# Propositional Logic

COMP1600 / COMP6260



Dirk Pattinson Victor Rivera
Australian National University



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# This Course

**Programming.** (Haskell, Java, Python, . . . )

- Tell the computer *how* to perform a certain task



**Logic.** (this course)

- Specify precisely *what* task should be performed



**Computation.** (this course)

- the (discrete) maths of computation: what can be done in principle?



You know how to program. We will explore logic to describe programs, and study fundamental (programming language independent) models of what can be computed in the first place.

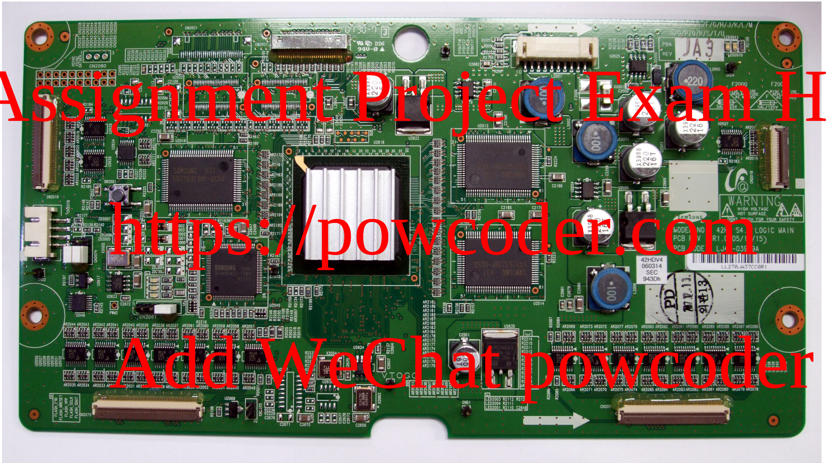
# Example: Vote Counting in the ACT

```
/* STEP 19 */
/* We haven't incremented count yet, so this applies to NEXT count */
mark_elected(cand, (void *)(count+1));
/* STEP 20 */
vote_value_of_surplus = cand->c[count].total - quota;
/* STEP 21 */
increment_count();
/* STEP 22 */
pile = cand->c[cand->count_when_quota_reached].pile;
non_exhausted_ballots
= (number_of_ballots(pile)
   - for_each_ballot(pile, &is_exhausted, candidates));
/* STEP 23 */
if (non_exhausted_ballots == 0)
new_vote_value = fraction_one;
else
new_vote_value = ((struct fraction) { vote_value_of_surplus,
                                                       non_exhausted_ballots });
/* STEP 24 */
update_vote_values(pile, new_vote_value);
/* STEP 24b */
/* Report actual value (may be capped) */
report_transfer(count,
pile->ballot->vote_value,
vote_value_of_surplus);
distribute_ballots(pile, candidates,vacating);
```

(Part of the EVACS code used for vote counting in the ACT in 2012)

Do you trust this? What does it do? Does it *really* implement the law?

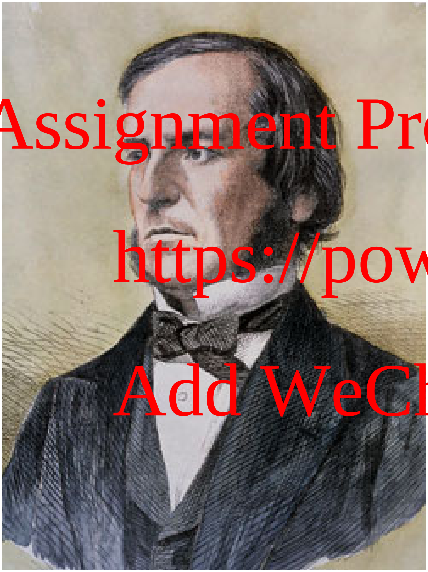
# The two faces of Boolean logic: Circuits





"Computation with boolean / binary values: 0 and 1"

# The two faces of Boolean logic: Formulae



George Boole (1815 – 1864)



- "Logic" as a way of reasoning about true statements
- at the time: mathematical statements
- here: statements about computer programs
- "All statements are either true or false: $T$ / $F$"

# Truth Values

Consider *binary*, or *boolean* values

- 0, or false (written $F$)
- 1, or true (written $T$)



**Boolean Functions** are functions that

- take boolean values as *inputs* (zero, or more)
- produce boolean values as *outputs* (generally, one or more)



**Example.**



| $x$ | $y$ | $f(x, y)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

| $x$ | $y$ | $f(x, y)$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | F |

# Two Interpretations

**Example.**

| $x$ | $y$ | $f(x, y)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

| $x$ | $y$ | $f(x, y)$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | F |



**Computation with Binary Values**

- the output is 1 if precisely one of the inputs (but not both) is 1
- or: considering $x$ and $y$ as one-bit numbers, the output is the sum $x + y$ taken modulo two.

**Truth of Statements**

Here, $x$ and $y$ represent whether certain statements are true or false, e.g.:

- $x$ could stand for "Joe was born in Melbourne"
- $y$ could stand for "Joe was born in Sydney"

Then $f(x, y)$ is the truth value of the statement "Joe was born either in Sydney or in Melbourne"

# More on the Two Interpretations



**Computing with Binary Values**

- inputs are either 0 or 1
- one output is prescribed for each combination
- used to describe *computation*



**Truth of Statements**

- inputs describe whether certain statements are true or false
- output describes the truth value of a compound statement
- used to describe *truth*

# Building Boolean Functions

As with writing programs, boolean functions are constructed from *basic building blocks*



$x$ AND $y$

| $x$ | $y$ | $x \wedge y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$x$ OR $y$

| $x$ | $y$ | $x \vee y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

NOT $x$

| $x$ | $\neg x$ |
|---|---|
| 0 | 1 |
| 1 | 0 |







- AND, OR and NOT are written as $\wedge$, $\vee$ and $\neg$, respectively and called *conjunction*, *disjunction* and *negation*.
- the symbols in the last row are circuit representations

# Example revisited

| $x$ | $y$ | $f(x,y)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

| $x$ | $y$ | $f(x,y)$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | F |



**Written as Logical Formulae**



$$(x \wedge \neg y) \vee (\neg x \wedge y)$$

**Reading of the Formula.** This formula is true in a *situation*



- where either the LHS (i.e. $x \wedge \neg y$) is true, or the RHS (i.e. $\neg x \wedge y$) is true, *or both* (!)
- the LHS is true if $x$ is true and $\neg y$ is true (so $y$ is false)
- the RHS is true if $\neg x$ is true (so $x$ is false) and $y$ is true
- so that the formula is true in a situation where *precisely one* of $x$ and $y$ are true (but not both)

# About Situations



**Recall.** Truth of the formula $(x \wedge \neg y) \vee (\neg x \wedge y)$ depends on a *situation*

- that tells us which variables are true and false
- can evaluate a formula to either true or false in any situation



**Definition.** If $V$ is a set of variables (e.g. $V = \{x, y\}$ for the formula above), then a situation for $V$ is a mapping $s : V \to \{\mathsf{T}, \mathsf{F}\}$.

# Formula View

**Q.** How do we align the formula and the boolean function?
**A.** Construct the value of the boolean function step by step



| $x$ | $y$ | $\neg x$ | $\neg y$ | $x \wedge \neg y$ | $\neg x \wedge y$ | $(x \wedge \neg y) \vee (\neg x \wedge y)$ |
|---|---|---|---|---|---|---|
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |



Left two columns: all possible truth values for $x$ and $y$



- given $y$, we know $\neg y$ (4th column)
- given $x$ and $\neg y$, we know $x \wedge \neg y$ (penultimate column)
- given $x$, we know $\neg x$ (3rd column)
- given $\neg x$ and $y$, we know $\neg x \wedge y$ (6th column)
- given $x \wedge \neg y$, and $\neg x \wedge y$, we know $(x \wedge \neg y) \vee (\neg x \wedge y)$ (last column)

Indeed, $(x \wedge \neg y) \vee (\neg x \wedge y)$ is exclusive-or!

# Formula View, Continued

$$(x \wedge \neg y) \vee (\neg x \wedge y)$$

**Q.** What columns do I need?



**A.** Look at the target formula ...

- to evaluate $(x \wedge \neg y) \vee (\neg x \wedge y)$, we need both $x \wedge \neg y$and $\neg x \wedge y$
- to evaluate $x \wedge \neg y$, we need $x$ (given) and $\neg y$
- to evaluate $\neg x \wedge y$, we need $\neg x$ and $y$ (given)



Coloured formulae indicate columns in the table.

**More systematically**



1. start with the target formula, and create entries for immediate subformulae
2. repeat step 1 for all subformulae until inputs are reached.

# Truth Tables

**Formulae.** Are constructed from $T$ (true), $F$ (false) and variables $(x, y, \ldots)$ using $\wedge, \vee, \neg$ (more connectives later).



**Truth Tables** for a formula

- list all $T/F$ combinations of the variables (*all situations*)
- list the truth value of the formula, given variable values
- possibly list truth values of subformulae, too



**Example.** (same as before)



| $x$ | $y$ | $\neg x$ | $\neg y$ | $x \wedge \neg y$ | $\neg x \wedge y$ | $(x \wedge \neg y) \vee (\neg x \wedge y)$ |
|---|---|---|---|---|---|---|
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |

# On Representation

**Boolean Functions** with $n$ inputs are simply functions:

$$f : \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}} \to \{0,1\}.$$



**Logical Formula** with $n$ variables *represent* boolean functions

- the truth table of the formula defines a boolean function
- many representations of the *same* boolean function!



**Circuits** with $n$ inputs also *represent* boolean functions

- one boolean output is determined for each combination of inputs
- again, can have many circuits representing the same function



**Summary.** Boolean functions can be represented by truth tables (uniquely), formulae and circuits.

# More Circuit Primitives

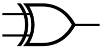
$x$ XOR $y$

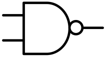
| $x$ | $y$ | $\mathrm{xor}(x, y)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$x$ NAND $y$

| $x$ | $y$ | $\mathrm{nand}(x, y)$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |







$(x \wedge \neg y) \vee (\neg x \wedge y)$ $\qquad$ $\neg(x \wedge y)$

- we don't introduce formula type connectives for NAND and XOR as they are not commonly used (and we will not use them in what follows), but they can be encoded
- the symbols in the middle row are the circuit representations

## Can We Express All Boolean Functions?

**Q.** Given a boolean function $f(x_1, \ldots, x_n)$, can we construct a formula that represents $f$?



**A.** Yes, this is a theorem and here is the proof. Recall that a formula represents a function if the truth table of the formula gives precisely the function table.



**Proof (Sketch).**
For boolean values $i_1, \ldots, i_n \in \{T, F\}$ we have a formula $\phi_{i_1 \ldots i_n}$ such that

$$\phi_{i_1 \ldots i_n}(x_1, \ldots, x_n) = T \iff x_1 = i_1 \text{ and } \ldots \text{ and } x_n = i_n.$$



(This formula is a large $\wedge$ with elements $x_\ell$ if $i_\ell = T$ and $\neg x_\ell$ if $i_\ell = F$.)

Then take $\phi_f = \phi_{r_1} \vee \cdots \vee \phi_{r_k}$ where $r_1, \ldots, r_k$ are precisely the vectors $(i_1, \ldots, i_n)$ for which $f(i_1, \ldots, i_n) = T$.

# Proof Detail



Suppose we have a line of the truth table

| $x$ | $y$ | $z$ | $f(x,y,z)$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ |



Consider the formula

$$\phi_{TTF} \equiv x \wedge y \wedge (\neg z)$$

then we have



$$\phi_{TTF} = T \text{ if and only if } x = T, y = T, z = F$$

# Proof Detail

Now consider all lines of the truth table

| $x$ | $y$ | $z$ | $f(x,y,z)$ |
|---|---|---|---|
| | | … | |
| $T$ | $T$ | $F$ | $T$ |
| | | … | |
| $T$ | $F$ | $F$ | $T$ |
| | | … | |
| $F$ | $F$ | $F$ | $T$ |
| | | … | |



that have a $T$ in the right hand column, where lines that are not shown have a $F$ in the right column. The formula

$$\phi_{TTF} \vee \phi_{TFF} \vee \phi_{FFF}$$

has precisely the truth table above.

# Expressively Complete Sets

**Definition.** A set of logical connectives is *expressively complete* if it allows us to build all boolean functions.



**Example.** The set consisting of $\wedge$, $\vee$ and $\neg$ expressively complete.

**Non-example.** The sets consisting just of $\neg$, and just of $\vee$, are *not* expressively complete.



**Theorem.** The set consisting of just nand is expressively complete.

**Proof (Sketch).** Using formula notation, we can use nand to



- express negation: $\neg x = \text{nand}(x, x)$
- express conjunction: $x \wedge y = \neg\text{nand}(x, y)$
- express disjunction: $x \vee y = \neg(\neg x \wedge \neg y)$

**Q1.** How many boolean functions with $n$ inputs (and one output) can we create using AND, OR and NOT circuits / using the logical connectives $\wedge$, $\vee$, and $\neg$?





**Q2.** How many boolean functions with $n$ inputs (and one output) can we create *just* with OR-gates and a constant wire with value 0 / the logical connectives $\vee$ and $F$?



**Q3.** How many gates do we need in the worst case to construct a boolean function?

# A Formula-Size Theorem

**Theorem.** Every boolean function with $n$ input bits (and one output) can be represented using at most $10 \cdot 2^n - 10$ logical connectives from the set $\{\neg, \wedge, \vee\}$.



**Proof (Sketch).** We use a technique called *induction* that we will analyse more closely later in the course.

*Base case: $n = 1$.* We can build every one-bit function using at most $10 \cdot 2^1 - 10 = 10$ connectives.



*Inductive Step: $n > 1$.* By fixing the first bit of $f$ to be 0 and 1, we have two $n-1$-bit functions $f_0$ and $f_1$:

$$f_0(x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n) \qquad f_1(x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)$$



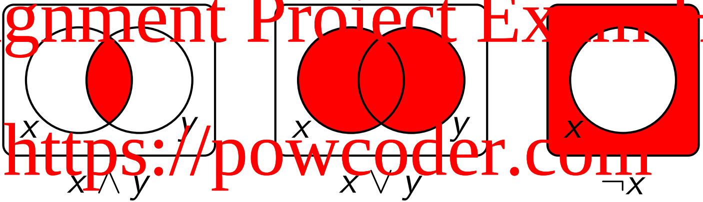
Both can be represented using at most $10 \cdot 2^{n-1} - 10$ connectives.

We can write $f = (\neg x_1 \wedge f_0) \vee (x_1 \wedge f_1)$. Using $f_0$ and $f_1$, we therefore need to add 4 more connectives to get a formula for $f$, and we have used

$$2 \cdot (10 \cdot 2^{n-1} - 10) + 4 = 10 \cdot 2^n - 20 + 4 \leq 10 \cdot 2^n - 10$$

connectives.

# Visualisation: Venn Diagrams






- the boxes are the space of all situations where $x$ and $y$ are true or false
- labelled circles describe those situations where $x$ and $y$ are true
- red area describes those situations where the formula is true.

(Every point in a Venn diagram describes a *situation*).

# Memories from High School . . .

**Analogy: Algebraic Terms.** Given a set $V$ of variables, algebraic terms are constructed as follows:



- $0, 1, \ldots$ and all variables $x \in V$ are algebraic terms
- if $s$ and $t$ are algebraic terms, then so is $s + t$ and $s \cdot t$.

**Example.**



$$5 + (3 \cdot x) \qquad x \qquad (x \cdot (3 \cdot (7 + 5))) + z$$

**Usual precedence.** $\cdot$ binds more strongly than $+$:



$$x \cdot 3 + y \text{ reads as } (x \cdot 3) + y$$

**Crucial Aspect.**

Terms can be *evaluated* given values for all variables.

# Back to Boolean Functions

**Definition.** Given a set $V$ of variables, boolean formulae are constructed as follows:

- $T$ (true) and $F$ (false) and all variables $x \in V$ are boolean formulae
- if $\phi$ and $\psi$ are boolean formulae, then so are $\phi \wedge \psi$ and $\phi \vee \psi$.
- if $\phi$ is a boolean formula, then so is $\neg\phi$.



**Examples.**



$$T \vee \neg(x \vee (\neg y)) \qquad\qquad x \qquad\qquad x \wedge \neg(T \vee (F \vee x))$$

**Precedence.** $\neg$ binds more strongly than $\wedge$ binds more strongly than $\vee$:



$$\neg x \wedge y \vee z \text{ reads as } ((\neg x) \wedge y) \vee z$$

**Crucial Aspect.**

Boolean formulae can be *evaluated* given (boolean) values for all variables.

# Equations

**Examples from Algebra.**

$$x \cdot (3 + y) = x \cdot 3 + x \cdot y$$
$$25 + (18 \cdot y) = 18 \cdot y + 25$$

**Boolean Equations.** have boolean formulae on the left and right:

$$x \wedge (y \vee x) = x$$
$$T \wedge (y \vee x) = x \vee y$$

**Valid Equations.**

For *all* values of variables, LHS and RHS evaluate to same number.

Applies to *both* algebraic terms and boolean formulae!

## Valid Boolean Equations.

**Associativity**

$a \vee (b \vee c) = (a \vee b) \vee c$ $\qquad$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$



**Commutativity**

$a \vee b = b \vee a$ $\qquad$ $a \wedge b = b \wedge a$

**Absorption.**



$a \vee (a \wedge b) = a$ $\qquad$ $a \wedge (a \vee b) = a$

**Identity.**



$a \vee F = a$ $\qquad$ $a \wedge T = a$

**Distributivity.**

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\qquad$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

**Complements.**

$a \vee \neg a = T$ $\qquad$ $a \wedge \neg a = F$

# Equational Reasoning in Ordinary Algebra



$$
\begin{aligned}
& (x + 12) \cdot 3(x + 17) & \\
&= (x + 12) \cdot (3 \cdot x + 3 \cdot 17) & \text{(distributivity)} \\
&= (x + 12) \cdot (3 \cdot x + 51) & \\
&= x \cdot (3 \cdot x + 51) + 12 \cdot (3 \cdot x + 51) & \text{(distributivity)} \\
&= x \cdot 3 \cdot x + x \cdot 51 + 12 \cdot 3 \cdot x + 12 \cdot 51 & \text{(distributivity)} \\
&= 3 \cdot x^2 + 51 \cdot x + 36 \cdot x + 612 & \text{(commutativity)} \\
&= 3 \cdot x^2 + (51 + 36) \cdot x + 612 & \text{(distributivity)} \\
&= 3 \cdot x^2 + 87 \cdot x + 612 &
\end{aligned}
$$

- each step (other than addition / multiplication of numbers) justified by a "law of arithmetic"
- "pattern matching" against algebraic laws

# Proving Boolean Equations

**Example.** We prove the law of *idempotence*:



$$
\begin{aligned}
x \vee x &= (x \vee x) \wedge T && \text{(identity)} \\
&= (x \vee x) \wedge (x \vee \neg x) && \text{(complements)} \\
&= x \vee (x \wedge \neg x) && \text{(distributivity)} \\
&= x \vee F && \text{(complements)} \\
& x && \text{(identity)}
\end{aligned}
$$





**Rules of Reasoning.**

- All boolean equations may be assumed (with variables substituted by formulae)
- may replace formulae with formulae that are proven equal
- equality is transitive!

# Two faces of boolean Equations

**Truth** of boolean equations:

A boolean equation $\phi = \psi$ (where $\phi, \psi$ are boolean formulae) is *true* if $\phi$ and $\psi$ evaluate to the same truth values in all situations (i.e. for all possible truth values of the variables that occur in $\phi$ and $\psi$).

**Equational Provability** of boolean equations:

A boolean equation is *provable* if it can be derived from associativity, commutativity, absorption, identity, distributivity and complements using the laws of equational reasoning.

**Q.** How do these two notions hang together?

# Soundness and Completeness

**Slightly Philosophical.**

- *Truth* of an equation relates to the *meaning* (think: truth tables) of the connectives $\wedge$, $\vee$ and $\neg$.



- Equational *provability* relates to a method that allows us to *establish* truth of an equation.

They are *orthogonal* and independent ways to think about equations.



**Soundness.** If a boolean equation $\phi = \psi$ is provable using equations, then it is true.

- all basic equations (associativity, distributivity, ...) are true



- the rules of equational reasoning preserve truth.

**Completeness.** If a boolean equation is true, then it is provable using equations.

- more complex proof (not given here), using the so-called *Lindenbaum Construction*.

# Challenge Problem: The De Morgan Laws

**De Morgan's Laws**

$$\neg(x \vee y) = \neg x \wedge \neg y \qquad \neg(x \wedge y) = \neg x \vee \neg y$$



**In English**

- if it is false that either $x$ or $y$ is true, they must both be false
- if it is false that both $x$ and $y$ are true, then one of them must be false.



**Truth** of De Morgan's Laws: Easy to establish via truth tables.



**Provability** of De Morgan's Laws

- if the completeness theorem (that we didn't prove!) is true, then an equational proof must exist
- however, it is quite difficult to actually find it!