Nondeterministic Finite Automata

COMP1600 / COMP6260

Victor Rivera Dirk Pattinson
Australian National University

Semester 2, 2021

DFA Minimisation

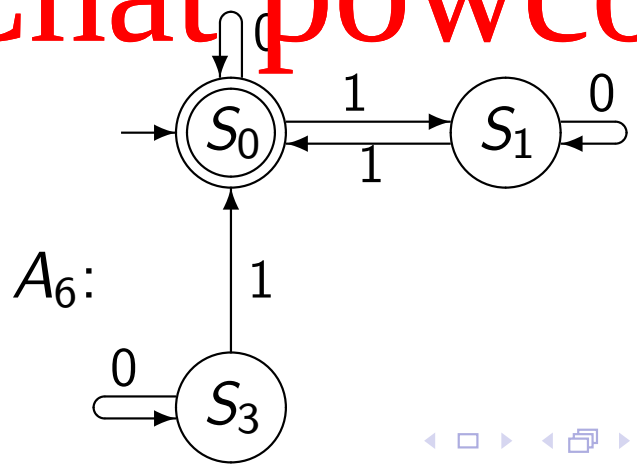
Elimination of equivalent states:

- if two states are equivalent, one can be eliminated

Elimination of Unreachable States:

- if a state cannot be reached from the initial state then it can also be eliminated.

Example. S_3 not reachable



The Standard Minimisation Algorithm

Main Idea.

- aggregate states into groups (of possibly equivalent states)
- initially, *all* states are possibly equivalent
- *split* a group of possibly equivalent states if we have *evidence* that they are not equivalent.
  - a non-final state is never equivalent to a final state
  - two states are non-equivalent if the transition function takes them into different groups (with the same letter)
- repeat until no more groups can be split.

Realisation.

- The working data structure for the algorithm is a list of lists ("groups") of states
- On each iteration, we test one of the groups with a symbol from the alphabet.
- If we notice differing behaviour, we **split the group.**

The Algorithm Details

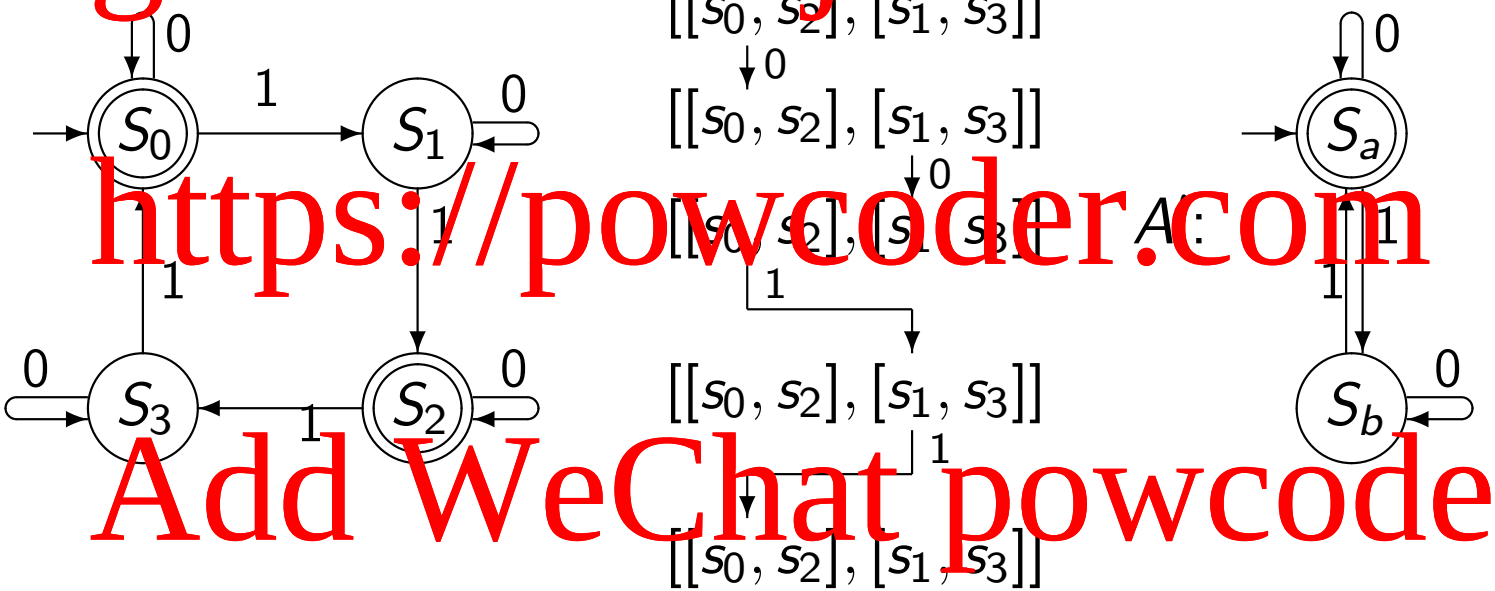
- **Input:** A list containing two "groups". (a group is represented as a list of states). One group consists of the Final states and the other consists of the non-final states.
- **Data:** The working data structure, $WDS : [[State]]$, is a list of groups of states. When two states are in different groups, we *know* they are not equivalent.

- **Loop:** Pick a group, $\{s_1, ...s_j\}$ and a symbol, x.
  - If the states $\{N(s_i, x) \mid i = 1, \ldots, j\}$ are all in the same group, then the group $\{s_1 ... s_j\}$ is not split.
  - If the states $\{N(s_i, x) \mid i = 1, \ldots, j\}$ belong to different groups of WDS, then the group $\{s_1, ...s_j\}$ should be split accordingly.

- **Continue until** we cannot, by *any* choice of letter, split *any* group.

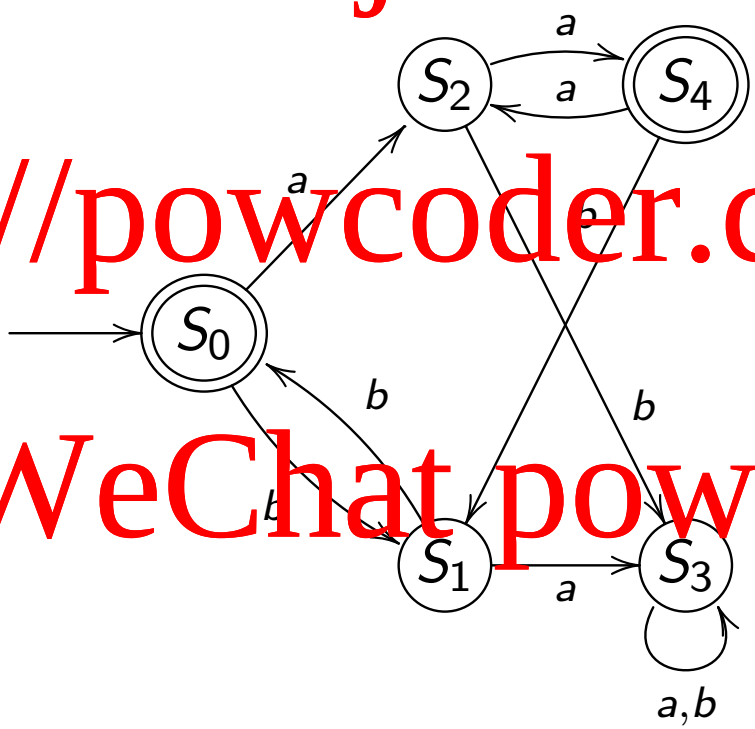
Our Previous Example

Our running example is trivial. The initial split is it.

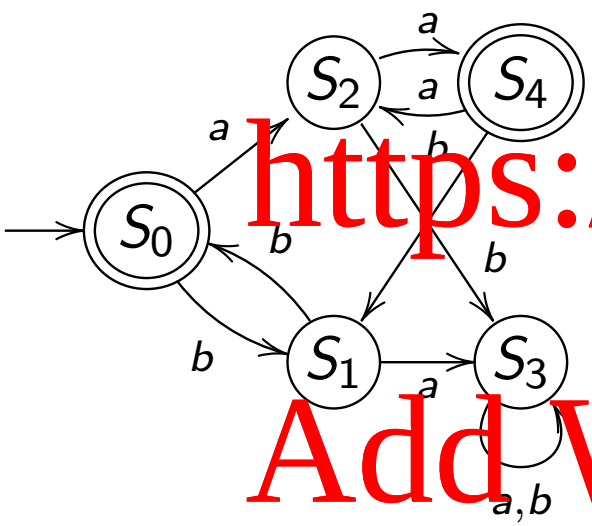


Minimisation: Second Example

Q. What is the language of this automaton? Can you find a simpler automaton with the same language?



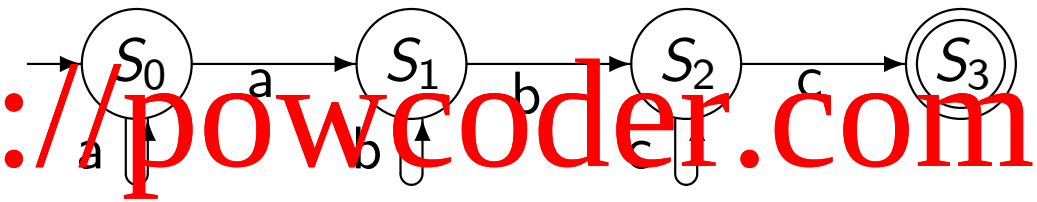
Minimisation Step by Step



- initial split: $[[0,4],[1,2,3]]$
  - check $[0,4]$: don't split
  - check $[1,2,3]$:
    - ★ $S_1 \xrightarrow{a} S_3$ and $S_2 \xrightarrow{a} S_4$ in different group, so split
    - ★ $S_1 \xrightarrow{b} S_3$ and $S_3 \xrightarrow{b} S_3$ in different group, so split
    - ★ $S_2 \xrightarrow{a} S_4$ and $S_3 \xrightarrow{a} S_3$ in different group, so split
- next split: $[[0,4],[1],[2],[3]]$
  - check $[0,4]$: don't split
  - check $[1]$, $[2]$ and $[3]$: don't split
- final split $[[0,4],[1],[2],[3]]$
  - as no more splits did occur in the last round

Non-Deterministic Finite State Automata — NFAs

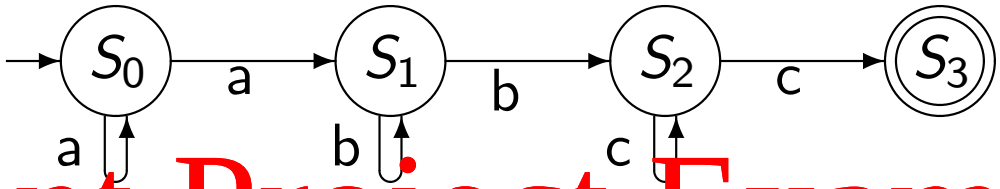
Consider this FSA:



Q. Is it intuitively clear what it does?

Q. Is it a DFA in the sense of our definition?

Is it legal, i.e. a "proper" DFA?



A. It makes sense, but it is *nondeterministic*: A nondeterministic finite automaton (NFA). So not a "legal" DFA, but a specimen of a different breed.

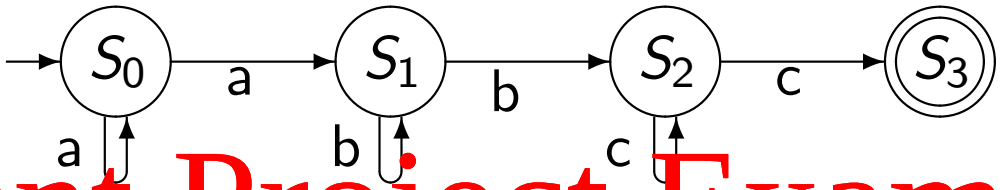
Differences to deterministic automata

- Multiple edges with the **same label** come out of states
- For some states, there is **not an edge** for every token

Formally. NFAs have a transition *relation* rather than a transition *function*.

- transition relation $R(s_1, x, s_2)$ obtains if there's an x-labelled edge from s_1 to s_2
- there can be *no* x-labelled edge between s_1 and *any* state
- there can be *many* states $s_2, s_3, \ldots$ that are connected to s_1 via an x-labelled edge.

Is it clear what it does?



Observations.

- Some states don't have an outgoing edge with a certain letter, so the NFA can "get stuck".
- In some states, there's more than one possible successor state with a certain letter.

Acceptance condition for NFAs given string α:

- can get from initial to final state, making the "right" choice of successor state
- without getting stuck

Example. $\alpha = aaabcc$

- need to "look ahead" to make the right choice
- (alternatively, try to backtrack if wrong choice has been made)

DFAs *vs* NFAs

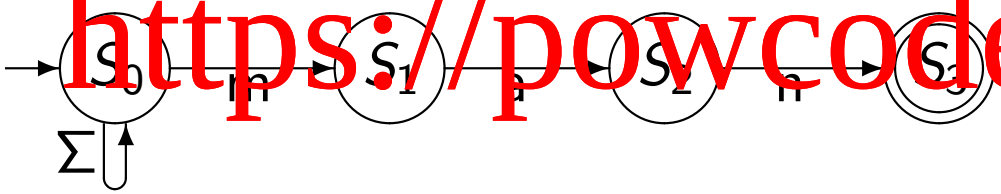
Key Differences

- For each state in a DFA and for each input symbol, there is a **unique** successor state.
- DFAs have a *transition function*.
- NFAs allow **zero, one or more** transitions from a state for the same input symbol.
- NFAs have a *transition* **relation**.
- An input sequence $a_1, a_2, \ldots, a_n$ is *accepted* by a NFA if there *exists* some sequence of transitions that leads from the initial state to a final state.

Why NFAs?

Example. NFAs are simpler.

A NFA recognising strings of letters ending in "man":
(Σ is the Latin alphabet)



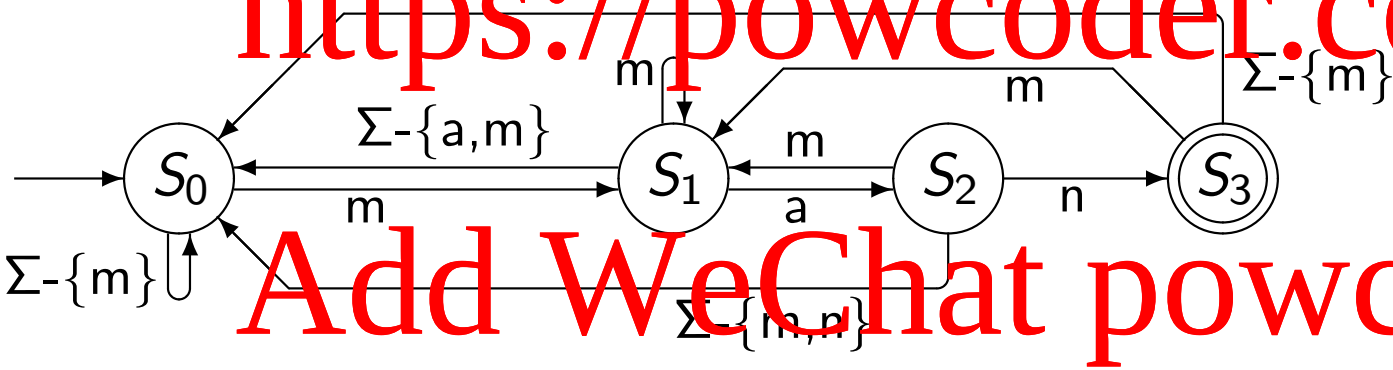
Note.

- *two* transitions from S_0 for the letter "m"
- *no* transition from S_1 for (e.g.) the letter "n"

An Equivalent DFA

Example. DFAs are (often) more complex.

A DFA that recognises strings of letters that end in "man".



NFAs: Formal Definition

A Nondeterministic Finite State Automaton (NFA) consists of five parts:

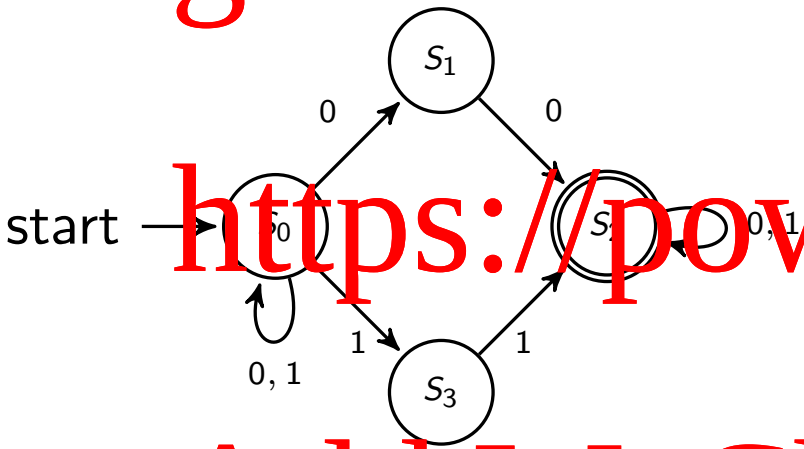
$A = (\Sigma, S, s_0, F, R)$

- an input **alphabet** Σ, the set of tokens
- a set of **states** S
- an **"initial"** state $s_0 \in S$ (we start here)
- a set of **"final"** states $F \subseteq S$ (we hope to finish in one of these)
- a **transition relation** $R \subseteq S \times \Sigma \times S$.

Aside. The transition *relation* is what makes the automaton nondeterministic. It can be seen as a function $\delta : S \times \Sigma \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the set of subsets of S.

Another Example

Transition Diagram



As a transition table

	0	1
$\rightarrow S_0$	$\{S_0, S_1\}$	$\{S_0, S_3\}$
S_1	$\{S_2\}$	$\emptyset$
$\odot S_2$	$\{S_2\}$	$\{S_2\}$
S_3	$\emptyset$	$\{S_2\}$

Both convey precisely the same information. What is the language of this automaton?

Acceptance for NFAs

Given. An NFA $A = (\Sigma, S, F, s_0, R)$. Then A *accepts* a word $w = a_1 a_2 \dots a_n$ (in symbols: $w \in L(A)$) if there exists a sequence of states

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_{n-1} \xrightarrow{a_n} s_n$$

where s_0 is the starting state, $s_n \in F$ is an accepting state, and $s \xrightarrow{a} t$ if $(s, a, t) \in R$.

Aside. This is like for deterministic automata, the only difference is that for

- *non-deterministic automata* we have $s \xrightarrow{a} t$ if $(s, a, t) \in R$ (that is, the automaton can make a transition)
- *deterministic automata* we have $s \xrightarrow{a} t$ if $N(s, a) = t$ (that is, the automaton makes the transition)

Eventual State Relation for NFAs

Basic Idea. The eventual state relation $R^*(s, w, s')$ is true if s' is a state that the NFA can reach, starting in state s and reading string w.

Formal Definition. The eventual state relation has type

$$R^* \subseteq S \times \Sigma^* \times S$$

$$\text{or } R^* : S \times \Sigma^* \times S \to Bool$$

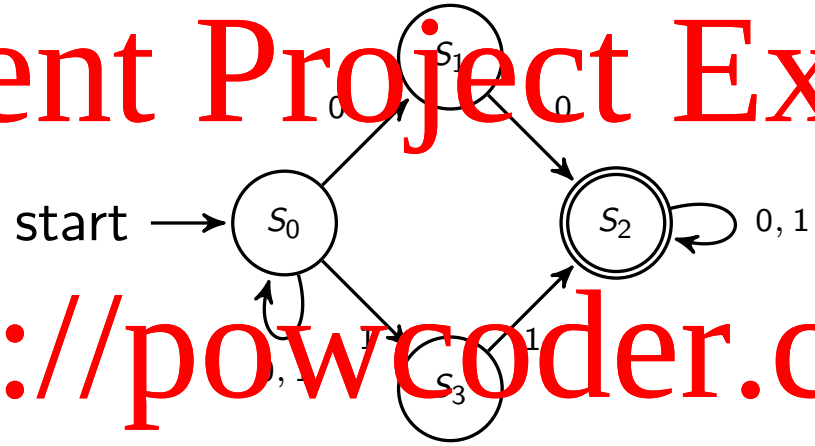
and is defined inductively as follows:

$$R^*(s, \epsilon, s)$$

$$R^*(s, x\alpha, s') = \exists s''.R(s, x, s'') \wedge R^*(s'', \alpha, s')$$

Eventual State Relation: Example

The "double digits" automaton



Eventual State Relation.

- $(S_0, \epsilon, S_0) \in R^*$ by definition.
- $S_0 \xrightarrow{0} S_0 \xrightarrow{0} S_0 \xrightarrow{1} S_0$, hence $(S_0, \text{“001”}, S_0) \in R^*$.
- $S_0 \xrightarrow{0} S_1 \xrightarrow{0} S_2 \xrightarrow{1} S_2$, hence $(S_0, \text{“001”}, S_2) \in R^*$.
- $S_1 \xrightarrow{0} S_2 \xrightarrow{0} S_2 \xrightarrow{1} S_2$, hence $(S_1, \text{“001”}, S_2) \in R^*$.

An Important (but Unsurprising) Theorem about R^*

For all states s, s' and for all strings $\alpha, \beta \in \Sigma^*$

$R^*(s, \alpha\beta, s')$ if and only if $\exists s''. R^*(s, \alpha, s'') \wedge R^*(s'', \beta, s')$

The proof is similar to the corresponding result for N^* in DFAs.

Language of a NFA

Let $A = (\Sigma, S, s_0, F, R)$ be a NFA.

Theorem. A string w is *accepted* by A if

$$\exists s \in F.\ R^*(s_0, w, s)$$

(Compare with the definition of acceptance for NFAs earlier)

Language of an NFA.

The *language* accepted by A is the set of all strings accepted by A

$$L(A) = \{w \in \Sigma^* \mid \exists s \in F.\ R^*(s_0, w, s)\}$$

Informally. That is, $w \in L(A)$ iff *there exists* a path through the diagram for A, from s_0 to a final state s ($s \in F$), such that the symbols on the path match the symbols in w

Power of Nondeterminism?

Q. Is there a language that is accepted by an NFA for which we *cannot* find a DFA that (also) accepts it?

- it seems easier to construct NFAs
- but in examples, DFAs did also exist

A. A simple "no".

Theorem. If language L is accepted by a NFA, then there is some DFA which accepts the same language.

Moreover, this DFA can be computed using an algorithm.

- just like the minimal automaton can be computed using state equivalence

Drawback. The resulting DFA may have exponentially many states

- Have to record a *set* of states that the NFA *could* be in.

Constructing the Equivalent DFA from an NFA

Assumption. We have an NFA with state set $\{q_0, \ldots, q_n\}$.

Basic Idea.

- consider all possible runs of the NFA in parallel
- as a consequence, can be in a *set* of states

Construction

- A *state* of the DFA is a *set of states* of the NFA
- e.g. $\{q_3, q_7\}$ or $\emptyset$
- signifies the states that the NFA *can* be in after reading some input
- transition function: records possible next states
- e.g. from $\{q_3, q_7\}$ with letter x, take union of transitions (with x) from q_3 and q_7
- *final states* are state sets that *contain* a final state.

Subset Construction: The Finer Points

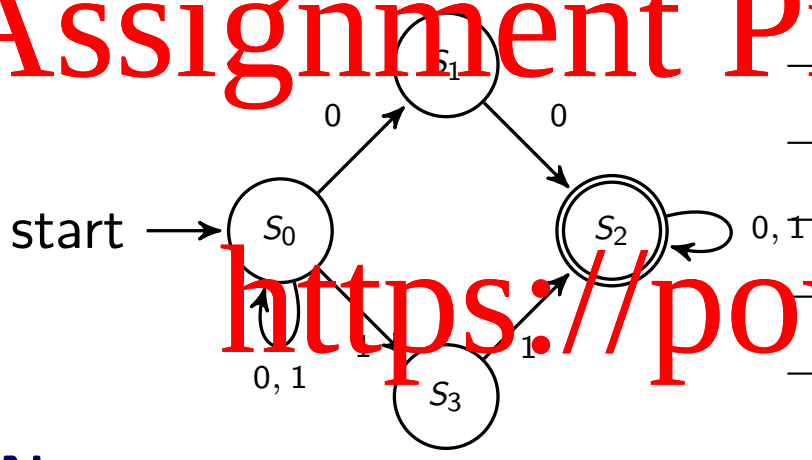
Given. NFA $A = (\Sigma, S, s_0, F, R)$.

Subset Construction.

- states are *subsets* of S but each subset plays the role of a single state!
- transitions: for a state $Q \subseteq S$ and a letter $a \in \Sigma$:

$$N(Q, a) = \{s_1 \in S \mid s \xrightarrow{a} s_1 \text{ for some } s \in Q\}$$
$$= \{s_1 \in S \mid (s, a, s_1) \in R \text{ for some } s \in Q\}$$

Determinisation: Example



Subset Construction: transition table

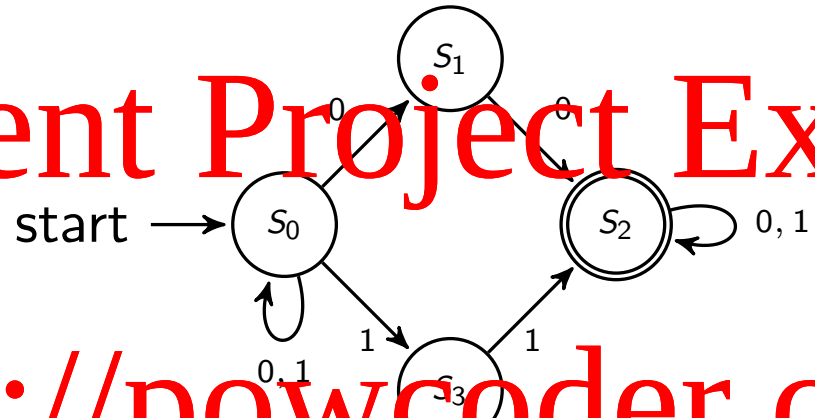
	0	1
$\rightarrow \{S_0\}$	$\{S_0, S_1\}$	$\{S_0, S_3\}$
$\{S_0, S_1\}$	$\{S_0, S_1, S_2\}$	$\{S_0, S_3\}$
$\{S_0, S_3\}$	$\{S_0, S_1\}$	$\{S_0, S_3, S_2\}$
$\circledcirc\{S_0, S_1, S_2\}$	$\{S_0, S_1, S_2\}$	$\{S_0, S_3, S_2\}$
$\odot\{S_0, S_3, S_2\}$	$\{S_0, S_1, S_2\}$	$\{S_0, S_3, S_2\}$

Note.

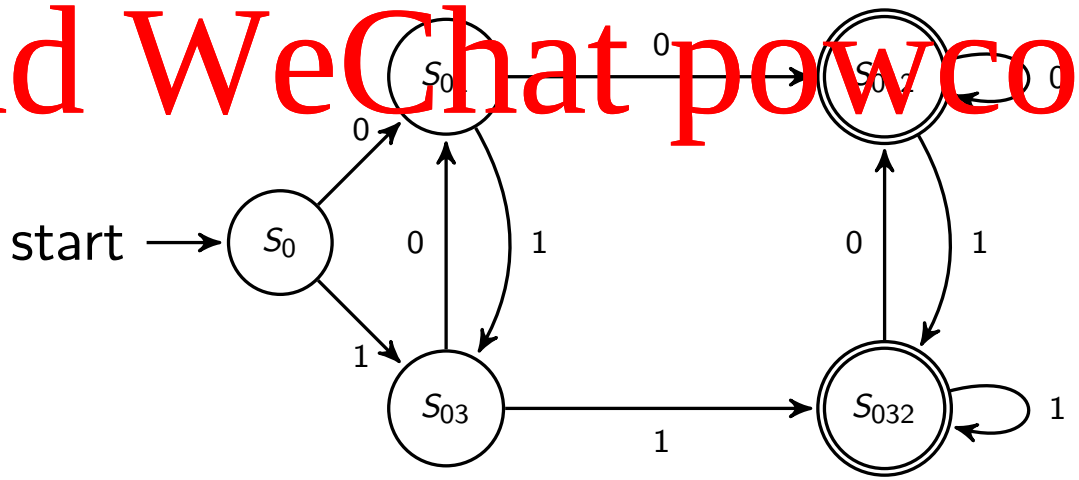
- don't have transition for *all* states, just those that are reachable from $\{S_0\}$
- all others are not relevant (cf. elimination of unreachable states)
- having all states would require $2^4 = 16$ entries.

Determinisation Example, as Diagrams

Double Digits, as NFA.

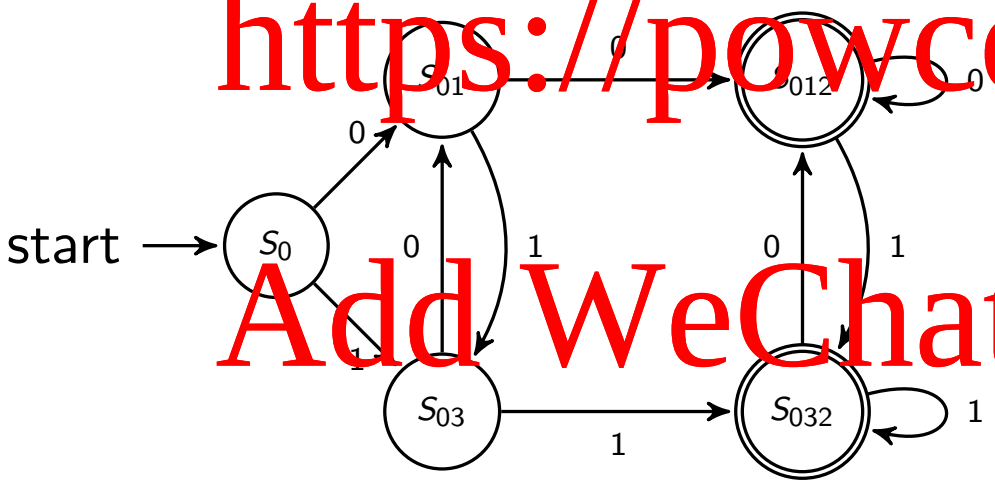


Double Digits as DFA.



Recall Minimisation ...

Q. Can there be a simpler DFA (with fewer states) that recognises the same language?



- initial split:
  $[[S_{012}, S_{032}], [S_0, S_{01}, S_{03}]]$
- next split:
  $[[S_{012}, S_{032}], [S_{01}], [S_0, S_{03}]]$
- next split:
  $[[S_{012}, S_{032}], [S_{01}], [S_0], [S_{03}]]$
- no more splits, so S_{012} and S_{032} can be merged.

More Expressive Power: ϵ-transitions

Extra Ingredient: Spontaneous transitions that don't "eat" a letter

- NFAs that may change state *without* consuming a symbol.
- NFAs of this kind are called *NFAs with ϵ-transitions*
- can convert NFAs with ϵ-transitions to (standard) NFAs

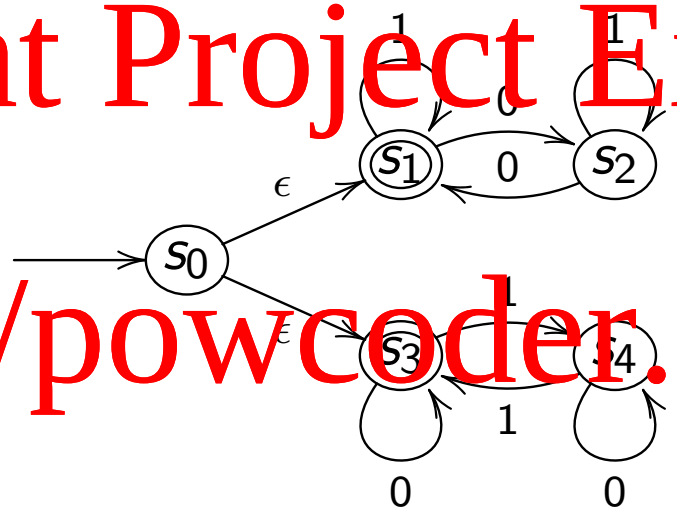
Formal Definition: An NFA with ϵ-transitions is an NFA, but the transition relation has the form

$$R \subseteq S \times \Sigma \cup \{\epsilon\} \times S$$

- cf. NFAs with transition relation $R \subseteq S \times \Sigma \times S$
- $R(s, \epsilon, s')$ is a spontaneous transition (without reading input symbol)
- ϵ is *not* an element of the alphabet!

ϵ-NFA: Example

General Pattern. ϵ-transitions say "or"

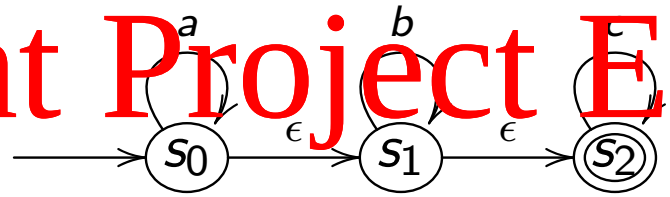


Interpretation.

- "top" automaton (with start state s_1) requires even number of 0's
- "bottom" automaton (with start state s_3) requires even number of 1's
- entire automaton (with start state s_0) accepts *either* an even number of 1's *or* an even number of 0's

Example and Acceptance

Language of this Automaton?



Acceptance. An ϵ-NFA A *accepts* a word $w = a_1 \dots a_n$ if there is a sequence of states

$$s_0 \xrightarrow{\epsilon^*} r_1 \xrightarrow{a_1} r_1' \xrightarrow{\epsilon^*} r_2 \xrightarrow{a_2} r_2' \dots r_n \xrightarrow{a_n} r_n' \xrightarrow{\epsilon^*} f$$

where s_0 is the starting state, $f \in F$ is an accepting state and

- $s \xrightarrow{a} t$ if there is an a-transition from s to t, i.e $(s, a, t) \in R$
- $s \xrightarrow{\epsilon^*} t$ if there is a sequence of ϵ-transitions (only!) from s to t.

In particular: the empty string $\epsilon \in L(A)$ if $s_0 \xrightarrow{\epsilon^*} f$ for a final state $f \in F$.

Eventual State Relation for ϵ-NFAs

Given. An ϵ-NFA (Σ, S, s_0, F, R) (i.e. $R \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$) then the *$\epsilon$-closure* of a state $s \in S$ is given by

$$\text{eclose}(s) = \{s' \in S \mid \text{there is a sequence of } \epsilon\text{-transitions from } s \text{ to } s'\}$$

and the *eventual state relation* is given by

$$R^*(s, \epsilon, s') \iff s' \in \text{eclose}(s)$$

$$R^*(s, aw, s') \iff \text{there are } s_0 \text{ and } s_1 \text{ such that}$$

$$s_0 \in \text{eclose}(s), (s_0, a, s_1) \in R, (s_1, w, s') \in R^*$$

As for DFAs / NFAs:
A string w is *accepted* by an ϵ-NFA A (in symbols: $w \in L(A)$) if $(s_0, w, f) \in R^*$ for some final state $f \in F$, that is

$$L(A) = \{w \in \Sigma^* \mid \exists f \in F.(s_0, w, f) \in R^*\}$$

Q. How does this relate to the notion of acceptance earlier?

Relationship Between NFAs and ϵ-NFAs

Q. Are there languages *only* accepted by ϵ-NFAs?

A. No. Every ϵ-NFA $A = (\Sigma, S, s_0, F, R)$ can be converted to an NFA A' without ϵ-transitions so that $L(A) = L(A')$.

Construction. Put $A' = (\Sigma, S, s_0, F', R')$ where

- Make $s \in S$ an accepting state *in* A' if s can reach an accepting state *in* A by ϵ-transitions:

$$F' = \{s \in S \mid \text{eclose}(s) \cap F \neq \emptyset\}$$

- Put an arc $s \xrightarrow{a} t$ *into* A' if there's a transition $s' \xrightarrow{a} t$ *in* A with $s' \in \text{eclose}(s)$:

$$R' = \{(s, a, t) \mid (s', a, t) \in R \text{ for some } s' \in \text{eclose}(s)\}$$

(and convince yourself that A and A' accept the same strings!)

Regular Expressions

Challenge. Understand the computational power of DFAs / NFAs.

Approach. Characterise the languages that can be accepted by an NFA in a different form.

One Characterisation. Regular expressions (cf. Perl, Ruby, grep)

Basic Operators used to construct new expressions from old:

- vertical bar (pipe): choose either the left or right expression
- Kleene star: repeat strings from an expression
- ϵ, the empty string, and every letter of the alphabet
- concatenation, for sequencing expressions
- parentheses, for grouping

Example.

- a^* indicates 0 or more as.
- $\mathrm{yes} \mid \mathrm{no}$ is the language with just the 2 given strings.
- $(0 \mid 1)^*$ indicates the set of binary numerals.

Regular Expressions — More Examples

- $0 \mid (1(0 \mid 1)^*)$ is the set of binary numerals with no leading zeros.
- $(a \mid b)^* c (a \mid b)^*$ is the set of strings over $\{a, b, c\}$ with just one c.
- $(0^*10^*10^*)^*$ is the language of bit-strings that have an even number of ones. (Alternatively $0^*(10^*10^*)^*$)
- $(z^*(x^* \mid y^*)\ z))^*$ is the set of strings over $\{x, y, z\}$ with no x and y adjacent.
- $1 \mid (0 \mid (0 \mid 1)^*1)$ is binary fractional numerals between 0 and 1 with no trailing zeroes. (e.g. 0.1, 0.110011 but not .1 or 0.10)

The Definition of Regular Expressions

Key Concept.

- regular expressions are purely *syntactical* – just like formulae
- *but:* every expression denotes a set of strings – this is the meaning.

Definition. The regular expressions over alphabet Σ and the sets that they denote are:

- $\emptyset$ is a regular expression and denotes the empty set $\emptyset$
- ϵ is a regular expression and denotes the set $\{\epsilon\}$
- for each $a \in \Sigma$, a is a regular expression and denotes the set $\{a\}$

If α and β are regular expressions denoting languages R and S, respectively, then:

- $\alpha \mid \beta$ denotes $R \cup S$
- $\alpha\,\beta$ denotes RS which is $\{xy \mid x \in R \wedge y \in S\}$
- α^* denotes R^*, ie, the set of *finitely* many $r_i \in R$, concatenated

R^* is (inductively) defined as $\{\epsilon\} \cup RR^*$

Regular Expressions and DFAs

Key Insight.

Regular expressions and NFAs / DFAs are equivalent.

- for every DFA A, have regular expression r with $L(A) = L(r)$
- for every regular expression r, have DFA A with $L(r) = L(A)$
- so the "power" of NFAs / DFAs are *completely described* by regular expressions.

Q. Can we "compute" more than what can be described by regular expressions?

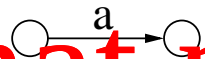
Regular Expressions to ϵ-NFAs

Key Insight.

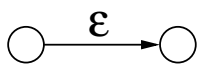
- regular expressions are an *inductively defined structure*
- e.g. representable by an inductive data type in Haskell
- as a consequence, we can give *inductive definition* of the corresponding automaton

Construction (start state on left, final state on right)

- When the regular expression is a symbol a of the alphabet (language is $\{a\}$) the automaton is



- When the regular expression is ϵ (language is $\{\epsilon\}$) the automaton is

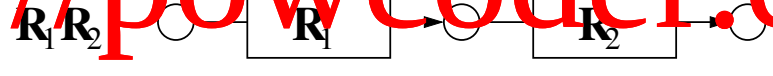


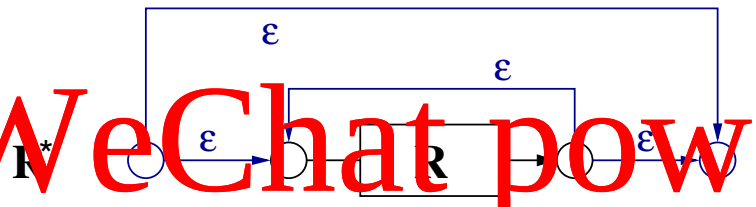
- When the regular expression is $\emptyset$ (language is $\emptyset$) the automaton has no edges

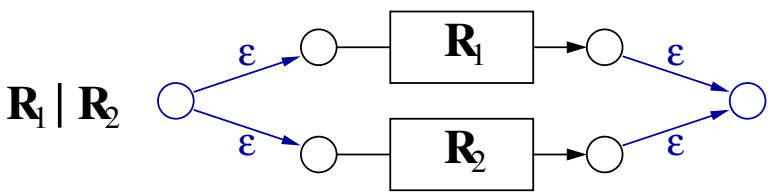
Regular Expressions to NFAs, ctd

Suppose the NFA corresponding to some R is:

Then NFAs corresponding to composite regular expressions are defined as follows:

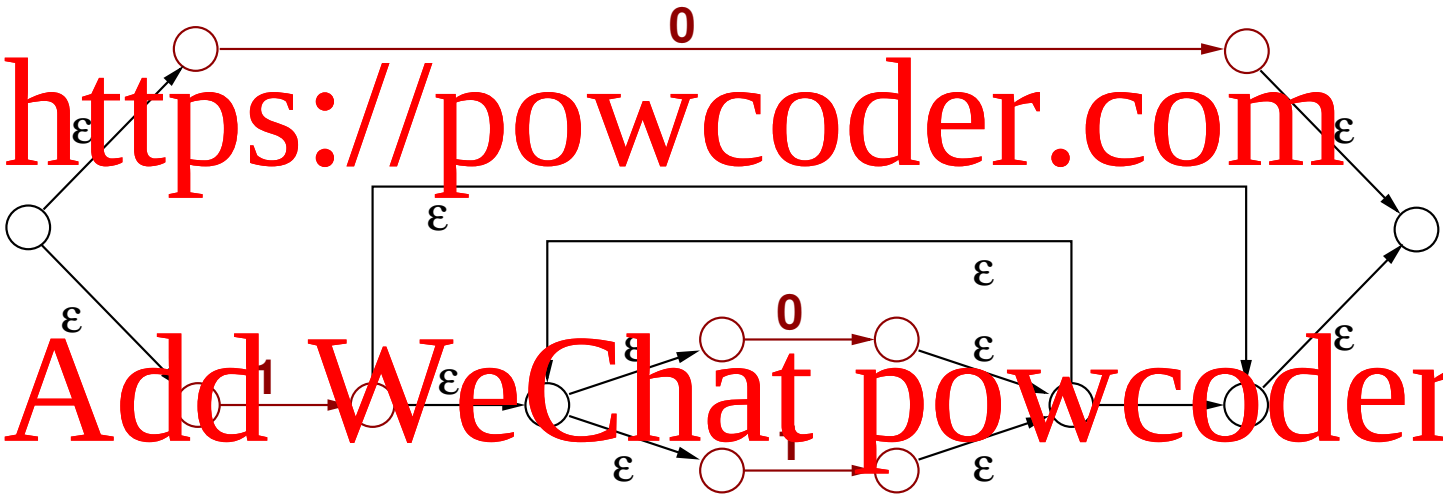






Example

Given the regular expression for binary numerals without leading zeros, $(0 \mid 1(0|1)^*)$, the above algorithm gives this NFA.



Closing the Loop

Given. A finite alphabet Σ and a language $L \subseteq \Sigma^*$. The following are equivalent:

- L can be described by a regular expression
- L can be recognised by an ϵ-NFA
- L can be recognised by an NFA
- L can be recognised by a DFA ...

as we can convert regular expressions into ϵ-NFAs into NFAs into DFAs.

Missing Link. Construction of regular expressions from DFAs (not covered in this course)

Summary.

Starting Point. Finite Automata

- motivated by computers having finite memory (only)
- solving simple problems: "is string s accepted?"

Limitations of Finite Automata

- e.g. cannot recognise $L = \{a^n b^n \mid n \geq 0\}$

Characterisation of expressive power

- can go back and forth between automata and regular expressions

Q. Are finite automata a "good" model of computation?

- if yes, why?
- if not, why not? What is missing?

Literature.

- Introduction to Automata Theory, Languages, and Computation By Hopcroft, Motwani, and Ullman.

  A classic text that has been re-worked from a standard textbook.

- Introduction To The Theory Of Computation by Michael Sipser

  The part on Automata and Languages covers (more than) what we have discussed here.