

COMP2610 / COMP6261 - Information Theory

Lecture 4: Bayesian Inference

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<https://powcoder.com>

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Australian  
National  
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July 31, 2018

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- Examples of joint, marginal and conditional distributions

- When can we say that  $X$ ,  $Y$  do not influence each other?

- What, if anything, does  $p(X = x, Y = y)$  tell us about  $p(Y = y|X = x)$ ?

## Review Exercise

Suppose we have binary random variables  $X, Y$  such that

$$p(X = 1) = 0.6$$

$$p(Y = 1|X = 0) = 0.7$$

$$p(Y = 1|X = 1) = 0.8$$

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Then,

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$$p(X = 1|Y = 1) =$$

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## Review Exercise

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$$p(Y = 1|X = 0) = 0.7$$

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Then,

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$$p(X = 1|Y = 1) = \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1)} \quad \text{Bayes' rule}$$

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## Review Exercise

Suppose we have binary random variables  $X, Y$  such that

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$$p(X = 1|Y = 1) = \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1)}$$

Bayes' rule

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$$= \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1|X = 1)p(X = 1) + p(Y = 1|X = 0)p(X = 0)}$$

## Review Exercise

Suppose we have binary random variables  $X, Y$  such that

$$p(X = 1) = 0.6$$

$$p(Y = 1|X = 0) = 0.7$$

$$p(Y = 1|X = 1) = 0.8$$

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Then,

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$$\begin{aligned} p(X = 1|Y = 1) &= \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1)} && \text{Bayes' rule} \\ &= \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1|X = 1)p(X = 1) + p(Y = 1|X = 0)p(X = 0)} \\ &= \frac{(0.8)(0.6)}{(0.8)(0.6) + (0.7)(0.4)} \\ &\approx 0.63 \end{aligned}$$

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This time

# Assignment Project Exam Help

- More examples on Bayes' theorem:
  - ▶ Eating hamburgers

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- ▶ Detecting terrorists

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- Are there notions of probability beyond frequency counting?

# Outline

## 1 Bayes' Rule: Examples

- Eating Hamburgers
- Detecting Terrorists
- The Monty Hall Problem

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## 2 Moments for functions of two discrete Random Variables

## 3 The meaning of Probability

## 4 Wrapping Up

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1 Bayes' Rule: Examples

- Eating Hamburgers
- Detecting Terrorists
- The Monty Hall Problem

# Assignment Project Exam Help

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2 Moment Generating Functions in Multivariate Random Variables

3 The meaning of Probability

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4 Wrapping Up

## Bayesian Inference:

Example 1 (Barber, BRML, 2011)

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- 90% of people with McD syndrome are frequent hamburger eaters

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# Bayesian Inference:

Example 1 (Barber, BRML, 2011)

## Assignment Project Exam Help

- 90% of people with McD syndrome are frequent hamburger eaters

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- Probability of someone having McD syndrome:  $1/10000$

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# Bayesian Inference:

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## Assignment Project Exam Help

- 90% of people with McD syndrome are frequent hamburger eaters

- <https://powcoder.com>  
Probability of someone having McD syndrome:  $1/10000$

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Proportion of hamburger eaters is about 50%

# Bayesian Inference:

Example 1 (Barber, BRML, 2011)

## Assignment Project Exam Help

- 90% of people with McD syndrome are frequent hamburger eaters

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Probability of someone having McD syndrome:  $1/10000$

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Proportion of hamburger eaters is about 50%

What is the probability that a hamburger eater will have McD syndrome?

# Bayesian Inference:

## Example 1: Formalization

# Assignment Project Exam Help

Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

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# Bayesian Inference:

## Example 1: Formalization

# Assignment Project Exam Help

Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

$$p(H = 1 | \neg McD = 1) = 9/10$$

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# Bayesian Inference:

## Example 1: Formalization

# Assignment Project Exam Help

Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

$$p(H = 1 | McD = 1) = 9/10 \quad p(McD = 1) = 10^{-4}$$

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# Bayesian Inference:

## Example 1: Formalization

# Assignment Project Exam Help

Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

$$\begin{aligned} p(H = 1 | McD = 1) &= 9/10 & p(McD = 1) &= 10^{-4} \\ p(H = 1) &= 1/2 \end{aligned}$$

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# Bayesian Inference:

## Example 1: Formalization

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Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

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We need to compute  $p(McD = 1 | H = 1)$ , the probability of a hamburger eater having McD syndrome.

# Bayesian Inference:

## Example 1: Formalization

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Let  $McD \in \{0, 1\}$  be the variable denoting having the McD syndrome and  $H \in \{0, 1\}$  be the variable denoting a hamburger eater. Therefore:

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We need to compute  $p(McD = 1 | H = 1)$ , the probability of a hamburger eater having McD syndrome.

Any ballpark estimates of this probability?

# Bayesian Inference:

## Example 1: Solution

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$$p(McD = 1 | H = 1) = \frac{p(H = 1 | McD = 1)p(McD = 1)}{p(H = 1)}$$
$$= 1.8 \times 10^{-4}$$

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## Bayesian Inference:

### Example 1: Solution

# Assignment Project Exam Help

$$p(McD = 1 | H = 1) = \frac{p(H = 1 | McD = 1)p(McD = 1)}{p(H = 1)}$$
$$= 1.8 \times 10^{-4}$$

Repeat the above computation if the proportion of hamburger eaters is rather small. (say in France) 0.001.

## Example 2: Detecting Terrorists:

From [understandinguncertainty.org](http://understandinguncertainty.org)

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- Scanner detects true terrorists with 95% accuracy

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## Example 2: Detecting Terrorists:

From [understandinguncertainty.org](http://understandinguncertainty.org)

# Assignment Project Exam Help

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- Scanner detects upstanding citizens with 95% accuracy

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# Assignment Project Exam Help

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- There is 1 terrorist on your plane with 100 passengers aboard

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- The shift looking man sitting next to you tests positive (terrorist)

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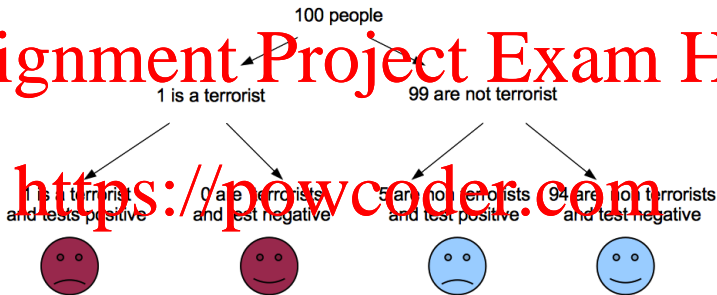
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- The shift looking man sitting next to you tests positive (terrorist)

What are the chances of this man being a terrorist?

## Example 2: Detecting Terrorists:

Simple Solution Using “Natural Frequencies” (David Spiegelhalter)

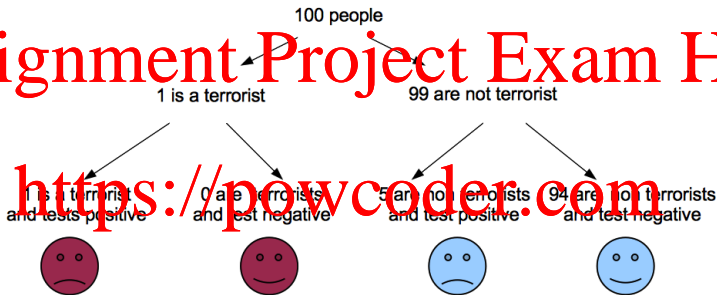


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Figure reproduced from [users.ics.forth.gr/~lazaros/teaching/uncertainty/](http://users.ics.forth.gr/~lazaros/teaching/uncertainty/)

## Example 2: Detecting Terrorists:

Simple Solution Using “Natural Frequencies” (David Spiegelhalter)



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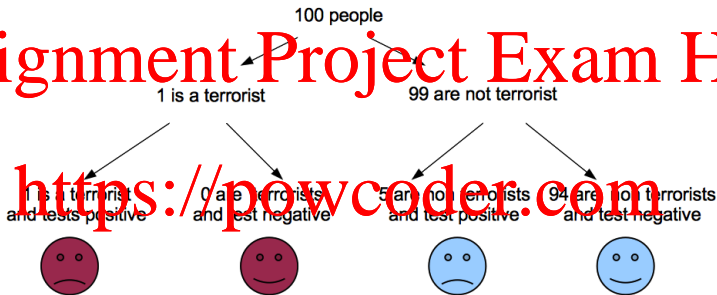
Figure reproduced from <https://www.powcoder.com>

The chances of the man being a terrorist are  $\approx \frac{1}{6}$

## Example 2: Detecting Terrorists:

Simple Solution Using “Natural Frequencies” (David Spiegelhalter)

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Figure reproduced from <https://www.powcoder.com>

The chances of the man being a terrorist are  $\approx \frac{1}{6}$

- Relation to disease example
- Consequences when catching criminals

## Example 2: Detecting Terrorists:

Formalization with Actual Probabilities

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Let  $T \in \{0, 1\}$  denote the variable regarding whether the person is a terrorist and  $S \in \{0, 1\}$  denote the outcome of the scanner.

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## Example 2: Detecting Terrorists:

Formalization with Actual Probabilities

# Assignment Project Exam Help

Let  $T \in \{0, 1\}$  denote the variable regarding whether the person is a terrorist and  $S \in \{0, 1\}$  denote the outcome of the scanner.

$$p(S=1|T=1)=0.95, \quad p(S=0|T=1)=0.05$$

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## Example 2: Detecting Terrorists:

Formalization with Actual Probabilities

# Assignment Project Exam Help

Let  $T \in \{0, 1\}$  denote the variable regarding whether the person is a terrorist and  $S \in \{0, 1\}$  denote the outcome of the scanner.

$$\begin{aligned} p(S = 1 | T = 1) &= 0.95 & p(S = 0 | T = 1) &= 0.05 \\ p(S = 0 | T = 0) &= 0.95 & p(S = 1 | T = 0) &= 0.05 \end{aligned}$$

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## Example 2: Detecting Terrorists:

Formalization with Actual Probabilities

# Assignment Project Exam Help

Let  $T \in \{0, 1\}$  denote the variable regarding whether the person is a terrorist and  $S \in \{0, 1\}$  denote the outcome of the scanner.

$$\begin{aligned}p(S = 1 | T = 1) &= 0.95 & p(S = 0 | T = 1) &= 0.05 \\p(S = 0 | T = 0) &= 0.95 & p(S = 1 | T = 0) &= 0.05 \\p(T = 1) &= 0.01 & p(T = 0) &= 0.99\end{aligned}$$

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## Example 2: Detecting Terrorists:

Formalization with Actual Probabilities

# Assignment Project Exam Help

Let  $T \in \{0, 1\}$  denote the variable regarding whether the person is a terrorist and  $S \in \{0, 1\}$  denote the outcome of the scanner.

$$\begin{aligned}p(S = 1 | T = 1) &= 0.95 & p(S = 0 | T = 1) &= 0.05 \\p(S = 0 | T = 0) &= 0.95 & p(S = 1 | T = 0) &= 0.05 \\p(T = 1) &= 0.01 & p(T = 0) &= 0.99\end{aligned}$$

We want to compute  $p(T = 1 | S = 1)$ , the probability of the man being a terrorist given that he has tested positive.

## Example 2: Detecting Terrorists:

Solution with Bayes' Rule

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$$p(T = 1|S = 1) = \frac{p(S = 1|T = 1)p(T = 1)}{p(S = 1|T = 1)p(T = 1) + p(S = 1|T = 0)p(T = 0)}$$

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## Example 2: Detecting Terrorists:

Solution with Bayes' Rule

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$$p(T = 1|S = 1) = \frac{p(S = 1|T = 1)p(T = 1)}{p(S = 1|T = 1)p(T = 1) + p(S = 1|T = 0)p(T = 0)}$$
$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.05)(0.99)}$$

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## Example 2: Detecting Terrorists:

Solution with Bayes' Rule

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$$p(T = 1|S = 1) = \frac{p(S = 1|T = 1)p(T = 1)}{p(S = 1|T = 1)p(T = 1) + p(S = 1|T = 0)p(T = 0)}$$
$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.05)(0.99)}$$

≈ 0.16

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## Example 2: Detecting Terrorists:

Solution with Bayes' Rule

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$$p(T = 1|S = 1) = \frac{p(S = 1|T = 1)p(T = 1)}{p(S = 1|T = 1)p(T = 1) + p(S = 1|T = 0)p(T = 0)}$$
$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.05)(0.99)}$$

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*The probability of the man being a terrorist is  $\approx \frac{1}{6}$*

## Example 2: Detecting Terrorists:

### Posterior Versus Prior Belief

While the man has a low probability of being a terrorist, our belief has

increased compared to our prior

$$\frac{p(T = 1|S = 1)}{p(T = 1)} = \frac{0.16}{0.01} = 16$$

i.e. our belief in him being a terrorist has gone up by a factor of 16

Since terrorists are so rare, a factor of 16 does not result in a very high (absolute) probability or belief

(Aside: They are indeed very rare. For an intriguing (and surprising) example of the implications of inability to take account of actual base rates (in the example above we made the numbers up), and the effect on people's subsequent decisions, see Gerd Gigerenzer, Dread Risk, September 11, and Fatal Traffic Accidents, *Psychological Science* 15(4), 286–287, (2004); Gerd Gigerenzer, Out of the Frying Pan into the Fire: Behavioural Reactions to Terrorist Attacks, *Risk Analysis* 26(2), 347–351 (2006). His calculation (which of course is based on some assumptions) is that in the year following 9/11, 6 times the number of people who were killed as passengers *additionally* died on roads (that is the increase in road deaths due to people choosing to drive instead of flying)! He calls the reaction to very low probability events with a bad outcome “dread risk”. )

## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty

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## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty
- Each box has equal probability of having the prize

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## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty
- Each box has equal probability of having the prize
- Your goal is to pick up the box with the prize in it

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## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty
- Each box has equal probability of having the prize
- Your goal is to pick up the box with the prize in it
- You select one of the boxes

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## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty
- Each box has equal probability of having the prize
- Your goal is to pick up the box with the prize in it
- You select one of the boxes
- The host, who knows the location of the prize, opens the empty box out of the other two boxes

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## Example 3: The Monty Hall Problem

### Problem Statement

- Three boxes, one with a prize and the other two are empty
- Each box has equal probability of having the prize
- Your goal is to pick up the box with the prize in it
- You select one of the boxes
- The host, who knows the location of the prize, opens the empty box out of the other two boxes

Should you switch to the other box? Would that increase your chances of winning the prize?

## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

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## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

- You have selected box  $r$

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## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

- You have selected box  $r$
- Denote the event: “the host opens box  $b$ ” with  $H=b$

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## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

- You have selected box  $r$
- Denote the event: "the host opens box  $b$ " with  $H=b$

$$P(C=r) = \frac{1}{3} \quad P(C=g) = \frac{1}{3} \quad P(C=b) = \frac{1}{3}$$

## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

- You have selected box  $r$
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$$P(C=r) = \frac{1}{3} \quad P(C=g) = \frac{1}{3} \quad P(C=b) = \frac{1}{3}$$

$$p(H=b|C=r) = \frac{1}{2} \quad p(H=b|C=g) = 1 \quad p(H=b|C=b) = 0$$

## Example 3: The Monty Hall Problem:

### Formalization

Let  $C \in \{r, g, b\}$  denote the box that contains the prize where  $r, g, b$  refer to the identity of each box.

WLOG assume the following:

- You have selected box  $r$
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$$p(C=r) = \frac{1}{3} \quad p(C=g) = \frac{1}{3} \quad p(C=b) = \frac{1}{3}$$

$$p(H=b|C=r) = \frac{1}{2} \quad p(H=b|C=g) = 1 \quad p(H=b|C=b) = 0$$

We want to compute  $p(C=r|H=b)$  and  $p(C=g|H=b)$  to decide if we should switch from our initial choice.

## Example 3: The Monty Hall Problem:

Solution

We have that:

$$P(H = b) = \sum_{c \in \{r, g, b\}} P(H = b | C = c) P(C = c)$$

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## Example 3: The Monty Hall Problem:

### Solution

We have that:

$$P(H = b) = \sum_{c \in \{r, g, b\}} P(H = b | C = c) P(C = c) \\ = (1/2)(1/3) + (1)(1/3) + (0)(1/3)$$

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## Example 3: The Monty Hall Problem:

### Solution

We have that:

$$P(H = b) = \sum_{c \in \{r, g, b\}} P(H = b | C = c) P(C = c)$$

$$= (1/2)(1/3) + (1)(1/3) + (0)(1/3)$$

$$= 1/2$$

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## Example 3: The Monty Hall Problem:

### Solution

We have that:

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Therefore:

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## Example 3: The Monty Hall Problem:

### Solution

We have that:

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Therefore:

$$p(C = r | H = b) = \frac{p(H = b | C = r) p(C = r)}{p(H = b)}$$



## Example 3: The Monty Hall Problem:

### Solution

We have that:

$$\begin{aligned} p(H = b) &= \sum_{c \in \{r, g, b\}} p(H = b | C = c) p(C = c) \\ &= (1/2)(1/3) + (1)(1/3) + (0)(1/3) \\ &= 1/2 \end{aligned}$$

Therefore:

$$p(C = r | H = b) = \frac{p(H = b | C = r) p(C = r)}{p(H = b)} = \frac{(1/2)(1/3)}{(1/2)} = 1/3$$

## Example 3: The Monty Hall Problem:

### Solution

We have that:

$$\begin{aligned} p(H = b) &= \sum_{c \in \{r, g, b\}} p(H = b | C = c) p(C = c) \\ &= (1/2)(1/3) + (1)(1/3) + (0)(1/3) \\ &= 1/2 \end{aligned}$$

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Therefore:

$$p(C = r | H = b) = \frac{p(H = b | C = r) p(C = r)}{p(H = b)} = \frac{(1/2)(1/3)}{(1/2)} = 1/3$$

Similarly,  $p(C = g | H = b) = 2/3$ .

## Example 3: The Monty Hall Problem:

### Solution

We have that:

$$\begin{aligned} p(H = b) &= \sum_{c \in \{r, g, b\}} p(H = b | C = c) p(C = c) \\ &= (1/2)(1/3) + (1)(1/3) + (0)(1/3) \\ &= 1/2 \end{aligned}$$

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Therefore:

$$p(C = r | H = b) = \frac{p(H = b | C = r) p(C = r)}{p(H = b)} = \frac{(1/2)(1/3)}{(1/2)} = 1/3$$

Similarly,  $p(C = g | H = b) = 2/3$ .

You should switch from your initial choice to the other box in order to increase your chances of winning the prize!

## Example 3: The Monty Hall Problem:

Illustration of the Solution

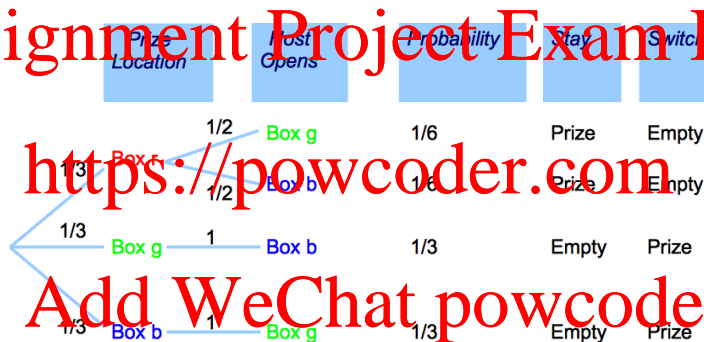


Illustration of the solution when you have initially selected box r.

## Example 3: The Monty Hall Problem:

Another Perspective

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Switching is bad if, and only if, we initially picked the prize box (because if not, the other remaining box must contain the prize)

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We picked the prize box with probability  $1/3$ . This is independent of the host's action

Hence, with probability  $2/3$  switching will reveal the prize box

## Example 3: The Monty Hall Problem:

Variants to Ponder

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Would switching be rational if

- The host only revealed a box when he knew we picked the right one?

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- The host only revealed a box when he knew we picked the wrong one?

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- The host is himself unaware of the prize box, and reveals a box at random, which by chance does not have the prize?

1 Bayes' Rule: Examples

- Eating Hamburgers
- Detecting heartbeats
- The Monty Hall Problem

# Assignment Project Exam Help

2 Moments for functions of two discrete Random Variables

<https://powcoder.com>

3 The meaning of Probability

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4 Wrapping Up

# The Expected Value of a Function of Two Discrete Random Variables

(Assuming you have met **Expectation**  $E[X]$  and **Variance**  $\text{Var}(X)$  before.)

The expected value of a function  $g(X, Y)$  of two discrete random variables is defined as

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p(X=x, Y=y). \quad (1)$$

In particular, the expected value of  $X$  is given by

$$E[X] = \sum_x \sum_y xp(X=x, Y=y). \quad (2)$$

It should be noted that if we have already calculated the marginal distribution of  $X$ , then it is simpler to calculate  $E[X]$  using this.



## Covariance and the Correlation Coefficient

The covariance between  $X$  and  $Y$ ,  $\text{Cov}(X, Y)$  is given by

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) \quad (3)$$

Note that by definition  $\text{Cov}(X, X) = E(X^2) - E(X)^2 = \text{Var}(X)$ .  
The coefficient of correlation between  $X$  and  $Y$  is given by

$$r(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \quad (4)$$

Always in  $[-1, 1]$ .

## Example

Discrete random variables  $X$  and  $Y$  have the following joint distribution:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$X = 1$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Calculate

- 1  $p(X > Y)$
- 2 marginal distributions of  $X$  and  $Y$
- 3 expected values and variances of  $X$  and  $Y$
- 4 coefficient of correlation between  $X$  and  $Y$

Are  $X$  and  $Y$  independent?

## Example

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To calculate the probability of such an event, note that we sum over all the cells which correspond to that event. Hence,

$$\begin{aligned} p(X \geq Y) &= p(X = 0, Y = -1) + p(X = 1, Y = -1) \\ &\quad + p(X = 1, Y = 0) = \frac{1}{3} \end{aligned}$$

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## Example

Recall that

$$p(X = x) = \sum_y p(X = x, Y = y).$$

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Hence,

$$p(X = 0) = \sum_{y=-1}^1 p(X = 0, Y = y) = 0 + \frac{1}{3} + 0 = \frac{1}{3}$$

$$p(X = 1) = \sum_{y=-1}^1 p(X = 1, Y = y) = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$$

Note that after obtaining  $p(X = 0)$ , we could calculate  $p(X = 1)$  by using the fact that

$$p(X = 1) = 1 - p(X = 0), \quad (5)$$

since  $X$  only takes the values 0 and 1.

## Example

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$$p(Y = -1) = \sum_{x=-1}^1 p(X = x, Y = -1) = 0 + \frac{1}{3} = \frac{1}{3}$$

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$$p(Y = 0) = \sum_{x=0}^1 p(X = x, Y = 0) = \frac{1}{3} + 0 = \frac{1}{3}$$

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$$p(Y = 1) = 1 - p(Y = -1) - p(Y = 0) = \frac{1}{3}$$

## Example

We then calculate the expected values and variances of  $X$  and  $Y$  from these marginal distributions.

$$E(X) = \sum_{x=0}^1 x p(X=x) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$$

$$E(Y) = \sum_{y=-1}^1 y p(Y=y) = -1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = 0$$

## Example

To calculate the variances of  $X$  and  $Y$ ,  $\text{Var}(X)$  and  $\text{Var}(Y)$ , we use the formula

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{x=0}^1 x^2 p(X=x) = 0^2 \times \frac{1}{3} + 1^2 \times \frac{2}{3} = \frac{2}{3}$$

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$$E(Y^2) = \sum_{y=-1}^1 y^2 p(Y=y) = (-1)^2 \times \frac{1}{3} + 0^2 \times \frac{1}{3} + 1^2 \times \frac{1}{3} = \frac{2}{3}.$$

Thus we get

$$\text{Var}(X) = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$$

$$\text{Var}(Y) = \frac{2}{3} - (0)^2 = \frac{2}{3}$$

## Example

To calculate the correlation coefficient, we first calculate the covariance between  $X$  and  $Y$ . We have

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

where

$$\begin{aligned} E(XY) &= \sum_{x=-1}^1 \sum_{y=-1}^1 xy p(X=x, Y=y) \\ &= 0(-1)0 + 0(0)\frac{1}{3} + 0(1)0 + 1(-1)\frac{1}{3} + 1(0)0 + 1(1)\frac{1}{3} = 0 \end{aligned}$$

Thus we get

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0 - \frac{2}{3} \times 0 = 0.$$

From the definition of the correlation coefficient,

$$\rho(X, Y) = 0.$$



Example - is  $X$  and  $Y$  independent

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We have that

$$p(X=0, Y=-1) = 0 \neq p(X=0)p(Y=-1) = \left(\frac{1}{3}\right)^2$$

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1 Bayes' Rule: Examples

- Eating Hamburgers

- Selecting Parents

- The Monty Hall Problem

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## The meaning of Probability

Frequentist : Frequencies of random repeatable experiments

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- E.g. Prob. of biased coin landing "Heads"

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**Bayesian** : Degrees of Belief

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## Cox Axioms

Given  $B(x)$ ,  $B(\bar{x})$ ,  $B(x, y)$ ,  $B(x|y)$ ,  $B(y)$ :

- 1 Degrees of belief can be ordered
- 2  $B(x) = f[B(x)]$
- 3  $B(x, y) = g[B(x|y), B(y)]$

# The meaning of Probability

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## Cox Axioms

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If a set of Beliefs satisfy these axioms they can be mapped onto probabilities satisfying the rules of probability.

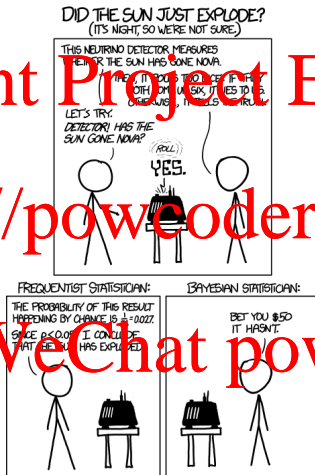


# Frequentists versus Bayesians: Round I

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## Frequentists versus Bayesians: Round II

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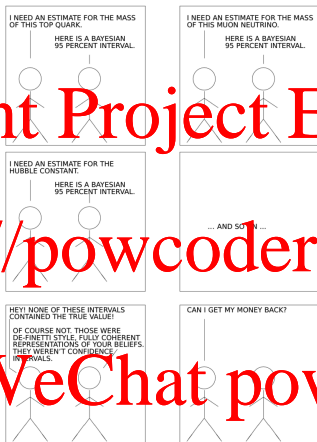


Image from <http://normaldeviate.wordpress.com/2012/11/09/anti-xkcd/>

In practice one needs to make use of both interpretations. Wise to be open to both. This is a huge topic which we can not get into further here. Note that Mackay was firmly in the Bayesian camp. . .

1 Bayes' Rule: Examples

- Eating Hamburgers
- Detecting Spam Emails
- The Monty Hall Problem

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2 Moment Generating Functions and Multivariate Normal Variables

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## Summary

- Examples of application of Bayes' rule
  - Formalization

- ▶ Solution by applying Bayes' theorem

- Intuition is usually helpful although it may sometimes deceive us

- Frequentist v Bayesian probabilities

- Cox axioms

Next time

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- Working through some useful probability distributions

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- More on Bayesian inference

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