Natural Deduction

COMP1600 / COMP6260

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Criticism of Equational Proofs

The good.

Completeness tells us that if an equation is true, we can prove it.

The bad.

Sometimes need *lots* of ingenuity to find a proof! E.g.

$$x \vee x = (x \vee x) \wedge T = (x \vee x) \wedge (x \vee \neg x) = x \vee (x \wedge \neg x) = x \vee F = x$$

The ugly.

Equational reasoning is not *natural*, i.e. it doesn't mirror the *meaning* of $\wedge, \vee$ and $\neg$.

Towards Propositional Formulae and Natural Deduction

New Connective. Implication, written $\rightarrow$

In English. $x \rightarrow y$ means "if x is true, then so is y".

Truth Table. Informally, think of $x \rightarrow y$ as a *promise*.

- the promise is that y is true if x is true

- $x \rightarrow y$ evaluates to F if the promise is broken

x	y	$x \rightarrow y$
F	F	T
F	T	T
T	F	F
T	T	T

Interlude: Logic to English

Exercise. Use the predicates I – I'm going surfing, Y – you're going surfing, and W – there'll be a big wave that kills us all, to translate the following statements to English:

1. $I \wedge Y \rightarrow W$
2. $(I \rightarrow W) \vee (Y \rightarrow W)$

Possible Answer.

Interlude: Logic to English

Exercise. Use the predicates I – I'm going surfing, Y – you're going surfing, and W – there'll be a big wave that kills us all, to translate the following statements to English:

1. $I \wedge Y \to W$
2. $(I \to W) \vee (Y \to W)$

Possible Answer.

1. If both of us are going surfing, then there'll be a big wave that kills us all.

Interlude: Logic to English

Exercise. Use the predicates I – I'm going surfing, Y – you're going surfing, and W – there'll be a big wave that kills us all, to translate the following statements to English:

1. $I \wedge Y \rightarrow W$
2. $(I \rightarrow W) \vee (Y \rightarrow W)$

Possible Answer.

1. If both of us are going surfing, then there'll be a big wave that kills us all.
2. If both of us are going surfing, then there'll be a big wave that kills us all.

(Both formulae have the same truth table!)

Propositional Formulae

Definition. Given a set V of variables, *propositional formulae* are constructed as follows:

- T (true) and F (false) and all variables $x \in V$ are boolean formulae

- if ϕ and ψ are boolean formulae, then so are $\phi \wedge \psi$ and $\phi \vee \psi$ and $\phi \to \psi$
- if ϕ is a boolean formula, then so is $\neg\phi$

Precedence. $\neg$ binds more strongly than $\wedge$ binds more strongly than $\vee$ binds more strongly than $\to$:

$$\neg x \to y \vee z \text{ reads as } (\neg x) \to (y \vee z)$$

Boolean Formulae vs Propositional Formulae

- propositional formulae are boolean formulae with addition of $\to$
- $\to$ is expressible using boolean formulae: $x \to y = \neg x \vee y$
- but included as implication is used *very* frequently

Contradictions and Contingencies

Types of Propositional Formulae. A propositional formula is

- *valid*, if it evaluates to T in all situations / under all truth value assignments.
- a *contradiction* if it evaluates to F in all situations, and
- a *contingency* if there are (necessarily different) situations for which it evaluates to T and to F.

Example.

- 'John had toast for breakfast' is a contingency.
- 'John had toast for breakfast' $\wedge \neg$ 'John had toast for breakfast' is a contradiction.
- $p \to (\neg q \vee p) \to (p \wedge q) \vee r$ – can be complicated

Example proof using truth tables

Statement to be proved:

$$\phi \equiv (p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee r))$$

For all 8 ($= 2^3$) possibilities of p, q, r, calculate truth value of the statement

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \wedge q) \vee r$	ϕ
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	T
F	T	T	T	F	F	T	T
F	T	F	T	F	F	F	T
F	F	T	T	F	F	T	T
F	F	F	F	F	F	F	T

Always exponential in size!

Natural Deduction

Truth Tables. Can be exponential

Equational Proofs. Can be *very* unintuitive

Natural Deduction

- formal system that imitates human reasoning
- explains one connective at a time: intro and elim rules
- used to prove *validity* of formulae
- also used in all formal theorem provers

Informal Proof

Goal. Show that $\phi \equiv (p \wedge (q \vee r)) \rightarrow (q \rightarrow s) \vee p$ is valid.

Informal Proof.

1. First assume that $p \wedge (q \vee r)$ (is true) and show that $(q \rightarrow s) \vee p$.
2. under this assumption, we have that p (is true).
3. still under this assumption, $(q \rightarrow s) \vee p$ (is true).
4. That is, $p \wedge (q \vee r) \rightarrow (q \rightarrow s) \vee p$ *without* assumptions.

Formal Natural Deduction Proof.

1	$\quad p \wedge (q \vee r)$	Assumption
2	$\quad p$	$\wedge$-E, 1
3	$\quad (q \rightarrow s) \vee p$	$\vee$-I, 2
4	$(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p)$	$\rightarrow$-I, 1–3

Conjunction rules

And Introduction ($\wedge$-I)

$$\frac{p \qquad q}{p \wedge q}$$

- as p is true, and q is true, we have that $p \wedge q$ is true.

And Elimination ($\wedge$-E)

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q}$$

- as $p \wedge q$ is true, we have that p is true.
- as $p \wedge q$ is true, we have that q is true.

Example

Example. Commutativity of conjunction (derived rule)

$$\frac{p \wedge q}{q \wedge p}$$

- assuming that $p \wedge q$ (is true), we (also) have that $q \wedge p$ (is true).

Informal Proof.

1. Assume that $p \wedge q$.
2. because of $p \wedge q$, we have p.
3. because of $p \wedge q$, we have q.
4. therefore, we also have $q \wedge p$.

Natural Deduction Proof.

1	$p \wedge q$	
2	p	$\wedge$-E, 1
3	q	$\wedge$-E, 1
4	$q \wedge p$	$\wedge$-I, 2, 3

Implication rules

Implication Introduction ($\rightarrow$-I)

$$\frac{\begin{array}{c}[\,p\,]\\ \vdots \\ q\end{array}}{p \rightarrow q}$$

- if q is true under the *assumption* p, then $p \rightarrow q$ is true *without* the assumption p.
- $[\ldots]$ means that the assumption p is discarded (no longer made).

Implication Elimination ($\rightarrow$-E)

$$\frac{p \qquad p \rightarrow q}{q}$$

- if both p and $p \rightarrow q$ hold (are true), then so does q.

Example - transitivity of implication (derived rule)

We prove $$\frac{p \to q \qquad q \to r}{p \to r}$$

Informal Proof

1. fix the assumption $p \to q$.
2. fix the assumption $q \to r$.
3. additionally assume p (and show r)
4. because p and $p \to q$, we have q.
5. because q and $q \to r$, we have r.
6. hence $p \to r$ holds *without* assuming p

Natural Deduction Proof

1	$p \to q$	
2	$q \to r$	
3	p	
4	q	$\to$-E, 1, 3
5	r	$\to$-E, 2, 4
6	$p \to r$	$\to$-I, 3–5

- lines 1 and 2 are assumptions, can be used anywhere
- line 3 is an assumption we make, can be used *only* in scope (l 3–5).

Aside: Justification of Proof Steps

Silly Proof. (we prove what we already know!)

1	$p \to q$	
2	$\quad p$	
3	$\quad q$	$\to$-E, 1, 2
4	$p \to q$	$\to$-I, 2–3

Line Number Notation

- $\to$-E,1,2 means that rule $\to$-E proves line 3 from lines 1 *and* 2
- $\to$-I,2-3 means rule $\to$-I proves line 4 from *the fact that* we could *assume* line 2 and (using that assumption) *prove* line 3.
- In $\to$-I, 2–3 is the *entire* scope of the assumption p.

Rules involving assumptions

1	$p \to q$	
2	$q \to r$	
3	$\quad p$	
4	$\quad q$	$\to$-E, 1, 3
5	$\quad r$	$\to$-E, 2, 4
6	$q \wedge r$ *WRONG*	$\wedge$-I, 4, 5

- statements inside the scope of an assumption depends on that assumption.
- we only know that they are true if the assumption is true!
- we have assumed p and "proved" $q \wedge r$, but $q \wedge r$ depends on p.
- Indentation and vertical lines indicate scoping
- Similar to programming: p is a "local variable".

Useless assumptions

You can assume anything, but it might not be useful.

1	$p \wedge q$	
2	q	$\wedge$-E, 1
3	$(p \wedge q) \to q$	$\to$-I, 1–2

1	$p \wedge$ You are a giraffe	
2	You are a giraffe	$\wedge$-E, 1
3	$p \wedge$ You are a giraffe $\to$ You are a giraffe	$\to$-I, 1–2

Disjunction rules

Or Introduction ($\vee$-I)

$$\frac{p}{p \vee q} \qquad \frac{p}{q \vee p}$$

- if p (holds), then so do $p \vee q$ and $q \vee p$

Or Elimination ($\vee$-E)

$$\frac{p \vee q \qquad \begin{matrix}[p] \\ \vdots \\ r\end{matrix} \qquad \begin{matrix}[q] \\ \vdots \\ r\end{matrix}}{r}$$

- assuming that we have a proof of $p \vee q$ and
- for the case that p holds, we have a proof of r
- for the case that q holds we have a proof of r
- then we have a proof of r *just* from $p \vee q$.

$\vee$-E template

1. know that $p \vee q$
2. in case p is true

   . . .

a. we know that r.

b. and in case that q is true

   . . .

c. we also know that r

d. so we know r as long as $p \vee q$!

1	$p \vee q$	
2	$\quad p$	
$\vdots$	$\quad \vdots$	
a	$\quad r$	
b	$\quad q$	
$\vdots$	$\quad \vdots$	
c	$\quad r$	
	r	$\vee$-E, 1, 2–a, b–c

Example: commutativity of disjunction (derived rule)

$$\frac{p \vee q}{q \vee p}$$

Informal Proof.

1. fix the assumption $p \vee q$.
2. first assume that p is true.
3. then also have that $q \vee p$.
4. now assume that q is true.
5. then also have that $q \vee p$.
6. hence $q \vee p$, without assuming either p or q.

Natural Deduction Proof.

1	$p \vee q$	
2	$\quad p$	
3	$\quad q \vee p$	$\vee$-I, 2
4	$\quad q$	
5	$\quad q \vee p$	$\vee$-I, 4
6	$q \vee p$	$\vee$-E, 1, 2–3, 4–5

Negation and Truth Rules

not introduction (¬-I)

$$\dfrac{\begin{matrix}[p]\\ \vdots \\ F\end{matrix}}{\neg p}$$

not elimination (¬-E)

$$\dfrac{p \qquad \neg p}{F}$$

- if assuming p gives a contradiction, p is wrong – so $\neg p$ must hold.

Proof by Contradiction (PC)

$$\dfrac{\begin{matrix}[\neg p]\\ \vdots \\ F\end{matrix}}{p}$$

Truth

$$\dfrac{}{T}$$

- to prove p, assume $\neg p$ and derive a contradiction.
- truth, i.e. T, can always be established without assumptions.

Example: double negation introduction (derived rule)

$$\frac{p}{\neg\neg p} \qquad \frac{\text{It is raining}}{\text{It is not the case that it is not raining}}$$

Informal Proof.

1. Fix the assumption p.
2. additionally assume that $\neg p$.
3. then F as p and $\neg p$ (under assn p)
4. hence $\neg p$ is contradictory, so $\neg\neg p$.

Natural Deduction Proof.

1	p	
2	$\quad \neg p$	
3	$\quad F$	$\neg$E, 1, 2
4	$\neg\neg p$	$\neg$I, 2–3

Example: contradiction elimination (derived rule)

"Anything follows form a contradiction"

$$\frac{F}{q}$$

1	F	
2	$\neg q$	
3	F	R, 1
4	q	PC, 2–3

- R stands for "repeat".
- F holds and continues to hold within the scope of the assumption $\neg q$.
- assuming $\neg q$ a "technical trick".

Example: double negation elimination (derived rule)

$$\frac{\neg\neg p}{p}$$

1	$\neg\neg p$	
2	$\quad\neg p$	
3	$\quad F$	$\neg$E, 1, 2
4	p	PC, 2–3

Equivalence

$p \leftrightarrow q$ means p is true if and only if q is true

We can make the definition

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

which would naturally give us these rules

introduction rule:

$$\frac{p \to q \qquad q \to p}{p \leftrightarrow q}$$

elimination rules:

$$\frac{p \leftrightarrow q}{p \to q} \qquad \frac{p \leftrightarrow q}{q \to p}$$

Which rule to use next?

- Guided by the "form" of your goal, and what you already have proved
- "form" — ie, look at the connective: $\wedge, \vee, \rightarrow, \neg$
- always can consider using PC (proof by contradiction)
- to prove $p \vee q$, $\vee$-I (or introduction) may not work

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q}$$

p may not be necessarily true, q may not be necessarily true

To prove $p \vee q$, sometimes you need to do this:

1. Using PC, assume $\neg(p \vee q)$ (hoping to prove some contradiction)
2. When is $\neg(p \vee q)$ true? When both p and q are false!
3. From $\neg(p \vee q)$ how to prove $\neg p$? (next slide)
4. Having proved both $\neg p$ and $\neg q$, prove some further contradiction

Tutorial Exercise. $\dfrac{\neg p \to q}{p \vee q}$

Not-or elimination (derived rule)

$$\frac{\neg(p \vee q)}{\neg p}$$

1	$\neg(p \vee q)$	
2	$\quad p$	
3	$\quad p \vee q$	$\vee$-I, 2
4	$\quad F$	$\neg$E, 1, 3
5	$\neg p$	$\neg$I, 2–4

Proving a contrapositive rule

In the same way, whenever you can prove any $\dfrac{p}{q}$

then you can prove $\dfrac{\neg q}{\neg p}$

1	$\neg q$	
2	$\quad p$	
3	$\quad q$	your proof of q from p
4	$\quad F$	$\neg$E, 1, 3
5	$\neg p$	$\neg$I, 2–4

Law of the excluded middle (derived)

$$p \vee \neg p$$

"Everything must either be or not be." — Russell

1	$\neg(p \vee \neg p)$	
2	$\neg p$	$\neg\vee$-E (previous slide), 1
3	$\neg\neg p$	$\neg\vee$-E (previous slide), 1
4	F	$\neg$E, 2, 3
5	$p \vee \neg p$	PC, 1–4

Summary: Major Proof Techniques

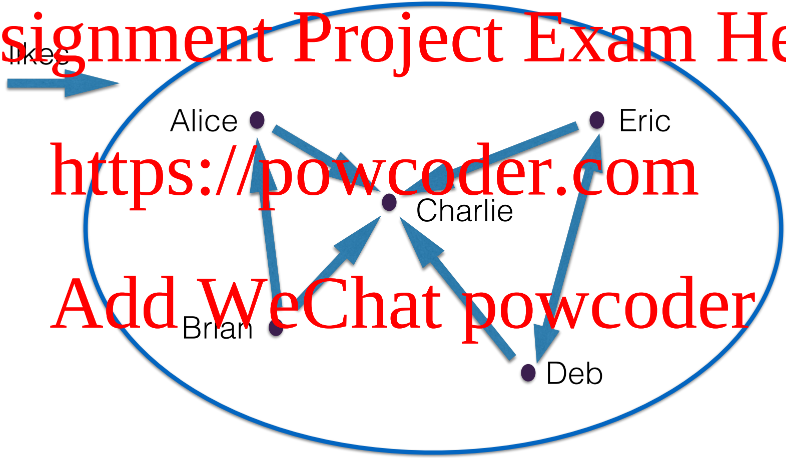
Three major styles of proof in logic and mathematics

- **Model based computation**: truth tables for propositional logic
- **Algebraic proof**: equational reasoning
- **Deductive reasoning**: rules of inference (e.g. Natural Deduction)

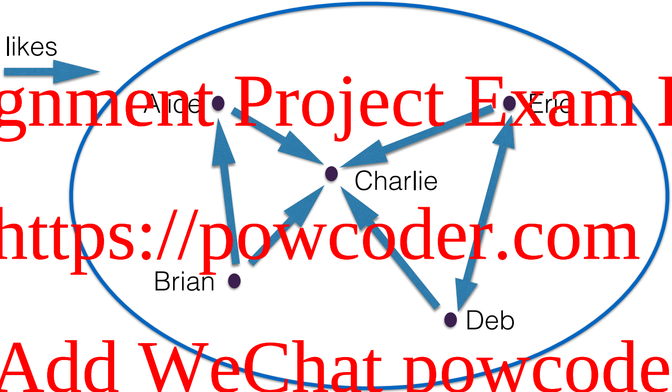
Q. Why bother? Why not write a program that does truth tables?

- propositional logic is *decidable*: can write a program
- other logics are *not*: first order logic (next)

What can we say about the following situation?

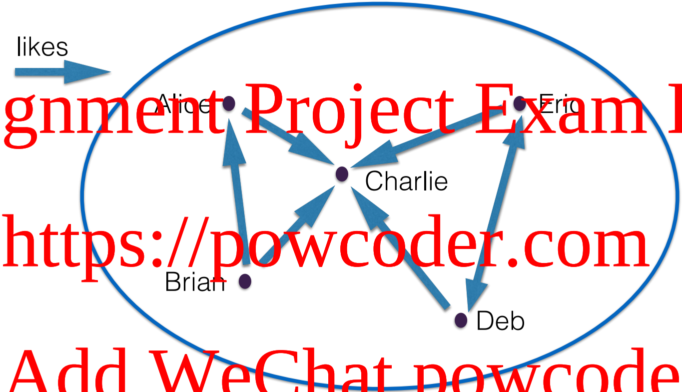


Some (English) Sentences



- Brian likes Alice.
- Eric and Deb like one another.
- Nobody likes Brian.
- Everybody likes Charlie.
- Two people like each other.
- Someone is liked by everyone.

Key Ingredients



Set

- $U = \{A, B, C, D, E\}$
- thought of as "individuals" (can be things)

Relation

- $R = \{(A, C), (B, A), (B, C), (D, C), (D, E), (E, C), (E, D)\}$
- set of ordered pairs (directional)

Limits of Propositional Logic

Propositional Logic.

- atomic propositions, no "inner structure"
- could be 'Brian likes Alice' and 'someone likes someone (else)'

How about . . .

- Alice likes someone who is liked by everyone
- Everybody likes someone who doesn't like anyone
- . . .

Propositional Logic is not enough!

- What is the limit of what we can say?
- What are the relationships between all these propositions?

Limits of Propositional Logic

Propositional logic talks (only) about *statements*, or *facts*

- e.g. 'I am going surfing'
- can be true or false.

Propositional Logic *cannot* talk about

- **Objects**, e.g. people, houses, cars, numbers, programs, variables
- **Relations**, e.g. red, prime, larger than

First Order Logic

- Predicates *depending* on variables (e.g. likes(x, y))
- Combined using universal ($\forall$) and existential ($\exists$) quantifiers
- In the example: everyone ($\forall$) and someone ($\exists$)
- more complex concepts by nesting (everybody loves someone who . . .)

Quantifiers, Informally

Universal Quantification. $\forall x$: "All x . . .

- (logic) $\forall x.\text{likes}(x, \text{Charlie})$
- (direct translation) For all x, x likes Charlie.
- (natural English) Charlie is liked by everyone.

Existential Quantification. $\exists x$: "There is an x s.t. . . .

- (logic) $\exists x.\text{likes}(x, \text{Alice})$
- (direct translation) There is an x s.t. x likes Alice.
- (natural English) Alice is liked by someone.

More Complex Sentences.

Nobody likes Brian.

$$\forall x(\neg \text{likes}(x, \text{Brian}))$$

Two people like each other.

$$\exists x\, \exists y(\text{likes}(x, y) \land \text{likes}(y, x))$$

Someone is liked by everyone.

$$\exists x\, \forall y\, (\text{likes}(y, x))$$

Q. What does the following translate to?

$$\forall x\, \exists y\, (\text{likes}(y, x))$$

First Order Logic: Vocabulary

Vocabulary. relation symbols

- have an *arity*: number of "things" that are related.
- Examples: is_elephant(x) (unary), likes(x, y) (binary)

Informal Interpretation.

- is_elephant: the set of all elephants
- likes: the set of all pairs (x, y) s.t. x likes y

Informal Interpretation: Syntax

- is_elephant: a proposition depending on an argument
- likes: a proposition depending on two arguments

First Order Logic: Official Syntax

Vocabulary. A *vocabulary* for first order logic is a set R (of relation symbols) where each relation symbol has an *arity* (number of arguments).

Syntax of First Order Logic. Let R be a vocabulary and V a set of variables. The *formulae* of first-order logic (over R and V) are constructed as follows:

1. If $r \in R$ is an n-ary and $x_1, \ldots, x_n \in V$, then $r(x_1, \ldots, x_n)$ is a formula.
2. If ϕ and ψ are formulae, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, and $\neg\phi$.
3. If ϕ is a formula, and $x \in V$, then $\exists x \phi$ and $\forall x \phi$ are formulae.

Dot Notation saves outermost parentheses:

$$\forall x.\text{very complex formula} \equiv \forall x(\text{very complex formula})$$

$$\exists x.\text{very complex formula} \equiv \exists x(\text{very complex formula})$$

Happy and Unhappy Dragons

Vocabulary.

unary predicates

$D(x)$ – x is a dragon, $H(x)$ – x is happy, $F(x)$ – x can fly

binary predicate

$C(x,y)$ – x is a child of y

English.

1. All dragons can fly unless they are unhappy.
2. At least one dragon can fly despite being unhappy.
3. A dragon is happy if all its children can fly.

Logic.

1. $\forall x. D(x) \to F(x) \lor \neg H(x)$
2. $\exists x. D(x) \land F(x) \land \neg H(x)$
3. $\forall x. D(x) \to (\forall y. C(y,x) \to F(y)) \to H(x)$

Common Patterns

Existential Quantifier. often goes with $\wedge$

- There is a dragon that ...
- There is an x that is a dragon and ...
- $\exists x.\text{dragon}(x) \wedge \ldots$

Universal Quantifier. often goes with $\rightarrow$

- All dragons ...
- For all x, if x is a dragon, then ...
- $\forall x.\text{dragon}(x) \rightarrow \ldots$

Situations for First Order Logic

Recall. Formulae of *propositional logic* depend on variables

- a situation tells us whether these variables are true or false
- given a situation, can evaluate a formula to true or false

Q. What do we need so that we can say that a first-order formula is true or false?

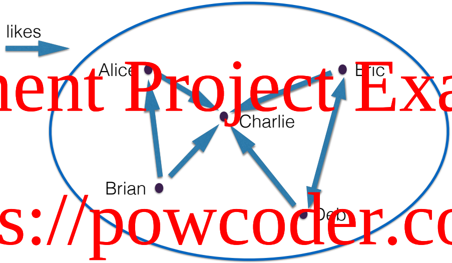
- Example: likes(x, y) – what are x and y, and what is “likes”?

Situations θ for first-order logic are given by:

- a domain of discourse U – simply a *set* (of things)
- for every $r \in R$ n-ary, an n-ary *relation* $\theta(r)$ on U
- for all variables x, an element $\theta(x)$ of U.

Notation: (U, θ)

Example: The Taxonomy of "likes"



Situation. $S = (U, \theta)$ where

- $U = \{A, B, C, D, E\}$
- $\theta(\text{likes}) = \{(A, C), (B, A), (B, C), (D, C), (D, E), (E, C), (E, D)\}$
- $\theta(x) = \theta(y) =$ Alice, $\theta(x) =$ Deb. (not required to be injective – many variables can point to same object)

Formal Semantics

Given. Vocabulary R, situation $S = (U, \theta)$

Truth of formula ϕ in situation (U, θ)

Base case. Truth of $r(x_1, \ldots, x_n)$

- if $r(x_1, \ldots x)$ is true in the situation
- that is, if $(\theta(x_1), \ldots, \theta(x_n))$ are related
- that is, if $(\theta(x_1), \ldots, \theta(x_n)) \in \theta(r)$.

Propositional Logic Cases.

- $\phi \wedge \psi$ is true $\iff$ ϕ is true and ψ is true
- $\phi \vee \psi$ is true $\iff$ ϕ is true or ψ is true
- $\phi \to \psi$ is true if ϕ is not true or ψ is true
- $\neg\phi$ is true $\iff$ ϕ is not true

Quantifier Cases.

- $\forall x.\phi$ is true if ϕ is true for *all* values of $\theta(x)$ (and everything else unchanged)
- $\exists x.\phi$ is true if ϕ is true for *some* value of $\theta(x)$ (and everything else unchanged)

Digression on Quantifiers

Universal Quantifier. To see that $\forall x.\phi$ is true in situation (U, θ):

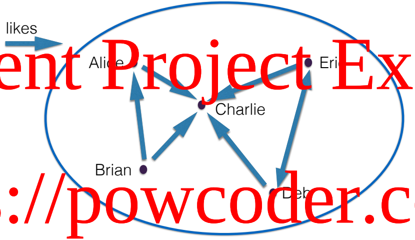
- need to take θ and vary the value of x
- show that ϕ is true in situation (U, θ')
- behaves like an "infinite and" over all elements of the domain

Existential Quantifier. To see that $\exists x.\phi$ is true in situation (U, θ):

- need to take θ and vary the value of x
- need to exhibit *one* varied θ' such that ϕ is true in situation (U, θ')
- behaves line an "infinite or" over all elements of the domain

Back to the Example

Variables $V = \{a, b, c, \ldots\}$ with $\theta(a) = \text{Alice}$, $\theta(b) = \text{Brian}$ etc.



Q. Does everyone like Charlie, i.e. is $\forall x(\text{likes}(x, c))$ true in the situation above?

- vary the value of x for θ, i.e. consider $\theta'(x) = a, b, \ldots, e$
- show $\text{likes}(x, c)$ is true for all such θ'
- show $\text{likes}(\text{Alice}, \text{Charlie})$ and $\text{likes}(\text{Brian}, \text{Charlie})$ and ...
- but it is *false* that $\text{likes}(\text{Charlie}, \text{Charlie})$!
- so $\forall x.\text{likes}(x, c)$ is not true in the situation above as Charlie doesn't like her/himself!

Special Case: Quantifiers over the Empty Set

Given.

- Situation $S = (U, \theta)$ for first order logic
- unary relation unicorn with $\theta(\text{unicorn}) = \emptyset$

Existential Quantifier

- "There is a tree-climbing unicorn": $\exists x.\text{unicorn}(x) \wedge \text{climbs\_trees}(x)$
- always *false*: $\exists x.\text{unicorn}(x) \wedge \text{climbs\_trees}(x)$ is false in S
- irrespective of value of x, unicorn(x) is always false.

Universal Quantifier.

- "All unicorns climb trees": $\forall x.\text{unicorn}(x) \rightarrow \text{climbs\_trees}(x)$
- always *true*: $\forall x.\text{unicorn}(x) \rightarrow \text{climbs\_trees}(x)$ is true in S.
- irrespective of value of x, unicorn(x) is *always* false.

Propositional vs First Order Logic

Propositional Logic.

- Given propositional logic formula ϕ, can decide validity by constructing all truth tables.

First Order Logic.

- To show (or check) that a first-order formula is valid, would need to construct all situations

- But there are infinitely many of them, and some of them infinite!

Decidability.

- Propositional logic is *decidable*: can write program that checks for validity.
- Naive checking for validity *doesn't* work for first order logic – but maybe there's a better way?
- Can *formally prove* that there *cannot* be a program that checks first-order validity!

Interlude: The Drinker's Paradox

Drinker's Paradox (Smullyan 1978, in "What is the name of this book?")

> *"In every non-empty pub there is someone so that if (s)he is drinking, so is everybody else."*

In Logic (with one unary predicate $D(x)$)

$$\phi \equiv \exists x(D(x) \rightarrow \forall y D(y))$$

Q. Is ϕ valid in all situations (U, θ) where U is not empty?

Laws for Quantifiers: Negating "there exists"

$$\neg(\exists x.\ P(x)) \leftrightarrow (\forall x.\ \neg P(x))$$

Examples.

- "No elephant is unhappy. "
- $\neg\exists x.\text{elephant}(x) \wedge \neg\text{happy}(x)$
- $\forall x.\neg(\text{elephant}(x) \wedge \neg\text{happy}(x))$
- $\forall x.\neg\text{elephant}(x) \vee \text{happy}(x)$
- "Everything is either happy or not an elephant"

De Morgan Laws. Consider domain $U = \{a_0, a_1, \dots\}$.

- $\neg\exists x.P(x)$ intuitively equivalent to $\neg(P(a_0) \vee P(a_1) \vee \dots)$
- $\neg(P(a_0) \vee P(a_1) \vee \dots)$ equiv to $\neg P(a_0) \wedge \neg P(a_1) \wedge \dots$ by De Morgan
- $\neg P(a_0) \wedge \neg P(a_1) \wedge \dots$ intuitively equivalent to $\forall x.\neg P(x)$

Negating $\exists$, Formally

Theorem. Let ϕ be a first-order formula, $S = (U, \theta)$ a situation. Then $\neg\exists x.\phi$ is true in S if and only if iff $\forall x.\neg\phi$ is true in S.

Proof (Sketch). Suppose that $\neg\exists x.\phi$ is true in S. To see that $\forall x.\neg\phi$ is true in S, let θ' agree with θ except possibly on the value of x.

We need to show that $\neg\phi$ is true in $S' = (U, \theta')$, i.e. ϕ is false in S'.

Suppose for a contradiction that ϕ is true in S'. As θ' agrees with θ except possibly on the value of x, this means that $\exists x.\phi$ is true in S, contradiction.

The reverse direction is analogous.

Negating "for all"

$$\neg(\forall x.\ P(x)) \leftrightarrow (\exists x.\ \neg P(x))$$

Examples

- "Not all elephants are happy"
- $\neg\forall x.\mathrm{elphant}(x) \rightarrow \mathrm{happy}(x)$
- $\exists x.\neg(\mathrm{elephant}(x) \rightarrow \mathrm{happy}(x))$
- $\exists x.\mathrm{elephant}(x) \wedge \neg\mathrm{happy}(x)$
- There exists an unhappy elephant.

Theorem. Let ϕ be a first-order formula, S a situation for first order logic. Then $\neg\forall x.\phi$ is true in S iff $\exists x.\neg\phi$ is true in S.

Mixed negated quantifiers

Here are four different expressions of the fact that there is no upper bound to the natural numbers.

We can shift from one to the other by negating the quantifiers.

$$\neg\exists\, m.\ \ \forall\, n.\ \ m \geq n$$

$$\forall\, m.\ \ \neg\forall\, n.\ \ m \geq n$$

$$\forall\, m.\ \ \exists\, n.\ \ \neg m \geq n$$

$$\forall\, m.\ \ \exists\, n.\ \ m < n$$