# COMP30026 Models of Computation

## Binary Relations and Functions





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# This Lecture is Being Recorded

A binary relation is a set of pairs, or 2-tuples.



"Being unifiable", "$\leq$", "$\subseteq$", "divides" are all binary relations.

For small relations we can tabulate:



| Beats | Paper | Scissors | Rock |
|---|---|---|---|
| Paper | 0 | 0 | 1 |
| Scissors | 1 | 0 | 0 |
| Rock | 0 | 1 | 0 |



We can express membership of a relation in many ways:
$(x, y) \in Beats$, $Beats(x, y)$, or $x$ $Beats$ $y$.

# Domain and Range of a Relation

The domain of $R$ is $dom(R) = \{x \mid \exists y\ R(x, y)\}$.



The range of $R$ is $ran(R) = \{y \mid \exists x\ R(x, y)\}$.

We say that $R$ is a relation from $A$ to $B$ if $dom(R) \subseteq A$ and $ran(R) \subseteq B$. Or, $R$ is a relation between $A$ and $B$.



A relation from $A$ to $A$ is a relation on $A$.

"Being unifiable" is a relation on *Term*.



"$<$" is a relation on integers.

"$\subseteq$" is a relation on $\mathcal{P}(A)$.

"Acted in" is a relation between actors and films.

# Identity and Inverse

$A \times B$ is a relation—the full relation from $A$ to $B$.



$\emptyset$ is a relation.

$\Delta_A = \{(x, x) \mid x \in A\}$ is a relation on $A$—the identity relation.



If $R$ is a relation from $A$ to $B$ then $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ is a relation from $B$ to $A$, called the inverse of $R$.

Clearly $(R^{-1})^{-1} = R$.



Since relations are sets, all the set operations, such as $\cap$ and $\cup$, are applicable to relations.

# Properties of Relations

Let $A$ be a non-empty set and let $R$ be a relation on $A$.



$R$ is reflexive iff $R(x, x)$ for all $x$ in $A$.

$R$ is irreflexive iff $R(x, x)$ holds for no $x$ in $A$.



$R$ is symmetric iff $R(x, y) \Rightarrow R(y, x)$ for all $x, y$ in $A$.

$R$ is asymmetric iff $R(x, y) \Rightarrow \neg R(y, x)$ for all $x, y$ in $A$.



$R$ is antisymmetric iff $R(x, y) \wedge R(y, x) \Rightarrow x = y$ for all $x, y$ in $A$.

$R$ is transitive iff $R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ for all $x, y, z$ in $A$.

Note that



1. The full relation is transitive.
2. Transitive relations are closed under intersection, that is, if $R_1$ and $R_2$ are transitive then so is $R_1 \cap R_2$.



Together, these two properties tell us that for any binary relation $R$, there is a unique smallest transitive relation $R^+$ which includes $R$.



We call $R^+$ the transitive closure of $R$.

Similarly we have the (unique) reflexive closure and the (unique) symmetric closure of $R$.

# Closures Quiz



What is the reflexive, transitive closure of $R = \{(n, n+1) \mid n \in \mathbb{N}\}$?

# Closures Quiz



What is the reflexive, transitive closure of $R = \{(n, n+1) \mid n \in \mathbb{N}\}$?



What is the symmetric closure of $<$ on $\mathbb{Z}$?

# Composing Relations



Let $R_1$ and $R_2$ be relations on $A$. The composition $R_1 \circ R_2$ is the relation on $A$ defined by



$$(x, z) \in (R_1 \circ R_2) \text{ iff } \exists y \, (R_1(x, y) \land R_2(y, z))$$

The $n$-fold composition $R^n$ is defined by



$$\begin{aligned} R^1 &= R \\ R^{n+1} &= R^n \circ R \end{aligned}$$

## Composition Quiz



If $R$ is $\{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$, what is $R^2$?

# Composition Quiz



If $R$ is $\{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$, what is $R^2$?

What is $R^3$?

# Composition Quiz



If $R$ is $\{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$, what is $R^2$?

What is $R^3$?



If $R$ is $<$ on $\mathbb{N}$, what is $R^2$?

The transitive closure of $R$ can be defined in terms of union and composition:



$$R^+ = \bigcup_{n \geq 1} R^n$$



$$R^+ = R \cup R^2 \cup R^3 \ldots$$

$(x, y), (y, z) \in R$, and hence $(x, y), (y, z) \in R^+$, but since $R^+$ is transitive, $(x, z) \in R^+$ ($R^2$ gives us all such pairs)



$(x, z) \in R^2, (z, w) \in R$, and hence $(x, z), (z, w) \in R^+$, but since $R^+$ is transitive, $(x, w) \in R^+$ ($R^3$ gives us all such pairs)

The reflexive, transitive closure is

$$R^* = R^+ \cup \Delta_A$$

# Equivalence Relations



A binary relation which is reflexive, symmetric and transitive is an equivalence relation.



The identity relation $\Delta_A$ is the smallest equivalence relation on a set $A$. The full relation $A^2$ is the largest equivalence relation on $A$.

# Quiz: Equivalence Relations?



Which of these binary relations are equivalence relations?



- $\leq$ on $\mathbb{Z}$?
- $\equiv_m$ on $\mathbb{Z}$, where $a \equiv_m b$ iff $a \bmod m = b \bmod m$?
- "are unifiable" on the set of terms (over some alphabet)?
- $\{(a, b) \mid |a - b| \leq 3\}$?
- "are compatriots" on the set of all people?
- "are logically equivalent" on the set of propositional formulas?

# Partial Orders

$R$ is a preorder iff $R$ is transitive and reflexive.



$R$ is a strict partial order iff $R$ is transitive and irreflexive.

$R$ is a partial order iff $R$ is an antisymmetric preorder.



$R$ is linear iff $R(x, y) \vee R(y, x) \vee x = y$ for all $x, y$ in $A$.

A linear partial order is also called a total order.



In a total order, every two elements from $A$ are comparable.

# Quiz: Partial Orders?



Which of these binary relations are partial orders?



- The relation $\leq$ on $\mathbb{N}$?
- The relation $\subseteq$ on $\mathcal{P}(\mathbb{N})$?
- The relation "divides" on $\mathbb{N}$?

# Functions



A function $f$ is a relation with the property that $(x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$. That is, for $x \in dom(f)$, there is exactly one $y \in ran(f)$ such that $(x, y) \in f$. We write this: $f(x) = y$.

# Domains and Co-Domains

We say that the function $f$ is from $X$ to $Y$, or



$$f : X \to Y$$

if $dom(f) = X$ (total function) and $ran(f) \subseteq Y$. We call $Y$ the co-domain of $f$.



Example: The range of the factorial function is $\{1, 2, 6, 24, 120, \ldots\}$, but we normally define it as having co-domain $\mathbb{N}$.



The domain/co-domain specification is integral to the function definition, as we define functions $f : X \to Y$ and $f' : X' \to Y'$ to be equal iff $X = X'$, $Y = Y'$, and for all $x \in X, f(x) = f'(x)$.

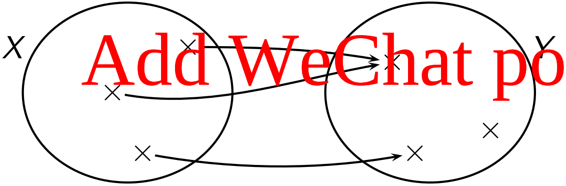
# Injections, Surjections and Bijections

A function $f : X \to Y$ is



- surjective (or onto) iff $ran(f)$ equals the co-domain of $f$.
- injective (or one-to-one) iff $f(x) = f(y) \Rightarrow x = y$.
- bijective iff it is both surjective and injective.





# Examples

$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$ is neither surjective nor injective.



$g : \mathbb{Z} \to \mathbb{N}$ defined by $g(n) = |n|$ is surjective but not injective.

$s : \mathbb{N} \to \mathbb{N}$ defined by $s(n) = n + 1$ is injective but not surjective.



$d : \mathbb{Z} \to \mathbb{N}$ defined by



$$d(n) = \begin{cases} 2n - 1 & \text{if } n > 0 \\ -2n & \text{if } n \leq 0 \end{cases}$$

is bijective. It establishes a one-to-one mapping between $\mathbb{Z}$ and $\mathbb{N}$.

# Function Composition

The composition of $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$ defined by



$$(g \circ f)(x) = g(f(x))$$



We assume that $g$'s domain coincides with $f$'s co-domain, although the composition makes sense as long as $ran(f) \subseteq dom(g)$.

Note the unfortunate inconsistency with the use of $\circ$ for composing relations. For functions, $g \circ f$ is best read as '$g$ after $f$'.



$\circ$ is associative, and for $f : X \to Y$, $f \circ 1_X = 1_Y \circ f = f$, where $1_X : X \to X$ is the identity function on $X$.