Hoare Logic: Partial Correctness

COMP1600 / COMP6260

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Programming Paradigms

Functional. (Haskell, SML, OCaml, ...)

- main paradigm: *functions* that *don't* rely on state
- main ingredient: *recursion*

Imperative. (C, Java, Algol, (Visual) Basic, ...)

- main paradigm: *operations* that *do* manipulate state.
- main ingredient: *loops*

Example: From Recursion to Loops

In Haskell.

```
fact_tr :: Int -> Int -> Int
fact_tr 0 acc = acc
fact_tr n acc = fact_tr (n-1) (acc * n)
fact n = fact_tr n 1
```

In Java.

```
public static int fact (int n) {
  int acc = 1;
  while (n > 0) { acc = acc * n; n = n-1; }
  return acc;
}
```

Main Difference.

- programs are not simple equations any more
- need to keep track of *changing* values of variables

Verification for Imperative Languages

Main Ingredients.

- properties of **program states**
- **commands** that modify state.

Description of both.

- *properties* of states: *formulae*, e.g. $x < 0 \wedge y > x$
- *commands* are taken from the programming language
- *formal rules* that tell us how to manipulate both.

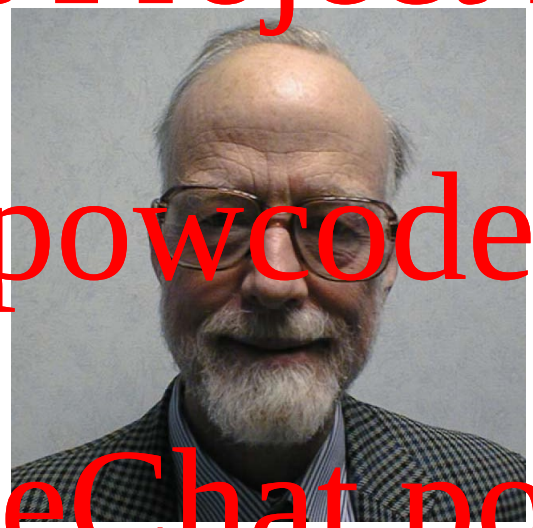
Hoare Logic ties in program states and formulae:

$$\boxed{\{P\} \quad \texttt{program} \quad \{Q\}}$$

"Running `program` in a state that satisfies P gives a state that satisfies Q"

C. A. R. (Tony) Hoare

The inventor of this week's logic is also famous for inventing the **Quicksort** algorithm in 1960 - when he was just **26**! A quote:



Computer programming is an **exact science** *in that all the properties of a program and all the consequences of executing it in any given environment can, in principle, be found out from the text of the program itself by means of purely* **deductive reasoning**.

Logic = Syntax + Semantics + Calculus

Example. Propositional Logic

- syntax: atomic propositions $p, q, r, \ldots$, $\bot$, $\wedge$, $\vee$, $\rightarrow$ and $\neg$
- semantics: truth tables
- calculus: natural deduction.

Hoare Logic.

- syntax: triples $\{P\}$ `program` $\{Q\}$
- semantics: P in pre-state implies Q in post-state
- calculus: Hoare Logic

Q. What are pre/post conditions *precisely*? what are the programs? What about termination?

Hoare Logic: A Simple Imperative Programming Language

Q. In a Hoare triple $\{P\}$ `program` $\{S\}$, what are the programs?

A. We roll our own: a very simple Java-like language.

Assignment – $\texttt{x} := \texttt{e}$

where $\texttt{x}$ is a variable, and $\texttt{e}$ is an expression built from variables and arithmetic that returns a number, e.g. $2 + 3$, $\texttt{x} * \texttt{y} + 1$...

Sequencing – $\texttt{S}_1; \texttt{S}_2$

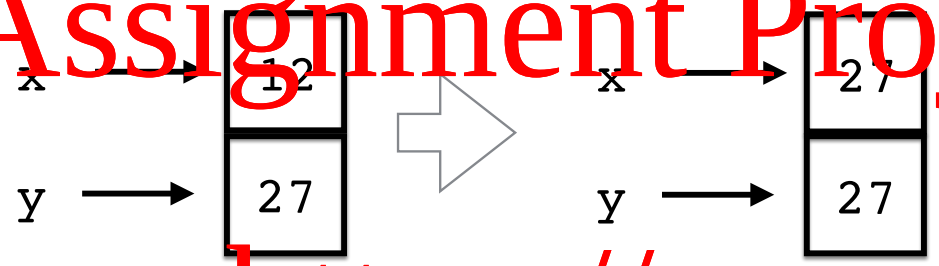
Conditional – `if b then` $\texttt{S}_1$ `else` $\texttt{S}_2$

where $\texttt{b}$ is an expression built from variables, arithmetic and logic that returns a boolean (true or false), e.g. $\texttt{y} < 0$, $\texttt{x} \neq \texttt{y} \wedge \texttt{z} = 0$...

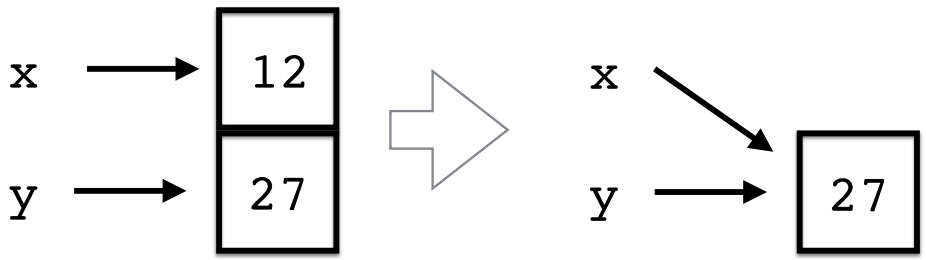
While – `while b do S`

A Note on (the lack of) Aliasing

Assignments `x := y` *copy values*



No Aliasing, i.e. `x` and `y` point to the same region in memory



Syntax of Hoare Logic: Assertions

Q. How do we describe *properties* of states?

- *states* are determined by the values of program variables
- here: states will store *numbers* only.

Properties of States. propositional formulae built from variables, numbers and basic arithmetic

- $x = 3$;
- $x = y$;
- $x \neq y$;
- $x > 0$;
- $x \leq (y^2 + 1\frac{3}{4})$;
- etc...

Syntax of Preconditions and Postconditions ctd.

Propositional Logic to combine simple assertions, e.g.

- $x = 4 \wedge y = 2$;
- $x < 0 \vee y < 0$;
- $x > y \leftrightarrow x = 2 * y$;
- *True*;
- *False*.

The last two logical constructions – *True* and *False* – will prove particularly useful, as we'll later see.

Alternative. Could use *first order logic* – more expressive power.

Anatomy of a Hoare Triple

$$\{P\}\ \texttt{program}\ \{Q\}$$

- `program` is a simple program written using assignments, conditionals, while and sequencing
- pre/post conditions are propositional formulae built from arithmetical relations

Semantics: A Hoare triple $\{P\}$ `program` $\{Q\}$ is *valid* if

- whenever we run `program` in a state that satisfies P
- *and* the program terminates, then the post-state satisfies Q

Example Statements in Hoare Logic

$$\{x > 0\}\ \mathtt{y := 0 - x}\ \{y < 0 \wedge x \neq y\}$$

If $(x > 0)$ *is true* *before* `y := 0-x` *is executed*
then $(y < 0 \wedge x \neq y)$ *is true afterwards*

Here:

- $(x > 0)$ is the precondition;
- `y := 0-x` is a (very simple) code fragment;
- $(y < 0 \ \wedge \ x \neq y)$ is the postcondition.

Hoare logic will provide the rules to *prove* this.

Hoare's Notation – the Definition

The **Hoare triple**:

$$\{P\}\ S\ \{Q\}$$

means:

> **If** *P is true in the initial state*
> **and** *S terminates*
> **then** *Q will hold in the final state.*

Examples:

1. $\{x = 2\}$ `x := x+1` $\{x = 3\}$
2. $\{x = 2\}$ `x := x+1` $\{x = 5000\}$
3. $\{x > 0\}$ `y := 0-x` $\{y < 0 \wedge x \neq y\}$

(Hoare Triples can be true or false)

Some Hoare Triples

Q. Under what conditions are the following Hoare Triples valid?

1. $\{True\}$ `program` $\{True\}$
2. $\{True\}$ `program` $\{False\}$
3. $\{False\}$ `program` $\{True\}$
4. $\{False\}$ `program` $\{False\}$

Some Hoare Triples

Q. Under what conditions are the following Hoare Triples valid?

1. $\{True\}$ `program` $\{True\}$
2. $\{True\}$ `program` $\{False\}$
3. $\{False\}$ `program` $\{True\}$
4. $\{False\}$ `program` $\{False\}$

A. Consider (precondition) $\wedge$ (termination) $\rightarrow$ (postcondition)

1. is always true (as RHS of $\rightarrow$ is true)
2. true if `program` never terminates
3. always true (as RHS of $\rightarrow$ is true)
4. always true (as LHS of $\rightarrow$ is false)

A Larger Hoare Triple

$\{n \geq 0\}$

```
fact := 1;
k := n;
while (k>0)
   fact := fact * k;
   k := k-1
```

$\{fact = n!\}$

Q1. is this Hoare triple true or false?

A Larger Hoare Triple


```
{n>=0}
fact := 1;
k := n;
while (k>0)
   fact := fact * k;
   k := k-1

{fact = n!}
```

Q1. is this Hoare triple true or false?

Q2. what if $n < 0$ initially?

Partial Correctness

Partial Correctness.

A program is *partially correct* if it gives the right answer whenever it terminates.

Hoare Logic (in the form discussed now) (only) proves partial correctness.

Total Correctness.

A program is *totally correct* if it always terminates *and* gives the right answer.

Example.

$$\{x = 1\} \quad \texttt{while x=1 do y:=2} \quad \{x = 3\}$$

is *true* in Hoare logic semantics (just because the loop never terminates).

Partial Correctness is OK

Why not insist on termination?

- We *may not want* termination.

$\{True\}$ `webserver {very good reason}`

- Not accounting for termination makes things simpler.
- We can add termination assertions (next week).

Specification vs Verification

Hoare triples allow us to say something about the *intended effect* of the code

Q. How do we *make sure* that the code validates these assertions?

A1. Testing. For example, for $\{P\}$ `program` $\{Q\}$:

```
assert (P);
  program;
assert (Q);
```

```
assert (n >= 0);
  do something;
assert (m = n * n);
```


- does this catch *all* possible errors?
- How to structure test cases? Changes of variable values?

A2. Proving. Show that $\{P\}$ `program` $\{Q\}$ is true *for all* states

Hoare Calculus.

- a collection of **rules and procedures** for (formally) manipulating the (language of) triples.

(Just like ND for classical propositional logic . . .)

The Assignment Axiom (Rule 1/6)

Rules for proving correctness of programs:

- one rule per construct (assignment, sequencing, if, while)
- two rules to glue things together

Assignment Rule.

- assignment x := e *change state*
- reflect this in pre/post condition

Terminology

- Suppose $Q(x)$ is a predicate involving a variable x,
- Then $Q(e)$ indicates the same formula with all occurrences of x replaced by the expression e.

The Rule.

$$\{Q(e)\}\ \mathtt{x} := \mathtt{e}\ \{Q(x)\}$$

The Assignment Axiom – Intuition

$$\{Q(e)\}\ \mathtt{x} := \mathtt{e}\ \{Q(x)\}$$

Before / After

- want x to have property Q *after* assignment
- then property Q must hold for the value e *before* assignment

Q. Why is this backwards? Shouldn't it be

$$\{Q(x)\}\ \mathtt{x} := \mathtt{e}\ \{Q(e)\}$$

Counterexample. precondition $x = 0$, assignment `x := 1`

$$\{x = 0\}\ \mathtt{x} := 1\ \{1 = 0\}$$

which says "if $x = 0$ initially and $\mathtt{x} := 1$ terminates then $1 = 0$ finally"

Work from the Goal, 'Backwards'

Forward Reasoning. Not usually helpful

- start at the precondition, work your way down to the postcondition
- not the best way – cf. e.g. doing natural deduction proofs

Backwards Reasoning

- start with the goal (postcondition)
- work your way back up to the precondition

Example.

$$\{Q(e)\}\ \mathtt{x} := \mathtt{e}\ \{Q(x)\}$$

- start with postcondition, *copy* it over to precondition
- **replace** all occurrences of `x` with `e`.
- postcondition may have no, one, or many occurrences of `x` in it; all get replaced

Example 1 of $\{Q(e)\}$ x := e $\{Q(x)\}$

Code Fragment. x := 2, postcondition $y = x$.

- copy $y = x$ over to the precondition
- replace all occurrences of x with 2

Formally.

$$\{y = 2\}\ \texttt{x:=2}\ \{y = x\}$$

is an instance of the assignment axiom.

Example 2 of $\{Q(e)\}$ x := e $\{Q(x)\}$

Code Fragment. `x := x + 1`, postcondition $y = x$.

As before.

$$\{y = x + 1\}\ \texttt{x:=x+1}\ \{y = x\}$$

is an instance of the assignment axiom.

Example 3 of $\{Q(e)\}$ x := e $\{Q(x)\}$

Q. How do we prove

$\{y > 0\}$ x := y + 2 $\{x \geq 3\}$?

Example 3 of $\{Q(e)\}$ x := e $\{Q(x)\}$

Q. How do we prove

$\{y > 0\}$ x := y+2 $\{x > 3\}$?

A.

1. Start with the postcondition $x > 3$ and apply the axiom:

$$\{y + 3 > 3\}\ \texttt{x:=y+3}\ \{x > 3\}$$

2. use the fact that $y + 3 > 3$ is equivalent to $y > 0$ to get our result.

Equivalent Predicates.

Can always replace predicates by equivalent predicates, label with *precondition equivalence*, or *postcondition equivalence*.

Proving the Assignment Axiom sound w.r.t. semantics

Assignment Axiom

$$\{Q(e)\}\ \mathtt{x} := \mathtt{e}\ \{Q(x)\}$$

Justification.

- Let v be the value of expression e in the initial state.
- If $Q(e)$ is true initially, then so is $Q(v)$.
- Since the variable x has value v after the assignment (*and nothing else is changed in the state*), $Q(x)$ must be true after that assignment.

The Assignment Axiom is Optimal

Proof Strength. The assignment axiom *is as strong as possible*.

$$\{Q(e)\}\ x := e\ \{Q(x)\}$$

Meaning?

If $Q(x)$ holds after the assignment then $Q(e)$ *MUST* have held before.

- Suppose $Q(x)$ is true after the assignment.
- If v is the value assigned, $Q(v)$ is true after the assignment.
- Since it is only the value of x that is changed, and the predicate $Q(v)$ does not involve x, $Q(v)$ must also be true before the assignment.
- Since v was the value of e before the assignment, $Q(e)$ is true initially.

A non-example

What if we wanted to prove

$\{y = 2\}\ x := y\ \{x > 0\}$?

A non-example

What if we wanted to prove

$$\{y = 2\}\ x := y\ \{x > 0\}\ ?$$

This is clearly true. But our assignment axiom doesn't get us there:

$$\{y > 0\}\ x := y\ \{x > 0\}$$

Problem.

cannot just replace $y > 0$ with $y = 2$ either - they are not equivalent.

Solution.

Need a new Hoare logic rule that allows for manipulation of pre (and post) conditions.

Weak and Strong Predicates

Stronger.

A predicate P is *stronger* than Q if P implies Q.

Weaker.

Q is *weaker* than P if P is stronger that Q.

Intuition. If P is stronger than Q, then

- P is *more restrictive*, i.e. holds in fewer situations
- Q holds in *more* cases than P, including all cases where P holds.
- stronger predicates convey *more* information than weaker predicates.

Q. Can you give me an example of a *really strong* predicate?

Example.

- *I will keep unemployment below 3%* is *stronger* than
- *I will keep unemployment below 15%*.
- The *strongest* possible statement is *False* (unemployment below 0%)
- The *weakest* possible statement is *True* (unemployment at or below 100%)

Weak and Strong in Pictures

Strong Postconditions

Example.

- $(x = 6) \implies (x > 0)$, so $(x = 6)$ is *stronger* than $(x > 0)$
- The Hoare triple

$$\{x = 5\}\ \mathtt{x} := \mathtt{x} + 1\ \{x = 6\}$$

says *more* about the code than does

$$\{x = 5\}\ \mathtt{x} := \mathtt{x} + 1\ \{x > 0\}$$

Strong Postconditions in general

- if postcondition Q_1 is *stronger* than Q_2, then $\{P\}\ \mathtt{S}\ \{Q_1\}$ is a *stronger* statement than $\{P\}\ \mathtt{S}\ \{Q_2\}$.
- if postcondition $x = 6$ is *stronger* than postcondition $x > 0$, then $\{P\}\ \mathtt{S}\ \{x = 6\}$ is a *stronger* statement than $\{P\}\ \mathtt{S}\ \{x > 0\}$

Weak Preconditions

Formula Example.

- condition $(x > 0)$ says *less* about a state than $x = 5$.
- so $x > 0$ is a *weaker* condition than $x = 5$ since $x = 5$ implies $x > 0$.

Hoare Triple Example

- the Hoare triple $\{x > 0\}\ \mathtt{x := x + 1}\ \{x > 1\}$ says *more* about the code than $\{x = 5\}\ \mathtt{x := x + 1}\ \{x > 1\}$
- this is because it says something about *more* situations

Weak Preconditions

- If precondition P_1 is *weaker* than P_2, then $\{P_1\}\ \mathtt{S}\ \{Q\}$ is *stronger* than $\{P_2\}\ \mathtt{S}\ \{Q\}$.
- if precondition $x > 0$ is *weaker* than precondition $x = 5$, then $\{x > 0\}\ \mathtt{S}\ \{Q\}$ is *stronger* than $\{x = 5\}\ \mathtt{S}\ \{Q\}$.

Weak/Strong Pre/Postconditions

Precondition Strengthening. If P_2 is *stronger* than P_1, then $\{P_2\}\ \mathtt{S}\ \{Q\}$ is true whenever $\{P_1\}\ \mathtt{S}\ \{Q\}$ is true.

Proof. Assume that $\{P_1\}\ \mathtt{S}\ \{Q\}$ is true.

- Assume that we run S in a state that satisfies P_2
- but since P_2 is stronger than P_1, we have $P_2 \rightarrow P_1$
- hence S also satisfies P_1 so that Q is true afterwards

Postcondition Weakening. If Q_1 is a stronger postcondition than Q_2, then $\{P\}\ \mathtt{S}\ \{Q_2\}$ is true whenever $\{P\}\ \mathtt{S}\ \{Q_1\}$ is true.

Proof. Assume that $\{P\}\ \mathtt{S}\ \{Q_1\}$ is true.

- assumes that we run S in a state that satisfies P and that S terminates
- this will lead to a post-state that satisfies Q_1
- but because Q_1 is stronger than Q_2, we have $Q_1 \rightarrow Q_2$
- hence the post-state will also satisfy Q_2.

Proof rule for Strengthening Preconditions (Rule 2/6)

Q. How do we reflect this in the Hoare calculus?

A. We codify this in terms of *proof rules* that we can apply

Precondition Strengthening.

Interpretation. If the premises are provable then so is the conclusion

$$\frac{P_s \rightarrow P_w \qquad \{P_w\}\ \mathtt{S}\ \{Q\}}{\{P_s\}\ \mathtt{S}\ \{Q\}}$$

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$$\frac{P_s \rightarrow P_w \qquad \{P_w\}\ \mathtt{S}\ \{Q\}}{\{P_s\}\ \mathtt{S}\ \{Q\}}$$

Example by pattern matching

$$\frac{y = 2 \rightarrow y > 0 \qquad \{y > 0\}\ \mathtt{x := y}\ \{x > 0\}}{\{y = 2\}\ \mathtt{x := y}\ \{x > 0\}}$$

Precondition Equivalence. If $P_1 \leftrightarrow P_2$ then both $P_1 \rightarrow P_2$ and $P_2 \rightarrow P_1$.

Proof rule for Weakening Postconditions (Rule 3/6)

Postcondition Weakening.

Interpretation. If the premises are provable, then so is the conclusion.

$$\frac{\{P\}\ \mathtt{S}\ \{Q_s\} \qquad Q_s \to Q_w}{\{P\}\ \mathtt{S}\ \{Q_w\}}$$

Example by pattern matching

$$\frac{\{x > 2\}\ \mathtt{x := x + 1}\ \{x > 3\} \qquad x > 3 \to x > 0}{\{x > 2\}\ \mathtt{x := x + 1}\ \{x > 0\}}$$

Postcondition Equivalence. If $Q_1 \leftrightarrow Q_2$ then $Q_1 \to Q_2$ and $Q_2 \to Q_1$.
i.e. $Q_s \to Q_w \wedge Q_w \to Q_s$

Sequencing (Rule 4/6)

Sequencing.

- execute commands one after another, each one manipulates the state
- need to think about the *overall* effect of state change

Sequencing as a proof rule

Interpretation. If the premises are provable then so is the conclusion

$$\frac{\{P\}\ \mathtt{S}_1\ \{Q\} \qquad \{Q\}\ \mathtt{S}_2\ \{R\}}{\{P\}\ \mathtt{S}_1; \mathtt{S}_2\ \{R\}}$$

Example.

$$\frac{\{x > 2\}\ \mathtt{x} := \mathtt{x} + 1\ \{x > 3\} \qquad \{x > 3\}\ \mathtt{x} := \mathtt{x} + 2\ \{x > 5\}}{\{x > 2\}\ \mathtt{x} := \mathtt{x} + 1; \mathtt{x} := \mathtt{x} + 2\ \{x > 5\}}$$

Interlude: Laying out a proof

Layout Problem. Assertions may depend on more than one premise.

Linear Layout.

1. $\{x + 2 > 5\}$ $\texttt{x := x + 2}$ $\{x > 5\}$ (Assignment)
2. $\{x > 3\}$ $\texttt{x := x + 2}$ $\{x > 5\}$ (Precondition Equivalence, 1)
3. $\{x + 1 > 3\}$ $\texttt{x := x + 1}$ $\{x > 3\}$ (Assignment)
4. $\{x > 2\}$ $\texttt{x := x + 1}$ $\{x > 3\}$ (Precondition Equivalence, 1)
5. $\{x > 2\}$ $\texttt{x := x + 1; x := x + 2}$ $\{x > 5\}$ (Sequence, 4, 2)

Note the *numbered proof steps* and *justifications*.

Finding a Proof

Q. Where do we get the "condition in the middle" from?

$$\frac{\{P\}\ \mathtt{S}_1\ \{Q\} \qquad \{Q\}\ \mathtt{S}_2\ \{R\}}{\{P\}\ \mathtt{S}_1; \mathtt{S}_2\ \{R\}}$$

- overall precondition P and overall postcondition R are given
- sequencing requires us to *find* a gluing condition Q

A. Start with the goal R and *work backwards* (as usual)

$$\frac{\{x > 2\}\ \ \mathtt{x := x + 1}\ \ \{Q\} \qquad \{Q\}\ \ \mathtt{x := x + 2}\ \ \{x > 5\}}{\{x > 2\}\ \ \mathtt{x := x + 1; x := x + 2}\ \ \{x > 5\}}$$

An example with precondition strengthening

Goal. Prove that the following is true:

$$\{x = 3\} \quad \texttt{x} := \texttt{x} + 1; \texttt{x} := \texttt{x} + 2 \quad \{x > 5\}$$

First Steps in Linear layout

5. $\{x > 2\} \quad \texttt{x} := \texttt{x} + 1; \texttt{x} := \texttt{x} + 2 \quad \{x > 5\}$ (See earlier slide)

Add the following

6. $x = 3 \rightarrow x > 2$ (Basic arithmetic)
7. $\{x = 3\} \quad \texttt{x} := \texttt{x} + 1; \texttt{x} := \texttt{x} + 2 \quad \{x > 5\}$ (Prec. Strength. 5, 6)

Soundness of Rule for Sequences

Lemma. If the premises of Sequencing rule are true then so is the conclusion.

Proof. Suppose the premises $\{P\}S_1\{Q\}$ and $\{Q\}S_2\{R\}$ are true and let σ_0 be an arbitrary state that satisfies P.

- if we run S_1 in state σ_0 we get a state σ_1 that satisfies Q
- if we run S_2 in state σ_1 we get a state σ_2 that satisfies R
- but executing $S_1; S_2$ just means execute S_1 first and then S_2

- hence we end up in a state that satisfies R

Q. What about termination?

Proof Rule for Conditionals (Rule 5/6)

Conditionals.

$$\texttt{if b then } S_1 \texttt{ else } S_2$$

- b is a *boolean condition* that evaluates to `true` or `false`
- the value of b may depend on the *program state*

Informal Reasoning. Case split

- if b evaluates to `true`, then run S_1
- if b evaluates to `false`, then run S_2.

Additional Precondition.

- in the `if`-branch, additionally know that b is `true`
- in the `then`-branch, additionally know that b is `false`

Q. What is / are the "right" premise(s) for the `if`-rule

$$\frac{?}{\{P\}\ \texttt{if b then } S_1 \texttt{ else } S_2\ \{Q\}}$$

Proof Rule for Conditionals

Proof Rule

$$\frac{\{P \wedge b\}\ \mathtt{S}_1\ \{Q\} \qquad \{P \wedge \neg b\}\ \mathtt{S}_2\ \{Q\}}{\{P\}\ \mathtt{if\ b\ then\ S_1\ else\ S_2}\ \{Q\}}$$

Justification.

- When a conditional is executed, either S_1 or S_2 is executed.
- Therefore, if the *conditional* is to establish Q, *both* S_1 and S_2 must establish Q.
- Similarly, if the precondition for the *conditional* is P, then it must also be a precondition for the two branches S_1 and S_2.

- The choice between S_1 and S_2 depends on evaluating b *in the initial state*, so we can also assume b to be a precondition for S_1 and $\neg b$ to be a precondition for S_2.

Example of Conditional Rule

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad \{P \wedge \neg b\}\ S_2\ \{Q\}}{\{P\}\ \texttt{if b then } S_1 \texttt{ else } S_2\ \{Q\}}$$

Example. We want to show that the following is true

$$\{x > 2\}\ \texttt{if x>2 then y:=1 else y:=-1}\ \{y > 0\}$$

Using the conditional rule (pattern matching)

$$\frac{\{x > 2 \wedge x > 2\}\ \texttt{y:=1}\ \{y > 0\} \qquad \{x > 2 \wedge \neg(x > 2)\}\ \texttt{y:=-1}\ \{y > 0\}}{\{x > 2\}\ \texttt{if x>2 then y:=1 else y:=-1}\ \{y > 0\}}$$

Precondition Equivalence means that we need to show:

(1) $\{x > 2\}\ \texttt{y:=1}\ \{y > 0\}$

(2) $\{\mathit{False}\}\ \texttt{y:=-1}\ \{y > 0\}$

Example In Full

Show. $\{x > 2\}$ `if x>2 then y:=1 else y:=-1` $\{y > 0\}$

Proof in linear layout:

1. $\{1 > 0\}$ `y:=1` $\{y > 0\}$ (Assignment)
2. $(1 > 0) \leftrightarrow \mathit{True}$ (Prop. Logic)
3. $\{\mathit{True}\}$ `y:=1` $\{y > 0\}$ (Prec. Equivalence, 2, 1)
4. $(x > 2) \rightarrow \mathit{True}$ (Prop. Logic)
5. $\{x > 2\}$ `y:=1` $\{y > 0\}$ *(premise (1))* (Prec. Stre., 3, 4)

---

6. $\{-1 > 0\}$ `y:=-1` $\{y > 0\}$ (Assignment)
7. $\mathit{False} \leftrightarrow (-1 > 0)$ (Prop. Logic)
8. $\{\mathit{False}\}$ `y:=-1` $\{y > 0\}$ *(premise(2))* (Prec. Eq)

---

9. $\{x > 2\}$ `if x>2 then y:=1 else y:=-1` $\{y > 0\}$ (Conditional, 5, 8)

Interlude: Conditionals Without 'Else'

Conditionals are complete in the sense that they include an `else`-branch:

$$\texttt{if b then } S_1 \texttt{ else } S_2$$

Q. Our language *could* have statements of the form

$$\texttt{if b then S}$$

What would be the rule?

Interlude: Conditionals Without 'Else'

Conditionals are complete in the sense that they include an `else`-branch:

$$\texttt{if b then } S_1 \texttt{ else } S_2$$

Q. Our language *could* have statements of the form

$$\texttt{if b then S}$$

What would be the rule?

A. Conditionals without else are equivalent to

$$\texttt{if b then S else (do nothing)}$$

Conditional Rule.

$$\frac{\{P \wedge b\}\ \texttt{S}\ \{Q\} \qquad \{P \wedge \neg b\}\ \texttt{do nothing}\ \{Q\}}{\{P\}\ \texttt{if b then S}\ \{Q\}}$$

Conditionals Without 'Else' ctd.

Q. How do we establish the following? **Conditional Rule.**

$$\frac{\{P \wedge b\}\ \mathtt{S}\ \{Q\} \qquad \{P \wedge \neg b\}\ \mathtt{do\ nothing}\ \{Q\}}{\{P\}\ \mathtt{if\ b\ then\ S}\ \{Q\}}$$

Q1. How about `do nothing`?

A. Easy: $\{P\}$ `do nothing` $\{P\}$ is always true.

Precondition Strengthening to the rescue:

$$\frac{\{P \wedge b\}\ \ S\ \ \{Q\} \qquad (P \wedge \neg b) \rightarrow Q}{\{P\}\ \ \textbf{if}\ \ b\ \ \textbf{then}\ \ S\ \ \textbf{else}\ x := x\ \ \{Q\}}$$

Finding a Proof

Q. How do we prove that

$$\{x = 3\} \;\; \mathtt{x := x + 1; x := x + 2} \;\; \{x > 5\}$$

A. Use sequencing rule

$$\frac{\{P\} \;\; S_1 \;\; \{Q\} \qquad \{Q\} \;\; S_2 \;\; \{R\}}{\{P\} \;\; S_1 \; ; \; S_2 \;\; \{R\}}$$

Concrete Instance.

$$\frac{\{x = 3\} \;\; \mathtt{x := x + 1} \;\; \{Q\} \qquad \{Q\} \;\; \mathtt{x := x + 2} \;\; \{x > 5\}}{\{x = 3\} \;\; \mathtt{x := x + 1; x := x + 2} \;\; \{x > 5\}} \; \text{Seq}$$

Finding a Proof

Goal. Prove that the following is true:

$$\{x = 3\}\ \ \mathtt{x := x + 1; x := x + 2}\ \ \{x > 5\}$$

First Take. Apply assignment axiom $\{Q(e)\}\mathtt{x := e}\{Q(x)\}$

Q. What rule could (?) be?

$$\frac{\{x = 3\}\ \ \mathtt{x := x + 1}\ \ \{Q\} \qquad \dfrac{\{x - 2 > 5\}\ \ \mathtt{x := x + 2}\ \ \{x > 5\}}{\{Q\}\ \ \mathtt{x := x + 2}\ \ \{x > 5\}}\ ?}{\{x = 3\}\ \ \mathtt{x := x + 1; x := x + 2}\ \ \{x > 5\}}\ \text{Seq}$$

Finding a Proof

Goal. Prove that the following is true:

$$\{x = 3\} \;\; \texttt{x := x + 1; x := x + 2} \;\; \{x > 5\}$$

A. Putting $Q \equiv x > 3$ would mean that (!) is precondition equivalence

$$\dfrac{\dfrac{\{x = 3\} \;\; \texttt{x := x + 1} \;\; \{x > 3\} \qquad \dfrac{\{x + 2 > 5\} \;\; \texttt{x := x + 2} \;\; \{x > 5\}}{\{x > 3\} \;\; \texttt{x := x + 2} \;\; \{x > 5\}}\,\text{PreEq}}{\{x = 3\} \;\; \texttt{x := x + 1; x := x + 2} \;\; \{x > 5\}}\,\text{Seq}$$

Finding a Proof

Goal. Prove that the following is true:

$$\{x = 3\} \;\; \texttt{x := x + 1; x := x + 2} \;\; \{x > 5\}$$

Second Take. Can apply the assignment axiom $\{Q(e)\}\texttt{x := e}\{Q(x)\}$

Q. What rule could (?) be?

$$\dfrac{\dfrac{\{x+1>3\} \;\; \texttt{x := x + 1} \;\; \{x>3\}}{\{x=3\} \;\; \texttt{x := x + 1} \;\; \{x>3\}} \; ? \; \dfrac{\{x+2>5\} \;\; \texttt{x := x + 2} \;\; \{x>5\}}{\{x>3\} \;\; \texttt{x := x + 2} \;\; \{x>5\}}\,\text{PreEq}}{\{x=3\} \;\; \texttt{x := x + 1; x := x + 2} \;\; \{x>5\}}\,\text{Seq}$$

Finding a Proof

A. Let's try precondition equivalence again:

$$x > 2 \leftrightarrow x + 1 > 3$$

$$\dfrac{\dfrac{\{x+1>3\}\ \texttt{x := x + 1}\ \{x>3\}}{\{x>2\}\ \texttt{x := x + 1}\ \{x>3\}}\ \text{PreEq} \quad ?}{\dfrac{\{x=3\}\ \texttt{x := x + 1}\ \{x>3\}}{\{x=3\}\ \texttt{x := x + 1; x := x + 2}\ \{x>5\}}\ \text{Seq} \qquad \dfrac{\{x+2>5\}\ \texttt{x := x + 2}\ \{x>5\}}{\{x>3\}\ \texttt{x := x + 2}\ \{x>5\}}\ \text{P}}$$

Q. There's still something missing. What is (?) now?

Finding a Proof

A. $x = 3$ implies $x > 2$ so "?" can be precondition strengthening.

Precondition Strengthening.

$$\frac{P_s \to P_w \qquad \{P_w\}\ \mathtt{S}\ \{Q\}}{\{P_s\}\ \mathtt{S}\ \{Q\}}$$

Complete Proof as a tree

$$\dfrac{\dfrac{\dfrac{\{x+1>3\}\ \ \mathtt{x := x+1}\ \ \{x>3\}}{\{x>2\}\ \ \mathtt{x := x+1}\ \ \{x>3\}}\ \text{PreEq}}{\{x=3\}\ \ \mathtt{x := x+1}\ \ \{x>3\}}\ \text{PreStr} \qquad \dfrac{\{x+2>5\}\ \ \mathtt{x := x+2}\ \ \{x>5\}}{\{x>3\}\ \ \mathtt{x := x+2}\ \ \{x>5\}}\ \text{PreE}}{\{x=3\}\ \ \mathtt{x := x+1; x := x+2}\ \ \{x>5\}}\ \text{Seq}$$

The Same Proof in Linear Form

1. $\{x+1>3\}$ $\texttt{x := x + 1}$ $\{x>3\}$ (Assignment)
2. $x>2 \leftrightarrow x+1>3$ (Basic arithmetic)
3. $\{x>2\}$ $\texttt{x := x + 1}$ $\{x>3\}$ (Prec. Equi. 1, 2)
4. $x=3 \rightarrow x>2$ (Basic arithmetic)
5. $\{x=3\}$ $\texttt{x := x + 1}$ $\{x>3\}$ (Prec. Stren. 3, 4)

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6. $\{x+2>5\}$ $\texttt{x := x + 2}$ $\{x>5\}$ (Assignment)
7. $x>3 \leftrightarrow x+2>5$ (Basic arithmetic)
8. $\{x>3\}$ $\texttt{x := x + 2}$ $\{x>5\}$ (Prec. Equiv. 6, 7)

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9. $\{x=3\}$ $\texttt{x := x + 1; x := x + 2}$ $\{x>5\}$ (Seq. 5, 8)

(sections separated by horizontal lines are both premises of the sequencing rule)