# COMP 250

# INTRODUCTION TO COMPUTER SCIENCE



Week 9-1 : Induction

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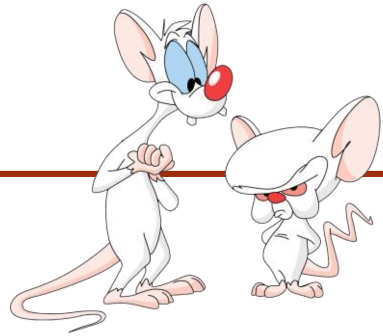
# WHAT ARE WE GOING TO DO IN THIS VIDEO?



- Inductive/Recursive definitions
- Inductive/Recursive proofs
  - Mathematical Induction

# INDUCTION

# PROOFS

For all $n \geq 1$,

$$1 + 2 + 3 + \dots + (n-1) + n = \frac{n(n+1)}{2}$$



How can we prove such a statement?



- By "proof", we mean a formal logical argument that convincingly demonstrate the truth of a given proposition.
- Note that "convincingly" is itself not well defined.

# EXAMPLE

$$1 + 2 + \ldots + (n-1) + n$$

Rewrite by considering $n/2$ pairs :





$$1 + 2 + \cdots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) \ldots + (n-1) + n$$



If $n$ is even, then adding up the $n/2$ pairs gives

$$n/2 * (n + 1)$$

- What if $n$ is odd?

## EXAMPLE

- What if $n$ is odd? Then, $n$-1 is even. So,



$$1 + 2 + \dots + (n-1) + n$$

$$= \left(\frac{n-1}{2} * n\right) + n$$

$$= \left(\frac{n-1}{2} + 1\right) * n$$

$$= \frac{n+1}{2} * n$$

which is the same formula as before.

# RECURSIVE (INDUCTIVE) DEFINITION

- Some set of elements can be define recursively/inductively.



- A recursive/inductive definition consists of the following:
  - A *base clause*



    Which one or more basic/initial element of the set.



  - One or more *inductive clauses*

    Rules on how to generate "new" elements of the set from "old" ones.
  - A *final clause*

    which simply states that no other element is part of the set.

# EXAMPLE – NATURAL NUMBERS

The set of natural numbers can be defined as follows:

- *Base clause*:

  $0$ is a natural number



- *Inductive clause*:

  If $n$ is a natural number, then $n + 1$ is also a natural number.

- *Final clause*: Nothing else is a natural number.

# MATHEMATICAL INDUCTION

Consider a statement of the form:

“For all $n \geq n_0$ : $P(n)$ is true”



where $n_0$ is some constant and proposition $P(n)$ has value true or false for each $n$.



- If n is an element of an inductively defined set, then the statement above can be proven using a technique called *mathematical induction*.

# (WEAK) MATHEMATICAL INDUCTION

To prove a property by mathematical induction, we proceed as follows:

- *Base case*
  Show that the property holds for the basic/initial elements of the set.



- *Induction step*
  Assume the property hold for some element $n$. (Induction Hypothesis)
  Show that the property also holds for any element generated from $n$ using the inductive clauses.

- *Conclusion*
  The property holds for all elements.

# EXAMPLE

"For all $n \geq n_0$, $P(n)$ is true"



For all $n \geq 1$,

$$1 + 2 + 3 + \ldots\ldots + (n-1) + n = \frac{n(n+1)}{2}$$

This is a property of natural numbers. Since this is a set that can be defined inductively, we can use mathematical induction to prove such property!

# PROOF BY MATHEMATICAL INDUCTION

We need to prove the following:

- Base case:
  
  $P(n_0)$ is true, i.e. the property holds for $n_0$ which in this case is $1$.
  

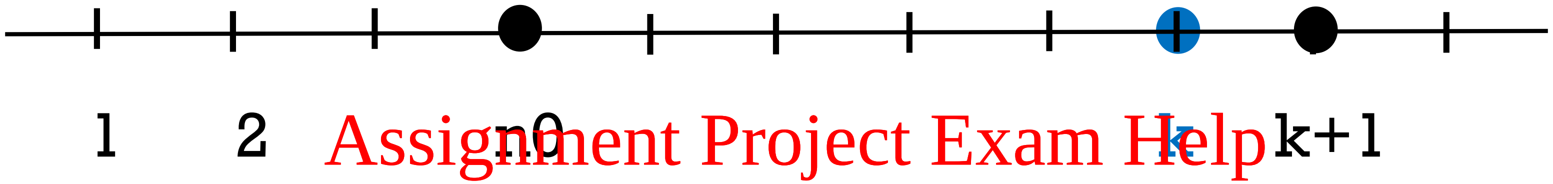
- Induction step:
  
  IH: Assume $P(k)$ is true, i.e. the property holds for an element $k$.
  Prove that $P(k+1)$ is true, i.e. the property holds for $k+1$.

***Base case:***

P($n_0$) is true.

***Induction step:***

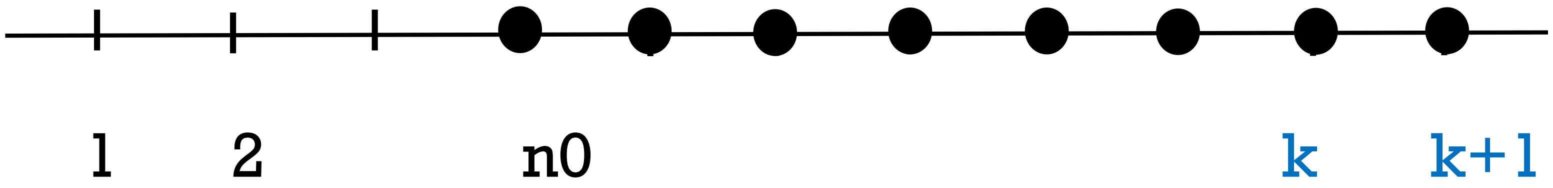
For any k >= $n_0$, if P(k) is true then P(k+1) is true.






Thus we have proved:

For any n >= $n_0$, P(n) is true.



# BACK TO THE PROOF

For all $n \geq 1$, $1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}$



- Base case: $n = 1$, to prove 

$$1 = \frac{1 * (1 + 1)}{2}$$



$$1 = \frac{2}{2} = 1$$ ✓

# BACK TO THE PROOF

- Induction step:

  IH: Assume that it holds for $k$, that is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Statement: For all $n \geq 1$, $1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$

# BACK TO THE PROOF

- Induction step:

  IH: Assume that it holds for $k$, that is



$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$



Prove it for $k+1$:



$$1 + 2 + \cdots + k + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1), \text{ by IH}$$

$$= (k+1) * \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

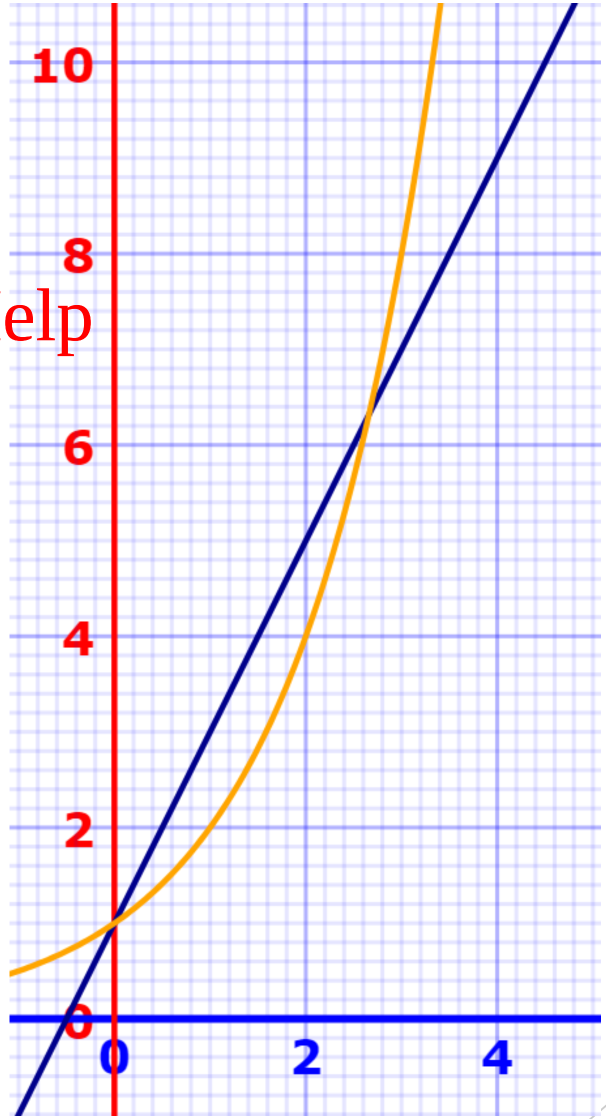
✓

# EXAMPLE 2

- Prove the following statement:

For all $n \geq 3$, $2n + 1 < 2^n$.





# EXAMPLE 2

<u>Statement:</u> For all $n \geq 3$, $2n + 1 < 2^n$.



- Note: P( $n$ ) is false for n= 1, 2.



*But that has nothing to do with what we need to prove.*

# EXAMPLE 2

<u>Statement:</u> For all $n \geq 3$, $2n + 1 < 2^n$.



Proof: (by mathematical induction)



- Base case ($n = 3$):



$$2 * 3 + 1 = 7 < 8 = 2^3$$ ✓

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

# EXAMPLE 2

- Induction step:

IH: Assume $2 * k + 1 < 2^k$.

<u>Statement:</u> For all $n \geq 3$, $2n + 1 < 2^n$.

# EXAMPLE 2

- Induction step:

IH: Assume $2 * k + 1 < 2^k$.



Prove it for $k + 1$:



$$2 * (k + 1) + 1$$



$$= 2 * k + 2 + 1$$

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

# EXAMPLE 2

- Induction step:

IH: Assume $2 * k + 1 < 2^k$.



Prove it for $k + 1$:



$$2 * (k + 1) + 1 = 2k + 2 + 1$$



$$= 2 * k + 1 + 2$$

$$< 2^k + 2, \text{ by IH}$$

$$< 2^k + 2^k, \text{ for } k \geq 3$$

$$= 2^{k+1}$$

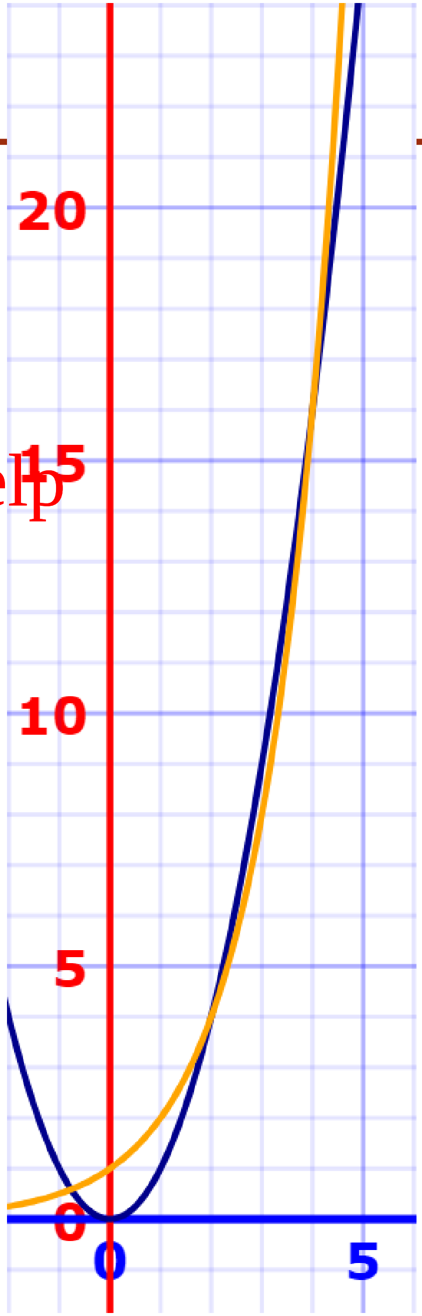
✓

# EXAMPLE 3



**Statement:** For all $n \geq 5$, $n^2 < 2^n$.





# EXAMPLE 3

**Statement:** For all $n \geq 5$, $n^2 < 2^n$.



Proof: (by mathematical induction)



- Base case ($n = 5$):



$$5^2 = 25 < 32 = 2^5$$

## EXAMPLE 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.

- Induction step.



What should we assume? 



What do we need to prove?

## EXAMPLE 3

**Statement:** For all $n \geq 5, \quad n^2 < 2^n$.

- Induction step.



What should we assume? $k^2 < 2^k$ for a $k \geq 5$



What do we need to prove? $(k+1)^2 < 2^{(k+1)}$

# EXAMPLE 3

**Statement:** For all $n \geq 5$, $n^2 < 2^n$.

- Induction step.

  IH: $k^2 < 2^k$ for a $k \geq 5$





$$(k+1)^2 = k^2 + 2k + 1$$

# EXAMPLE 3

**Statement:** For all $n \geq 5$, $n^2 < 2^n$.

- Induction step.

  IH: $k^2 < 2^k$ for a $k \geq 5$



$$(k+1)^2 = k^2 + 2k + 1$$

$$< 2^k + 2k + 1, \text{ by IH}$$

$$< 2^k + 2^k, \text{ by Example 2}$$

$$= 2^{k+1}$$

# (STRONG) MATHEMATICAL INDUCTION

- Sometimes one would like to assume the induction hypothesis not only for the previous element, but also for smaller elements. This leads to a logically equivalent proof method called *strong (or complete) mathematical induction*.
- To prove a property by strong mathematical induction, we proceed as follows:
  - Induction step
    Assume the property hold *for all elements* less than an arbitrary $k$. (Induction Hypothesis)
    Show that the property also holds for the $k$ element which was generated using the inductive clauses.
  - Conclusion
    The property holds for all elements.

## FIBONACCI NUMBERS

- The Fibonacci sequence is one of the most common example of a recursively-defined set.



- Consider the following sequence of numbers:



$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

Let $f_n$ denote the nth Fibonacci number. How can we define the sequence above?

# FIBONACCI NUMBERS – INDUCTIVE DEFINITION

- Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …



- Base clause:
  $f_0 = f_1 = 1$ are Fibonacci numbers.





- Inductive clause:
  If $f_{n-1}$ and $f_{n-2}$ are Fibonacci numbers, then $f_n = f_{n-1} + f_{n-2}$ is a Fibonacci number.

## EXAMPLE 4

**Statement:** For all $n \geq 0$, $f_n \leq \left(\frac{7}{4}\right)^n$

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Statement: For all $n \geq 0$, $f_n \leq \left(\frac{7}{4}\right)^n$



Proof: (by strong mathematical induction)



- Induction step

  IH: Let $k$ be $\geq 0$, and assume that for any number $i$ such that $0 \leq i < k$ then



$$f_i \leq \left(\frac{7}{4}\right)^i$$

# EXAMPLE 4

Statement: For all $n \geq 0$, $f_n \leq \left(\frac{7}{4}\right)^n$



Proof: (by strong mathematical induction)



- Induction step

  IH: Let $k$ be $\geq 0$, and assume that for any number $i$ such that $0 \leq i < k$ then



$$f_i \leq \left(\frac{7}{4}\right)^n$$

To show: $f_k \leq \left(\frac{7}{4}\right)^k$

# EXAMPLE 4

There are 3 possible cases:

1. $k = 0$

   $f_0 = 1$ and $\left(\frac{7}{4}\right)^0 = 1$, so the claim holds.

2. $k = 1$

   $f_1 = 1$ and $\left(\frac{7}{4}\right)^1 > 1$, so the claim holds.

## EXAMPLE 4

There are 3 possible cases:

3. $k > 1$

$$f_k = f_{k-1} + f_{k-2}$$



$$\leq \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2}$$ , by IH



$$= \left(\frac{7}{4}\right)^{k-2}\left(1 + \frac{7}{4}\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{11}{4}\right)$$



$$= \left(\frac{7}{4}\right)^{k-2}\left(\frac{44}{16}\right)$$

$$< \left(\frac{7}{4}\right)^{k-2}\left(\frac{49}{16}\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}\right)^{2}$$

$$= \left(\frac{7}{4}\right)^{k}$$

# RECOMMENDED EXERCISES

1. Prove that for all $n \geq 0, \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Prove that for all $n \geq 0, \sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$



3. Consider the following recursive definition of addition ('+') on natural numbers:



   - Base clause:

$$0 + m = m$$

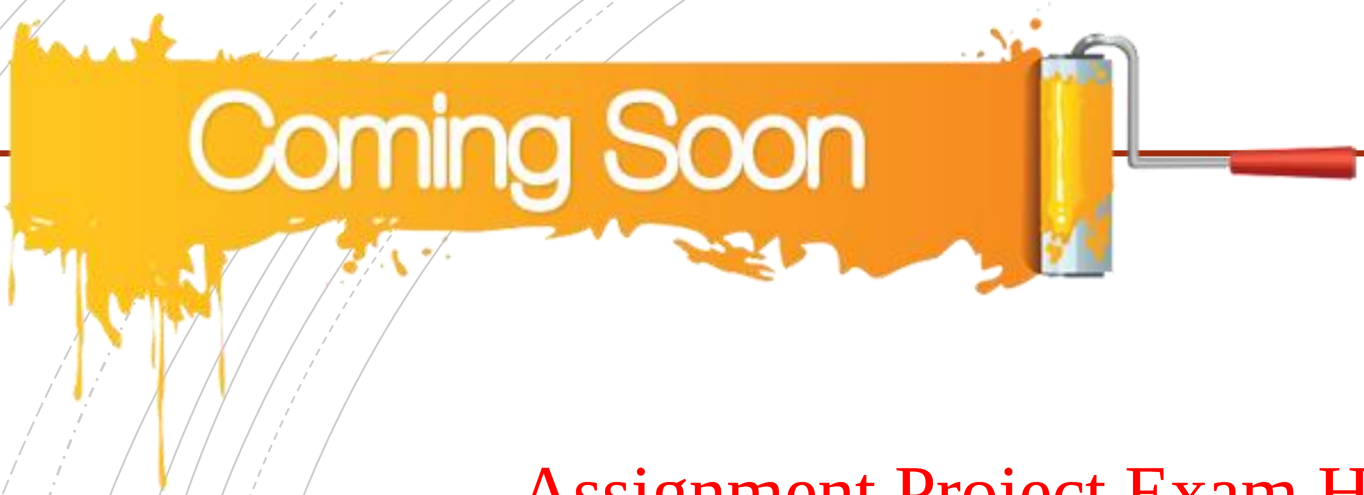


   - Inductive clause:

$$(n + 1) + m = (n + m) + 1$$

Prove that addition is associative, i.e. for all natural numbers $(a + b) + c = a + (b + c)$

Hint: use mathematical induction on $a$




In the next videos:

- Recursive algorithms #1