# COMP2610 / COMP6261 - Information Theory

## Lecture 5: Bernoulli, Binomial, Maximum Likelihood and MAP



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# Last time



- Examples of application of Bayes' rule
  - Formalizing problems in language of probability
  - Eating hamburgers, detecting terrorists, ...
- Frequentist vs Bayesian probabilities

# The Bayesian Inference Framework

## Bayesian Inference

Bayesian inference provides us with a a mathematical framework explaining how to change our (prior) beliefs in the light of new evidence



$$\underbrace{p(Z|X)}_{\text{posterior}} = \frac{\overbrace{p(X|Z)}^{\text{likelihood}}\overbrace{p(Z)}^{\text{prior}}}{\underbrace{p(X)}_{\text{evidence}}}$$



$$= \frac{p(X|Z)p(Z)}{\sum_{Z'} p(X|Z')p(Z')}$$



**Prior**: Belief that someone is sick

**Likelihood**: Probability of testing positive given someone is sick

**Posterior**: Probability of being sick given someone tests positive

## This time



- The Bernoulli and binomial distribution (we will make much use of this henceforth in studying binary channels)



- Estimating probabilities from data



- Bayesian inference for parameter estimation

# Outline

Consider a binary variable $X \in \{0, 1\}$. It could represent many things:



- Whether a coin lands heads or tails
- The presence/absence of a word in a document



- A transmitted bit in a message
- The success of a medical trial



Often, these outcomes (0 or 1) are not equally likely

What is a general way to model such an $X$?

# The Bernoulli Distribution

## Definition



The variable $X$ takes on the outcomes

$$X = \begin{cases} 1 & \text{probability } \theta \\ 0 & \text{probability } 1-\theta \end{cases}$$



Here, $0 \leq \theta \leq 1$ is a parameter representing the probability of success

For higher values of $\theta$, it is more likely to see 1 than 0



- e.g. a biased coin

# The Bernoulli Distribution

## Definition

By definition,



$$p(X = 1|\theta) = \theta$$

$$p(X = 0|\theta) = 1 - \theta$$



More succinctly,

$$p(X = x|\theta) = \theta^x (1 - \theta)^{1-x}$$

# The Bernoulli Distribution

## Definition

By definition,



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More succinctly,



$$p(X = x|\theta) = \theta^x (1 - \theta)^{1-x}$$

This is known as a Bernoulli distribution over binary outcomes:



$$p(X = x|\theta) = \mathrm{Bern}(x|\theta) = \theta^x (1 - \theta)^{1-x}$$

Note the use of the conditioning symbol for $\theta$; will revisit later

# The Bernoulli Distribution

## Mean and Variance

The expected value (or mean) is given by:

$$
\begin{aligned}
\mathbb{E}[X|\theta] &= \sum_{x \in \{0,1\}} x \cdot p(x|\theta) \\
&= 1 \cdot p(X = 1|\theta) + 0 \cdot p(X = 0|\theta) \\
&= \theta.
\end{aligned}
$$

The variance (or squared standard deviation) is given by:

$$
\begin{aligned}
\mathbb{V}[X|\theta] &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[(X - \theta)^2] \\
&= (0 - \theta)^2 \cdot p(X = 0|\theta) + (1 - \theta)^2 \cdot p(X = 1|\theta) \\
&= \theta(1 - \theta).
\end{aligned}
$$

# Example: Binary Symmetric Channel

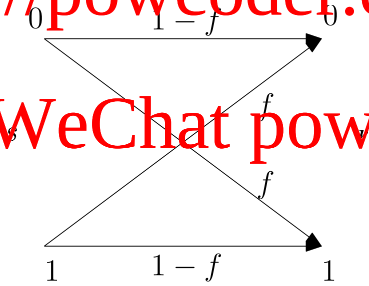
Suppose a sender transmits messages *s* that are sequences of bits

The receiver sees the bit sequence (message) *r*



Due to noise in the channel, the message is flipped with probability $0 \leq f \leq 1$

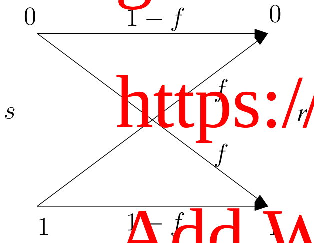




# Example: Binary Symmetric Channel

We can think of $r$ as the outcome of a random variable, with conditional distribution given by:








$$p(r = 0 | s = 0) = 1 - f \qquad p(r = 0 | s = 1) = f$$
$$p(r = 1 | s = 0) = f \qquad p(r = 1 | s = 1) = 1 - f$$



If $E$ denotes whether an error occurred, clearly

$$p(E = e) = \mathrm{Bern}(e | f), \qquad e \in \{0, 1\}.$$

Suppose we perform $N$ independent Bernoulli trials

- e.g. we toss a coin $N$ times
- e.g. we transmit a sequence of $N$ bits across a noisy channel

Each trial has probability $\theta$ of success

What is the distribution of the number of times ($m$) that $X = 1$?

- e.g. the number of times we obtained $m$ heads
- e.g. the number of errors in the transmitted sequence

# The Binomial Distribution

## Definition

Let

$$Y = \sum_{i=1}^{N} X_i$$



where $X_i \sim \text{Bern}(\theta)$.



Then $Y$ has a binomial distribution with parameters $N, \theta$:

$$p(Y = m) = \text{Bin}(m|N, \theta) = \binom{N}{m} \theta^m (1-\theta)^{N-m}$$



for $m \in \{0, 1, \dots, N\}$. Here

$$\binom{N}{m} = \frac{N!}{(N-m)!m!}$$

is the # of ways we can we obtain $m$ heads out of $N$ coin flips

# The Binomial Distribution:

## Mean and Variance

It is easy to show that:



$$\mathbb{E}[Y] = \sum_{m=0}^{N} m \cdot \mathrm{Bin}(m|N,\theta) = N\theta$$



$$\mathbb{V}[Y] = \sum_{m=0}^{N} (m - \mathbb{E}[m])^2 \cdot \mathrm{Bin}(m|N,\theta) = N\theta(1-\theta)$$

- Follows from linearity of mean and variance



$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathbb{E}\left[X_i\right] = N\theta$$

$$\mathbb{V}[Y] = \mathbb{V}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathbb{V}\left[X_i\right] = N\theta(1-\theta)$$

# The Binomial Distribution:

## Example

Ashton is an excellent off spinner. The probability of him getting a wicket during a cricket match is $\frac{1}{4}$. (That is, on each attempt, there is a 1/4 chance he will get a wicket.)



His coach commands him to make 10 attempts of wickets in a particular game.



1. What is the probability that he will get exactly three wickets?
   $\text{Bin}(3|10, 0.25)$



2. What is the expected number of wickets he will get?
   $\mathbb{E}[Y]$, where $Y \sim \text{Bin}(\cdot|10, 0.25)$.
3. What is the probability that he will get at least one wicket?
   $\sum_{m=1}^{10} \text{Bin}(m|N = 10, \theta = 0.25) = 1 - \text{Bin}(m = 0|N = 10, \theta = 0.25)$

# The Binomial Distribution:

## Example: Distribution of the Number of Wickets




Histogram of the binomial distribution with $N = 10$ and $\theta = 0.25$. From Bishop (PRML, 2006)

# The Bernoulli Distribution: Parameter Estimation

Consider the set of observations $\mathcal{D} = \{x_1, \ldots, x_N\}$ with $x_i \in \{0, 1\}$



- The outcomes of a sequence of coin flips



- Whether or not there are errors in a transmitted bit string

Each observation is the outcome of a random variable $X$, with distribution



$$p(X = x) = \text{Bern}(x|\theta) = \theta^x (1 - \theta)^{1-x}$$

for some parameter $\theta$

## The Bernoulli Distribution: Parameter Estimation

We know that



$$X \sim \text{Bern}(x|\theta) = \theta^x (1-\theta)^{1-x}$$

But often, we don't know what the value of $\theta$ is



- The probability of a coin toss resulting in heads

- The probability of the word *defence* appearing in a document about sports



What would be a reasonable estimate for $\theta$ from $\mathcal{D}$?

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood



Say that we observe



$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

Intuitively, which seems more plausible: $\theta = \frac{1}{2}$? $\theta = \frac{1}{5}$?

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood

Say that we observe



$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

**If** it were true that $\theta = \frac{1}{2}$, **then** the probability of this sequence would be



$$\begin{aligned} p(\mathcal{D}|\theta) &= \prod_{i=1}^{10} p(x_i|\theta) \\ &= \prod_{i=1}^{10} \frac{1}{2} \\ &= \frac{1}{2^{10}} \\ &\approx 0.001. \end{aligned}$$

## The Bernoulli Distribution: Parameter Estimation:

### Maximum Likelihood

Say that we observe



$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

**If** it were true that $\theta = \frac{1}{5}$, **then** the probability of this sequence would be



$$\begin{aligned} p(\mathcal{D}|\theta) &= \prod_{i=1}^{10} p(x_i|\theta) \\ &= \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^8 \\ &\approx 0.007. \end{aligned}$$

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood

We can write down how likely $\mathcal{D}$ is under the Bernoulli model. Assuming independent observations:



$$p(\mathcal{D}|\theta) = \prod_{i=1}^{N} p(x_i|\theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$$



We call $L(\theta) = p(\mathcal{D}|\theta)$ the likelihood function

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood



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We call $L(\theta) = p(\mathcal{D}|\theta)$ the likelihood function



Maximum likelihood principle: We want to maximize this function wrt $\theta$

The parameter for which the observed sequence has the highest probability

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood

Maximising $p(\mathcal{D}|\theta)$ is equivalent to maximising $\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta)$



$$\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta) = \sum_{i=1}^{N} \log p(x_i|\theta) = \sum_{i=1}^{N} [x_i \log \theta + (1 - x_i) \log(1 - \theta)]$$

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood



Maximising $p(\mathcal{D}|\theta)$ is equivalent to maximising $\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta)$



$$\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta) = \sum_{i=1}^{N} \log p(x_i|\theta) = \sum_{i=1}^{N} [x_i \log \theta + (1 - x_i) \log(1 - \theta)]$$

Setting $\frac{d\mathcal{L}}{d\theta} = 0$ we obtain:



$$\theta_{\text{ML}} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

# The Bernoulli Distribution: Parameter Estimation:

## Maximum Likelihood



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Setting $\frac{d\mathcal{L}}{d\theta} = 0$ we obtain:



$$\theta_{\text{ML}} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

The proportion of times $x = 1$ in the dataset $\mathcal{D}$!

# The Bernoulli Distribution:

## Parameter Estimation — Issues with Maximum Likelihood

Consider the following scenarios:

- After $N = 3$ coin flips we obtained 3 'tails'
  - What is the estimate of the probability of a coin flip resulting in 'heads'?
- In a small set of documents about sports, the words *defence* never appeared.
  - What are the consequences when predicting whether a document is about sports (using Bayes' rule)?

# The Bernoulli Distribution:

## Parameter Estimation — Issues with Maximum Likelihood

Consider the following scenarios:

- After $N = 3$ coin flips we obtained 3 'tails'
  - What is the estimate of the probability of a coin flip resulting in 'heads'?
- In a small set of documents about sports, the words *defence* never appeared.
  - What are the consequences when predicting whether a document is about sports (using Bayes' rule)?

These issues are usually referred to as overfitting

- Need to "smooth" out our parameter estimates

- Alternatively, we can do Bayesian inference by considering priors over the parameters

# The Bernoulli Distribution:

## Parameter Estimation: Bayesian Inference

Recall:

$$\underbrace{p(\theta|X)}_{\text{posterior}} = \frac{\overbrace{p(X|\theta)}^{\text{likelihood}} \overbrace{p(\theta)}^{\text{prior}}}{\underbrace{p(X)}_{\text{evidence}}}$$

If we treat $\theta$ as a random variable, we may have some prior belief $p(\theta)$ about its value

- e.g. we believe $\theta$ is probably close to 0.5

Our prior on $\theta$ quantifies what we believe $\theta$ is likely to be, before looking at the data

Our posterior on $\theta$ quantifies what we believe $\theta$ is likely to be, after looking at the data

The likelihood of X given $\theta$ is

$$\text{Bern}(x|\theta) = \theta^x (1-\theta)^{1-x}$$



For the prior, it is mathematically convenient to express it as a Beta distribution:

$$\text{Beta}(\theta|a,b) = \frac{1}{Z(a,b)} \theta^{a-1} (1-\theta)^{b-1},$$

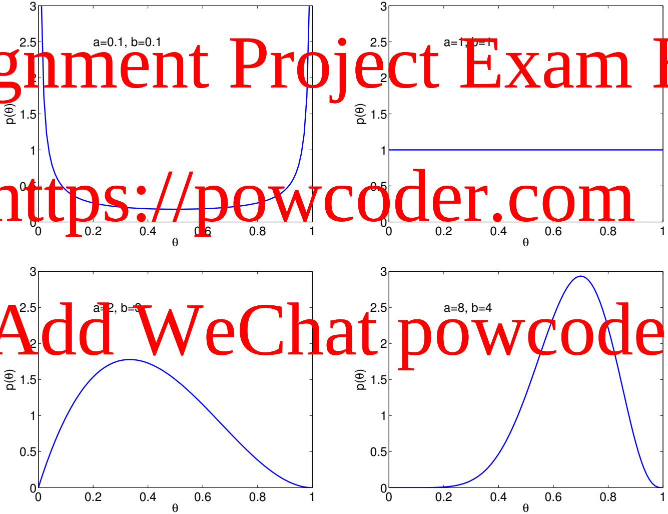


where $Z(a,b)$ is a suitable normaliser

We can tune $a, b$ to reflect our belief in the range of likely values of $\theta$

# Beta Prior

## Examples



Recall that for $\mathcal{D} = \{x_1, \ldots, x_N\}$, the likelihood under a Bernoulli model is:

$$p(\mathcal{D}|\theta) = \theta^m (1-\theta)^\ell,$$

where $m = \sharp(x = 1)$ and $\ell \stackrel{\text{def}}{=} N - m = \sharp(x = 0)$.

# Beta Prior and Binomial Likelihood:

## Beta Posterior Distribution

Recall that for $\mathcal{D} = \{x_1, \ldots, x_N\}$, the likelihood under a Bernoulli model is:



$$p(\mathcal{D}|\theta) = \theta^m (1-\theta)^\ell,$$

where $m = \sharp(x = 1)$ and $\ell \stackrel{\text{def}}{=} N - m = \sharp(x = 0)$.



For the prior $p(\theta|a, b) = \text{Beta}(\theta|a, b)$ we can obtain the posterior:

$$\begin{aligned} p(\theta|\mathcal{D}, a, b) &= \frac{p(\mathcal{D}|\theta)p(\theta|a, b)}{p(\mathcal{D}|a, b)} \\ &= \frac{p(\mathcal{D}|\theta)p(\theta|a, b)}{\int_0^1 p(\mathcal{D}|\theta)p(\theta|a, b)d\theta} \\ &= \text{Beta}(\theta|m + a, \ell + b). \end{aligned}$$

# Beta Prior and Binomial Likelihood:

## Beta Posterior Distribution

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Can use this as our new prior if we see more data!

# Beta Prior and Binomial Likelihood:

## Beta Posterior Distribution

Now suppose we choose $\theta_{\text{MAP}}$ to maximise $p(\theta|\mathcal{D})$
(MAP= Maximum *A Posteriori*)



One can show that

$$\theta_{\text{MAP}} = \frac{m + a - 1}{N + a + b - 2}$$



cf. the estimate that did not use any prior,

$$\theta_{\text{ML}} = \frac{m}{N}$$



The prior parameters *a* and *b* can be seen as adding some "fake" trials!

What values of *a* and *b* ensure $\theta_{\text{MAP}} = \theta_{\text{ML}}$? $a = b = 1$. Make sense?
(Note that the choice of the beta distribution was not accidental here — it is the "conjugate prior" for the binomial distribution. )

# Summary



- Distributions involving binary random variables
  - Bernoulli distribution



  - Binomial distribution
- Bayesian inference: Full posterior on the parameters
  - Beta prior and binomial likelihood $\rightarrow$ Beta posterior



- Reading: Mackay §23.1 and §23.5; Bishop §2.1 and §2.2

## Next time



- Entropy