**School of Computing and Information Systems**
**COMP30026 Models of Computation Tutorial Week 7**

16–18 September 2020

## The exercises

51. Let $A$, $B$, and $C$ be sets. Show:

    (a) $A \not\subseteq B \Leftrightarrow A \setminus B \neq \emptyset$.

    (b) $A \cap B = A \setminus (A \setminus B)$.

    Hint: Use the formal (logical) definitions of the concepts involved.

52. Recall that the *symmetric difference* of sets $A$ and $B$ is the set $A \oplus B = (A \setminus B) \cup (B \setminus A)$. For each of the following set equations, give an equivalent equation that does not use $\oplus$. However, do not simply replace $\oplus$ by its definition, instead try to find the simplest equivalent equation.



    (a) $A \oplus B = A$

    (b) $A \oplus B = A \setminus B$

    (c) $A \oplus B = A \cup B$

    (d) $A \oplus B = A \cap B$

    (e) $A \oplus B = A^c$



53. Consider this statement: For all sets $S$ and $T$, $S \times T = T \times S$ iff $S = T$.

    If the statement is true, prove it. Otherwise provide a counter-example.

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54. Show that a relation $R$ on $A$ is transitive iff $R \circ R \subseteq R$. Then give an example of a transitive relation $R$ for which $R \circ R = R$ fails to hold.

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55. Relations are sets. To say that $R(x, y) \wedge S(x, y)$ holds is the same as saying that $(x, y)$ is in the relation $R$ and also in the relation $S$, that is, $(x, y) \in R \cap S$.

    Suppose $R$ and $S$ are reflexive relations on a set $A$. Then $\Delta_A \subseteq R$ and $\Delta_A \subseteq S$, so $\Delta_A \subseteq R \cap S$. That is, $R \cap S$ is also reflexive. We say that intersection *preserves* reflexivity. It is easy to see that union also preserves reflexivity. Similarly, if $R$ is reflexive then so is $R^{-1}$, but the complement $A^2 \setminus R$ is clearly not. The following table lists these results. Complete the table, indicating which operations on relations preserve symmetry and transitivity.

| Property | Reflexivity | Symmetry | Transitivity |
|---|---|---|---|
| preserved under $\cap$? | yes | | |
| preserved under $\cup$? | yes | | |
| preserved under inverse? | yes | | |
| preserved under complement? | no | | |

56. Continuing from the previous question, now assume that $R$ and $S$ are equivalence relations. From your table's first two rows, determine whether $R \cap S$ necessarily is an equivalence relation, and whether $R \cup S$ is.

57. Suppose we know about functions $f : A \to B$ and $g : B \to A$ that $f(g(y)) = y$ for all $y \in B$. What, if anything, can be deduced about $f$ and/or $g$ being injective and/or surjective?

58. Suppose $h : X \to X$ satisfies $h \circ h \circ h = 1_X$. Show that $h$ is a bijection. Also give a simple example of a set $X$ and a function $h : X \to X$ such that $h \circ h \circ h = 1_X$, but $h$ is not the identity function (hint: think paper-scissors-rock).

59. (Drill.) The *Cartesian product* of two sets $A$ and $B$ is defined $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. That is, a pair whose first component comes from $A$ and whose second component comes from $B$ is an element of $A \times B$ (and no other pairs are). Recall that $\cap$ and $\cup$ are absorptive, commutative and associative. Does $\times$ have any of those properties?

60. (Drill.) Consider this conjecture: If a binary relation $R$ on some set $A$ is both symmetric and anti-symmetric then $R$ is reflexive. Prove or disprove the conjecture.

61. (Drill.) Suppose $A$ is a set of cardinality 42, that is, $A$ has 42 elements. What, if anything, can we say about $B$'s cardinality if we know that some function $f : A \to B$ is injective? What, if anything, can we say about $B$'s cardinality if we know that some function $f : A \to B$ is surjective?

62. (Optional.) Let $\leq$ be a partial order on a set $X$. We say that a function $h : X \to X$ is:

    - *idempotent* iff $\forall x \in X\ (h(h(x)) = h(x))$
    - *monotone* iff $\forall x, y \in X\ (x \leq y \Rightarrow h(x) \leq h(y))$
    - *increasing* iff $\forall x \in X\ (x \leq h(x))$



    Note that an idempotent function does all of its work "in one go"; repeated application will not change its result. A monotone function is one that respects order: if its input grows, its output must grow too (or stay the same).



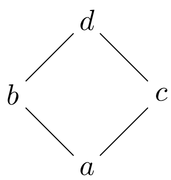
    A function which is idempotent and monotone is a *closure operator*. If it is also increasing, we call it an *upper* closure operator. Closure operators are important and appear in many different contexts. We have met several—let $\mathcal{R}$ be the set of all binary relations. Then the functions *refl*, *symm*, and *trans*, in $\mathcal{R} \to \mathcal{R}$, producing a relation's reflexive, symmetric, and transitive closure respectively, are all upper closure operators. Soon we will meet an "$\epsilon$-closure" function that is part of the algorithm for turning a non-deterministic automaton into an equivalent deterministic automaton—yet another upper closure operator.



    Consider $D = \{a, b, c, d\}$ and the partial order $\leq$ on $D$, defined by

$$x \leq y \text{ iff } x = a \vee x = y \vee y = d$$

    Below is the so-called Hasse diagram for $D$. A Hasse diagram provides a helpful way of depicting a partially ordered set. The nodes are the elements of the set, and the order is given by the edges: $x \leq y$ if and only if there is a path from $x$ to $y$ travelling upwards only, along edges (and the path can have length 0).




    Define eight functions $f_1, \ldots, f_8 : D \to D$, exhibiting all possible combinations of the three properties. That is, find some

    (a) $f_1$ which is idempotent, monotone, and increasing;
    (b) $f_2$ which is idempotent and monotone, but not increasing;
    (c) $f_3$ which is idempotent and increasing, but not monotone;
    (d) $f_4$ which is monotone and increasing, but not idempotent;
    (e) $f_5$ which is idempotent, but neither monotone nor increasing;
    (f) $f_6$ which is monotone, but neither idempotent nor increasing;
    (g) $f_7$ which is increasing, but neither idempotent nor monotone;
    (h) $f_8$ which is neither idempotent, monotone, nor increasing.