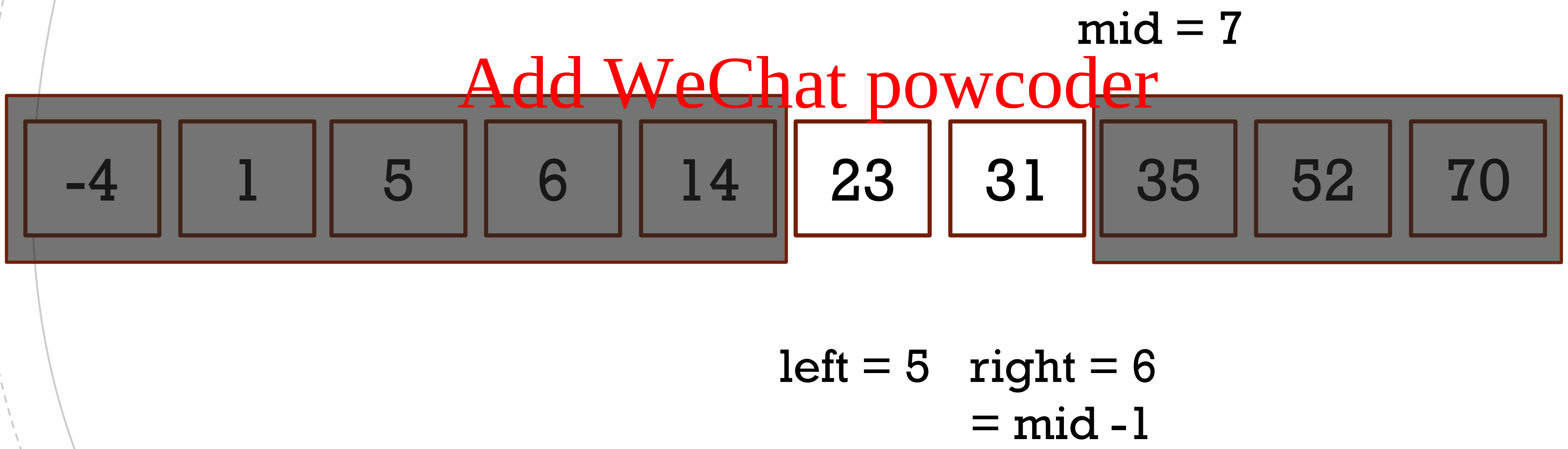


left = 5

right = 9

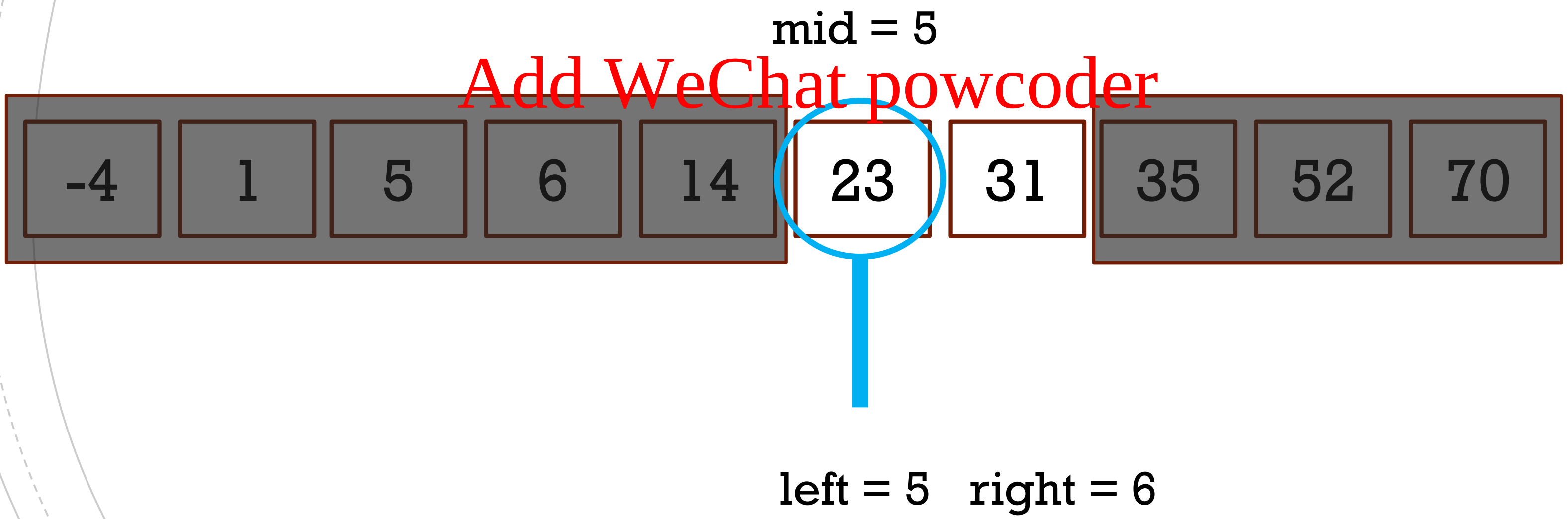
EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update right)



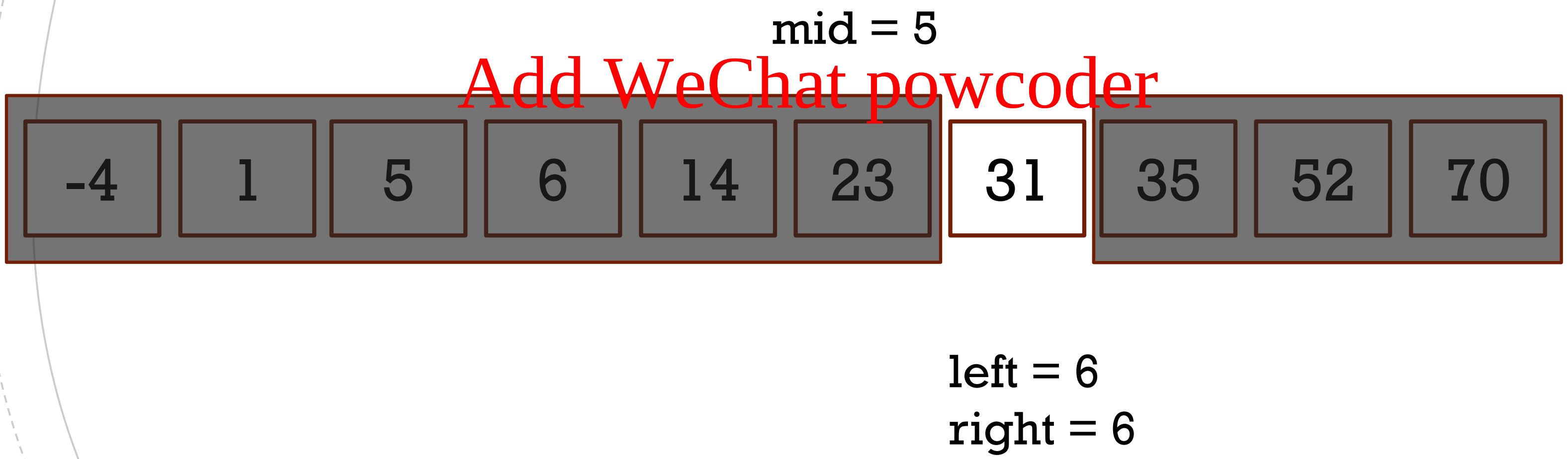
EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)



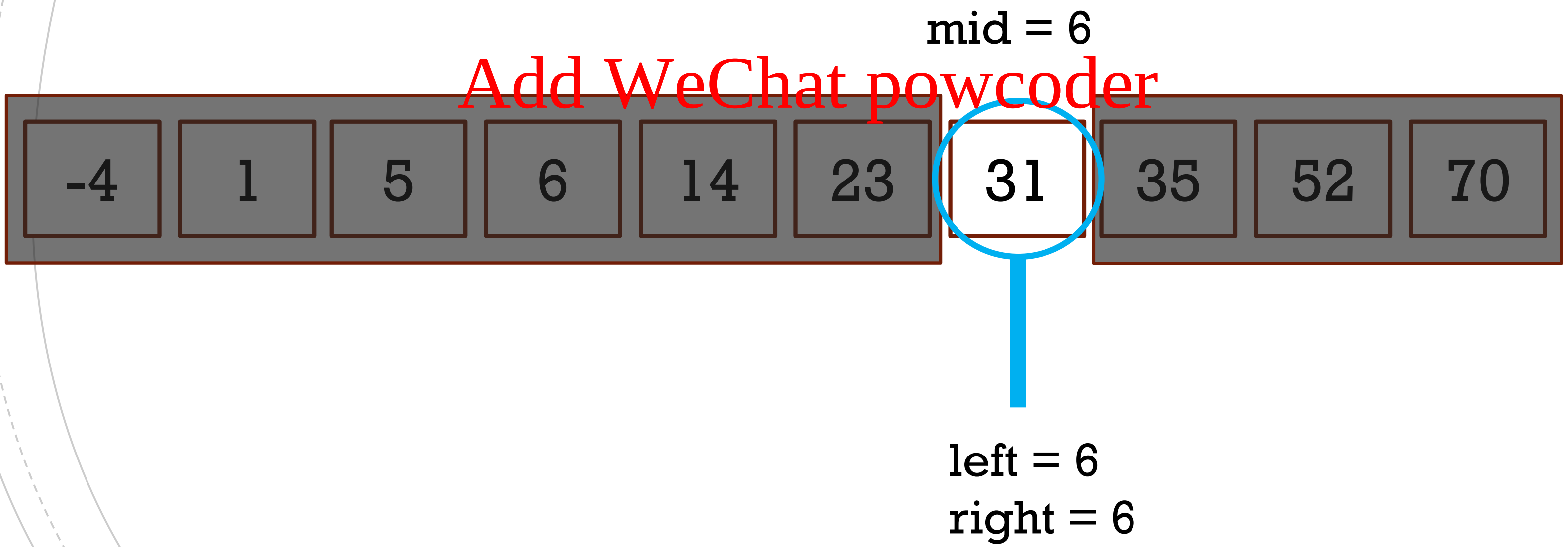
EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update left)



EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)



EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update right)

mid = 6

-4	1	5	6	14	23	31	35	52	70

right= 5 left = 6

EXAMPLE

- Search for 25
- There are no more elements in the list (right<left)
  → the element is not there! Return -1.

-4	1	5	6	14	23	31	35	52	70

right= 5   left = 6

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
    left = 0                        initialize left and right
    right = list.size() - 1
    while(left <= right){ // until there are elements to search

                   ...

    }
    return -1               // key not in list
}
```

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
   left = 0                               initialize left and right
   right = list.size() - 1
   while(left <= right){ // until there are elements to search
      mid = (left + right)/2     // compute mid
      if(list[mid]==key)         // compare element with key
         return mid
      else {

            // update either left or right


      }
   }
   return -1              // key not in list
}
```

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
    left = 0                              initialize left and right
    right = list.size() - 1
    while(left <= right){ // until there are elements to search
       mid = (left + right)/2     // compute mid
       if(list[mid]==key)         // compare element with key
          return mid
       else {
          if(key<list[mid])
             right = mid -1       // update right
          else
             left = mid + 1       // update left
       }
    }
    return -1              // key not in list
}
```

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key){
   left = 0
   right = list.size() - 1
   while(left <= right){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            right = mid -1
         else
            left = mid + 1
      }
   }
   return -1
}
```


What should change?

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
   while(left <= right){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            right = mid -1
         else
            left = mid + 1
      }
   }
   return -1
}
```

Pass left and right as parameters to the method

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
    if(left <= right){
        mid = (left + right)/2
        if(list[mid]==key)
            return mid
        else {
            if(key<list[mid])
                right = mid -1
            else
                left = mid + 1
        }
    }
    return -1
}
```

Replace the while with an if

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
    if(left <= right){
        mid = (left + right)/2
        if(list[mid]==key)
            return mid
        else {
            if(key<list[mid])
                return binarySearch(list, key, left, mid-1)
            else
                return binarySearch(list, key, mid+1, right)
        }
    }
    return -1
}
```

Add recursive calls

OBSERVATIONS

Q: How many times through the while loop ? (iterative)

How many recursive calls? (recursive)

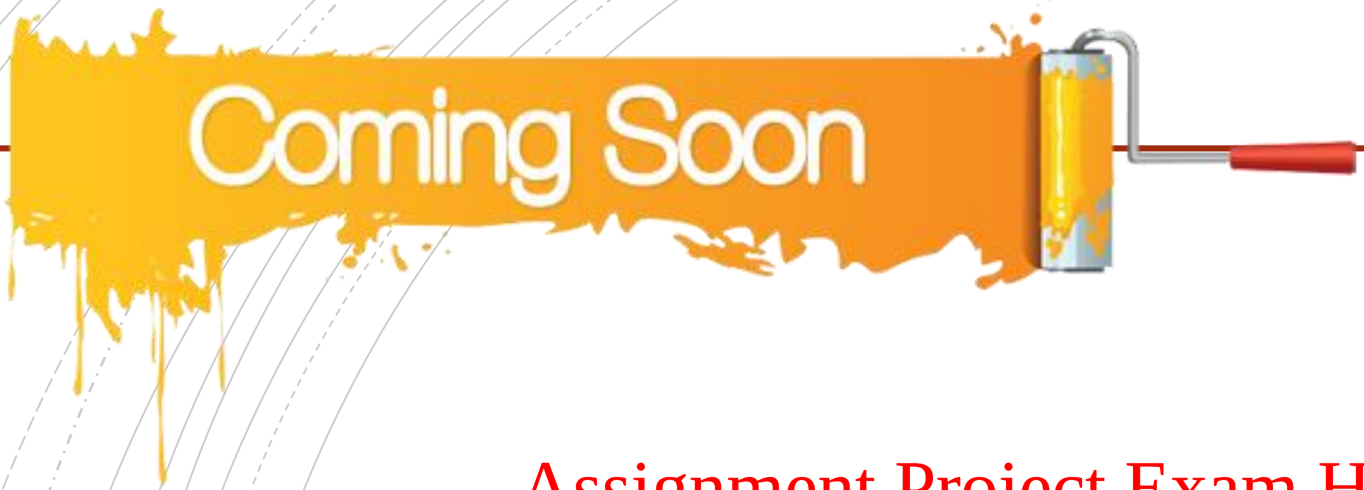
A:

OBSERVATIONS

Q: How many times through the while loop ? (iterative)

How many recursive calls? (recursive)

A: Worst case: the element cannot be found. Then, worst case is $O(\log n)$ where n is size of the list. Why? Because each time we are approximately halving the size of the list.



In the next videos:

- Quick sort
- Merge sort
- Recurrences