COMP2610 / 6261 — Information Theory

Lecture 13: Symbol Codes for Lossless Compression

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Last time

Proof of the source coding theorem

- Foundational theorem, but impractical
- Requires potentially very large block sizes

The theorem also only considers uniform coding schemes

- Could variable length coding help?
- Does entropy turn up for such codes as well?

This time

Variable-length codes

Prefix codes

Kraft's inequality

Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length N*.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
- $\{0,1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$

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Binary Symbol Code

Let X be an ensemble with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.
A function $c : \mathcal{A}_X \mapsto \{0,1\}^+$ is a **code** for X.

- The binary string $c(x)$ is the **codeword** for $x \in \mathcal{A}_X$

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  Shorthand: $\ell_i = \ell(a_i)$ for $i = 1 \ldots, I$.

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  Shorthand: $\ell_i = \ell(a_i)$ for $i = 1 \ldots, I$.
- The **extension** of c assigns codewords to any sequence $x_1 x_2 \ldots x_N$ from $\mathcal{A}^+$ by $c(x_1 \ldots x_N) = c(x_1) \ldots c(x_N)$

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

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Codes: A Review

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- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$

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Codes: A Review

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Unique Decodeability

Recall that a code is lossless if for all $x, y \in \mathcal{A}_X$

$$x \neq y \implies c(x) \neq c(y)$$

This ensures that if we work with a single outcome, we can uniquely decode the outcome

When working with variable-length codes, it will be convenient to also require the following:

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When working with variable-length codes, it will be convenient to also require the following:

Uniquely Decodable

A code c for X is **uniquely decodable** if no two strings from $\mathcal{A}_X^+$ have the same codeword. That is, for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_X^+$

$$\mathbf{x} \neq \mathbf{y} \implies c(\mathbf{x}) \neq c(\mathbf{y})$$

This ensures that if we work with a sequence of outcomes, we can still uniquely decode the individual elements

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- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because
  $c(aaa) = c(d) = 111$ and $c(ab) = c(c) = 110$

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- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable

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- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable
  - We can easily segment a given code string scanning left to right

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- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable
  - We can easily segment a given code string scanning left to right
  - e.g. $0110010 \rightarrow 0, 110, 0, 10$

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Trivial to segment a given code string into individual codewords

- Keep scanning until we match a codeword
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Trivial to segment a given code string into individual codewords

- Keep scanning until we match a codeword
- Once matched, add new segment boundary, and proceed to rest of string

Once our current segment matches a codeword, no ambiguity to resolve

- Why? No codeword is a prefix of any other

Not true for every uniquely decodable code, e.g. $C_4 = \{0, 01, 011\}$

- First bit 0 $\rightarrow$ no certainty what the symbol is

Prefix Codes

a.k.a *prefix-free* or *instantaneous* codes

A simple property of codes **guarantees** unique decodeability

Prefix property

A codeword $\mathbf{c} \in \{0,1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0,1\}^+$ if there exists a string $\mathbf{t} \in \{0,1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

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Can you create $\mathbf{c}'$ by gluing something to the end of $\mathbf{c}$?

- **Example**: 01101 has prefixes 0, 01, 011, 0110.

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Prefix Codes

A code $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$ is a **prefix code** if for every codeword $\mathbf{c}_i \in C$ there is no prefix of $\mathbf{c}_i$ in C.

In a stream, no confusing one codeword with another

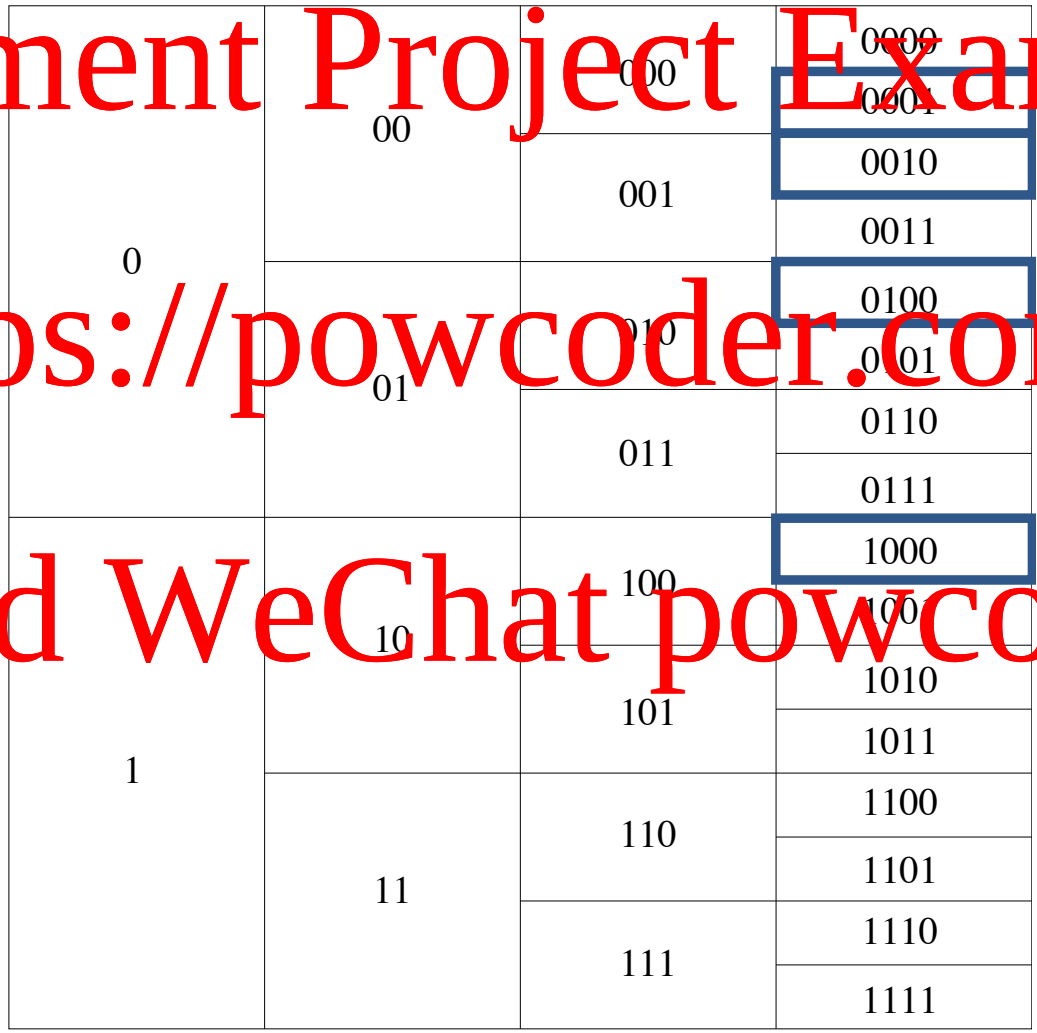
Prefix Codes: Examples

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix-free
- $C_2 = \{0, 10, 110, 111\}$ is prefix-free
- $C_2' = \{1, 10, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 c_2$
- $C_2'' = \{1, 01, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 10$

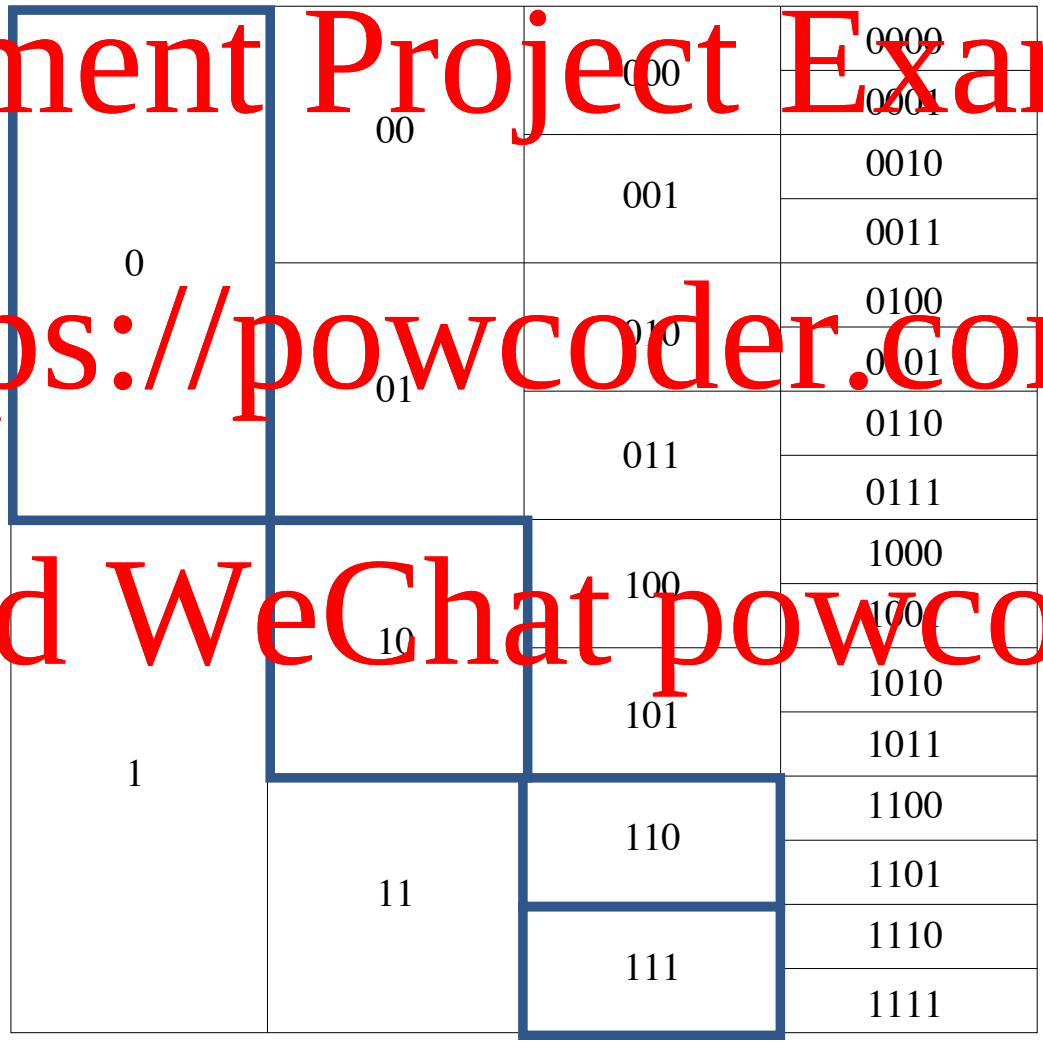
Prefix Codes as Trees

$$C_1 = \{0001, 0010, 0100, 1000\}$$



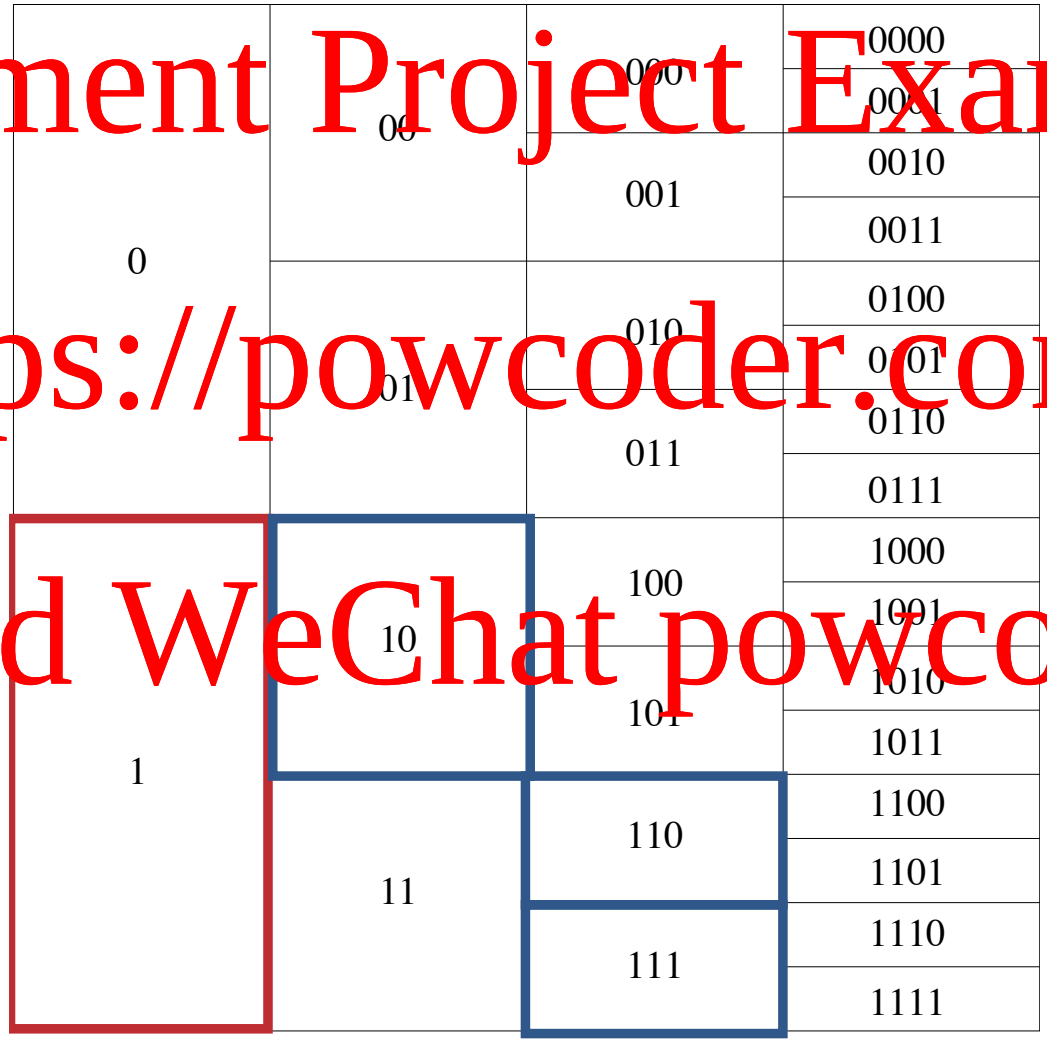
Prefix Codes as Trees

$$C_2 = \{0, 10, 110, 111\}$$



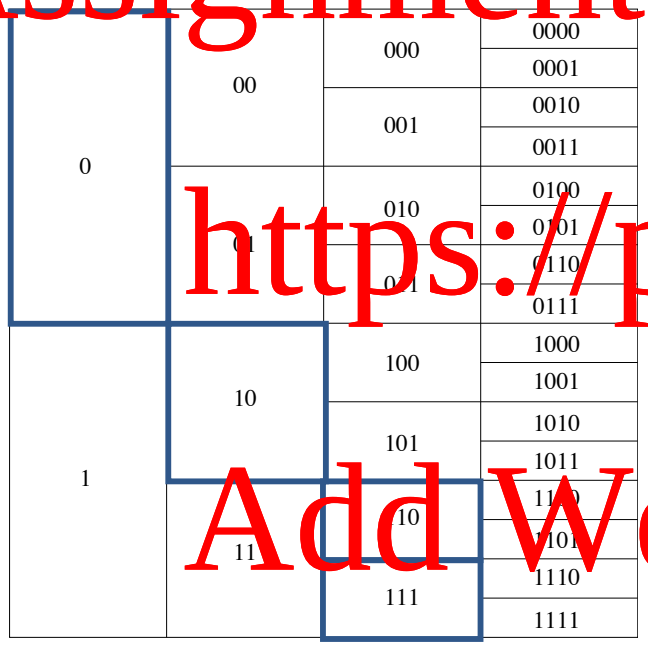
Prefix Codes as Trees

$$C_2' = \{1, 10, 110, 111\}$$



Each codeword choice eliminates its descendants

Prefix Codes are Uniquely Decodeable



- If $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$ then symbol is decodeable after seeing at most ℓ^* bits
- Consider $C_2 = \{0, 10, 110, 111\}$
  - If $c(\mathbf{x}) = 0 \ldots$ then $x_1 = \mathtt{a}$
  - If $c(\mathbf{x}) = 1 \ldots$ then $x_1 \in \{\mathtt{b}, \mathtt{c}, \mathtt{d}\}$
  - If $c(\mathbf{x}) = 10 \ldots$ then $x_1 = \mathtt{b}$
  - If $c(\mathbf{x}) = 11 \ldots$ then $x_1 \in \{\mathtt{c}, \mathtt{d}\}$

Uniquely Decodeable Codes are Not Always Prefix Codes

A uniquely decodeable code is not necessarily a prefix code

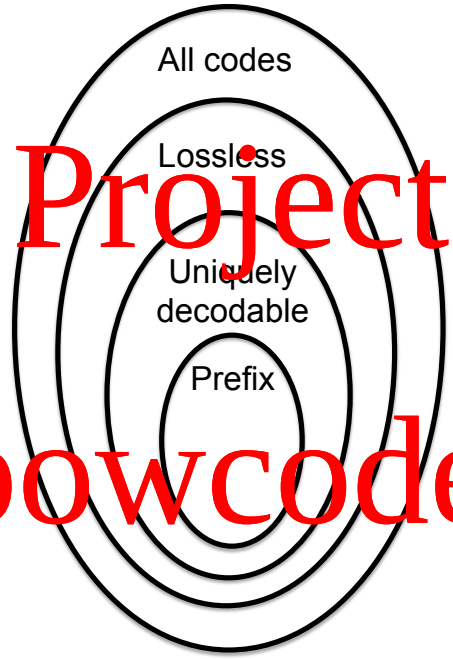
Example: $C_1 = \{0, 01, 011\}$

- $00\ldots \rightarrow$ first codeword
- $0101\ldots \rightarrow$ second codeword
- $011\ldots \rightarrow$ third codeword

Example: $C_2 = \{0, 01, 011, 111\}$

- This is the reverse of the prefix code $C_2' = \{0, 10, 110, 111\}$

Relating various types of codes



Note that e.g.

$$\text{Prefix} \implies \text{Uniquely Decodable}$$

but

$$\text{Not Prefix} \not\implies \text{Not Uniquely Decodable}$$

Why prefix codes?

While prefix codes do not represent **all** uniquely decodable codes, they are convenient to work with

It will be easy to generate prefix codes (Huffman coding, next lecture)

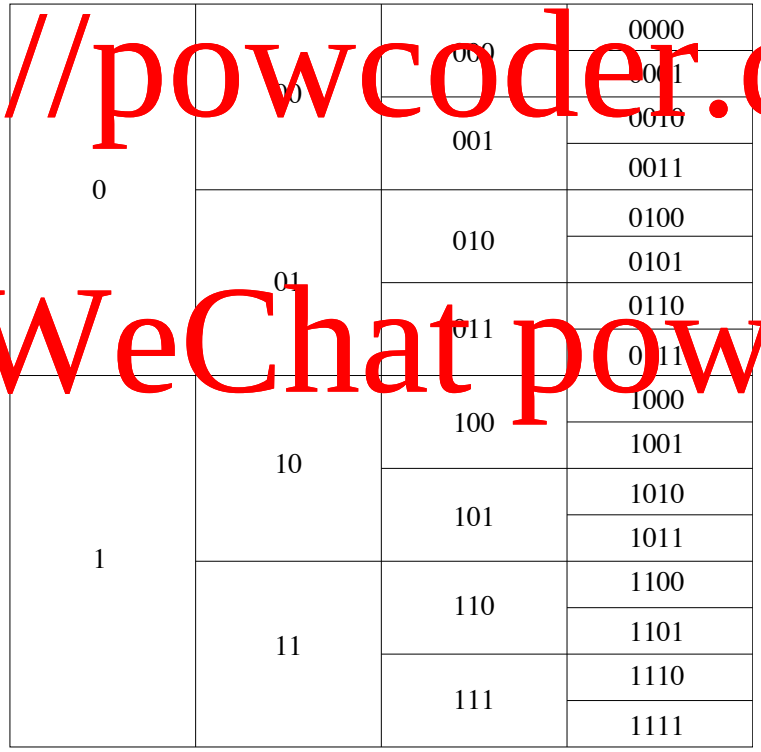
Further, we can quickly establish if a given code is **not** prefix

- Testing for unique decodability is non-trivial in general

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

- $L_1 = \{4, 4, 4, 4\}$
- $L_2 = \{1, 2, 3, 3\}$
- $L_3 = \{2, 2, 3, 4, 4\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



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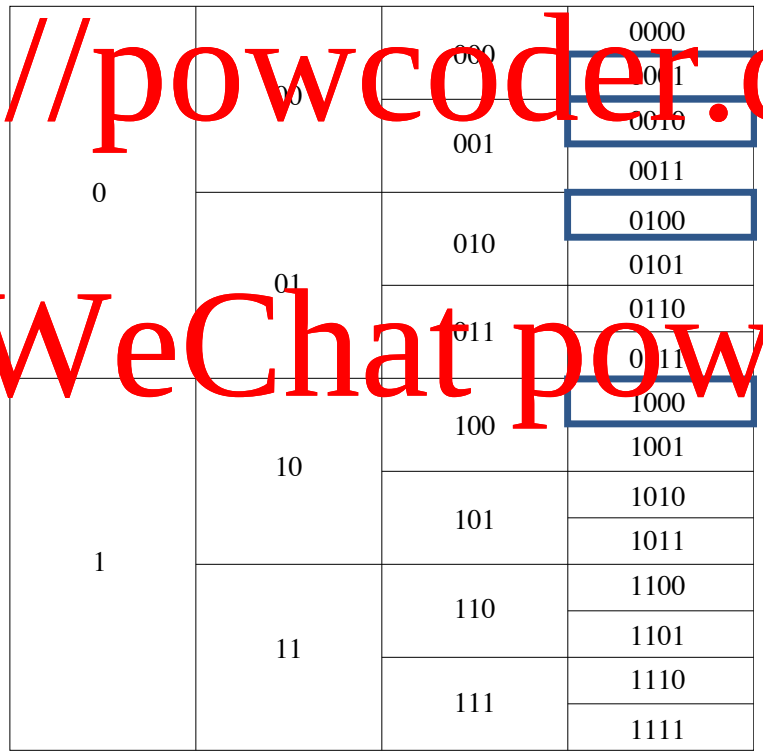

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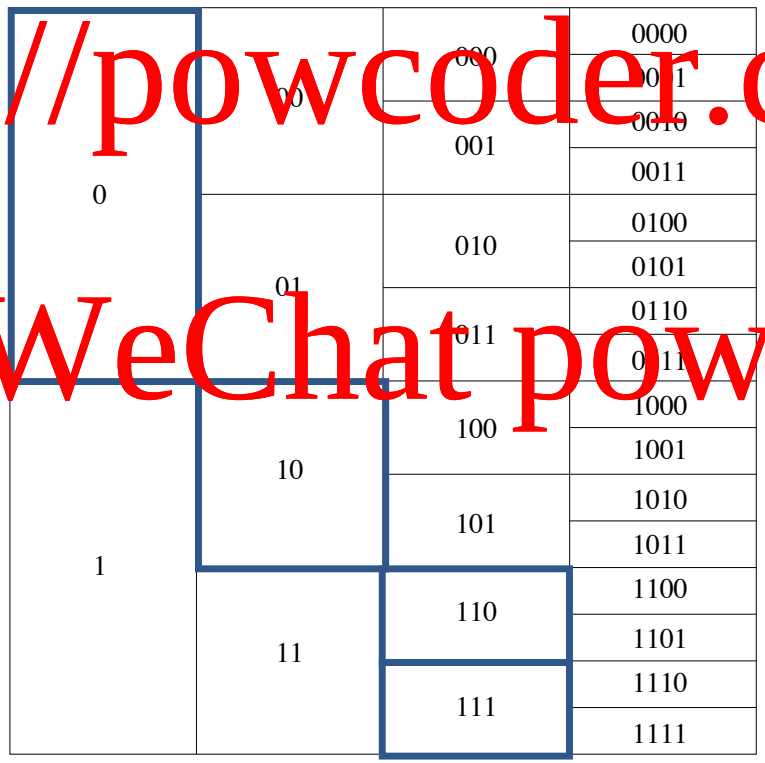

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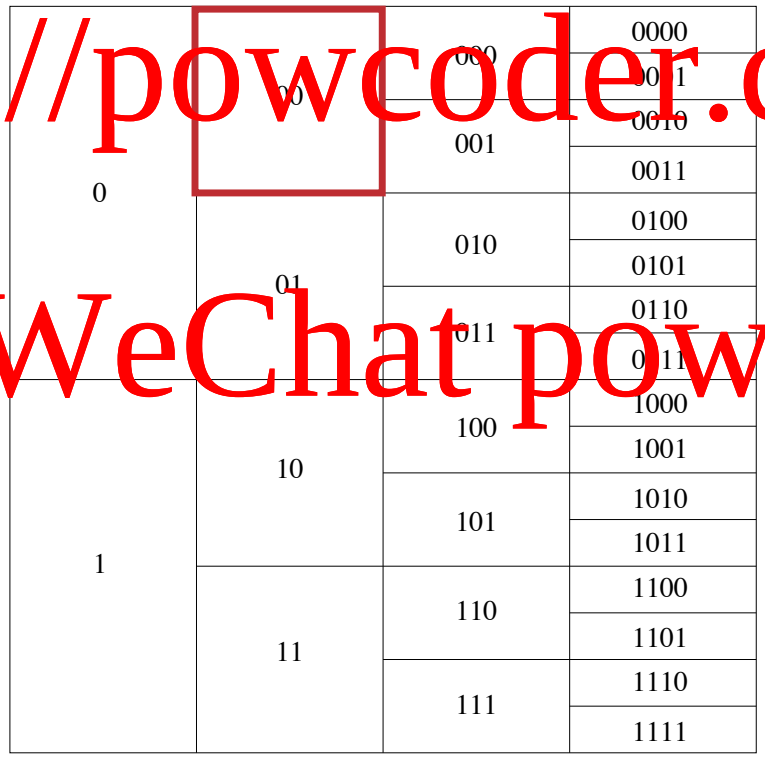

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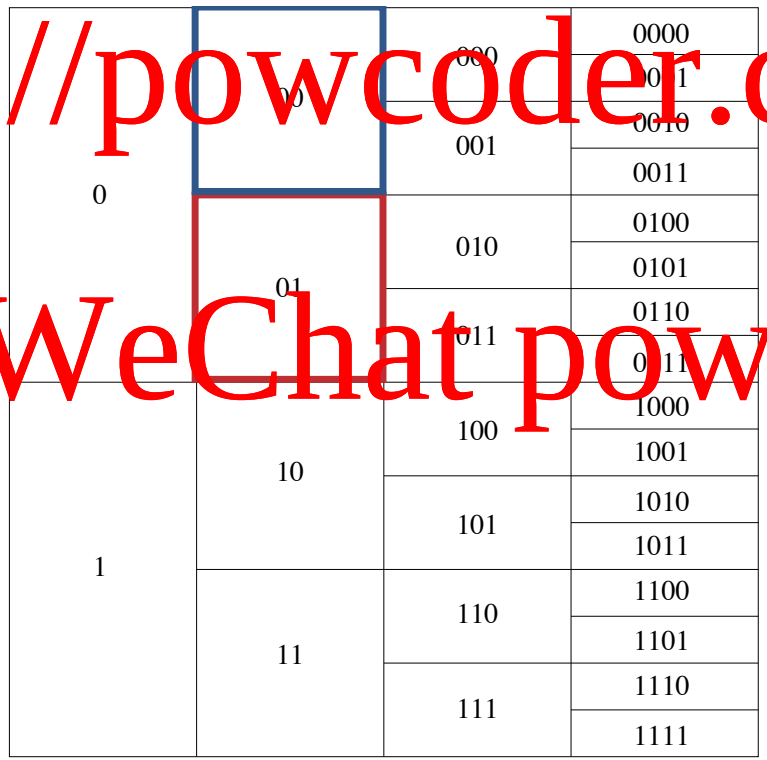

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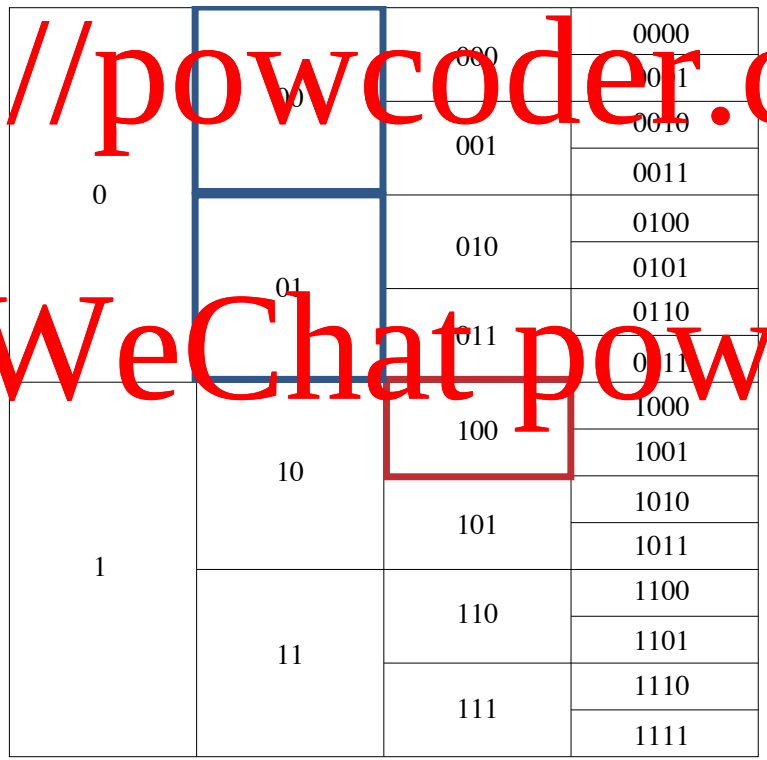

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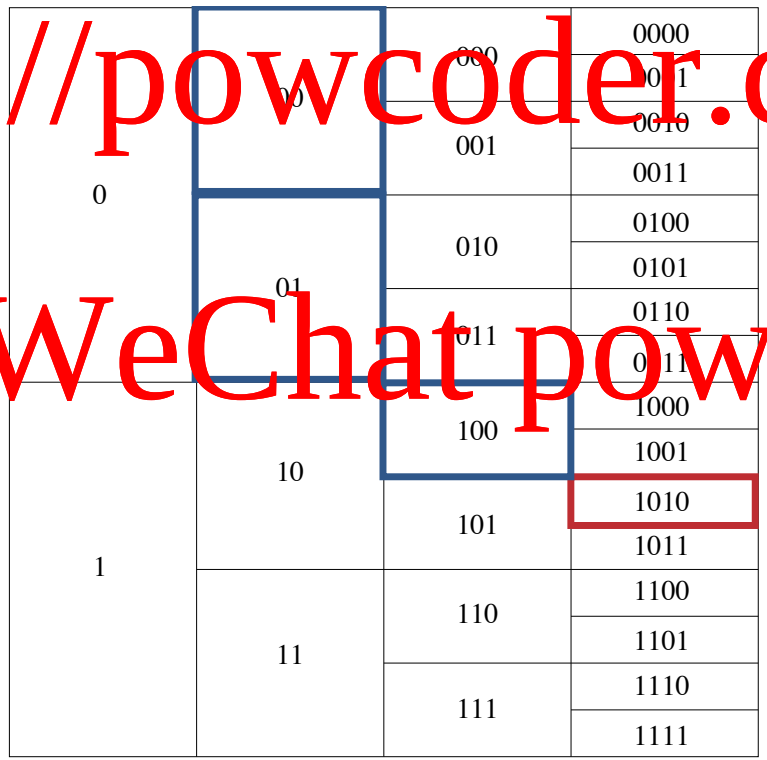
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

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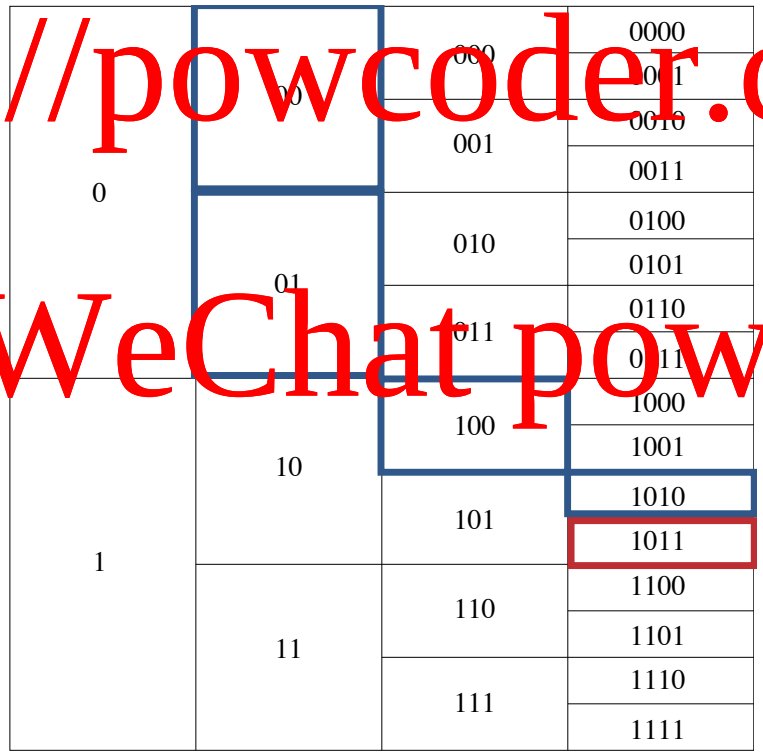

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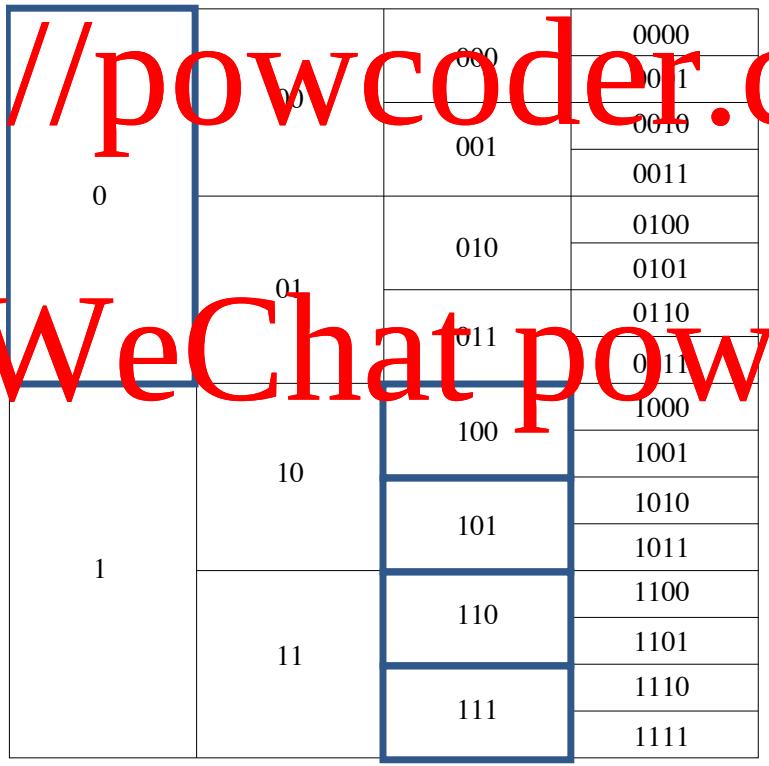

The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

Kraft Inequality

For any prefix (binary) code C, its codeword lengths $\{\ell_1, \ldots, \ell_I\}$ satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1 \tag{1}$$

Conversely, if the set $\{\ell_1, \ldots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

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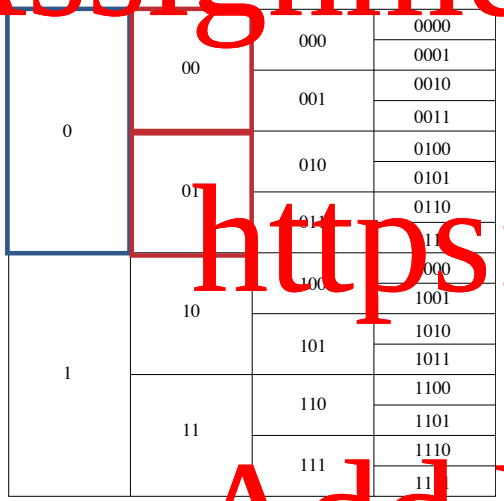
3. Lengths $\{1, 2, 2, 3\}$ give $\sum_{i=1}^{4} 2^{-\ell_i} = \frac{1}{2} + \frac{2}{4} + \frac{1}{8} > 1$ so no prefix code

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:



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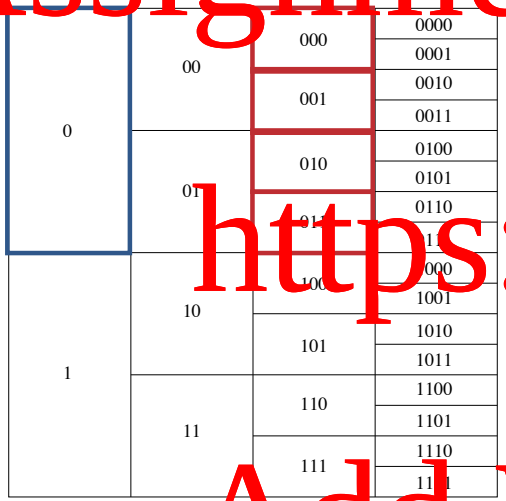

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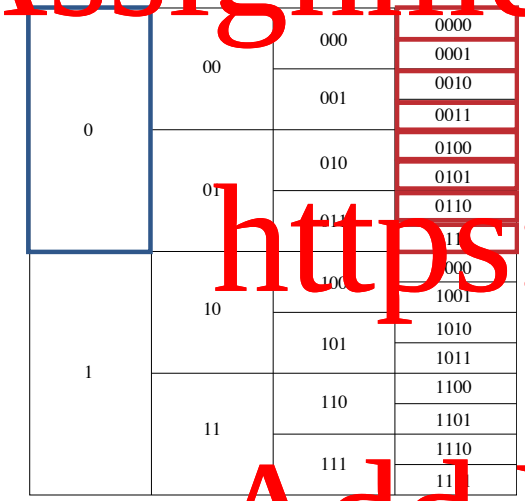

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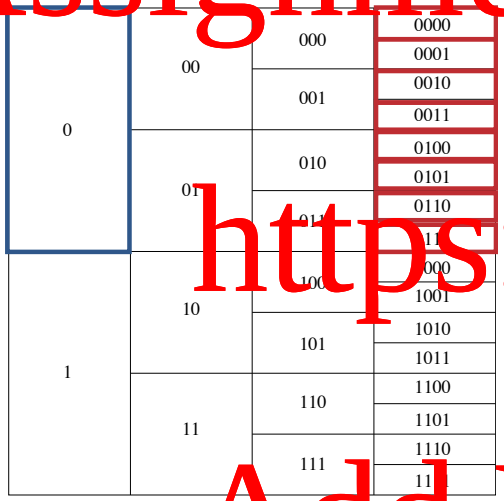

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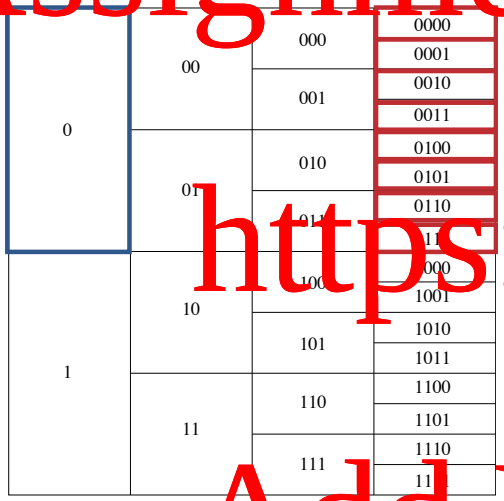

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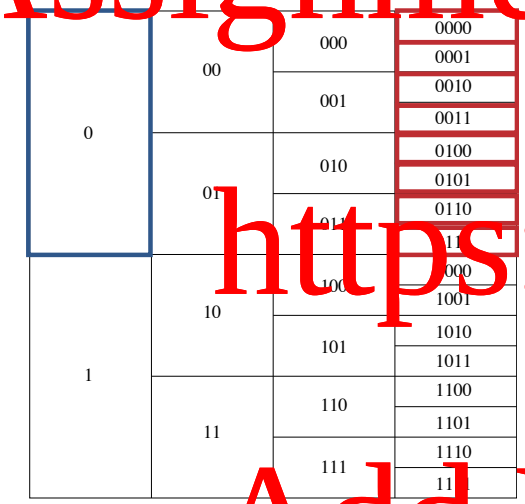

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$$\sum_{i=1}^{I} 2^{\ell^* - \ell_i}$$

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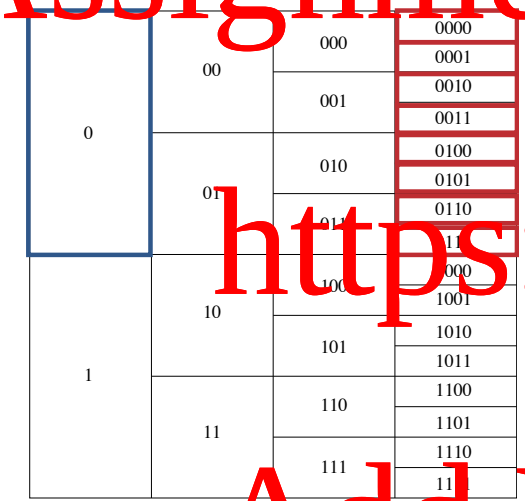

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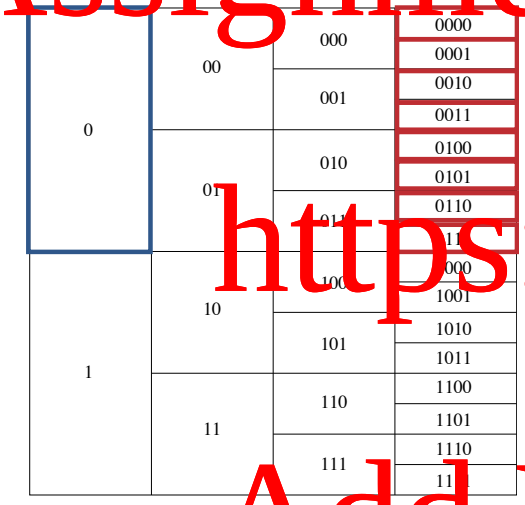

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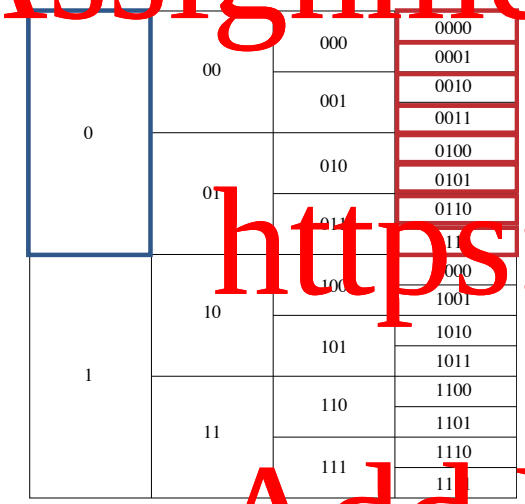

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Kraft inequality: other direction

Suppose we are given lengths satisfying

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

Then, we can construct a code by:

- Picking the first (remaining) node at depth ℓ_1, and using it as the first codeword
- Removing all descendants of the node (to ensure the prefix condition)
- Picking the next (remaining) node at depth ℓ_2, and using it as the second codeword
- Removing all descendants of the node (to ensure the prefix condition)
- $\vdots$

Kraft inequality: comments

Kraft's inequality actually holds more generally for uniquely decodable codes

- Harder to prove

Note that if a **given** code has lengths that satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

it does not mean the **given** code necessarily is prefix

- Just that we can **construct** a prefix code with these lengths

Summary

Key ideas from this lecture:

- **Prefix** and **Uniquely Decodeable** variable-length codes
- Prefix codes are tree-like
- Every Prefix code is Uniquely Decodeable but not *vice versa*
- The **Kraft Inequality**:
  - Code lengths satisfying $\sum_i 2^{-\ell_i} \leq 1$ implies Prefix/U.D. code exists
  - Prefix/U.D. code implies $\sum_i 2^{-\ell_i} \leq 1$

Relevant Reading Material:

- MacKay: §5.1 and §5.2
- Cover & Thomas: §5.1, §5.2, and §5.5

Next time

Bound on expected length for a prefix code

Shannon codes

Huffman coding