lecture 1

- two's complement
- floating point numbers
- hexadecimal

Car odometer (fixed number of digits)

$$
\begin{array}{rccccccc}
 & 0 & 0 & 0 & 0 & 9 & 9 & 9 \\
+ & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{r}
999999 \\
+\ 000001 \\
\hline
000000
\end{array}
$$

$$\Rightarrow \quad 999999 \equiv -1$$

328769

$+$?

000000

$$\begin{array}{r} 3\,2\,8\,7\,6\,9 \\ +\quad 6\,7\,1\,2\,3\,1 \\ \hline 0\,0\,0\,0\,0\,0 \end{array}$$

$$= \; - \; 3\,2\,8\,7\,6\,9$$

If you know what "modular arithmetic" is (MATH 240), then you recognize this: addition of integers mod 10^6.

Q: How to represent negative numbers in binary ?

A: Given an 8 bit binary number m, define -m so that m + (-m) = 0.

$$
\begin{array}{rcccccccc l}
 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & m \\
+ & & & & ? & & & & & -m \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}
$$

Two's complement representation of integers

Example: How to represent -26 ?

Use a trick!

$$\begin{array}{rl} 0\,0\,0\,1\,1\,0\,1\,0 & n = 26 \\ +\;1\,1\,1\,0\,0\,1\,0\,1 & \leftarrow \text{invert bits} \\ \hline 1\,1\,1\,1\,1\,1\,1\,1 & \end{array}$$

```
  0 0 0 1 1 0 1 0      n = 26
+ 1 1 1 0 0 1 0 1      ← invert bits
  ---------------
  1 1 1 1 1 1 1 1
+               1      ← add 1
  ---------------
  0 0 0 0 0 0 0 0
```



```
  1 1 1 0 0 1 0 1      ← inverted bits
+               1      ← add 1
  ---------------
  1 1 1 0 0 1 1 0      ← -26
```

Another example: What is -0 ?

0 0 0 0 0 0 0 0 m = 0

1 1 1 1 1 1 1 1 invert bits

+ 1 add 1

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1
+ 1
0 0 0 0 0 0 0 0

We have verified that -0 = 0.

What about m = 128 ? What is -128 ?

```
   1 0 0 0 0 0 0 0        m = 128

   0 1 1 1 1 1 1 1        invert bits

+                1        add 1
 ------------------
   0 0 0 0 0 0 0 0
```



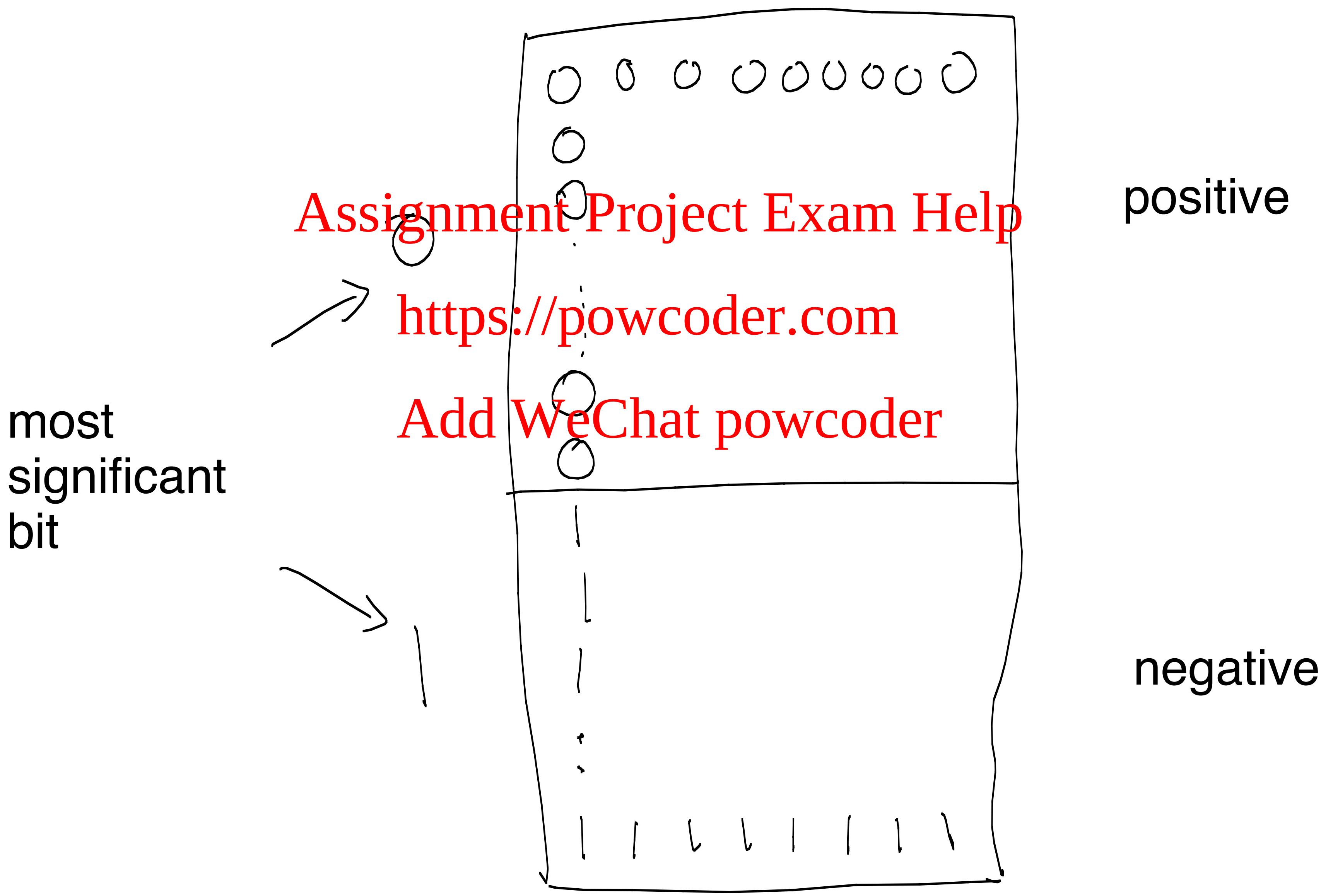
```
   0 1 1 1 1 1 1 1

+                1
 ------------------
   1 0 0 0 0 0 0 0        m = - 128
```

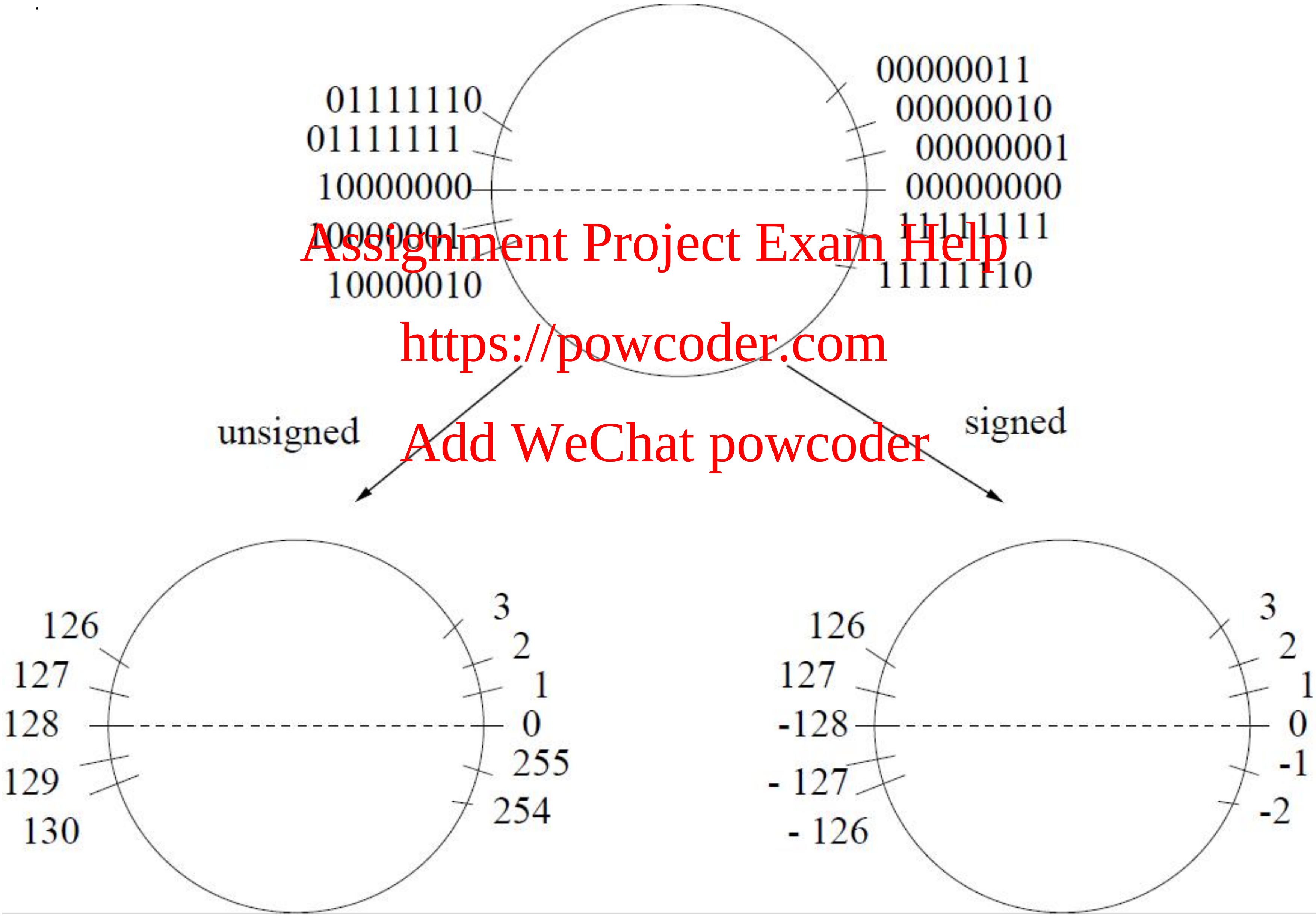
Thus, 128 is equivalent to -128.

binary	"unsigned"	"signed"
0 0 0 0 0 0 0 0	0	0
0 0 0 0 0 0 0 1	1	1
0 0 0 0 0 0 1 0	2	2
0 0 0 0 0 0 1 1	3	3
0 0 0 0 0 1 0 0	4	4
⋮	⋮	⋮
0 1 1 1 1 1 1 1	127	127
1 0 0 0 0 0 0 0	128	−128
⋮	⋮	⋮
1 1 1 1 1 1 1 0	254	−2
1 1 1 1 1 1 1 1	255	−1

signed integers



8 bit integers (unsigned vs. signed)



n bits defines 2^n integers

unsigned

$0, 1, \ldots, 2^n - 1$

signed

$-2^{n-1}, \ldots, 0, \ldots, 2^{n-1} - 1$

Take n = 32.

The largest signed integer is 2^31 - 1.

2 ^ 10 = 1024 ~ 10 ^ 3 = one thousand.

2 ^ 20 ~ 10 ^ 6 = one million

2 ^ 30 ~ 10 ^ 9 = one billion

2 ^ 31 ~ 2,000,000,000 = two billion

Java Example

```
int  j = 4000000000;   // 4 billion  >  2^31
```

This gives a compiler error. "The literal of type int is out of range."


```
int  j = 2000000000;  //   2 billion <  2^31
```



```
System.out.println( 2 * j );
```

// This prints out -294967296.
// To understand why these particular digits are printed, you
// would need to convert 4000000000 to binary, which I don't
// recommend.)

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Floating Point

"decimal point"

$$26.375 = 2 \times 10^{1} + 6 \times 10^{0} + 3 \times 10^{-1} + 7 \times 10^{-2} + 5 \times 10^{-3}$$

"binary point"

$$(11010.011)_2 = 2^{4} + 2^{3} + 2^{1} + 2^{-2} + 2^{-3}$$

$$= 16 + 8 + 2 + 0.25 + 0.125$$

$$= 26.375$$

Convert from binary to decimal

We must use both positive and negative powers of 2.

n	2^n
-1	$.5$
-2	$.25$
-3	$.125$
-4	$.0625$
-5	$.03125$
-6	$.015625$
-7	$.0078125$
$\vdots$	etc.

Sum up the contributing 1 bits as on previous slide.

How to convert from decimal to binary ?

$$26.375 = (\underline{\quad ? \quad} . \underline{\quad ? \quad})_2$$

To find the bits for the positive powers of 2, use the algorithm from last lecture ("repeated division") .

m	b_i
26	
13	0
6	1
3	0
1	1
0	1

$$\Rightarrow \quad 26 = (11010)_2$$

What about negative powers of 2 ?

In general, note that multiplying by 2 shifts bits to the left (or shifts binary point to the right)

Example:

$$(11010.011)_2 \times 2$$

$$= (110100.11)_2$$

Similarly....dividing by 2 and not ignoring remainder shifts bits to the right (or shifts binary point to the left)

$$(11010.011)_2 / 2$$

$$= (1101.0011)_2$$

For the negative powers of 2, use "repeated multiplication"

$$.375$$

$$= .375 \times 2^{1} \times 2^{-1}$$

$$= .75 \times 2^{-1}$$

$$= 1.5 \times 2^{-2}$$

$$= 3.0 \times 2^{-3}$$

convert decimal to binary

$$= (11)_2 \times 2^{-3}$$

$$= (.011)_2$$

A more subtle example:

$$19.243 = (\; ? \;)_2$$

First, find the bits for the positive powers of 2 using "repeated division" (last lecture).

m	b_i
19	
9	1
4	1
2	0
1	0
0	1

$$\therefore \; 19 = (10011)_2$$

Then find the bits for the negative powers of 2 using repeated multiplication.

$$.243$$

$$= .243 \times 2^{1} \times 2^{-1}$$

$$= (0).486 \times 2^{-1}$$

$$= ?$$

Then find the bits for the negative powers of 2 using repeated multiplication.

$$.243$$

$$= (0)_2 \; .486 \times 2^{-1}$$

$$= (00)_2 \; .972 \times 2^{-2}$$

$$= (001)_2 \; .944 \times 2^{-3}$$

$$= (0011)_2 \; .888 \times 2^{-4}$$

Thus $(.243)_{10} = (.0011)_2 + \sum_{i=-5}^{-\infty} b_i 2^i$

Note the summation is over bits bi from -5, -6, ..., -infinity.

$$19.243 = (10011.0011\ \_\_\_)_2$$

We cannot get an exact representation using a finite number of bits for this example.

Can we say anything more general about what happens ?

$$.(05)_{10}$$

$$= (0)_2 . 1 \times 2^{-1}$$

$$= (00)_2 . 2 \times 2^{-2}$$

$$= (000)_2 . 4 \times 2^{-3}$$

$$= (0000)_2 . 8 \times 2^{-4}$$

$$= (00001)_2 . 6 \times 2^{-5}$$

$$= (000011)_2 . 2 \times 2^{-6}$$

$$= .00\,\underline{0011}\,0011\,\underline{0011}\ \text{etc.}$$

This will repeat over and over again.

When we convert a floating point decimal number with a finite number of digits into binary, we get:

- a finite number of non-zero bits to left of binary point

- an infinitely repeating sequence of bits to the right of the binary point

Why ?

[Note: sometimes the infinite number of repeating bits are all 0's, as in the case of 0.375 a few slides back.]

Recall previous example...

$$.243$$
$$= (0)_2 \; .486 \times 2^{-1}$$
$$= (00)_2 \; .972 \times 2^{-2}$$
$$= (001)_2 \; .944 \times 2^{-3}$$
$$= (0011)_2 \; .888 \times 2^{-4}$$
$$= \text{etc}$$

Eventually, the three digits to the right of the decimal point will enter a cycle that repeats forever. This will produce a bit string that repeats forever.

Hexadecimal

Writing down long strings of bits is awkward and error prone.

Hexadecimal simplifies the representation.

Hexadecimal	(base 16)
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
a	1010
b	1011
c	1100
d	1101
e	1110

Examples of hexadecimal

1) 0010 1111 1010 0011

2 f a 3

We write 0x2fa3 or 0X2FA3.

2) 101100

We write 0x2c (10 1100), not 0xb0 (1011 00)