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School of Computing
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Semester 2, 2021
Practical Session 2

Foundations of Computation

The practical contains a number of exercises designed for the students to practice the course content. During the practical session, the tutor will work through some of these exercises while students will be responsible for completing the remaining exercises in their own time. There is no expectation that all the exercises will be covered in the practical session.

Covers: Lecture Material Week 2

At the end of this tutorial, you will be able to

- determine whether a propositional formula is valid, a contradiction or a contingency;
- apply natural deduction to prove (establish) the validity of formulae;
- understand First Order Logic formulae.

Exercise 1 — Types of Propositional Formulae

Determine the nature of the following propositional formulae. Remember that a propositional formula is

- *valid*, if it evaluates to T under all truth value assignments.
- a *contradiction*, if it evaluates to F under all truth value assignments, and
- a *contingency*, if there are (necessarily different) truth value assignments for which it evaluates to T and to F.

Formulae:

1. $a \wedge \neg a$

   Solution. Using truth tables:

a	$\neg a$	$a \wedge \neg a$
T	F	F
F	T	F

   It is a *contradiction*: the formula evaluates to F under all truth value assignments.

2. $(a \wedge (a \to b)) \to \neg b$

   Solution. Using truth tables:

a	b	$a \to b$	$a \wedge (a \to b)$	$\neg b$	$(a \wedge (a \to b)) \to \neg b$
T	T	T	T	F	F
T	F	F	F	T	T
F	T	T	F	F	T
F	F	T	F	T	T

   It is a *contingency*: there are truth value assignments for which the formula evaluates to T and to F.

   We can use another method using Boolean Algebra:

$$
\begin{array}{rl}
(a \wedge (a \to b)) \to \neg b = & \\
(a \wedge (\neg a \vee b)) \to \neg b = & \text{(Logical equivalence } p \to q = \neg p \vee q) \\
\neg(a \wedge (\neg a \vee b)) \vee \neg b = & \text{(Logical equivalence } p \to q = \neg p \vee q) \\
\neg a \vee \neg(\neg a \vee b) \vee \neg b = & \text{(De Morgan Laws)} \\
\neg a \vee (a \wedge \neg b) \vee \neg b = & \text{(De Morgan Laws)} \\
((\neg a \vee a) \wedge (\neg a \vee \neg b)) \vee \neg b = & \text{(Distributivity)} \\
(T \wedge (\neg a \vee \neg b)) \vee \neg b = & \text{(Complements)} \\
\neg a \vee \neg b \vee \neg b = & \text{(Identity)} \\
\neg a \vee \neg b &
\end{array}
$$

   Clearly, the result is neither T nor F. So, the given formula is a *Contingency*.

3. $((a \to b) \wedge (b \to c)) \wedge (a \wedge \neg c)$

Solution. Using truth tables:

a	b	c	$a \to b$	$b \to c$	$(a \to b) \wedge (b \to c)$	$a \wedge \neg c$	$((a \to b) \wedge (b \to c)) \wedge (a \wedge \neg c)$
T	T	T	T	T	T	F	F
T	T	F	T	F	F	T	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	T	F
F	T	T	T	T	T	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F

It is a *contradition*: the formula evaluates to F under all truth value assignments.

We can use another method using Boolean Algebra:

$$\begin{aligned}
((a \to b) \wedge (b \to c)) \wedge (a \wedge \neg c) &= \\
((\neg a \vee b) \wedge (\neg b \vee c)) \wedge (a \wedge \neg c) &= && \text{(Logical equivalence } p \to q = \neg p \vee q\text{)} \\
(((\neg a \vee b) \wedge \neg b) \vee ((\neg a \vee b) \wedge c)) \wedge (a \wedge \neg c) &= && \text{(Distributivity)} \\
(((\neg b \wedge \neg a) \vee (\neg b \wedge b)) \vee ((c \wedge \neg a) \vee (c \wedge b))) \wedge (a \wedge \neg c) &= && \text{(Distributivity)} \\
((\neg b \wedge \neg a) \vee ((c \wedge \neg a) \vee (c \wedge b))) \wedge (a \wedge \neg c) &= && \text{(Complements, Identity)} \\
(a \wedge \neg c \wedge \neg b \wedge \neg a) \vee ((a \wedge \neg c) \wedge ((c \wedge \neg a) \vee (c \wedge b))) &= && \text{(Distributivity)} \\
(a \wedge \neg a \wedge \neg c \wedge \neg b) \vee ((a \wedge \neg c) \wedge ((c \wedge \neg a) \vee (c \wedge b))) &= && \text{(Commutativity)} \\
(F \wedge \neg c \wedge \neg b) \vee ((a \wedge \neg c) \wedge ((c \wedge \neg a) \vee (c \wedge b))) &= && \text{(Complements)} \\
((a \wedge \neg c) \wedge ((c \wedge \neg a) \vee (c \wedge b))) &= && \text{(Identity)} \\
(a \wedge \neg c \wedge c \wedge \neg a) \vee (a \wedge \neg c \wedge c \wedge b) &= && \text{(Commutativity)} \\
(a \wedge F \wedge \neg a) \vee (a \wedge F \wedge b) &= && \text{(Complements)} \\
F \wedge F = F & && \text{(Identity)}
\end{aligned}$$

The given formula is a *Contradiction*.

4. $\neg(a \to b) \vee (\neg a \vee (a \wedge b))$

Solution. Using truth tables:

a	b	$\neg(a \to b)$	$a \wedge b$	$\neg a \vee (a \wedge b)$	$\neg(a \to b) \vee (\neg a \vee (a \wedge b))$
T	T	F	T	T	T
T	F	T	F	F	T
F	T	F	F	T	T
F	F	F	F	T	T

It is *valid*: the formula evaluates to T under all truth value assignments.

We can use another method using Boolean Algebra:

$$\begin{aligned}
\neg(a \to b) \vee (\neg a \vee (a \wedge b)) &= \\
\neg(\neg a \vee b) \vee (\neg a \vee (a \wedge b)) &= && \text{(Logical equivalence } p \to q = \neg p \vee q\text{)} \\
\neg(\neg a \vee b) \vee ((\neg a \vee a) \wedge (\neg a \vee b)) &= && \text{(Distributivity)} \\
\neg(\neg a \vee b) \vee (\neg a \vee b) &= && \text{(Complements, Identity)} \\
(a \wedge \neg b) \vee (\neg a \vee b) &= && \text{(De Morgan Laws)} \\
(\neg a \vee b \vee a) \wedge (\neg a \vee b \vee \neg b) &= && \text{(Distributivity)} \\
(T \vee b) \wedge (T \vee \neg a) &= && \text{(Complements)} \\
T \wedge T = T & && \text{(Identity)}
\end{aligned}$$

The given formula is *Valid*.

Exercise 2 **Natural Deduction Problems**

Construct a natural deduction proof for the following alleged propositional logic theorems. Use *only* the rules of natural deduction.

1. $p \to (q \to p)$

   Solution.

1	p	
2	q	
3	p	R, 1
4	$q \to p$	$\to$-I, 2–3
5	$p \to (q \to p)$	$\to$-I, 1–4

2. $(p \wedge q) \to (r \to (q \wedge r))$

   Solution.

1	$p \wedge q$	
2	r	
3	q	$\wedge$-E, 1
4	$q \wedge r$	$\wedge$-I, 3, 2
5	$r \to (q \wedge r)$	$\to$-I, 2–4
6	$(p \wedge q) \to (r \to (q \wedge r))$	$\to$-I, 1–5

These problems are simple because the form of the expression in each case suggests a rule to apply.

Exercise 3 **Natural Deduction Proofs**

1. Establish that $(p \to q) \to (\neg q \to \neg p)$ is valid using natural deduction. This is a well-known theorem of logic and you should recognise that it is used in everyday reasoning. There is a simple proof using ($\neg$-I) and ($\neg$-E).

   Solution.

1	$p \to q$	
2	$\neg q$	
3	p	
4	q	$\to$-E, 1, 3
5	F	$\neg$E, 2, 4
6	$\neg p$	$\neg$I, 3–5
7	$\neg q \to \neg p$	$\to$-I, 2–6
8	$(p \to q) \to (\neg q \to \neg p)$	$\to$-I, 1–7

2. Prove the derived rule $\frac{p \vee (q \wedge r)}{p \vee q}$ using natural deduction. This is a theorem which is easily proved using $\vee$-E.

   Solution.

1	$p \vee (q \wedge r)$	
2	p	
3	$p \vee q$	$\vee$-I, 2
4	$q \wedge r$	
5	q	$\wedge$-E, 4
6	$p \vee q$	$\vee$-I, 5
7	$p \vee q$	$\vee$-E, 1, 2–3, 4–6

Exercise 4 **More Natural Deduction Proofs**

1. $\dfrac{((p \vee q) \to q)}{(p \to (p \wedge q))}$

   Solution.

1	$(p \vee q) \to q$	
2	p	
3	$p \vee q$	$\vee$-I, 2
4	q	$\to$-E, 1, 3
5	$p \wedge q$	$\wedge$-I, 2, 4
6	$p \to (p \wedge q)$	$\to$-I, 2–5

2. $((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$

   Solution.

1	$(p \to q) \wedge (p \to r)$	
2	$p \to q$	$\wedge$-E, 1
3	$p \to r$	$\wedge$-E, 1
4	p	
5	q	$\to$-E, 2, 4
6	r	$\to$-E, 3, 4
7	$q \wedge r$	$\wedge$-I, 5, 6
8	$p \to (q \wedge r)$	$\to$-I, 4–7
9	$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$	$\to$-I, 1–8

Exercise 5 **Harder Natural deduction proofs**

Establish the validity of the following formulae using natural deduction.

1. $(p \wedge q \to r) \leftrightarrow (p \to (q \to r))$, that is you need to prove $\dfrac{p \wedge q \to r}{p \to (q \to r)}$ and $\dfrac{p \to (q \to r)}{p \wedge q \to r}$.

   Solution.

1	$(p \wedge q) \to r$	
2	p	
3	q	
4	$p \wedge q$	$\wedge$-I, 2, 3
5	r	$\to$-E, 1, 4
6	$q \to r$	$\to$-I, 3–5
7	$p \to (q \to r)$	$\to$-I, 2–6

1	$p \to (q \to r)$	
2	$\quad p \land q$	
3	$\quad p$	$\land$-E, 2
4	$\quad q \to r$	$\to$-E, 1, 3
5	$\quad q$	$\land$-E, 2
6	$\quad r$	$\to$-E, 4, 5
7	$(p \land q) \to r$	$\to$-I, 2–6

Exercise 6 **Understanding FOL formulae**

The following sentences talk about a solar power system, which consists of one or more *installations* of solar panels. Each installation of solar panels consists of one or more *panels*. Each panel consists of one or more *cells*.

The following predicates are given:

- $L(x)$ – x receives less than 50% of expected light
- $S(x)$ – x is shaded
- $F(x)$ – x is fully operational
- $E(x)$ – x is producing enough energy
- $B(x, y)$ – x belongs to y

Translate the following sentences into first-order logic:

1. A system is producing enough energy if all its installations are fully operational

   Solution. $\forall s.(\forall i.B(i, s) \to F(i)) \to E(s)$

2. An installation is fully operational if no solar panel in that installation is shaded

   Solution. $\forall i.\neg(\exists p.B(p, i) \land S(p)) \to F(i)$

3. A solar panel is shaded if some cell of the panel receives less than 50 percent of expected light

   Solution. $\forall p.(\exists c.B(c, p) \land L(c)) \to S(p)$

Appendix 1: Natural Deduction Rules

Propositional Calculus

$$(\wedge I)\quad \frac{p \qquad q}{p \wedge q} \qquad\qquad (\wedge E)\quad \frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q}$$

$$(\vee I)\quad \frac{p}{p \vee q} \qquad \frac{p}{q \vee p} \qquad\qquad (\vee E)\quad \frac{p \vee q \qquad \begin{array}{c}[p]\\ \vdots \\ r\end{array} \qquad \begin{array}{c}[q]\\ \vdots \\ r\end{array}}{r}$$

$$(\rightarrow I)\quad \frac{\begin{array}{c}[p]\\ \vdots \\ q\end{array}}{p \rightarrow q} \qquad\qquad (\rightarrow E)\quad \frac{p \qquad p \rightarrow q}{q}$$

$$(\neg I)\quad \frac{\begin{array}{c}[p]\\ \vdots \\ F\end{array}}{\neg p} \qquad\qquad (\neg E)\quad \frac{p \qquad \neg p}{F}$$

$$(\mathsf{PC})\quad \frac{\begin{array}{c}[\neg p]\\ \vdots \\ F\end{array}}{p} \qquad\qquad (T)\quad \frac{}{T}$$

Predicate Calculus

$$(\forall I)\quad \frac{P(a) \qquad (a \text{ arbitrary})}{\forall x.\ P(x)}$$

$$(\forall E)\quad \frac{\forall x.\ P(x)}{P(a)}$$

$$(\exists I)\quad \frac{P(a)}{\exists x P(x)}$$

$$(\exists E)\quad \frac{\exists x P(x) \qquad \begin{array}{c}[P(a)]\\ \vdots \\ q\end{array} \qquad (a \text{ arbitrary})}{q \quad (a \text{ is not free in } q)}$$

Appendix 2: Truth Table Values

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$\neg p$	$p \leftrightarrow q$
T	T	T	T	T	F	T
T	F	T	F	F	F	F
F	T	T	F	T	T	F
F	F	F	F	T	T	T

Appendix 3: Valid Boolean Equations

Associativity

$a \vee (b \vee c) = (a \vee b) \vee c$ $\qquad$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Commutativity

$a \vee b = b \vee a$ $\qquad$ $a \wedge b = b \wedge a$

Absorption.

$a \vee (a \wedge b) = a$ $\qquad$ $a \wedge (a \vee b) = a$

Identity.

$a \vee F = a$ $\qquad$ $a \wedge T = a$

Distributivity

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\qquad$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Complements.

$a \vee \neg a = T$ $\qquad$ $a \wedge \neg a = F$

Appendix 4: De Morgan Laws

De Morgan Laws

$\neg(x \vee y) = \neg x \wedge \neg y$ $\qquad$ $\neg(x \wedge y) = \neg x \vee \neg y$