The University of Melbourne
School of Computing and Information Systems
COMP30026 Models of Computation

Problem Set Solutions, Week 7

P7.1 (i) The transitive closure of $\{(2,3),(5,4),(0,3),(2,1),(1,5)\}$ is

$$\{(2,3),(5,4),(0,3),(2,1),(1,5),(2,5),(2,4),(1,4)\}$$

The symmetric transitive closure is $\{0,1,2,3,4,5\} \times \{0,1,2,3,4,5\}$. Neither of these are reflexive. The first doesn't have any (x,x) pairs, so it's actually *irreflexive*, and the second doesn't have, for example, $(6,6)$. Remember that this is a relation on the set $\mathbb{Z}$, so although it relates $0,1,2,3,4$ and 5 to themselves, it doesn't do this for the rest of the integers, so it's not reflexive.

(ii) The transitive closure and also symmetric transitive closure of

$$\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid |x-y| \leq 2\}$$

is the full relation, $\mathbb{Z} \times \mathbb{Z}$, which is reflexive.

P7.2 These are simpler expressions:

(a) $A \oplus B = A$ is equivalent to $B = \emptyset$.

(b) $A \oplus B = A \setminus B$ is equivalent to $B \subseteq A$.

(c) $A \oplus B = A \cup B$ is equivalent to $A \cap B = \emptyset$.

(d) $A \oplus B = A \cap B$ is equivalent to $A \cup B = \emptyset$.

(e) $A \oplus B = A^c$ is equivalent to $B = X$, assuming a universal set X.

(f) $A \oplus B = \emptyset$ is equivalent to $A = B$.

P7.3 Let $R \subseteq A \times B$ be a relation.

- Define $\chi_R : A \times B \to \{0,1\}$ such that

$$\chi_R(a,b) = \begin{cases} 1, & (a,b) \in R \\ 0, & (a,b) \notin R \end{cases}$$

We can determine whether a and b are related under this representation of R, by checking if $\chi_R(a,b) = 1$. Given any function $f : A \times B \to \{0,1\}$, we can form the relation

$$\{(a,b) \in A \times B \mid f(a,b) = 1\}$$

This recovers the original relation R from χ_R, since

$$\begin{aligned} \{(a,b) \in A \times B \mid \chi_R(a,b) = 1\} &= \{(a,b) \in A \times B \mid (a,b) \in R\} \\ &= R \end{aligned}$$

- Define $\alpha_R : A \to \mathcal{P}(B)$ such that

$$\alpha_R(a) = \{b \in B \mid (a,b) \in R\}$$

We can determine whether a and b are related under this representation of R, by checking if $b \in \alpha_R(a)$. Given any function $f : A \to \mathcal{P}(B)$, we can form the relation

$$\{(a, b) \in A \times B \mid b \in f(a)\}$$

This recovers the original relation R from α_R, since

$$\begin{aligned}\{(a, b) \in A \times B \mid b \in \alpha_R(a)\} &= \{(a, b) \in A \times B \mid b \in \{x \in B \mid (a, x) \in R\}\} \\ &= \{(a, b) \in A \times B \mid (a, b) \in R\} \\ &= R\end{aligned}$$

- Define $\beta_R : B \to \mathcal{P}(A)$ such that

$$\beta_R(b) = \{a \in A \mid (a, b) \in R\}$$

We can determine whether a and b are related under this representation of R, by checking if $a \in \beta_R(b)$. Given any function $f : B \to \mathcal{P}(A)$, we can form the relation

$$\{(a, b) \in A \times B \mid a \in f(b)\}$$

This recovers the original relation R from β_R, since

$$\begin{aligned}\{(a, b) \in A \times B \mid a \in \beta_R(b)\} &= \{(a, b) \in A \times B \mid a \in \{x \in A \mid (x, b) \in R\}\} \\ &= \{(a, b) \in A \times B \mid (a, b) \in R\} \\ &= R\end{aligned}$$

P7.4 Here is the complete table:

Property	Reflexivity	Symmetry	Transitivity
preserved under $\cap$?	yes	yes	yes
preserved under $\cup$?	yes	yes	no
preserved under inverse?	yes	yes	yes
preserved under complement?	no	yes	no

To see how transitivity fails to be preserved under union, consider two relations on $\{a, b, c\}$, namely $R = \{(a, a), (a, b), (b, b)\}$ and $S = \{(c, a)\}$, both transitive. $R \cup S$ is not transitive, because in the union we have (c, a) and (a, b), but not (c, b). And R's complement, $\{(a, c), (b, a), (b, c), (c, a), (c, b), (c, c)\}$ is not transitive either, as it contains, for example, (a, c) and (c, a), but not (a, a).

P7.5 From the first row of the last question's table, it follows that, if R and S are equivalence relations, then so is their intersection. But their union may not be. As an example, take the reflexive, symmetric, transitive closures of R and S from the previous answer, to get these two equivalence relations:

$$R' = \{(a, a), (a, b), (b, a), (b, b), (c, c)\} \quad \text{and} \quad S' = \{(a, a), (a, c), (b, b), (c, a), (c, c)\}.$$

Their union fails to be transitive, as it contains (c, a) and (a, b) but not (c, b).

P7.6 We certainly do not have $A \times A = A$. In fact, no member of A is a member of $A \times A$, and no member of $A \times A$ is a member if A. So $\times$ is not absorptive.

Neither is it commutative. Let $A = \{0\}$ and $B = \{1\}$. Then $A \times B = \{(0,1)\}$ while $B \times A = \{(1,0)\}$, and those singleton sets are different, because the members are.

If we also define $C = \{2\}$ then $A \times (B \times C) = \{(0,(1,2))\}$ while $(A \times B) \times C = \{((0,1),2)\}$. Again, these are different. However, it is not uncommon to identify both of $(0,(1,2))$ and $((0,1),2)$ with the triple $(0,1,2)$ ("flattening" the nested pairings). If we agree to do that then $\times$ is associative, and we can simply write $A \times B \times C$ for the set of triples.

P7.7 If f is injective then B has at least 42 elements. If f is surjective then B has at most 42 elements. (So if f is bijective, B has exactly 42 elements.)

P7.8 The conjecture is false. For a counter-example, take A to be $\{0,1\}$ and $R = \{(0,0)\}$. Then R is symmetric, and also anti-symmetric, but R is not reflexive, as it does not include $(1,1)$.

P7.9 The statement is false, as we have, for example, $\{42\} \times \emptyset = \emptyset \times \{42\} = \emptyset$, but $\emptyset \neq \{42\}$.

P7.10 The conjecture is false. Take A to be $\{a,b,c\}$ and $R = \{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c)\}$ The R is reflexive, but not symmetric, since $(c,a) \in R$ but $(a,c) \notin R$. And R is not anti-symmetric either, as we have $(a,b) \in R$ as well as $(b,a) \in R$.

P7.11 The reflexive closure of R is the smallest reflexive relation K such that $R \subseteq K$. We know that $R \cup \Delta_A$ is reflexive, since for any $a \in A$, $(a,a) \in \Delta_A$, so $(a,a) \in R \cup \Delta_A$. We also know that $R \subseteq R \cup \Delta_A$, so we just need to know that it's the smallest relation with these properties. So suppose K is reflexive and $R \subseteq K$. Then since K is reflexive we must have $\Delta_A \subseteq K$. So both R and Δ_A are subsets of K, so their union is, i.e. $R \cup \Delta_A \subseteq K$ (you can verify this by expanding the definition of union and subset). Hence $R \cup \Delta_A$ is in fact the smallest such reflexive relation containing R.

P7.12 No, for example, take the transitive closure of R from T7.3. This is a transitive binary relation on $\{1,2,3,4,5,6,7,8\}$, and if you take the symmetric reflexive closure of it, you will have $(2,1)$ and $(1,5)$ in the closure but not $(2,5)$, so it is not transitive.

P7.13 Here is the table, with appropriate check-marks:

	$<$	$\leq$	successor	divides	coprime
irreflexive	✓		✓		
reflexive		✓		✓	
asymmetric	✓		✓		
antisymmetric	✓	✓	✓	✓	
symmetric					✓
transitive	✓	✓		✓	
linear	✓	✓			

P7.14 Here are some functions that satisfy the requirements. We show $f_i(x)$ in the table's row x, column i:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
a	a	a	b	b	b	a	c	b
b	b	a	b	d	b	a	b	a
c	c	a	c	d	c	a	d	d
d	d	a	d	d	c	c	d	c

Maybe you skipped this optional exercise; but you may still want to verify, for each of these eight functions, that it really does satisfy its specification.