The Australian National University
School of Computing
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Semester 2, 2021
Practical Session 7

Foundations of Computation

The practical contains a number of exercises designed for the students to practice the course content. During the practical session, the tutor will work through some of these exercises while students will be responsible for completing the remaining exercises in their own time. There is no expectation that all the exercises will be covered in the practical session.

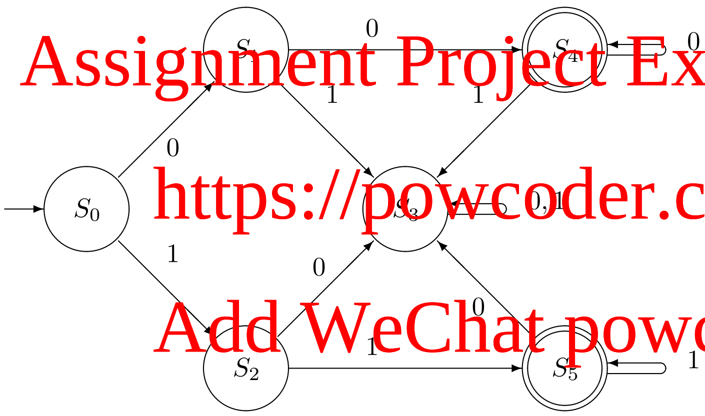
Covers: Lecture Material Week 7

At the end of this tutorial, you will be able to

- design Deterministic Finite Automata given a language;
- determine whether a word is recognised by an automaton;
- describe the language that an automaton recognises.

Exercise 1 **DFAs and Acceptance**

The automaton A_1 is specified by the following diagram.



1. Translate the graphical representation of A_1 into the form of a 5-tuple, (Σ, S, s_0, F, N).

   Solution.

$$\begin{aligned} \Sigma &= \{0,1\}; \\ S &= \{S_0, S_1, S_2, S_3, S_4, S_5\}; \\ s_0 &= S_0; \\ F &= \{S_4, S_5\}; \end{aligned}$$

$N =$

State	0	1
S_0	S_1	S_2
S_1	S_4	S_3
S_2	S_3	S_5
S_3	S_3	S_3
S_4	S_4	S_3
S_5	S_3	S_5

2. The next state function is N and so N^* is the eventual state function. What is $N^*(S_2, 00010)$? Please show your solution step by step.

Solution.

$$\begin{aligned} N^*(S_2, 00010) =& N^*(N(S_2, 0), 0010) && \text{you can skip this step} \\ =& N^*(S_3, 0010) \\ =& N^*(S_3, 010) \\ =& N^*(S_3, 10) \\ =& N^*(S_3, 0) \\ =& N^*(S_3, \epsilon) \\ =& S_3 \end{aligned}$$

3. What is the purpose of the state S_3?

   Solution. S_3 is a 'failure' state; any string that enters S_3 will stay in S_3 and never be accepted.

4. What language does A_1 accept? Phrase your answer in the form

   $$L(A_1) = \{w \in \Sigma^* \mid P(w)\}$$

   where $P(w)$ is some predicate on words.

   Solution. $L(A) = \{w \in \Sigma^* \mid \exists n \in \mathbb{N}.(w = 0^n \vee w = 1^n) \wedge n \geq 2\}$.

5. Consider the statement $\forall n \in \mathbb{N}.\ N^*(S_5, 1^n) = S_5$. Express it in English and give a proof. Which technique would you be using?

   Solution. In English, this reads that "the automaton remains in state S_5 if it processes an arbitrary long sequence of 1's.". We prove this statement by induction on n.

   For $n = 0$, we have that $N^*(S_5, 1^0) = N^*(S_5, \epsilon) = S_5$ (by def. of N^*). Now let $n > 0$. Then

   $$\begin{aligned} & N^*(S_5, 1^n) \\ =& N^*(N(S_5, 1), 1^{n-1}) (\text{by def. of } N^*) \\ =& N^*(S_5, 1^{n-1}) (\text{by def. of the transition state function}) \\ =& S_5 (\text{by IH}) \end{aligned}$$

   .

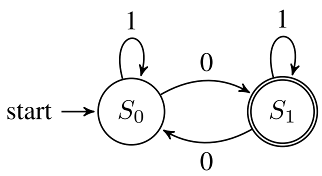
Exercise 2 **Deterministic Finite Automata**

Consider the alphabet $\Sigma = \{0, 1\}$.

1. Construct a DFA that recognises the language

   $\{w \in \Sigma^* \mid w \text{ has an odd number of 0's}\}$;

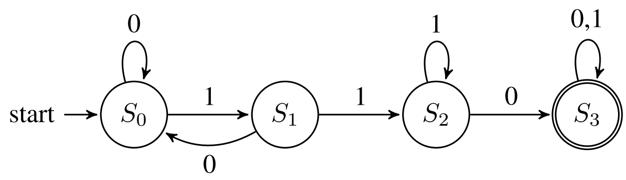
   Solution.



2. Construct a DFA that recognises the language

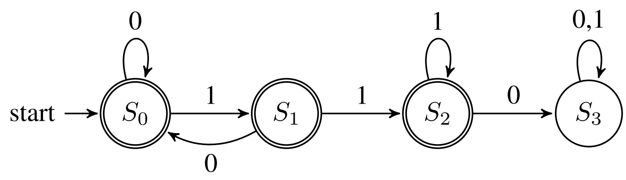
   $\{w \in \Sigma^* \mid w \text{ contains the substring } 110\}$;

   Solution.



3. Construct a DFA that recognises the language
   $\{w \in \Sigma^* \mid w$ does not contains the substring $110\}$;

   Solution.



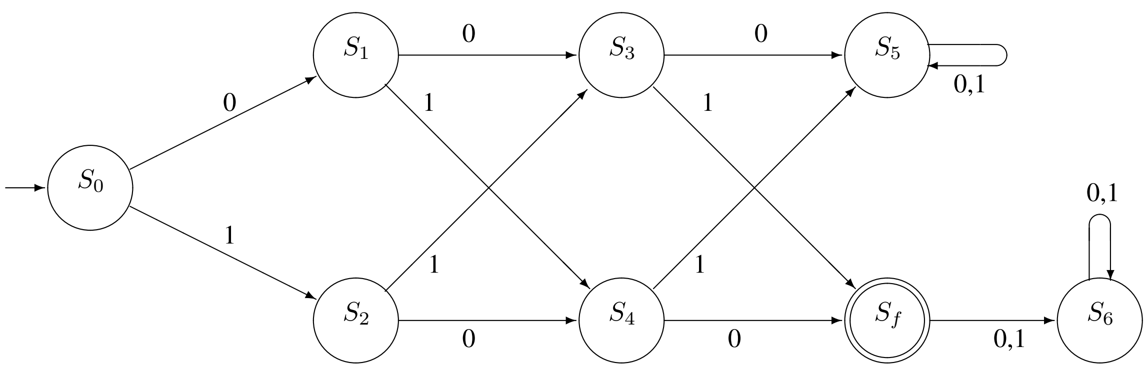
Exercise 3

Let A_2 be an automaton with next state function given in the following table. The initial state of the automaton is S_0 and the only accepting state is S_f.

State	0	1
S_0	S_1	S_2
S_1	S_3	S_4
S_2	S_4	S_3
S_3	S_5	S_f
S_4	S_f	S_5
S_5	S_5	S_5
S_f	S_6	S_6
S_6	S_6	S_6

1. Draw the transition diagram for A_2.

   Solution.



2. Describe the language $L(A_2)$ accepted by A_2.

   Solution. The language accepted by this automaton is the finite set of strings of length equal to 3 over the alphabet $\{0, 1\}$ with an odd number of 1's: $\{001, 010, 100, 111\}$.

3. State what you need to prove to establish that every word of the language you have described is accepted by A_2. (You do not need to give proofs.)

   Solution. We have to show that
   $N^*(S_0, 001) = S_f$,

$N^*(S_0, 010) = S_f$,
$N^*(S_0, 100) = S_f$,
$N^*(S_0, 111) = S_f$.

Give a proof of the fact that every word accepted by A_2 is in the language you have described.

Solution. We first show that every word accepted by A_2 has length exactly three. So consider that $N^*(S_0, w) = S_f$. Looking at the transition diagram, this can only happen if

- w is of the form xd where $d \in \{0, 1\}$ and x is a string,
- $N^*(S_0, x) \in \{S_3, S_4\}$.

Again, if $N^*(S_0, x) \in \{S_3, S_4\}$, this can only happen if

- x is of the form ye where $e \in \{0, 1\}$ and y is a string,
- and $N^*(S_0, y) \in \{S_1, S_2\}$

Again, the latter is only possible if y is a single letter $y \in \{0, 1\}$ so that in total $w = yed$ where all of $y, e, d \in \{0, 1\}$ are single letters, so that w is of length three.

There are eight words of length three, and by tracing the transition diagram, we have that A_2 accepts *exactly* the words in the language described above.

In summary: if w is accepted by A_2, then w is of length 3 and one of the words $\{001, 010, 100, 111\}$ as the other length-3 words are *not* accepted.

Exercise 4 **Non-Existence of Automata**

Let L be the language of strings over $\{0, 1\}$ that contain more 0's than 1's.

Prove that there is *no* finite state automaton that can recognise L.

Solution.

Suppose there exists a DFA A which accepts L. Let S_0 be A's start state, we derive a contradiction.

There are only finitely many different states of A, so by the pigeon-hole principle there exist numbers n, m, with $n > m$, such that

$$N^*(S_0, 0^n) \;=\; N^*(S_0, 0^m)$$

The string $0^n 1^m$ contains more 0's than 1's, so it is in L and is accepted by A. Hence

$$N^*(N^*(S_0, 0^n), 1^m) \;=\; N^*(S_0, 0^n 1^m) \;\in F$$

And therefore

$$N^*(S_0, 0^m 1^m) \;=\; N^*(N^*(S_0, 0^m), 1^m) \;=\; N^*(N^*(S_0, 0^n), 1^m) \;\in F$$

so $0^m 1^m$ is accepted. But $0^m 1^m \notin L$, giving us a contradiction.

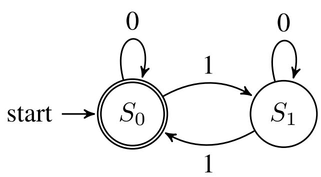
Exercise 5 **Deterministic Finite Automata**

Consider the alphabet $\Sigma = \{0, 1\}$.

1. Construct a DFA that recognises the language

   $\{w \in \Sigma^* \mid w$ has an even number of 1's$\}$;

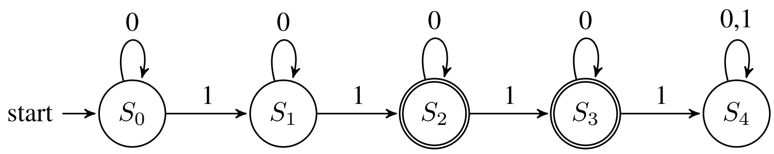
   Solution.



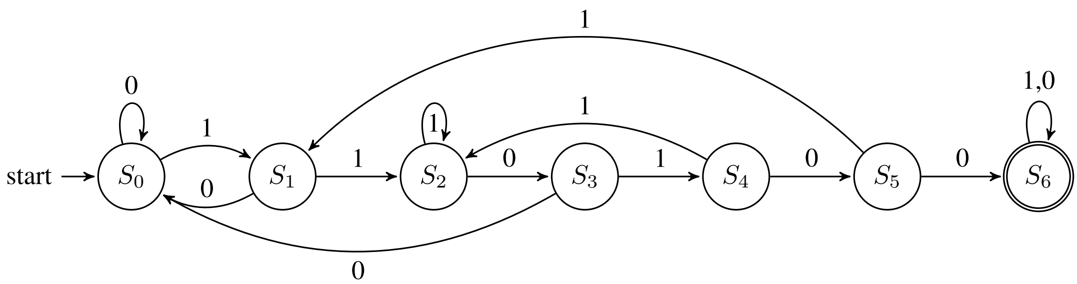
2. Construct a DFA that recognises the language

   $\{w \in \Sigma^* \mid w$ has two or three 1's$\}$;

   Solution.

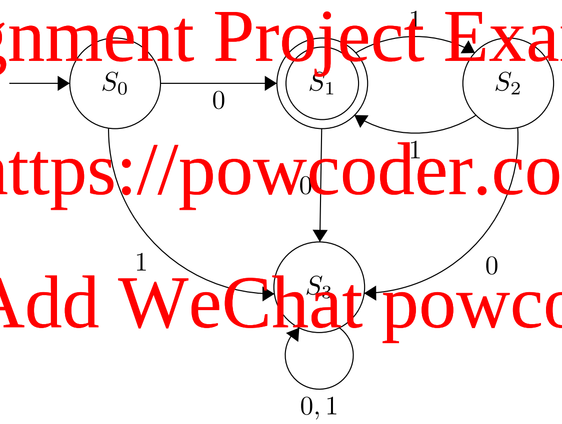


3. Construct a DFA that recognises the language
   $\{w \in \Sigma^* \mid w \text{ contains the substring } 110100\}$.
   Solution.



Exercise 6 **Formal Proofs of Automaton Languages I**

Consider the following finite automaton B:



We would like to prove that the language $L(B)$ that is accepted by B is equal to

$$L = \{0(11)^n : n \geq 0\}$$

Prove that $L = L(B)$. Ensure that you clearly state your two main proof obligations, and that the proof is given in full rigorous detail.

Solution. We first prove that B accepts all strings in L, that is,

$$x \in L \Rightarrow N^*(S_0, x) = S_1$$

Since all strings in L are of the form $0(11)^n$ for some $n \geq 0$, we do a proof by induction on n.

Base case, $n = 0$.

$$N^*(S_0, 0(11)^0) = N^*(S_0, 0\epsilon) = N^*(S_0, 0) = S_1$$

Step case. Assume that $N^*(S_0, 0(11)^n) = S_1$ and prove that $N^*(S_0, 0(11)^{n+1}) = S_1$.

$$N^*(S_0, 0(11)^{n+1}) = N^*(S_0, 011(11)^n) = N^*(S_1, 11(11)^n) = N^*(S_2, 1(11)^n) = N^*(S_1, (11)^n) = S_1$$

Where the last step is justified by since $N^*(S_0, 0(11)^n) = S_1$ by assumption, and $N^*(S_0, 0(11)^n) = N^*(S_1, (11)^n)$, we have that $N^*(S_1, (11)^n) = S_1$.

Hence $x \in L \Rightarrow N^*(S_0, x) = S_1$.

We now prove the converse, that if a string is accepted by B, then it must be in L.

$$N^*(S_0, x) = S_1 \Rightarrow x \in L$$

It is easier instead to prove the contrapositive: If a string is not in L, then B rejects it.

$$x \notin L \Rightarrow N^*(S_0, x) \neq S_1$$

Now, what kinds of strings are not in L? Well, L is the set of all strings that start with a zero, and are followed by an even number of ones. So strings that don't satisfy this condition must be either

- The empty string.
- Any string that starts with a one.
- Any string that starts with a zero, and is followed by another string that contains at least one zero.
- Any string that starts with a zero, and is followed by a string of odd many ones.

The first case is trivial, as $N^*(S_0, \epsilon) = S_0 \neq S_1$.

The second case is also trivial, as by observing the DFA we can see that S_3 is a trap state, so $N^*(S_3, \delta) = S_3$ for all strings δ. Then, $N^*(S_0, 1\delta) = N^*(S_3, \delta) = S_3 \neq S_1$

The third case requires some work, but we can use the append theorem to help us. A string of this form looks like $0\alpha0\beta$ for some strings α, β. Then,

$$N^*(S_0, 0\alpha0\beta) = N^*(S_1, \alpha0\beta) = N^*(N^*(S_1, \alpha), 0\beta)$$

Now, we are not sure where $N^*(S_1, \alpha)$ is, but we know that $N^*(S_1, \alpha) \neq S_0$, as there is no way to enter S_0 from any other state. So we check all the combinations.

- If $N^*(S_1, \alpha) = S_1$, then $N^*(N^*(S_1, \alpha), 0\beta) = N^*(S_1, 0\beta) = N^*(S_3, \beta) = S_3$.
- If $N^*(S_1, \alpha) = S_2$, then $N^*(N^*(S_1, \alpha), 0\beta) = N^*(S_2, 0\beta) = N^*(S_3, \beta) = S_3$.
- If $N^*(S_1, \alpha) = S_3$, then $N^*(N^*(S_1, \alpha), 0\beta) = N^*(S_3, 0\beta) = S_3$.

So in all cases, we end up in the non-final state S_3.

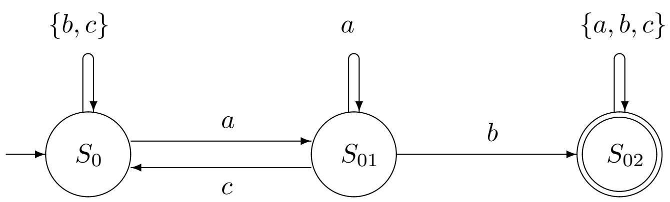
The forth case is a string of the form $0(11)^n1$, which is a zero followed by odd many ones. We can use the property proven before, that $N^*(S_0, 0(11)^n) = S_1$ together with the append theorem to obtain

$$N^*(S_0, 0(11)^n1) = N^*(N^*(S_0, 0(11)^n), 1) = N^*(S_1, 1) = S_2 \neq S_1$$

All the cases are shown to be rejected by the machine, as required.

Exercise 7 **Formal Proofs of Automaton Languages II**

(Difficult.) Consider the following finite automaton



and prove that the DFA C recognises the language $L = \{\gamma ab\delta \mid \gamma, \delta \in \{a, b, c\}^*\}$.

Ensure that you clearly state your two main proof obligations. Make sure that you give your proof in full rigorous detail. For example, be explicit about any use of the append theorem.

Solution.

We need to show two things:

1. C accepts all strings $\gamma ab\delta$, for any $\gamma, \delta \in \{a, b, c\}^*$, i.e.,
$$N^*(S_0, \gamma ab\delta) = S_{02}$$
2. if C accepts a string w, i.e., $N^*(S_0, w) = S_{02}$, then there exists $\gamma, \delta \in \{a, b, c\}^*$ such that $w = \gamma ab\delta$.

We tackle both in turn.

1. We establish the following facts that we will use later.

   Lemma 1
   $N^*(S_0, \gamma ab) = S_{02}$

   Proof: the left hand side equals $N^*(N^*(S_0, \gamma), ab)$ by the append theorem. But we have no way of knowing what $N^*(S_0, \gamma)$ is, so we consider every possibility:
$$\begin{aligned} N^*(S_0, ab) =& S_{02} \\ N^*(S_{01}, ab) =& S_{02} \\ N^*(S_{02}, ab) =& S_{02} \end{aligned}$$
   Hence $N^*(S_0, \gamma ab) = N^*(N^*(S_0, \gamma), ab) = N^*(S_{02} ab) = S_{02}$.

   We next prove

   Lemma 2
   $N^*(S_{02}, \delta) = S_{02}$

   Proof. This is an easy induction on the length of δ. For the base case ($\delta = \epsilon$) we have $N^*(S_{02}, \epsilon) = S_{02}$ by definition of N^*. For the step case, we calculate
$$\begin{aligned} N^*(S_{02}, x\delta) =& N^*(N(S_{02}, x), \delta) && \text{(Def. of } N^*\text{)} \\ =& N^*(S_{02}, \delta) && \text{(Holds for all } x \in \{a, b, c\}\text{)} \\ =& S_{02} && \text{(IH)} \end{aligned}$$
   We can now show that C recognises all strings of the form $\gamma ab\delta$ by looking at the eventual state function:
$$\begin{aligned} N^*(S_0, \gamma ab\delta) =& N^*(N^*(S_0, \gamma ab), \delta) && \text{(Append Theorem)} \\ =& N^*(S_{02}, \delta) && \text{(Lemma 1)} \\ =& S_{02} && \text{(Lemma 2)} \end{aligned}$$

2. As in the first part, we isolate some observations that we will use later.

   Lemma 3
   $N^*(S_0, w) = S_{02} \Rightarrow \exists \gamma', \delta. (w = \gamma' b\delta \wedge N^*(S_0, \gamma') = S_{01})$

   Proof: by induction on the length of w. Base case, $w = \epsilon$, which follows because $N^*(S_0, \epsilon) \neq S_{02}$, so the LHS of the implication is false, and the implication is vacuously true. Inductive case, suppose $N^*(S_0, wx) = S_{02}$. Now $N^*(S_0, wx) = N(N^*(S_0, w), x)$ by the corollary to the append theorem. We cannot know what $N^*(S_o, w)$ is, but can eliminate one possibility: there is no $x \in \{a, b, c\}$ such that $N(S_0, x) = S_{02}$, so $N^*(S_0, w) \neq S_0$. If $N^*(S_0, w)$ were S_{01} then x can only be b, and so Lemma 3 holds by setting $\gamma' = w$ and $\delta = \epsilon$. If $N^*(S_0, w)$ were S_{02}, then by the IH $w = \gamma' b\delta$ and $N^*(S_0, \gamma') = S_{01}$. Therefore $wx = \gamma' b(\delta x)$, whatever x is. So Lemma 3 holds in this case too.

   We next observe that

   Lemma 4
   $N^*(S_0, w) = S_{01} \Rightarrow \exists \gamma. w = \gamma a$

   Proof: simply because all arcs into S_{01} are labelled by a.

   We now show that any string accepted by C has necessarily the form $w = \gamma ab\delta$ for some strings γ and δ. So assume that C accepts w, that is, $N^*(S_0, w) = S_{02}$. By Lemma 3 there are strings γ' and δ such that
$$w = \gamma' b\delta \text{ and } N^*(S_0, \gamma') = S_{01}.$$
   Using Lemma 4 we now obtain γ such that $\gamma' = \gamma a$. Putting everything together, we have that $w = \gamma ab\delta$ as required.