

THE AUSTRALIAN NATIONAL UNIVERSITY

Assignment 3

COMP2610/COMP6261

Information Theory, Semester 2 2022

Release Date: Wednesday 28 September 2022

Due Date: Monday 24 October 2022, 9:00 a.m

Cut-off Date: Friday 28 October 2022, 5:00 p.m

No submission allowed after Friday 28 October 2022, 5:00 p.m.

Assignment 3 weighting is 20% of the course mark.

Assignment Project Exam Help

Instructions: <https://powcoder.com>

Marks:

- The mark for each question is indicated next to the question. For questions where you are asked to prove results, if you can not prove a precedent part, you can still attempt subsequent parts of the question assuming the truth of the earlier part.
- **COMP2610 students:** Answer Questions 1, 2-I, 2-II, 3-5, and 2-III-A. You are not expected to answer 2-III-B. You will be marked out of 100.
- **COMP6261 students:** Answer Questions 1, 2-I, 2-II, 3-5, and 2-III-B. You are not expected to answer 2-III-A. You will be marked out of 100..

Submission:

- Submit your assignment together **with a cover page** as a single PDF on Wattle.
- **Clearly mention whether you are a COMP2610 student or COMP6261 student in the cover page.**
- Submission deadlines will be strictly enforced. A late submission attracts a penalty of 5% per working day. If you submit after the cut-off date, you get zero marks (100% penalty), unless you are ill, in which case you will need to present a doctor's certificate, or have undergone severe trauma of some kind.
- All assignments must be done individually. Plagiarism is a university offence and will be dealt with according to university procedures <http://academichonesty.anu.edu.au/UniPolicy.html>.

Question 1: Entropy and Joint Entropy [10 marks total]

****All students are expected to attempt this question.**

An ordinary deck of cards containing 13 clubs, 13 diamonds, 13 hearts, and 13 spades cards is shuffled and dealt out one card at time without replacement. Let X_i be the suit of the i th card.

(a) Determine $H(X_1)$. [4 marks]

(b) Determine $H(X_1, X_2, \dots, X_{52})$. [6 marks]

Assignment Project Exam Help

<https://powcoder.com>

Add WeChat powcoder

Question 2: Source Coding [30 marks total]

Question 2-I [6 marks total]

****All students are expected to attempt this question.**

Consider the code $\{0, 01, 011\}$.

- (a) Is it instantaneous? [2 marks]
- (b) Is it uniquely decodable? [2 marks]
- (c) Is it nonsingular? [2 marks]

Question 2-II [12 marks total]

****All students are expected to attempt this question.**

Construct a binary Huffman code and Shannon code (not Shannon-Fano-Elias code) for the following distribution on 5 symbols $p = (0.3, 0.3, 0.2, 0.1, 0.1)$. What is the average length of these codes?

Question 2-III-A [For COMP2610 Students Only] [12 marks total]

****Only COMP2610 students are expected to attempt this question.**

Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

- (a) Find a binary Huffman code for X . [4 marks]
- (b) Find the expected codeword length for this encoding. [3 marks]
- (c) Find a ternary Huffman code for X . [5 marks]

Question 2-III-B [For COMP6261 Students Only] [12 marks total]

****Only COMP6261 students are expected to attempt this question.**

A random variable X takes on three values, e.g., a , b , and c , with probabilities 0.55, 0.25, and 0.2.

- (a) What are the lengths of the binary Huffman codewords for X ? What are the lengths of the binary Shannon codewords for X ? [4 marks]
- (b) What is the smallest integer D such that the expected Shannon codeword length with a D -ary alphabet equals the expected Huffman codeword length with a D -ary alphabet? [3 marks]
- (c) Here X_1 and X_2 are independent with each other and take on three values, e.g., a , b , and c , with probabilities 0.55, 0.25, and 0.2. We define $Y = \overline{X_1 X_2}$, e.g., $Y = ab$ if $X_1 = a$ and $X_2 = b$. Find the binary Huffman codewords for Y . [5 marks]

Question 3: Channel Capacity [30 marks total]

Question 3-I [20 marks total]

****All students are expected to attempt this question.**

There is a discrete memoryless channel (DMC) with the channel input $X \in \mathcal{X} = \{1, 2, 3, 4\}$. The channel output Y follows the following probabilistic rule.

$$Y = \begin{cases} X & \text{probability } \frac{1}{2} \\ 2X & \text{probability } \frac{1}{2} \end{cases}$$

Answer the following questions.

- (a) Draw the schematic of the channel and clearly show possible channel outputs and the channel transition probabilities. [5 marks]
- (b) Write the mutual information $I(X;Y)$ as a function of the most general input probability distribution. [10 marks]
- (c) Find a way of using only a subset of the channel inputs such that the channel turns into a noiseless channel and the maximum mutual information (you need to quantify its value) can be achieved with zero error. [5 marks]

Question 3-II [10 marks total]

****All students are expected to attempt this question.**

The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$p(y|x) = \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \quad x, y \in \{0, 1\}$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

Question 4: Joint Typical Sequences [30 marks total]

Question 4-I [15 marks total]

****All students are expected to attempt this question.**

Let (x^n, y^n, z^n) be drawn according to the joint distribution $p(x, y, z)$ in an independent and identically distributed (i.i.d.) manner. We say that (x^n, y^n, z^n) is jointly ϵ -typical if all the following conditions are met

- $|\tilde{H}(x^n) - H(X)| \leq \epsilon$
- $|\tilde{H}(y^n) - H(Y)| \leq \epsilon$
- $|\tilde{H}(z^n) - H(Z)| \leq \epsilon$
- $|\tilde{H}(x^n, y^n) - H(X, Y)| \leq \epsilon$
- $|\tilde{H}(x^n, z^n) - H(X, Z)| \leq \epsilon$
- $|\tilde{H}(y^n, z^n) - H(Y, Z)| \leq \epsilon$
- $|\tilde{H}(x^n, y^n, z^n) - H(X, Y, Z)| \leq \epsilon$

where $\tilde{H}(x^n) = -\frac{1}{n} \log_2(p(x^n))$. Now suppose that $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ is drawn i.i.d. according to $p(x)$, $p(y)$, and $p(z)$. Therefore, $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ have the same marginals as $p(x^n, y^n, z^n)$, but are independent. Find upper and lower bounds on the probability that $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ is jointly typical in terms of $H(X, Y, Z)$, $H(X)$, $H(Y)$, $H(Z)$, ϵ , and n .

Question 4-II [15 marks total]

****All students are expected to attempt this question.**

Let $\mathbf{p} = [0.43, 0.32, 0.25]$ be the distribution of a random variable X that takes symbols from $\{a, b, c\}$, respectively.

(a) Find the empirical entropy of the i.i.d. sequence

$$\mathbf{x} = aabaabbcbaccab$$

[5 marks]

(Hints: the empirical entropy $\tilde{H}(x^n) = -\frac{1}{n} \log_2(p(x^n))$.)

(b) Find whether it is a ϵ -typical sequence with $\epsilon = 0.05$

[5 marks]

(c) Now assume the following joint probability distribution between X and Y that take symbols from $\{a, b, c\}$ and $\{d, e, f\}$ respectively.

$$p(x, y) = \begin{bmatrix} 0.2 & 0.08 & 0.15 \\ 0.1 & 0.15 & 0.07 \\ 0.1 & 0.1 & 0.05 \end{bmatrix}$$

where in each row, x is fixed. We observe two i.i.d. sequences

$$\mathbf{x} = aabaabbcbaccab$$

$$\mathbf{y} = dfffdfedddeeffdd$$

Determine whether (\mathbf{x}, \mathbf{y}) are jointly ϵ -typical.

[5 marks]